Search algorithms for substructures in generalized quadrangles

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To Cimo and our baby

Preface

In this work we develop algorithms to investigate special substructures in combinatorial objects appearing in the areas of finite projective geometry and graph theory. Particularly, we search for maximal partial ovoids, maximal partial spreads and minimal blocking sets in finite classical generalized quadrangles.

We hope to convince the reader that cooperation and combination of theoretical and computer approaches can be fruitful. It will be nice if our work will be motivating and will lead to future research.

In Chapter 1 we describe different objects and related terminology that is essential for the thesis. Hence, we first recall some definitions from elementary graph theory and briefly introduce some basic concepts about backtracking algorithms, heuristic algorithms and greedy algorithms, since these are often mentioned when describing our research. Then we give some necessary definitions from projective geometry, i.e., incidence structures, projective spaces, quadrics and Hermitian varieties, and finally polar spaces.

In Chapter 2 we introduce finite generalized quadrangles. We restrict ourselves to the definitions and properties most relevant to our purposes.

For a computer search, an appropriate representation of a classical generalized quadrangle is needed. We represent it by an incidence matrix. In Chapter 3 we first explain *coordinatization*, a process of labeling the elements of a generalized polygon, in particular of a generalized quadrangle.

Then, a process of indexing points in a classical generalized quadrangle is briefly described, which leads to a construction of an incidence matrix of the generalized quadrangle.

A lot of attention has been paid to the (non-)existence of ovoids and spreads in finite generalized quadrangles. If a generalized quadrangle is known to have no ovoid, the problem of the largest partial ovoid naturally arises. We can also search for the largest partial ovoid different from an ovoid in generalized quadrangles having ovoids. The question in the dual terms deals with partial spreads. Recently, special attention has been paid to the smallest maximal partial ovoids and to the smallest maximal partial spreads of finite generalized quadrangles.

Our results on (partial) ovoids and (partial) spreads were mainly obtained by a computer search. In Chapter 4 we present the techniques used. Section 4.2 describes exhaustive search algorithms, where we use standard clique searching algorithms and add standard pruning strategies. In Sections 4.3, 4.4, 4.5, we describe new techniques based on specific properties of the generalized quadrangles. This approach leads to exact answers concerning e.g. the size of the largest/smallest maximal partial ovoid or spread, or the classification of all maximal partial ovoids and spreads of a given size. These results improve the best known bounds.

One can also ask, whether for a given size maximal partial ovoids and maximal partial spreads exist. In particular, we are interested in the spectrum of sizes for maximal partial ovoids and maximal partial spreads. The class of algorithms, described in Section 4.6, is based on heuristic techniques and turns out to be effective for exploring the spectra.

Chapters 5, 6 and 7 deal with partial ovoids of classical generalized quadrangles. For each quadrangle we give a survey of known results: information about existence of ovoids and spreads, theoretical upper and lower bounds on the size of maximal partial ovoids, as well as earlier computer results. We compare these with the results obtained by our own computer searches. In many cases we could improve earlier results or theoretical bounds. In some cases we were able to extend the results obtained by our computer searches to a general construction.

Next, we turn to the following question. What is the smallest set of points $\mathcal{B}$, such that each line of the generalized quadrangle $\mathcal{S}$ is incident with at least one point of $\mathcal{B}$? In Chapter 8 we introduce minimal blocking sets and we give an overview of necessary definitions. We describe our search algorithms and finally we present some known results on the small minimal blocking sets.

In our work we focused on the small minimal blocking sets of generalized quadrangle $Q(4, q)$ and the results are treated separately in Chapter 9.

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Contents

1 Introduction

This first chapter briefly describes different objects and related terminology.

1.1 Combinatorial algorithms

In this work we develop algorithms to investigate special substructures in combinatorial objects appearing in the areas of finite geometry and graph theory. We call such algorithms *combinatorial algorithms.* This section introduces some basic concepts, which will often be mentioned when describing our research. More about this subject can be found in e.g. [48].

Generation, enumeration and search

Combinatorial algorithms can be classified according to their purpose as follows.

- **Generation.** Construct all the combinatorial structures of a particular type.
- **Enumeration.** Compute the number of different structures of a particular type. Every generation algorithm is also an enumeration algorithm, since each object can be counted as it is generated. However, the converse is not true.

There are many situations in which two objects are different representations of the "same" structure. Those objects are considered *isomorphic*. Enumeration of the number of non-isomorphic structures of a given type often involves algorithms for isomorphism testing.

- **Searching.** Find at least one example of a structure of a particular type if it exists.

  A variation of a search problem is an *optimization problem*, where we want to find the optimal structure of a given type.

Many important search and optimization problems belong to the class of NP-hard problems ([33]), for which no efficient (i.e., polynomial-time) algorithm exist. For NP-hard problems, we often use algorithms based on the idea of backtracking. An alternative approach are heuristic algorithms.

Backtracking algorithms

For problems in which no efficient solution method is known, it might be necessary to test each possibility, in order to determine if it is the solution, so we perform an *exhaustive search.*

The idea of a *backtracking algorithm* is to try each possibility until we get the right one. During the search, if we try an alternative that doesn't work, we backtrack to the choice point. This is the place where different alternatives appeared, and we try the next alternative. When we have exhausted the alternatives, we return to the previous choice point and try the next alternative there. If there are no more choice points, the search fails.

Backtracking search is an exhaustive search and it will thus always find the optimal solution. In order to speed up the search, *pruning methods* can be used to avoid considering possible solutions that are not optimal.

Heuristic algorithms

Backtracking algorithms may be not efficient when searching for one optimal solution.

The term *heuristic algorithm* is used to describe an algorithm (sometimes a randomized algorithm) which tries to find a certain combinatorial structure or solve an optimization problem by use of heuristics, i.e., proceeding by "trial and error". Our heuristic algorithms consist in performing a minor modification, or a sequence of modifications, of a given (partial) solution in order to obtain a different (partial) solution.

These algorithms do not guarantee that an optimal solution will be found, therefore they are considered to be approximate algorithms. However, these algorithms often find a solution close to the best one and they find it fast and easily.

Greedy algorithms

An example of a heuristic algorithm is a *greedy algorithm*. The idea behind such an algorithm is to perform a single procedure over and over again, until it can not be done anymore and then see the produced results. A greedy algorithm makes the best immediate choice at each stage; it never reconsiders this decision, whatever situation may arise later. In general, it does not always find optimal solutions. However, these algorithms are easy to apply and may succeed in finding a solution that is close to the optimal one.

1.2 Graphs and their representation

Since graphs play an important role in this work, we will recall some definitions from elementary graph theory.

Definitions

An *undirected graph* or *graph* is an ordered pair $G = (V, E)$ with a *vertex* set V and a set E of unordered pairs of distinct vertices, called *edges*. The *order* of a graph G is the number of its vertices, the *size* of G is the number of its edges. We write uv for an edge $e = \{u, v\}$. If $uv \in E$, then u and v are *adjacent*. A vertex v is *incident* with an edge e if $v \in e$.

A graph with an empty edge set is known as an *empty graph* or *null graph*. A *complete* graph is a graph in which every pair of vertices forms an edge. A graph is *finite* if its vertex set and edge set are finite. In what follows, every graph mentioned is finite.

The *complement* $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ and the edge set is defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; we write $H \subseteq G$. An *induced subgraph* on a subset $V(H)$ of V is a graph $(V(H), E(H))$ such that $E(H)$ consists of all edges of E between vertices which are contained in $V(H)$.

A *clique* is a set of pairwise adjacent vertices, while an *independent set* or *co-clique* is a set of pairwise non-adjacent vertices. A clique in a graph G is an independent set in its complement $\overline{G}$. A *maximal* clique is a clique that is not contained in a larger clique. A *maximum* clique is a clique of maximum cardinality in the graph. Maximal and maximum independent sets are defined similarly.

A graph G is *regular* of degree $k > 0$, or k-regular, if each vertex is adjacent to exactly k vertices. A graph G of order v is *strongly regular* with parameters (v, k, λ, μ) if it is not complete or edgeless and

(i) each vertex is adjacent to exactly k vertices,

(ii) for each pair of adjacent vertices there are λ vertices adjacent to both,

(iii) for each pair of non-adjacent vertices there are μ vertices adjacent to both.

It can be proved that the complement of a strongly regular graph with parameters (v, k, λ, μ) is also strongly regular with parameters $(v, v-k-1, v-2k-2+\mu, v-2k+\lambda)$.

An *isomorphism* between graphs is a bijection between the vertex sets such that edges are mapped to edges and non-edges are mapped to non-edges. An *automorphism* of a graph is an isomorphism from the graph to itself.

Adjacency matrix

There are several different ways to represent a graph. Although graphs are usually shown using diagrams, this is only possible when the number of vertices and edges is small.

A convenient way to represent graphs is by their *adjacency matrix*. This is a symmetric matrix A, in which $A_{i,j} = 1$ if vertices i and j are adjacent, otherwise $A_{i,j} = 0$. We note that A has 0's on the diagonal.

1.3 Finite fields

A *finite field* is a field with a finite *field order* (number of elements). The finite field of order q is also called a Galois field and is denoted by $\mathrm{GF}(q)$. The order of a finite field is always a prime or a power of a prime. It is well known that the finite field of order q is unique up to isomorphism.

$\mathrm{GF}(p)$, p prime, consists of the residue classes of the integers modulo p under the natural addition and multiplication, and is a finite field of p elements. If $h > 1$, $\mathrm{GF}(p^h)$ can be represented as the field of equivalence classes modulo an irreducible polynomial of degree h whose coefficients belong to $\mathrm{GF}(p)$. Any irreducible polynomial of degree h yields the same field up to isomorphism.

The prime p is called the *characteristic* of the field.

1.4 Incidence structures and their representation

An *incidence structure* or *point-line geometry* $\mathcal{S}$ is a triple $\mathcal{S} = (P, B, \mathrm{I})$, with P and B non-empty disjoint sets and with I a symmetric relation between P and B called the *incidence relation*. The elements of P are called *points* of $\mathcal{S}$ and the elements of B are called the *lines* of $\mathcal{S}$. If for a pair $(x, b) \in (P \times B) \cup (B \times P)$ it holds that $x\mathrm{I}b$, then we say that x is *incident* with b. The *dual* of an incidence structure (P, B, I) is the incidence structure (B, P, I).

An incidence structure can be represented by its *incidence matrix* N.

It has rows indexed by the points, columns indexed by the lines, with $N_{p,L} = 1$ if the point p is incident with the line L, otherwise $N_{p,L} = 0$. Clearly, dualizing corresponds to transposing incidence matrices.

1.5 Projective spaces

To introduce projective spaces one usually starts with models and derives properties in the model itself. We define projective spaces first in an axiomatic way. The only reason is, that we need to introduce generalized quadrangles using axioms. We find it more illustrating, if we do both in the same way.

This section and Section 1.7 are mainly based on *Projective Geometries over Finite Fields* by J.W.P. Hirschfeld [42].

The axioms of a projective space

A *non-degenerate projective space* is an incidence structure $\mathcal{S} = (P, B, \mathrm{I})$, where I satisfies the following axioms:

(i) for any two distinct points x and y, there is exactly one line that is incident with both, denoted by xy;

(ii) if a, b, c, d are four distinct points such that the line ab intersects the line cd, then the line ac also intersects the line bd;

(iii) any line is incident with at least three points;

(iv) there are at least two lines.

The structure of a projective space

In what follows, the notion "projective space" means "non-degenerate projective space".

Since every line of $\mathcal{S}$ is uniquely defined by the points incident with it, a line can be identified with the set of points it contains.

Suppose now that $A \subset P$ is a subset of the point set of $\mathcal{S}$. The set A is called *linear* if every line meeting A in at least two points is completely contained in A. Define B' as the set of lines contained in the linear set A, $|A| \geq 2$, then the incidence structure $\mathcal{S}(A) = (A, B', \mathrm{I}')$ is a projective space or a line with I' the incidence relation I restricted to the set A. This induced space $\mathcal{S}(A)$ is called a *linear subspace* of $\mathcal{S}$.

Since the intersection of an arbitrary number of linear sets is a linear set, we can define, for any subset A of points, $\mathrm{span}(A) = \langle A \rangle = \bigcap\{S \mid A \subseteq S,\ S \text{ is a linear set}\}$, i.e., $\langle A \rangle$ is the smallest linear set containing A. A set A of points is called *linearly independent* if for any subset $A' \subset A$ and any point $p \in A \setminus A'$ we have $p \notin \langle A' \rangle$. A linearly independent set of points that spans the whole space $\mathcal{S}$ is called a *basis* of $\mathcal{S}$. It can be proved, that any two bases of a projective space $\mathcal{S}$ have the same number of elements. If the number of elements in a basis of $\mathcal{S}$ (and hence all bases) is $d + 1$, then d is called the *dimension* of $\mathcal{S}$ and denoted by $\dim(\mathcal{S})$.

Subspaces of dimension 2 are called *planes* and subspaces of dimension $d - 1$ are called *hyperplanes*.

The finite projective space $\mathrm{PG}(n, q)$

From now, we will only consider finite projective spaces. We start by constructing a particular projective space starting from a vector space. It can be shown (for instance in [5]), that for dimension at least three, this example is the only example (however, for dimension 2 there are many other examples of projective planes [45]).

Let $\mathrm{GF}(q)$ denote the finite field of order q, q a prime power, and let $V(n+1, q)$ denote the $(n+1)$-dimensional vector space over $\mathrm{GF}(q)$. Denote by D the set of all subspaces of $V(n + 1, q)$. Define the point set P as the set of 1-dimensional subspaces of $V(n + 1, q)$ and the line set B as the set of 2-dimensional subspaces of $V(n + 1, q)$.

The usual way to add structure to this set is by means of an incidence relation defined by containment of the corresponding subspaces. This incidence structure satisfies the axioms of a projective space and we denote it by $\mathrm{PG}(n, q)$.

A subspace $\alpha \in D$ of dimension $i+1$, $i \geq -1$, of $V(n+1,q)$ is said to have *geometric dimension* i considered as element of $\mathrm{PG}(n,q)$. It is called an *i-dimensional subspace*, or simply an *i-space*, of $\mathrm{PG}(n,q)$. We use familiar geometric terminology for subspaces of low dimension: *points*, *lines*, *planes* have vector space dimension 1,2,3 (and geometric dimension 0,1,2), respectively, and *hyperplanes* are subspaces of co-dimension 1. The (-1)-space is called the *empty space*.

We also use familiar words for incidence (a point lies on a line, a line passes through a point) and related properties (two lines are concurrent, etc.) In order to avoid confusion between the two dimensions, we use the term "rank" for vector space dimension, while unqualified "dimension" will be geometric dimension.

Homogeneous coordinates

It is easily seen from the construction of $\mathrm{PG}(n,q)$, that the points of the geometry have a natural relation to coordinates. Since a point of $\mathrm{PG}(n,q)$ corresponds to a vector line in $V(n+1,q)$, a point p in $\mathrm{PG}(n,q)$ can be represented by a nonzero vector $\mathbf{X}$ in $V(n+1,q)$; this point is denoted by $p(\mathbf{X})$. If we do this, one point has several possible coordinates, but they are all related by being scalar multiples of each other. Such coordinates are called *projective* or *homogeneous coordinates.* It is often convenient to select a standard representative from the set of equivalent coordinates for a point. One common convention is to select the coordinate whose last non-zero entry is 1.

In terms of these point coordinates, the set of points on a hyperplane is described by a linear equation. A hyperplane is a subspace of $V(n+1,q)$ of rank n, and so, its orthogonal complement is a subspace of rank 1. So, if $(u_0, u_1, \ldots, u_n)$ is a fixed non-zero vector, and $(x_0, x_1, \ldots, x_n)$ represents a variable non-zero vector of $V(n+1,q)$, then the solutions of $u_0x_0 + u_1x_1 + \ldots + u_nx_n = 0$ are the vectors in the hyperplane which are the orthogonal complements of the fixed vector. An m-space is a set of points whose representing vectors $\mathbf{X} = (x_0, x_1, \ldots, x_n)$ satisfy an equation $\mathbf{X}A = 0$, where A is an $(n+1) \times (n-m)$ - matrix over $\mathrm{GF}(q)$ of rank $n-m$ with coefficients in $\mathrm{GF}(q)$.

Let us briefly introduce *Grassman coordinates*, which seem to be more efficient than equations for describing the subspaces. An m-dimensional subspace α of $\mathrm{PG}(n,q)$ can be specified by an $(m+1)\times(n+1)$ matrix whose rows are the coordinates of a basis of the subspace α. The set of all $\binom{n+1}{m+1}$ $(m+1)\times(m+1)$-minors of this matrix are then called the Grassman coordinates of α. Specifically, when the subspace is a line, Grassman coordinates are as follows. Let $p(\mathbf{X})$, $p(\mathbf{Y})$ be distinct points determining a line L of $\mathrm{PG}(n,q)$. The representing vectors of two points are $\mathbf{X}=(x_0,x_1,\dots,x_n)$ and $\mathbf{Y}=(y_0,y_1,\dots,y_n)$ and determine $n(n+1)/2$ elements $p_{i,j}=\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$, $0\le i<j\le n$, which are the Grassman coordinates of the line L.

We note that the Grassman coordinates of a line of $\mathrm{PG}(3,q)$ are also called *Plücker coordinates.*

Combinatorics

The following information on number of points, lines, (subspaces, in general) can be useful in many counting arguments.

Theorem 1.5.1 (Hirschfeld [42], Theorem 3.1) *Let* $\mathrm{PG}^{(r)}(n,q)$ *denote the set of all r-spaces in* $\mathrm{PG}(n,q)$. *For* $0\le r\le n$,

$$|\mathrm{PG}^{(r)}(n,q)| = \frac{\prod_{i=n-r+1}^{n+1}(q^i-1)}{\prod_{i=1}^{r+1}(q^i-1)}.$$

Let $\chi(s,r;n,q)$ *denote the number of r-spaces containing a given s-space of* $\mathrm{PG}(n,q)$. *For* $0\le s\le r\le n$,

$$\chi(s,r;n,q) = \frac{\prod_{i=r-s+1}^{n-s}(q^i-1)}{\prod_{i=1}^{n-r}(q^i-1)}.$$

Often, subspaces of $\mathrm{PG}(n,q)$ are identified with the set of points contained in them. Then $|\mathrm{PG}(n,q)|$ will denote $|\mathrm{PG}^{(0)}(n,q)|$. It is clear that $|\mathrm{PG}(n,q)|=(q^{n+1}-1)/(q-1)$.

The principle of duality

For any space $\mathcal{S} = \mathrm{PG}(n, q)$ there is a dual space denoted by $\mathcal{S}^D$, whose *points* and *hyperplanes* are respectively the hyperplanes and points of $\mathcal{S}$, and incidence is reverse containment. For any theorem true in $\mathcal{S}$, there is an equivalent theorem true in $\mathcal{S}^D$. In particular, if T is a theorem in $\mathcal{S}$ stated in terms of points, hyperplanes and incidence, the same theorem is true in $\mathcal{S}^D$ and gives a dual theorem T^D by substituting "hyperplane" for "point" and "point" for "hyperplane". Hence the dual of an r-space in $\mathrm{PG}(n, q)$ is an $(n - r - 1)$-space.

Collineations and polarities

Let $\mathcal{S}$ and $\mathcal{S}'$ be two projective spaces $\mathrm{PG}(n, q)$, $n \geq 2$. A *collineation* $\varphi : S \to S'$ is a bijection which preserves incidence; that is, if $\alpha \subset \beta$, then $\alpha^\varphi \subset \beta^\varphi$. A collineation from a projective space to itself is called an *automorphism.*

Consider now $\mathrm{PG}(n, q)$ with underlying vector space $V(n+1, q)$. Every semi-linear mapping of $V(n + 1, q)$, which is the composition of a linear mapping and a field automorphism, induces a collineation of $\mathrm{PG}(n, q)$. Also the converse is true and is called the *Fundamental theorem of projective geometry.* Hence, we can make use of a coordinate description of $V(n+1, q)$, if we describe collineations by semi-linear mappings of $V(n + 1, q)$.

Consider a collineation $\varphi : \mathcal{S} \to \mathcal{S}'$, then every point $p(\mathbf{X})$ is mapped onto a point $p(\mathbf{X}')$, and the relation between the two coordinate vectors of these points can be expressed by a field automorphism θ and a non-singular $(n + 1) \times (n + 1)$ matrix A as: $t\mathbf{X}' = \mathbf{X}^\theta A$, where $\mathbf{X}^\theta = (x_0^\theta, \ldots, x_n^\theta)$ and $t \in \mathrm{GF}(q) \setminus \{0\}$.

Now suppose that $\mathcal{S}^D$ is the dual space of $\mathcal{S}$. Consider a collineation $\varphi : \mathcal{S} \to \mathcal{S}^D$. If φ is involutory, that is $\varphi^2 = I$, where I is identity, then φ is called a *polarity* of $\mathcal{S}$. With this definition, a polarity is a containment reversing bijection of the projective space; that is, if $\alpha \subset \beta$, then $\beta^\varphi \subset \alpha^\varphi$.

In a polarity, a point p is mapped onto hyperplane p^φ, also called the *polar* of the point p. Conversely, a hyperplane π is mapped onto the point π^φ, also called the *pole* of the hyperplane π. If r lies in p^φ, the polar of p, then p lies in r^φ the polar of r. In this case, p and r are *conjugate* points, and p^φ and r^φ are *conjugate* hyperplanes. A point p is *self-conjugate* or *absolute* if it is contained in its own polar; a hyperplane π is *self-conjugate* if it contains its own pole. Similarly, a subspace π is *self-conjugate* if $\pi \subseteq \pi^\varphi$ or $\pi^\varphi \subseteq \pi$.

Since a polarity φ is a collineation from $\mathrm{PG}(n,q)$ to its dual, it is determined by a non-singular $(n+1) \times (n+1)$ matrix A and a involutory field automorphism θ. It can be derived that a point $p(\mathbf{X})$ is self-conjugate if and only if $\mathbf{X}A(\mathbf{X}^\theta)^T = 0$. A self-conjugate subspace is also called *isotropic*. The *projective index* of the polarity is then the dimension of a maximal isotropic subspace.

Now the different types of polarities are examined (see also [42]). Note that $\mathrm{GF}(q)$ has a nontrivial involutory automorphism if and only if q is a square. If q is a square, then $\mathrm{GF}(q)$ has a unique non-trivial involutory automorphism $\theta : \mathrm{GF}(q) \to \mathrm{GF}(q) : x \to x^\theta = x^{\sqrt{q}}$.

1. θ is the identity and q is odd
   - If $A = A^T$ then φ is called an *ordinary polarity* or *orthogonal polarity*. The set of self-conjugate points forms a quadric.
   - If $A = -A^T$, then φ is called a *null polarity* or a *symplectic polarity*. All points of $\mathrm{PG}(n,q)$ are self-conjugate. It only occurs for n odd.
2. θ is the identity and q is even
   - If $A = A^T$, and not all diagonal elements of A equal zero, then φ is called a *pseudo-polarity*. The set of self-conjugate points forms a hyperplane.
   - If $A = A^T$, and all diagonal elements of A equal zero, then φ is called a *null polarity* or *symplectic polarity*. All points of $\mathrm{PG}(n,q)$ are self-conjugate. It only occurs for n odd.
3. θ is non-trivial

- then A can be chosen with respect to the given basis such that $(A^T)^\theta = A$. Then φ is called a *Hermitian polarity* or a *unitary polarity*. The set of self-conjugate points forms a Hermitian variety.

1.6 Blocking sets in projective spaces

In this section, only a few results are mentioned. We give definitions and a survey on theorems which will be crucial for us. A more extensive overview can be found in [36].

A *blocking set with respect to t-spaces* in $\mathrm{PG}(n, q)$ is a set of points having a non-empty intersection with every t-space.

Theorem 1.6.1 (Bose and Burton [11]) *If $\mathcal{B}$ is a blocking set with respect to t-spaces in $PG(n, q)$, then $|\mathcal{B}| \geq |\mathrm{PG}(n-t, q)|$. Equality holds if and only if $\mathcal{B}$ is an $(n-t)$-space.*

A blocking set with respect to t-spaces that contains an $(n-t)$-space is called *trivial*. A blocking set that has no proper subset which is also a blocking set is called *minimal*. In $\mathrm{PG}(n, q)$, a blocking set with respect to hyperplanes is simply called a *blocking set*.

In our study, blocking sets in $\mathrm{PG}(2, q)$ and in $\mathrm{PG}(3, q)$ will play an important role.

Blocking sets of $\mathrm{PG}(2, q)$

A *blocking set* $\mathcal{B}$ in $\mathrm{PG}(2, q)$ is a set of points in $\mathrm{PG}(2, q)$ such that each line is incident with at least one point of $\mathcal{B}$. It is a consequence of Theorem 1.6.1 that a blocking set contains at least $q+1$ points and that a blocking set of size $q+1$ is necessarily a line. A blocking set containing a line is called *trivial*, otherwise it is called *non-trivial*.

Clearly, for the size of a non-trivial blocking set $\mathcal{B}$ of $\mathrm{PG}(2, q)$, the inequality $q + 2 \leq |\mathcal{B}| \leq q^2 + q + 1$ necessarily holds. The following theorems give lower bounds for the size of a non-trivial minimal blocking set in $\mathrm{PG}(2,q)$.

Theorem 1.6.2 (Bruen [14]) *Let $\mathcal{B}$ be a minimal non-trivial blocking set in $\mathrm{PG}(2,q)$. Then $\mathcal{B} \geq q+\sqrt{q}+1$, with equality if and only if $\mathcal{B}$ is a Baer subplane.*

Since this lower bound can only be reached for q a square, some improvements for q not a square are presented in the following theorem. Let p be a prime. Let $c_p = 2^{-1/3}$ when $p \in \{2,3\}$ and $c_p = 1$ when $p \geq 5$.

Theorem 1.6.3 (Blokhuis [6],[7], Blokhuis et al. [10]) *Let $\mathcal{B}$ be a non-trivial blocking set of $\mathrm{PG}(2,q)$, $q > 2$.*

1. *If q is a prime, then $\mathcal{B} \geq 3(q+1)/2$.*
2. *If $q = p^{2e+1}$, p prime, $e \geq 1$, then $|\mathcal{B}| \geq \max(q+1+p^{e+1}, q+1+c_p q^{2/3})$.*

In both cases, examples attaining the bounds exist (see e.g. [36]).

Blocking sets of $\mathrm{PG}(3,q)$

Let $s(q)$ denote the cardinality of the second smallest non-trivial minimal blocking sets in $\mathrm{PG}(2,q)$.

Theorem 1.6.4 (Storme and Weiner [63]) *Let K be a blocking set of $\mathrm{PG}(3,q^2)$, $q=p^h$, $p>3$ prime, $h \geq 1$, of cardinality smaller than or equal to $s(q^2)$. Then K contains a line or a planar blocking set of $\mathrm{PG}(3,q^2)$.*

Theorem 1.6.5 (Storme and Weiner [63]) *A minimal blocking set of $\mathrm{PG}(3,q^3)$, $q=p^h$, $p \geq 7$ prime, $h \geq 1$, of size at most q^3+q^2+q+1 is one of the following:*

- *a line,*
- *a Baer-subplane if q is a square,*

- *a minimal planar blocking set of size* q^3+q^2+1,
- *a minimal planar blocking set of size* q^3+q^2+q+1,
- *a subgeometry* $\mathrm{PG}(3,q)$.

1.7 Quadrics and Hermitian varieties

Varieties

A homogeneous polynomial F in $\mathrm{GF}(q)[X_0,X_1,\ldots,X_n]$ is also called a *form*. Let $F_1,F_2,\ldots,F_r$ be forms in $\mathrm{GF}(q)[X_0,X_1,\ldots,X_n]$. Define $\mathcal{V}=\{p(X)\in \mathrm{PG}^{(0)}(n,k)\mid F_1(X)=\cdots=F_r(X)=0\}$.

Let $\mathcal{I}$ be the ideal of $\mathrm{GF}(q)[X_0,X_1,\ldots,X_n]$ generated by $F_1,F_2,\ldots,F_r$. A *variety* is a pair $\mathcal{F}=(\mathcal{V},\mathcal{I})$; also write $\mathcal{F}=\mathbf{v}(F_1,\ldots,F_r)$. The variety $\mathbf{v}(F)$ is called a *primal* or *hypersurface*. The *degree* of a primal $\mathbf{v}(F)$ is the degree of F.

Canonical forms for varieties

A *quadric (variety)* $\mathcal{Q}$ in $\mathrm{PG}(n,q)$ is a primal of degree two. So $\mathcal{Q}=\mathbf{v}(Q)$, where Q is a quadratic form; that is,

$$\begin{aligned} Q &= \sum_{\substack{i,j=0\\ i\le j}}^{n} a_{ij}X_iX_j \\ &= a_{00}X_0^2+a_{01}X_0X_1+\ldots. \end{aligned}$$

A *Hermitian variety* $\mathcal{U}$ in $\mathrm{PG}(n,q^2)$ is a variety $\mathbf{v}(H)$, where H is a Hermitian form; that is,

$$\begin{aligned} H &= \sum_{i,j=0}^{n} t_{ij}X_i^qX_j, \quad \text{where} \quad t_{ji}=t_{ij}^q \\ &= t_{00}X_0^qX_0+t_{01}X_0^qX_1+t_{01}^qX_0X_1^q\ldots. \end{aligned}$$

Proj. space	Variety	Canonical form
PG$(2n,q)$	$Q(2n,q)$	$x_0^2 + x_1x_2 + \ldots + x_{2n-1}x_{2n}$
PG$(2n+1,q)$	$Q^+(2n+1,q)$	$x_0x_1 + x_2x_3 + \ldots + x_{2n-2}x_{2n-1}$
PG$(2n+1,q)$	$Q^-(2n+1,q)$	$f(x_0,x_1) + x_2x_3 + \ldots + x_{2n-2}x_{2n-1}$ where f is an irreducible quadratic form
PG(n,q^2)	$H(n,q^2)$	$x_0^{q+1} + \ldots + x_n^{q+1}$

Table 1.1: Canonical forms of nonsingular quadrics and Hermitian varieties.

In each case, the form and variety are *singular* if there exists a change of coordinate system which reduces the form to one in fewer variables; otherwise, the form and the variety are *non-singular*.

We mention now the following results concerning the classification of non-singular quadrics and Hermitian varieties. The canonical forms for non-singular quadrics and Hermitian varieties are given in Table 1.1. In PG(n,q^2), there is, up to collineation, only one non-singular Hermitian variety, denoted by $H(n,q^2)$. In PG$(2n,q)$, there is, up to collineation, only one non-singular quadric, called the *parabolic quadric*, denoted by $Q(2n,q)$. In PG$(2n+1,q)$, there are, up to collineation, exactly two non-singular quadrics, the *hyperbolic quadric*, denoted by $Q^+(2n+1,q)$ and the *elliptic quadric*, denoted by $Q^-(2n+1,q)$.

If $n=2$, a non-singular quadric is also called a *conic* and a non-singular Hermitian variety is called a *Hermitian curve*.

When q is even, every non-singular parabolic quadric $Q(2n,q)$ has a *nucleus*, a point on which every line has exactly one point in common with $Q(2n,q)$.

1.8 Polar spaces

A *polar space* of *rank* r, $r \geq 2$, is a point set P together with a family of subsets of P called *subspaces*, satisfying:

(i) A subspace, together with the subspaces it contains, is isomorphic to a $\mathrm{PG}(d,q)$ with $-1 \leq d \leq r-1$; d is called the *dimension* of the subspace.

(ii) The intersection of any two subspaces is a subspace.

(iii) Given a subspace V of dimension $r-1$ and a point $p \in P \setminus V$, there is a unique subspace W such that $p \in W$ and $V \cap W$ has dimension $r-2$; W contains all points of V that are joined to p by a line (a *line* is a subspace of dimension 1).

(iv) There exist two disjoint subspaces of dimension $r-1$.

There is an alternative approach to polar spaces. F. Buekenhout and E.E. Shult [15] reformulated these axioms in terms of points and lines.

In a point-line geometry $\mathcal{S}$, a *subspace* X is a nonempty set of pairwise collinear points such that any line meeting X in more than one point is contained in X. Two distinct points are said to be *collinear* if there is a line incident with the two points.

Theorem 1.8.1 (Buekenhout and Shult [15]) *Suppose that a point-line geometry has the following properties:*

(i) if p is a point not on a line L, then p is collinear with one or all points of L;

(ii) any line contains at least three points;

(iii) no point is collinear with all others;

(iv) any chain of subspaces is finite.

Then the subspaces constitute a polar space.

The *finite classical polar spaces* are the following structures.

1. $W_{2n+1}(q)$, the polar space arising from a non-singular symplectic polarity of $\mathrm{PG}(2n+1,q)$, $n \geq 1$. It consists of the isotropic subspaces of $\mathrm{PG}(2n+1,q)$ with respect to the symplectic polarity (i.e. the point set is the point set of $\mathrm{PG}(2n+1,q)$).

Polar space	Number of points	Number of generators	Pr. index
$W_{2n+1}(q)$	$\frac{q^{2n+2}-1}{q-1}$	$(q+1)(q^2+1)\dots(q^{n+1}+1)$	n
$Q(2n,q)$	$\frac{q^{2n}-1}{q-1}$	$(q+1)(q^2+1)\dots(q^n+1)$	$n-1$
$Q^+(2n+1,q)$	$\frac{(q^n+1)(q^{n+1}-1)}{q-1}$	$2(q+1)(q^2+1)\dots(q^n+1)$	n
$Q^-(2n+1,q)$	$\frac{(q^n-1)(q^{n+1}+1)}{q-1}$	$(q^2+1)(q^3+1)\dots(q^{n+1}+1)$	$n-1$
$H(2n,q^2)$	$\frac{(q^{2n}-1)(q^{2n+1}+1)}{q^2-1}$	$(q^3+1)(q^5+1)\dots(q^{2n+1}+1)$	$n-1$
$H(2n+1,q^2)$	$\frac{(q^{2n+2}-1)(q^{2n+1}+1)}{q^2-1}$	$(q+1)(q^3+1)\dots(q^{2n+1}+1)$	n

Table 1.2: Number of points, number of generators and projective index of finite classical polar spaces.

2. $Q^-(2n+1,q)$, the polar space arising from a non-singular elliptic quadric of $\mathrm{PG}(2n+1,q)$, $n \geq 2$. It consists of the subspaces of $\mathrm{PG}(2n+1,q)$ completely contained in $Q^-(2n+1,q)$.

3. $Q(2n,q)$, the polar space arising from a non-singular parabolic quadric of $\mathrm{PG}(2n,q)$, $n \geq 2$. It consists of the subspaces of $\mathrm{PG}(2n,q)$ completely contained in $Q(2n,q)$.

4. $Q^+(2n+1,q)$, the polar space arising from a non-singular hyperbolic quadric of $\mathrm{PG}(2n+1,q)$, $n \geq 1$. It consists of the subspaces of $\mathrm{PG}(2n+1,q)$ completely contained in $Q^+(2n+1,q)$.

5. $H(n,q^2)$, the polar space arising from a non-singular Hermitian variety in $\mathrm{PG}(n,q^2)$, $n \geq 3$. It consists of the subspaces of $\mathrm{PG}(n,q^2)$ completely contained in $H(n,q^2)$.

A maximal subspace of a finite classical polar space $\mathcal{P}$ is called a *generator*. All generators have the same dimension, the dimension is called the *projective index*. In Table 1.2, we list the finite classical polar spaces with their projective index, number of points and number of generators.

Next, we specialize to the case $n = 2$. (A polar space of rank 1 is just an unstructured collection of points.) A polar space of rank 2 is a point-line geometry satisfying

(i) if p is a point not on a line L, then p is collinear with a unique point of L;

(ii) any line contains at least three points;

(iii) no point is collinear with all others;

(iv) two points lie on at most one line.

Such a geometry is called *generalized quadrangle*.

It was proved by F.D. Veldkamp [73], [74] and J. Tits [71], that all finite polar spaces of rank at least three are classical. However, this is not true for generalized quadrangles, since many non-classical examples are known. For more information, we refer to [55].

2 Generalized quadrangles

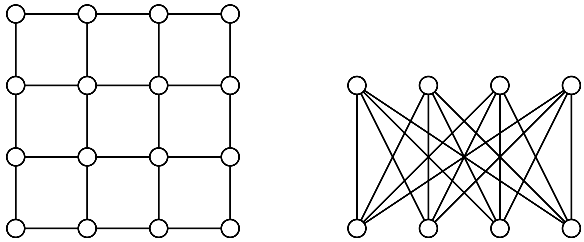
Generalized quadrangles were first introduced by J. Tits in 1959 [69] as a subclass of a larger class of incidence structures called generalized polygons. We have already seen another approach to generalized quadrangles, namely as polar spaces of rank 2. So, generalized quadrangles can be studied as a class of polar spaces or as a class of generalized polygons.

The main results on finite generalized quadrangles (up to 1984) are contained in the monograph *Finite generalized quadrangles* by S.E. Payne and J.A. Thas [55]. We follow their approach to introduce finite generalized quadrangles. This research field is too large even to give an overview of the most important properties. Hence, we will restrict ourselves to the definitions and properties most relevant to our purposes.

2.1 Finite generalized quadrangles

A *finite generalized quadrangle* (GQ) of order (s, t) is an incidence structure $\mathcal{S} = (P, B, \mathrm{I})$, for which I is a symmetric point-line incidence relation satisfying the following axioms:

(i) each point is incident with $1 + t$ lines $(t \geq 1)$ and two distinct points are incident with at most one line;



(a) A grid with parameters $(3,1)$

(b) A dual grid with parameters $(1,3)$

Figure 2.1: Grid and dual grid

(ii) each line is incident with $1+s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;

(iii) if x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in P \times B$ for which $x \mathrm{~I~} M \mathrm{~I~} y \mathrm{~I~} L$.

If $s = t$, $\mathcal{S}$ is said to have *order* s. A generalized quadrangle of order $(s,1)$ is called a *grid* and a generalized quadrangle of order $(1,t)$ is called a *dual grid*. In Figure 2.1, examples of a grid and a dual grid are given.

A generalized quadrangle with $s > 1$ and $t > 1$ is called *thick*. The smallest thick generalized quadrangle is shown in Figure 2.2.

There is a point-line duality for generalized quadrangle $\mathcal{S}$ of order (s,t) for which in any definition or theorem the words "point" and "line" are interchanged and its dual is a generalized quadrangle of order (t,s), denoted by $\mathcal{S}^D$. Normally, we assume without further notice that the dual of a given theorem has also been given.

Given two (not necessarily distinct) points x and y of $\mathcal{S}$, we write $x \sim y$ and say that x and y are *collinear* if there is some line L of $\mathcal{S}$ incident with both. If this is not the case, we write $x \nsim y$. Dually, for $L, M \in B$, we

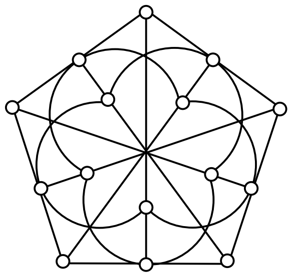


Figure 2.2: The smallest thick generalized quadrangle

write $L \sim M$ or $L \nsim M$ when L and M are *concurrent* or *non-concurrent*, respectively.

Since generalized quadrangles are point-line geometries, isomorphisms, automorphisms and polarities are defined in the usual way as follows. An *isomorphism* between generalized quadrangles is a bijection between the point sets together with a bijection between the line sets, such that incidence is preserved. An *automorphism* of a generalized quadrangle is an isomorphism from the generalized quadrangle to itself. A generalized quadrangle is called *self-dual* if it is isomorphic to its dual. A *polarity* of a generalized quadrangle is a duality φ such that $\varphi^2 = 1$. A point or a line is *absolute* with respect to the polarity φ if it is incident with its image under φ.

More definitions

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a generalized quadrangle of order (s,t). For $x \in P$ denote $x^\perp = \{y \in P \mid y \sim x\}$ and note that $x \in x^\perp$.

The *trace* of a pair (x,y) of distinct points is defined as $x^\perp \cap y^\perp$ and is denoted by $\mathrm{tr}(x,y)$ or $\{x,y\}^\perp$. We get $|\{x,y\}^\perp| = s+1$ or $t+1$ according to $x \sim y$ or $x \nsim y$. More generally, if A is an arbitrary subset of P, we define $A^\perp$ as $A^\perp = \bigcap\{x^\perp \mid x \in A\}$.

For $x \neq y$ the *span* of the pair (x,y) is defined as $\{x,y\}^{\perp\perp} = \{u \in P \mid u \in z^\perp,\ \forall z \in x^\perp \cap y^\perp\}$. If $x \nsim y$, then $\{x,y\}^{\perp\perp}$ is also called the *hyperbolic line* defined by x and y.

A *triad* (of points) is a triple of pairwise non-collinear points. Given a triad $T = (x, y, z)$, a *center* of T is just a point of $T^\perp$. We say T is *acentric, centric* or *unicentric* according to $|T^\perp|$ is zero, non-zero, or equal to 1.

Restrictions on the parameters

We continue with properties, which describe important restrictions on the parameters of a generalized quadrangle. For the proofs we refer to [55].

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a generalized quadrangle of order (s, t), and set $v = |P|,\ b = |B|$.

Theorem 2.1.1 ([55], Theorems 1.2.1. - 1.2.3., Theorem 1.2.5.)

(i) $v = (s+1)(st+1)$ *and* $b = (t+1)(st+1)$.

(ii) $s + t$ *divides* $st(s+1)(t+1)$.

(iii) If $s > 1$ *and* $t > 1$*, then* $t \leq s^2$*, and dually* $s \leq t^2$.

(iv) If $s \neq 1,\ t \neq 1,\ s \neq t^2$*, and* $t \neq s^2$*, then* $t \leq s^2 - s$ *and dually* $s \leq t^2 - t$.

Regularity and antiregularity

We continue with the same notation as in 2.1.

The pair of points (x, y) is called *regular* if $x \sim y,\ x \neq y$, or if $x \not\sim y$ and $|\{x, y\}^{\perp\perp}| = t + 1$. A point x is *regular* provided (x, y) is regular for all $y \in P, y \neq x$. A point x is *coregular* provided each line incident with x is regular. A pair $(x, y),\ x \not\sim y$, is *antiregular* provided $|z^\perp \cap \{x, y\}^\perp| \leq 2$ for all $z \in P - \{x, y\}$. A point x is *antiregular* provided (x, y) is antiregular for all $y \in P - x^\perp$.

The notions regularity and antiregularity play an important role in many characterization theorems. More about these properties can be found in [55]. We restrict ourselves to the properties which will be needed in this thesis.

Theorem 2.1.2 ([55], Theorem 1.3.6.) *Let x and y be fixed, non-collinear points of the generalized quadrangle $\mathcal{S} = (P, B, \mathrm{I})$ of order (s, t).*

(i) If $1 < s < t$, then (x, y) is neither regular nor antiregular.

(ii) The pair (x, y) is regular (with $s = 1$ or $s \geq t$) if and only if each triad (x, y, z) has exactly 0,1 or $t + 1$ centers. When $s = t$, (x, y) is regular if and only if each triad is centric.

(iii) If $s = t$, the pair (x, y) is antiregular if and only if each triad (x, y, z) has 0 or 2 centers.

(iv) If $s = t$ and each point in $x^{\perp} \setminus \{x\}$ is regular, then every point is regular.

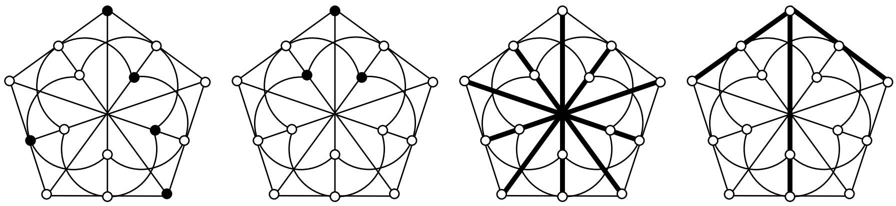
2.2 Ovoids and spreads

An *ovoid* of $\mathcal{S}$ is a set $\mathcal{O}$ of points of $\mathcal{S}$ such that each line of $\mathcal{S}$ is incident with a unique point of $\mathcal{O}$. A *partial ovoid* of $\mathcal{S}$ is a set $\mathcal{O}$ of points of $\mathcal{S}$ such that each line of $\mathcal{S}$ is incident with at most one point of $\mathcal{O}$. A *spread* of $\mathcal{S}$ is a set $\mathcal{R}$ of lines of $\mathcal{S}$ such that each point of $\mathcal{S}$ is incident with a unique line of R. A *partial spread* of $\mathcal{S}$ is a set $\mathcal{R}$ of lines of $\mathcal{S}$ such that each point of $\mathcal{S}$ is incident with at most one line of R. A (partial) ovoid in $\mathcal{S}$ is a (partial) spread in $\mathcal{S}^D$. A partial ovoid (or spread) is called *maximal* or *complete* if it is not contained in a larger partial ovoid (or spread). Two (partial) ovoids (or spreads) are called *equivalent* if there is an automorphism of $\mathcal{S}$ which transforms one into the other.

In Figure 2.3 an example of an ovoid, partial ovoid, spread and partial spread in the generalized quadrangle of Figure 2.2 are shown.

In what follows, some general theorems on ovoids and spreads are given. For the proofs we refer again to [55]. Later, in Section 2.3, we deal with these objects more specifically on examples of classical generalized quadrangles.

Theorem 2.2.1 ([55], Theorem 1.8.1.) *If $\mathcal{O}$ (resp. $\mathcal{R}$) is an ovoid (resp. spread) of the GQ $\mathcal{S}$ of order (s, t), then $|\mathcal{O}| = 1 + st$ (resp. $|\mathcal{R}| = 1 + st$).*



(a) An ovoid (b) A partial ovoid (c) A spread (d) A partial spread

Figure 2.3: Substructures of the generalized quadrangle of order 2

Theorem 2.2.2 ([55], Theorem 1.8.2.) *If the GQ $\mathcal{S} = (P, B, \mathrm{I})$ of order s admits a polarity, then $2s$ is a square. Moreover, the set of all absolute points (resp. lines) of a polarity θ of $\mathcal{S}$ is an ovoid (resp. a spread) of $\mathcal{S}$.*

Theorem 2.2.3 ([55], Theorem 1.8.3.) *A GQ $\mathcal{S} = (P, B, \mathrm{I})$ of order (s, t), with $s > 1$ and $t > s^2 - s$, has no ovoid.*

Theorem 2.2.4 ([55], Theorem 1.8.4.) *Let $\mathcal{S} = (P, B, \mathrm{I})$ be a GQ of order s, having a regular pair (x, y) of non-collinear points. If $\mathcal{O}$ is an ovoid of $\mathcal{S}$, then $|\mathcal{O} \cap \{x, y\}^{\perp\perp}|$, $|O \cap \{x, y\}^{\perp}| \in \{0, 2\}$, and $|O \cap (\{x, y\}^{\perp} \cup \{x, y\}^{\perp\perp})| = 2$. If the GQ $\mathcal{S}$ of order s, $s \neq 1$, contains an ovoid $\mathcal{O}$ and a regular point z not on $\mathcal{O}$, then s is even.*

The following is a related result for the case $s \neq t$.

Theorem 2.2.5 ([55], Theorem 1.8.6.) *Let $\mathcal{S}$ be a GQ of order (s, t), $1 \neq s \neq t$, and suppose that there is a hyperbolic line $\{x, y\}^{\perp\perp}$ of cardinality $s^2/t + 1$. Then any ovoid $\mathcal{O}$ of $\mathcal{S}$ has empty intersection with $\{x, y\}^{\perp\perp}$.*

A partial ovoid of size k is often called *k-cap* or *k-arc* of the GQ $\mathcal{S}$. A partial spread of size k is often called *k-span* of the GQ $\mathcal{S}$.

Theorem 2.2.6 ([55], Theorem 2.7.1.) *Any $(st-\rho)$-cap of $\mathcal{S}$ with $0 \leq \rho < t/s$ is contained in a uniquely defined ovoid of $\mathcal{S}$. Hence if $\mathcal{S}$ has no ovoid, then any k-cap of $\mathcal{S}$ necessarily satisfies $k \leq st - t/s$.*

The following results give an absolute lower bound for the size of a maximal partial ovoid, respectively spread, of a GQ. Simple proofs can be found in e.g. [28].

Theorem 2.2.7 ([28], Theorem 4.1) *A maximal partial ovoid of a finite GQ $\mathcal{S}$ of order (s,t) contains at least $s+1$ points; dually a maximal partial spread of $\mathcal{S}$ contains at least $t+1$ lines.*

Theorem 2.2.8 ([28], Theorem 4.3) *A GQ $\mathcal{S}$ of order s admits a complete $(s+1)$-cap if and only if $\mathcal{S}$ has a regular pair of non-collinear points.*

2.3 Finite classical generalized quadrangles

Description of the classical generalized quadrangles

We start by giving a brief description of the families of examples known as the finite classical generalized quadrangles. Let us recall that finite classical generalized quadrangles are finite polar spaces of rank 2.

(i) The points of $\mathrm{PG}(3,q)$, together with the totally isotropic lines with respect to a symplectic polarity, form a generalized quadrangle $W_3(q)$, shortly $W(q)$, with parameters

$s = t = q,\ v = b = (q+1)(q^2+1).$

A symplectic polarity of $\mathrm{PG}(3,q)$ has the following canonical bilinear form

$x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2 = 0.$

(ii) Consider a nonsingular quadric $\mathcal{Q}^+$ of projective index 1 in the projective space $\mathrm{PG}(3,q)$. Then the points of the quadric together with the lines of the quadric form a generalized quadrangle $Q^+(3,q)$ with parameters

$s = q,\ t = 1,\ v = (q+1)^2,\ b = 2(q+1).$

Since $Q^+(3,q)$ has $t = 1$, it is a grid and its structure is trivial. The quadric has the following canonical equation:

$x_0x_1 + x_2x_3 = 0.$

(iii) Consider a nonsingular quadric $\mathcal{Q}$ of projective index 1 in the projective space PG$(4,q)$. Then the points of the quadric together with the lines of the quadric form a generalized quadrangle $Q(4,q)$ with parameters

$s = t = q,\ v = b = (q+1)(q^2+1).$

The quadric has the following canonical equation:

$x_0^2 + x_1x_2 + x_3x_4 = 0.$

(iv) Consider a nonsingular quadric $\mathcal{Q}^-$ of projective index 1 in the projective space PG$(5,q)$. Then the points of the quadric together with the lines of the quadric form a generalized quadrangle $Q^-(5,q)$ with parameters

$s = q,\ t = q^2,\ v = (q+1)(q^3+1),\ b = (q^2+1)(q^3+1)$

The quadric has the following canonical equation:

$f(x_0,x_1) + x_2x_3 + x_4x_5 = 0$, where f is an irreducible homogeneous polynomial in x_0, x_1 over GF(q).

(v) Let $\mathcal{H}$ be a nonsingular Hermitian variety of the projective space $PG(d,q^2),\ d = 3$ or $d = 4$. Then the points of H together with the lines on H form a generalized quadrangle $H(d,q^2)$ with parameters

$s = q^2,\ t = q,\ v = (q^2+1)(q^3+1),\ b = (q+1)(q^3+1),$ when $d = 3$,
$s = q^2,\ t = q^3,\ v = (q^2+1)(q^5+1),\ b = (q^3+1)(q^5+1),$ when $d = 4$,

Recall that $\mathcal{H}$ has the canonical equation

$x_0^{q+1} + x_1^{q+1} + \ldots + x_d^{q+1} = 0.$

Isomorphisms between classical generalized quadrangles

To continue, we give some isomorphism results collected in [55]. We recall that if $\mathcal{S} = (P, B, \mathrm{I})$ is a GQ(s,t), then $\mathcal{S}^D$ denotes the dual of $\mathcal{S}$, i.e., $\mathcal{S}^D = (B, P, \mathrm{I})$ is a GQ(t,s).

Theorem 2.3.1

(i) $Q(4,q)$ *is isomorphic to the dual of* $W(q)$*. Moreover,* $Q(4,q)$*, (or* $W(q)$*) is self-dual if and only if* q *is even.*

(ii) $Q^-(5,q)$ *is isomorphic to the dual of* $H(3,q^2)$*.*

Regularity and antiregularity

Here, we consider only the classical generalized quadrangles and we give important properties in the following theorem. By Theorem 2.3.1 it is sufficient to consider $Q^+(3,q)$, $Q(4,q)$, $Q^-(5,q)$, and $H(4,q^2)$. Of course, the structure of $Q^+(3,q)$ is trivial.

Theorem 2.3.2

(i) In the generalized quadrangle $Q(4,q)$ *all lines are regular and all points are regular if and only if* q *is even. All points are antiregular if and only if* q *is odd.*

(ii) In the generalized quadrangle $Q^-(5,q)$ *all lines are regular.*

(iii) In the generalized quadrangle $H(4,q^2)$*, for any two non-collinear points* x, y *we have* $|\{x,y\}^{\perp\perp}| = q+1$*; for any two non-concurrent lines* L, M *we have* $|\{L,M\}^{\perp\perp}| = 2$*, but* (L,M) *is not antiregular.*

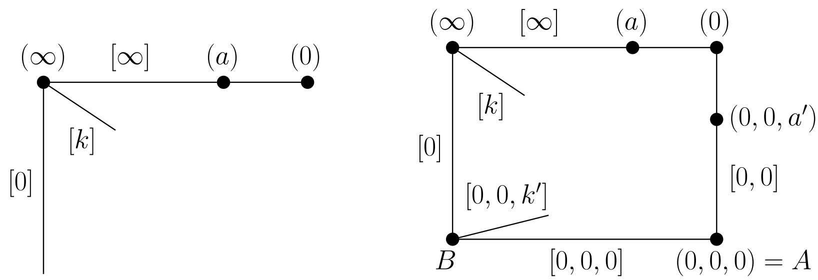
3 Computer representation

For a computer search, an appropriate representation of a classical generalized quadrangle is needed. We will represent it by an incidence matrix, which was already defined in Section 1.4.

In the first section we explain *polygon coordinatization*, a process of labeling the elements of a generalized polygon, in particular of a generalized quadrangle. For a detailed explanation of the coordinatization of generalized polygons, we refer to [72]. Here we focus on classical generalized quadrangles and we describe how the labeling of the elements works. Our description is mainly based on the labeling introduced by G. Hanssens and H. Van Maldeghem in [38].

In the second section, a process of indexing points in a classical generalized quadrangle is briefly described, which leads to a construction of an incidence matrix of the generalized quadrangle.

In the third section we introduce a collinearity graph as a possible representation of a generalized quadrangle.



(a) The points with one polygon coordinate

(b) Some points with three polygon coordinates

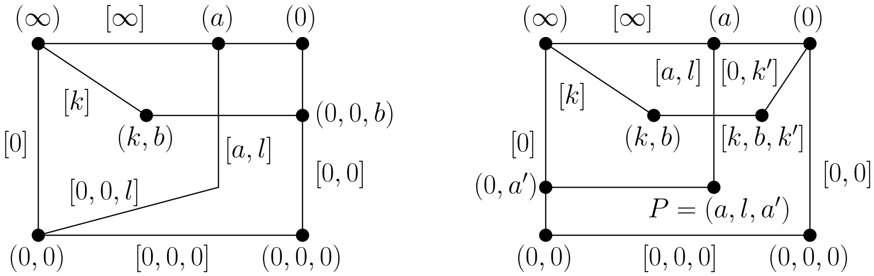
Figure 3.1: Polygon coordinatization

3.1 Coordinates of classical generalized quadrangles

Consider a finite generalized quadrangle of order (s, t). Let R_1 and R_2 be sets of cardinality s and t, respectively, both not containing the symbol (∞), but containing two distinct elements 0 and 1.

The first steps of the labeling can be followed in Figure 3.1(a). We choose a point (∞) and a line $[\infty]$ of $\mathcal{S}$, such that $(\infty)\mathrm{I}[\infty]$. We give coordinates (a), $a \in R_1$ to the remaining points on $[\infty]$ distinct from (∞) and coordinates $[k]$, $k \in R_2$ to the remaining lines on (∞) distinct from $[\infty]$.

Next, we complete the elements (∞), $[\infty]$, (0), $[0]$ to a quadrangle (∞), $[\infty]$, (0), $(0)A$, A, AB, B, $[0]$. As before we choose a bijection between R_1 and the points of the line $(0)A$ with the only restriction that A corresponds to 0. The point of $(0)A$ corresponding to $a' \in R_1$ will have coordinates $(0, 0, a') \in R_1 \times R_2 \times R_1$. Hence, the point A has coordinates $(0, 0, 0)$. Dually, the lines on B different from $[0]$ are given coordinates $[0, 0, k'] \in R_2 \times R_1 \times R_2$ with the restriction that BA has coordinates



(a) The points with two polygon coordinates

(b) Some points with three polygon coordinates

Figure 3.2: Polygon coordinatization

$[0, 0, 0]$. This is shown in Figure 3.1(b)

Now the points with two coordinates are defined. A point P collinear with (∞), but not lying on $[\infty]$ has coordinates $(k, b) \in R_2 \times R_1$ if and only if P lies on $[k]$ and is collinear with $(0, 0, b)$. Dually, lines meeting $[\infty]$ not passing through (∞) are given coordinates $[a, l] \in R_1 \times R_2$, see Figure 3.2(a).

Finally, consider a point P not collinear with (∞). Because $\mathcal{S}$ is a generalized quadrangle, there is exactly one line on P meeting $[\infty]$. This line must have two coordinates, say $[a, l]$. On the other hand, P is collinear with exactly one point $(0, a')$ on $[0]$. Now P are given the coordinates (a, l, a'). Conversely, let (a, l, a') be an element of $R_1 \times R_2 \times R_1$, then we construct a point P having this element as coordinate. Indeed, given the line $[a, l]$ and the point $(0, a')$ not incident with it, then there is exactly one point collinear with $(0, a')$ and lying on $[a, l]$. The coordinate of a line $[k, b, k']$ is defined dually. We illustrate this in Figure 3.2(b).

In the following subsections we assign projective coordinates to the points of the classical generalized quadrangles labeled by polygon coordinates. For our purposes it is easier to treat the points of $Q(4, q)$, $Q^-(5, q)$

and $H^D(4,q^2)$, as the lines of $W(q)$, $H(3,q^2)$ and $H(4,q^2)$, respectively. This means that the Grassman coordinates of the lines of $W(q)$, $H(3,q^2)$ or $H(4,q^2)$ are assigned to the points of $Q(4,q)$, $Q^-(5,q)$ or $H^D(4,q^2)$, respectively. Let us recall that if $\mathbf{X} = (x_0, x_1, \ldots, x_n)$ and $\mathbf{Y} = (y_0, y_1, \ldots, y_n)$ are the representing vectors of two points determining a line L of PG(n,q) then $p_{i,j} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$, $0 \leq i < j \leq n$, are the Grassman coordinates of the line L. We use the following fixed order of these coordinates:

$$(p_{01}, p_{02}, p_{12}, \ldots, p_{0n}, p_{1n}, p_{2n}, \ldots, p_{n-1,n}).$$

3.1.1 The symplectic generalized quadrangle $W(q)$

Recall that a symplectic polarity of PG(3,q) has the following canonical bilinear form

$$x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2 = 0.$$

We choose $R_1 = R_2 = \mathrm{GF}(q)$ and

$$\begin{aligned} (\infty) &= (1,0,0,0), \\ (0) &= (0,0,1,0). \end{aligned}$$

Then we take

$$\begin{aligned} (a) &= (a,0,1,0), \\ [k] &= \text{ the line on } (1,0,0,0) \text{ and } (0,0,k,1). \end{aligned}$$

Finally, we choose

$$\begin{aligned} (0,0,a') &= (0,1,-a',0), \\ [0,0,k'] &= \text{ the line on } (0,0,0,1) \text{ and } (k',1,0,0). \end{aligned}$$

We compute the points and lines with two coordinates

$$\begin{aligned} (k,b) &= (-b,0,k,1), \\ [a,l] &= \text{ the line on } (a,0,1,0) \text{ and } (l,1,0,-a). \end{aligned}$$

Finally, we obtain the points and lines with three coordinates

$$\begin{aligned}(a,l,a') &= (l-aa',1,-a',-a),\\ [k,b,k'] &= \text{ the line on } (-b,0,k,1) \text{ and } (k',1,-b,0).\end{aligned}$$

3.1.2 The quadric $Q(4,q)$

Take $R_1 = R_2 = \mathrm{GF}(q)$. Without further details, we compute the Grassman coordinates of the lines of $W(q)$ and these are assigned to polygon coordinates of the points of $Q(4,q)$.

$$\begin{aligned}(\infty) &= (0,1,0,0,0,0),\\ (a) &= (0,a,0,1,0,0),\\ (k,b) &= (k,-b,-1,-k^2,0,-k),\\ (a,l,a') &= (-l,l^2-aa',-a,-a',-1,l).\end{aligned}$$

3.1.3 The Hermitian variety $H(3,q^2)$

Though the canonical form of $\mathcal{H}$ given in Sections 1.7 and 2.3 is more used, we continue to follow the reference [38]. We suppose that $H(3,q^2)$ has the following equation:

$$x_0x_2^q + x_0^qx_2 + x_1x_3^q + x_1^qx_3 = 0.$$

We take $R_1 = \mathrm{GF}(q)^2$ and $R_2 = \mathrm{GF}(q)$. First we choose

$$\begin{aligned}(\infty) &= (1,0,0,0),\\ (a) &= (a,0,0,1),\\ [k] &= \text{ the line on } (1,0,0,0) \text{ and } (0,1,0,k).\\ (0,0,a') &= (0,0,1,-a'^q),\\ [0,0,k'] &= \text{ the line on } (0,1,0,0) \text{ and } (k',0,1,0).\end{aligned}$$

We compute the points and lines with two coordinates

$$\begin{aligned}(k,b) &= (b,1,0,k),\\ [a,l] &= \text{the line on } (a,0,0,1) \text{ and } (l,-a^q,1,0).\end{aligned}$$

Finally, we obtain the points and lines with three coordinates

$$\begin{aligned}(a,l,a') &= (l-aa'^q,-a^q,1,-a'^q),\\ [k,b,k'] &= \text{the line on } (b,1,0,k) \text{ and } (k',0,1,-b^q).\end{aligned}$$

3.1.4 The quadric $Q^-(5,q)$

We consider $Q^-(5,q)$ as the dual of $H(3,q^2)$. Hence take $R_1 = \mathrm{GF}(q)$ and $R_2 = \mathrm{GF}(q)^2$ and the projective coordinates of the points of $Q^-(5,q)$ are as follows.

$$\begin{aligned}(\infty) &= (0,0,0,1,0,0),\\ (a) &= (1,0,0,a,0,0),\\ (k,b) &= (-kk^q,k,0,-b,k^q,-1),\\ (a,l,a') &= (-a',l,1,-ll^q-aa',-l^q,-a).\end{aligned}$$

3.1.5 The Hermitian variety $H(4,q^2)$

We may suppose that $H(4,q^2)$ has the following equation:

$$x_0x_2^q + x_0^q x_2 + x_1x_3^q + x_1^q x_3 + x_4^{q+1} = 0.$$

We choose

$$\begin{aligned}(\infty) &= (1,0,0,0,0),\\ (0) &= (0,0,0,1,0).\end{aligned}$$

Let $R_1 = \mathrm{GF}(q^2)$ and for R_2 we take $\mathrm{GF}(q^2) \times K$ where K is the set of q solutions of the equation $t^q + t = 0$. Further, let σ be a fixed non zero element of $\mathrm{GF}(q^2)$ satisfying the equation $(1+\sigma)^{q+1} = 1$.

Then we put

$$\begin{aligned}(a) &= (a,0,0,1,0),\\ [k] &= [(k_0,k_1)] \text{ the line on } (1,0,0,0,0) \text{ and } (0,1,0,\sigma k_0^{q+1}+k_1,\sigma k_0).\end{aligned}$$

Finally, we take

$$\begin{aligned}(0,0,a') &= (0,0,1,-a'^q\sigma^q,0),\\ [0,0,k'] &= [0,0,(k'_0,k'_1)] \text{ the line on } (0,1,0,0,0) \text{ and}\\ &\quad (\sigma k'^{q+1}_0+k'_1,0,1,0,-\sigma k'_0).\end{aligned}$$

We compute the elements with two coordinates

$$\begin{aligned}(k,b) &= (\sigma b,1,0,\sigma k_0^{q+1}+k_1,\sigma k_0),\\ [a,l] &= \text{ the line on } (a,0,0,1,0) \text{ and}\\ &\quad (\sigma l_0^{q+1}+l_1,-a^q,1,0,-\sigma l_0).\end{aligned}$$

Finally, we look for the points and lines with three coordinates

$$\begin{aligned}(a,l,a') &= (\sigma l_0^{q+1}+l_1-\sigma^q a a'^q,-a^q,1,-\sigma^q a'^q,-\sigma l_0),\\ [k,b,k'] &= [(k_0,k_1),b,(k'_0,k'_1)] \text{ the line on } (\sigma b,1,0,\sigma k_0^{q+1}+k_1,\sigma k_0) \text{ and}\\ &\quad (\sigma k'^{q+1}_0+k'_1,0,1,-\sigma^q b^q+\sigma^{q+1}k_0^q k'_0,-\sigma k'_0).\end{aligned}$$

3.1.6 $H(4,q^2)^D$, the dual of Hermitian variety $H(4,q^2)$

Hence, we put $R_1 = \mathrm{GF}(q^2)\times K$, where K denotes the same set as mentioned before. Take $R_2 = \mathrm{GF}(q^2)$. The projective coordinates of the points are the Grassman coordinates of the lines of $H(4,q^2)$.

$$\begin{aligned}
(\infty) &= (0,0,0,1,0,0,0,0,0,0),\\
(a) &= (1,0,0,\sigma a_0^{q+2}+a_1,0,0,\sigma a_0,0,0,0),\\
(k,b) &= (-kk^q,k,0,-\sigma b_0^{q+1}-b_1,k^q,-1,-\sigma k b_0,0,0,-\sigma l_0),\\
(a,l,a') &= (-\sigma {a'}_0^{q+1}-a'_1,\sigma l,1,\\
&\quad -\sigma^{q+1}l^{q+1}+\sigma^{q+2}la_0^q a'_0-\sigma^2 a_0^{q+1}{a'}_0^{q+1}-\sigma a_0^{q+1}a'_1-\sigma {a'}_0^{q+1}a_1\\
&\quad -a_1a'_1,\\
&\quad -\sigma^q l^q+\sigma^{q+1}a_0^q a'_0,-\sigma a_0^{q+1}-a_1,-\sigma^2 la'_0-\sigma^2 a_0{a'}_0^{q+1}-\sigma a_0a'_1,\\
&\quad -\sigma a'_0,-\sigma a_0,-\sigma^2 a_0^{q+1}a'_0-\sigma a'_0a_1-\sigma^{q+2}a_0{a'}_0^{q+1}+\sigma^{q+1}l^q a_0).
\end{aligned}$$

3.2 Incidence matrix

In the previous section, a connection between polygon coordinates in generalized quadrangles and projective coordinates in PG(n,q) was given. From this, we can create the incidence matrix corresponding to a generalized quadrangle.

We recall the definition of an incidence matrix N of a generalized quadrangle $\mathcal{S}$. It has rows indexed by the points, columns indexed by the lines, with $N_{p,L}=1$ if the point p is incident with the line L, otherwise $N_{p,L}=0$. Hence, we need to order the points and lines of the generalized quadrangle $\mathcal{S}$.

First, we show our way of assigning an index to a point in PG(n,q). This helps us to give an index to a point of the generalized quadrangle $\mathcal{S}$. By indexing the points in $\mathcal{S}^D$ we immediately get indexing of lines in $\mathcal{S}$, hence the incidence matrix can be created.

Point indexing in PG(n,q)

A point of PG(n,q) can be represented by a non-zero vector in $V(n+1,q)$, it was introduced in Section 1.5. We also mentioned a convention to select the coordinate whose last non-zero entry is a 1.

In our implementation we do not work directly with the elements of GF(q). To each of the q elements of GF(q) we assign a natural number j, $0 \leq j \leq q-1$, such that 0 is assigned to element 0 from GF(q) and 1 is assigned to element 1 from GF(q). We will omit the zero entries which come after the last 1 and will call such coordinates *short coordinates.* The way we assign indexes is illustrated for $n = 3$ in Table 3.1.

Let a point p have short coordinates $(x_0, x_1, x_2, \ldots x_k, 1)$, $k < n$ and j be the index of the point p. Then the relation between the index and the non-zero coordinates is:

$$j = (x_0 + 1)q^k + (x_1 + 1)q^{k-1} + \ldots + (x_k + 1)q^0.$$

Point indexing in $W(q)$

We will not discuss the indexing of points of all classical generalized quadrangles. We sketch it for the points of $W(q)$. The indexing of the other classical generalized quadrangles can be treated in a similar way.

Recall that the polygon coordinates of $W(q)$ have one of the following forms: (∞), (a), (k, b), (a, l, a'), for $R_1 = R_2 = \mathrm{GF}(q)$. Now take a look at the short coordinates of PG(n, q) in Table 3.1. If we omit the last 1 in the coordinates and treat () as (∞), a similarity with the polygon coordinates in $W(q)$ is obvious. Hence, we can easily assign the indexes to the projective coordinates of points in $W(q)$ and we illustrate it in Table 3.2. Because of the duality of $W(q)$ and $Q(4, q)$, by indexing the points of $Q(4, q)$ we immediately get an indexing of the lines of $W(q)$.

3.3 Collinearity graph

With a generalized quadrangle $\mathcal{S}$ a so-called *collinearity graph* or *point graph* $G_\mathcal{S}$ can be associated as follows: the points of P correspond to the vertices of $G_\mathcal{S}$ and two vertices are adjacent if and only if the corresponding points are collinear. The graph $G_\mathcal{S}$ is a strongly regular graph [55] with parameters

$$v = (s+1)(st+1), \quad k = s(t+1), \quad \lambda = s-1, \quad \mu = t+1\,.$$

Considering the point graph, an ovoid of a generalized quadrangle $\mathcal{S}$ is a maximum independent set of size $st+1$ in $G_{\mathcal{S}}$, or equivalently, a maximum clique in its complement $\overline{G_{\mathcal{S}}}$. A spread of $\mathcal{S}$ is a maximum independent set of size $st+1$ in the collinearity graph $G_{\mathcal{S}^D}$ of the dual $\mathcal{S}^D$. Maximal partial ovoids and spreads are maximal cliques in $\overline{G_{\mathcal{S}}}$ or $\overline{G_{\mathcal{S}^D}}$.

In the next chapter we explain techniques used to search for maximal partial ovoids and maximal partial spreads in a generalized quadrangle $\mathcal{S}$, or equivalently to search for maximal cliques in the corresponding collinearity graph $\overline{G_{\mathcal{S}}}$ or $\overline{G_{\mathcal{S}^D}}$.

Index	Projective coordinates	Short coordinates
0	$(1,0,0,0)$	(1)
$0+1$	$(0,1,0,0)$	$(0,1)$
$1+1$	$(1,1,0,0)$	$(1,1)$
$2+1$	$(2,1,0,0)$	$(2,1)$
$\vdots$	$\vdots$	$\vdots$
$(q-1)+1$	$(q-1,1,0,0)$	$(q-1,1)$
$0+(q+1)$	$(0,0,1,0)$	$(0,0,1)$
$1+(q+1)$	$(0,1,1,0)$	$(0,1,1)$
$2+(q+1)$	$(0,2,1,0)$	$(0,2,1)$
$\vdots$	$\vdots$	$\vdots$
$q-1+(q+1)$	$(0,q-1,1,0)$	$(0,q-1,1)$
$\vdots$	$\vdots$	$\vdots$
$q^2-1+(q+1)$	$(q-1,q-1,1,0)$	$(q-1,q-1,1)$
$0+(q^2+q+1)$	$(0,0,0,1)$	$(0,0,0,1)$
$1+(q^2+q+1)$	$(0,0,1,1)$	$(0,0,1,1)$
$2+(q^2+q+1)$	$(0,0,2,1)$	$(0,0,2,1)$
$\vdots$	$\vdots$	$\vdots$
$q-1+(q^2+q+1)$	$(0,0,q-1,1)$	$(0,0,q-1,1)$
$\vdots$	$\vdots$	$\vdots$
$q^3-1+(q^2+q+1)$	$(q-1,q-1,q-1,1)$	$(q-1,q-1,q-1,1)$

Table 3.1: Indexing of the points of $\mathrm{PG}(3,q)$.

Index	Polygon coordinates in $W(q)$	Projective coordinates in $\mathrm{PG}(3,q)$
0	(∞)	$(1,0,0,0)$
$\mathbf{1}$ to $\mathbf{q}$	$(\mathbf{a})$	$(\mathbf{a},\mathbf{0},\mathbf{1},\mathbf{0})$
1	(0)	$(0,0,1,0)$
2	(1)	$(1,0,1,0)$
3	(2)	$(2,0,1,0)$
$\vdots$	$\vdots$	$\vdots$
(q)	$(q-1)$	$(q-1,0,1,0)$
$\mathbf{q}+\mathbf{1}$ to $\mathbf{q}^\mathbf{2}+\mathbf{q}$	$(\mathbf{k},\mathbf{b})$	$(-\mathbf{b},\mathbf{0},\mathbf{k},\mathbf{1})$
$q+1$	$(0,0)$	$(0,0,0,1)$
$q+2$	$(0,1)$	$(-1,0,0,1)$
$q+3$	$(0,2)$	$(-2,0,0,1)$
$\vdots$	$\vdots$	$\vdots$
q^2+q	$(q-1,q-1)$	$(1-q,0,q-1,1)$
$\mathbf{q}^\mathbf{2}+\mathbf{q}+\mathbf{1}$ to $\mathbf{q}^\mathbf{3}+\mathbf{q}^\mathbf{2}+\mathbf{q}$	$(\mathbf{a},\mathbf{l},\mathbf{a}')$	$(\mathbf{l}-\mathbf{a}\mathbf{a}',\mathbf{1},-\mathbf{a}',-\mathbf{a})$
q^2+q+1	$(0,0,0)$	$(0,1,0,0)$
q^2+q+2	$(0,0,1)$	$(0,1,-1,0)$
q^2+q+3	$(0,0,2)$	$(0,1,-2,0)$
$\vdots$	$\vdots$	$\vdots$
q^3+q^2+q	$(q-1,q-1,q-1)$	$((q-1)(2-q),1,1-q,1-q)$

Table 3.2: Indexing of the points of $W(q)$.

4 Algorithms for searching maximal partial ovoids

In this chapter we describe algorithms to find (partial) ovoids. The new techniques are to appear in *Clique algorithms for finding substructures in generalized quadrangles* [18].

4.1 Introduction

A lot of attention has been paid to the (non-)existence of ovoids and spreads in finite generalized quadrangles [67, 68]. If a generalized quadrangle is known to have no ovoid, the following question naturally arises. What is the largest set of points $\mathcal{O}$, such that each line of the generalized quadrangle $\mathcal{S}$ is incident with at most one point of $\mathcal{O}$? We can also search for the largest partial ovoid different from an ovoid in generalized quadrangles having ovoids. The question in the dual terms deals with partial spreads.

A lot of research has been done on partial spreads and partial ovoids of size $st + 1 - d$, with small deficiency d, with special emphasis on the extendability of such partial spreads and partial ovoids to spreads and ovoids [13, 37]. Some theoretical upper bounds on the size of a maximal partial ovoid are known.

Recently, special attention has been paid to the smallest maximal partial ovoids and to the smallest maximal partial spreads of finite generalized quadrangles.

Our results on (partial) ovoids were mainly obtained by a computer search. Section 4.2 describes exhaustive search algorithms, where we use standard clique searching algorithms and add standard pruning strategies.

In Sections 4.3, 4.4, 4.5, we describe new techniques based on specific properties of the generalized quadrangles. This approach leads to exact answers concerning e.g. the size of the largest/smallest maximal partial ovoid or spread, or the classification of all maximal partial ovoids and spreads of a given size. These results improve the best known bounds.

One can also ask, whether for a given size maximal partial ovoids and maximal partial spreads exist. In particular, we are interested in the spectrum of sizes for maximal partial ovoids and maximal partial spreads. The class of algorithms, described in Section 4.6, is based on heuristic techniques and turns out to be very effective for exploring the spectra.

4.2 Exhaustive search algorithms

The basic form of most published algorithms (e.g. [16]) for the maximal or maximum clique problem is a backtracking search which tries in every recursion step to extend a partial clique by adding the vertices of a set A_N of allowed vertices in a systematic way (Figure 4.1). When reaching a point where the set A_N is empty, a new maximal clique has been found.

4.2.1 Standard backtracking algorithm for a maximal clique problem

We will discuss a backtracking search for a maximal clique of given size in a given graph with vertices $v_1, \ldots, v_n$. Clearly, the allowed set A_N mentioned in the recursive step can be restricted to vertices from the neighborhood $N(v_i)$ of the vertex v_i which is currently added to the partial clique C. However, if a clique has size k, then an algorithm based on the

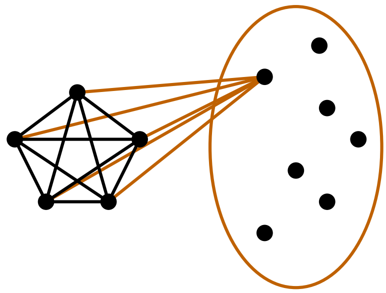


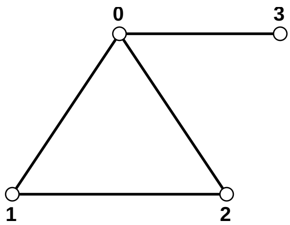
Figure 4.1: Adding of allowed vertex

above allowed set could generate it $k!$ times, once for each possible ordering of vertices.

We illustrate this on a simple example. Assume we search for a maximal clique of size 2 in the graph in Figure 4.2. While we found a maximal clique of the desired size, we generated two identical maximal cliques of size 3.

To avoid this duplication of work, we define a set A of allowed remaining vertices, i.e., in each recursive call the added vertex is removed from this set. In our example, by defining the set A we will avoid generating the maximal clique $\{0, 2, 1\}$. However, we must keep the allowed set A_N. Otherwise, in our example, the algorithm would see the clique $\{0, 2\}$ as maximal, which is false.

Algorithm 4.1 shows the pseudo-code of a backtracking search for a maximal clique of the given size *desired* in a given graph with vertices $v_1, \ldots, v_n$. The recursive function *clique* takes as parameters the sets A and A_N of allowed vertices and the partial clique C constructed at the current step in the recursive process. It also takes the desired size of a maximal clique (*desired*) as an extra parameter. In the original call both sets A and A_N are the full vertex set V of the graph and C is the empty set. When the allowed set A_N is empty, a new maximal clique has been found and it is checked whether this clique is of the desired size. If the set A_N is not empty, all



partial clique	allowed set	
$\emptyset$	$\{0,1,2,3\}$	
$\{0\}$	$\{1,2,3\}$	
$\{0,1\}$	$\{2\}$	
$\{\mathbf{0},\mathbf{1},\mathbf{2}\}$	$\emptyset$	maximal, size 3
$\{0\}$	$\{1,2\}$	
$\{0,2\}$	$\{1\}$	
$\{\mathbf{0},\mathbf{2},\mathbf{1}\}$	$\emptyset$	maximal, size 3
$\{0\}$	$\{1,2,3\}$	
$\{\mathbf{0},\mathbf{3}\}$	$\emptyset$	maximal, size 2
		FOUND

Figure 4.2: Repeating of maximal cliques

vertices of A are used in a systematic way to extend the partial clique C and for each case a recursive call is performed. The new allowed sets A_N and A in the recursive call are restricted to vertices from the neighborhood $N(v_i)$ of the vertex v_i which is currently added to C. Note that before calling the recursive function *clique*, the currently added vertex v_i is removed from A.

4.2.2 Standard backtracking algorithm for a maximum clique problem

In the previous subsection we described an algorithm for generating a maximal clique. This algorithm can easily be modified to find a maximum clique. Note, that we no longer need to maintain the set A_N. We simply check whether each constructed clique is larger than any previously constructed (not necessarily maximal) clique.

Algorithm 4.2 shows the pseudo-code of a backtracking search for the maximum clique size in a given graph with vertices $v_1, \ldots, v_n$. The global variable *max* keeps the value of the currently largest clique size found. The recursive function *clique* takes as parameters the set A of allowed vertices

Algorithm 4.1 Backtracking search for a maximal clique of desired size

function *clique* (*desired*)
  clique (V, V, $\emptyset$)
function *clique* (A, A_N, C, *desired*)
  if $A_N = \emptyset$ **then**
    {Found a new maximal clique}
    if $size(C) = desired$ **then**
      {Found a maximal clique of the desired size}
      return
  else
    while $A \neq \emptyset$ **do**
      if $size(C) + bound(A) \leq desired$ **then**
        return
      $i \leftarrow \min\{j \mid v_j \in A\}$
      $A \leftarrow A \setminus \{v_i\}$
      clique ($A \cap N(v_i)$, $A_N \cap N(v_i)$, $C \cup \{v_i\}$, *desired*)
function *bound* (A)
  {Determines a bound on the number of vertices from A that might be added to the partial clique}

Algorithm 4.2 Backtracking search for a maximum clique

function *clique* ()
  $max \leftarrow 0$
  clique $(V, \emptyset)$
function *clique* (A, C)
  if $A = \emptyset$ **then**
    {Found a new clique}
    if $size(C) > max$ **then**
      {Found a larger clique than the current largest}
      $max \leftarrow size(C)$
  else
    while $A \neq \emptyset$ **do**
      if $size(C) + bound(A) \leq max$ **then**
        return
      $i \leftarrow \min\{j \mid v_j \in A\}$
      $A \leftarrow A \setminus \{v_i\}$
      clique $(A \cap N(v_i), C \cup \{v_i\})$
function *bound* (A)
  {Determines a bound on the number of vertices from A that might be added to the partial clique}

and the partial clique C constructed at the current step in the recursive process. In the original call the set A is the full vertex set V of the graph and C is the empty set. When the allowed set A is empty, a new maximal clique has been found and it is checked whether this clique is larger than the currently largest clique; if so, then the value of *max* is updated. If the set A is not empty, all its vertices are used in a systematic way to extend the partial clique C and for each case a recursive call is performed. The new allowed set in the recursive call is restricted to vertices from the neighborhood $N(v_i)$ of the vertex v_i which is currently added to C.

4.2.3 Pruning strategies

Pruning strategies are used to avoid going through every single clique of the graph. Typically this consists in a *bounding function* (called *bound* in Algorithms 4.1 and 4.2) which gives an upper bound on the number of vertices that can still be added to the current partial clique.

E.g. when searching for maximum cliques, a straightforward idea is to backtrack when the set A becomes so small that even if all its vertices could be added to form a clique, the size of that clique would not exceed the size of the largest clique found so far; in that case $bound(A)$ is simply $|A|$.

Other pruning strategies involve vertex colorings. In a *vertex coloring*, adjacent vertices must be assigned different colors, so if a graph or an induced subgraph can be colored with c colors, then the graph or subgraph cannot contain a clique of size $c + 1$; in this case $bound(A)$ is the number of colors used to color the vertices of A. In practice a fixed coloring of the original graph is used, since determining a coloring for the induced subgraph $\langle A \rangle$ each time usually is too expensive.

4.2.4 Östergård's algorithm

Recently, P.J. Östergård [54] presented a new maximum clique algorithm that allows to introduce a new pruning strategy. Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of the graph, let $S_i = \{v_i, \ldots, v_n\}$ and let $c(i)$ denote the size of the largest clique in S_i. For any $1 \le i \le n-1$, either $c(i) = c(i+1)$ or $c(i) = c(i+1)+1$. Moreover $c(i) = c(i+1)+1$ if and only if there is a clique of size $c(i+1)+1$ in S_i that contains v_i. The algorithm starts with $c(n) = 1$ and computes $c(i)$, $i = n-1, \ldots, 1$ by searching for such a clique. Finally the size of a maximum clique is given by $c(1)$. The values of $c(i)$ can be used for pruning the search as follows. Searching for a clique of size larger than s, the search can be pruned if $j + c(i) \le s$, where j denotes the size of the current partial clique and i is the index of the next vertex v_i to be added to the current partial clique.

Algorithm 4.3 Östergård's algorithm for finding a maximum clique

function *clique* ()
- $max \leftarrow 0$
- $c_n \leftarrow 1$
- **for all** i **from** $n-1$ **downto** 1 **do**
  - $found \leftarrow false$
  - *clique* $(S_i \cap N_G(v_i),\, 1)$
  - $c_i \leftarrow max$

function *clique* $(A,\, size)$
- **if** $A = \emptyset$ **then**
  - **if** $size > max$ **then**
    - $max \leftarrow size$
    - $found \leftarrow true$
- **else**
  - **while** $A \neq \emptyset$ **do**
    - **if** $size + |A| \leq max$ **then**
      - return
    - $i \leftarrow \min\{j \mid v_j \in A\}$
    - **if** $size + c_i \leq max$ **then**
      - return
    - $A \leftarrow A \setminus \{v_i\}$
    - *clique* $(A \cap N_G(v_i),\, size + 1)$
    - **if** $found$ **then**
      - return

4.2.5 Isomorph rejection

Since the classical generalized quadrangles have automorphism groups that act transitively on the pairs of non-collinear points [44], every (partial) ovoid is equivalent to a (partial) ovoid containing a given pair of non-collinear points. Hence in the clique finding algorithm we can restrict the search to cliques containing a certain fixed edge. This reduces the search space with a factor of $O(vk')$, where $v = (s+1)(st+1)$ is the number of points of $\mathcal{S}$ and $k' = s^2t$ is the number of points not collinear with an arbitrary point. In some cases it is possible to fix even more pairwise adjacent vertices, e.g. for the generalized quadrangle $Q^-(5,q)$, 3 vertices can be fixed. This straightforward approach of fixing a certain number of vertices is already a very effective way to reduce the search space.

More advanced *isomorph-rejection* techniques, such as the techniques described in [61], allow to reduce the search space even further. Having determined in a step of the search process the set stabilizer of the current partial clique in the automorphism group of the quadrangle, it suffices to try only one point of each orbit of the stabilizer for extending the current partial clique in the next recursive steps instead of trying to add all vertices of the allowed set. An existing software package, such as *nauty* [49], can be used to compute the set stabilizer and its orbits.

Algorithm 4.4 gives the pseudo-code for a maximum clique search algorithm using isomorph pruning. The function *clique* now takes as an extra parameter the current level in the recursion tree. In case this level is smaller than a predefined maximum number of levels (*maxlevel*), the set stabilizer for the partial clique C is determined and C is extended with one representative of every orbit in a systematic way: the first vertex v of the orbit tried by the recursive process is added effectively, after handling v all vertices in the same orbit as v are removed from the set A of allowed vertices. At deeper levels the recursive process extends the current partial clique with every vertex from the allowed set, as described earlier in Algorithm 4.2. Algorithm 4.1 for searching for a maximal clique can be modified in the same way.

Algorithm 4.4 Maximum clique searching with isomorph pruning

function *clique* (A, C, *level*)
  if $A = \emptyset$ **then**
    {Found a new clique}
    if $size(C) > max$ **then**
      $max \leftarrow size(C)$
  else if $level < maxlevel$ **then**
    {Add only one vertex per orbit of the set stabilizer of C}
    compute set stabilizer of C and determine its orbits
    while $A \neq \emptyset$ **do**
      if $size(C) + bound(A) \leq max$ **then**
        return
      $i \leftarrow \min\{j \mid v_j \in A\}$
      $A \leftarrow A \setminus \{v_i\}$
      clique $(A \cap N(v_i), C \cup \{v_i\}, level + 1)$
      for all v_j in orbit of v_i **do**
        $A \leftarrow A \setminus \{v_j\}$
  else
    {Add all vertices in allowed set A}
    while $A \neq \emptyset$ **do**
      if $size(C) + bound(A) \leq max$ **then**
        return
      $i \leftarrow \min\{j \mid v_j \in A\}$
      $A \leftarrow A \setminus \{v_i\}$
      clique $(A \cap N(v_i), C \cup \{v_i\}, level + 1)$

$$\begin{array}{llllllll}
\mathrm{GQ}(s,t)^D & \rightarrow & l_1 & l_2 & \dots & l_{st+1} & = & \text{OVOID} \\
 & & \downarrow & \downarrow & \dots & \downarrow & & \\
 & & p_{1,1} & p_{2,1} & \dots & p_{st+1,1} & & \\
 & & p_{1,2} & p_{2,2} & \dots & p_{st+1,2} & & \\
 & & p_{1,s+1} & p_{2,s+1} & \dots & p_{st+1,s+1} & & \\
 & & \downarrow & \downarrow & \dots & \downarrow & & \\
\mathbf{GQ(s,t)} & \rightarrow & \mathbf{1.color} & \mathbf{2.color} & \dots & \mathbf{st+1.color} & = & \text{COLOR CLASSES}
\end{array}$$

Figure 4.3: Assigning colors to the points of a GQ

4.3 Spread coloring

When using a coloring bound in a maximum clique algorithm (see Subsection 4.2.3), one is faced with the problem that determining the chromatic number of a graph is also an NP-hard problem. However, an upper bound for the chromatic number can also be used as a coloring bound. Hence an approximation algorithm, often a simple greedy algorithm, is used to obtain a reasonable upper bound for the chromatic number. But in the case of generalized quadrangles theoretical arguments lead to good – even optimal – colorings for some types of generalized quadrangles.

For instance, classical constructions for ovoids in $Q(4,q)$ are known [55]. Since $Q(4,q)$ is isomorphic to the dual of $W(q)$, the points of an ovoid in $Q(4,q)$ correspond to the lines of a spread in $W(q)$, hence to a partitioning of the vertices of $\overline{G_{W(q)}}$ into classes of pairwise non-adjacent vertices. In other words, this is a partitioning of $\overline{G_{W(q)}}$ into color classes which can be used for pruning in a maximum clique algorithm. It is obvious that this construction uses $st+1$ colors to color the graph. In the same way a coloring with $st+1$ colors for $\overline{G_{Q^-(5,q)}}$ can be obtained from classical constructions of ovoids in its dual $H(3,q^2)$ [55].

The process of assigning colors to the points of a generalized quadrangle is illustrated in Figure 4.3.

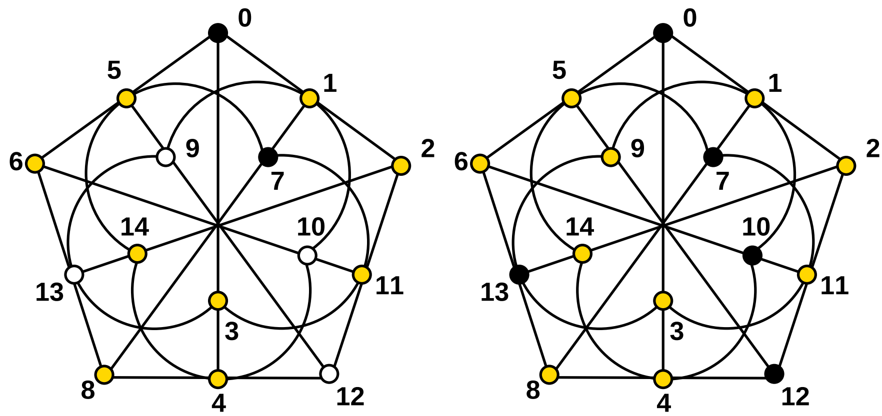
It is easy to see that the obtained vertex coloring is optimal, i.e. uses a minimum number of colors. This can be proved as follows. From the geometry of generalized quadrangles it follows that the points of a line in a generalized quadrangle $\mathcal{S}$ form a clique of maximum size in the collinearity graph $G_{\mathcal{S}}$. Indeed, the points of a line are pairwise collinear and hence the corresponding vertices form a clique of size $s + 1$, while a set of more than $s + 1$ pairwise collinear points would require a triangle in the generalized quadrangle, which is not allowed. Hence a color class in the graph $\overline{G_{\mathcal{S}}}$ has size at most $s + 1$. Since there are $(s + 1)(st + 1)$ vertices, at least $st + 1$ colors are needed to color $\overline{G_{\mathcal{S}}}$. The coloring obtained as described above uses exactly this number of colors and thus is an optimal coloring.

4.4 Forcing vertices using look-ahead

In some situations the pruning in a clique finding algorithm in a collinearity graph can be improved by using the information about the incidence structure of the generalized quadrangle as well as the graph. For instance, when classifying the ovoids in a generalized quadrangle or when checking whether a generalized quadrangle has an ovoid, the following idea proves to be useful.

Consider a step in the recursive process where the current partial ovoid gives rise to a line for which only one point still belongs to the allowed set A. If that point is not added to the current partial ovoid, then the resulting partial ovoid can never be extended to an ovoid, so we can prune these possibilities and force the point to be added to the current partial ovoid.

As an example, Figure 4.4 illustrates this idea on the quadrangle $W(2)$. After fixing the points 0 and 7 (see Figure 4.4(a)) we get the allowed set $A = \{9, 10, 12, 13\}$. This means that on the line $(6, 10, 11)$ there is only one point, i.e. 10, left which can be added to the partial ovoid. Similarly on the line $(2, 11, 12)$ only point 12 is left and on the line $(2, 13, 14)$ only 13 is left. Hence, when searching for ovoids, we can forcedly add the points 10, 12 and 13, thus pruning the possible extensions of $\{0, 7\}$ that do not contain these points (see Figure 4.4(b)). For this example this concludes the search.



(a) Partial ovoid $\{0, 7\}$

(b) Forcing 10, 12 and 13 which results in the ovoid

Figure 4.4: Illustrating the idea of forcing points in $W(2)$

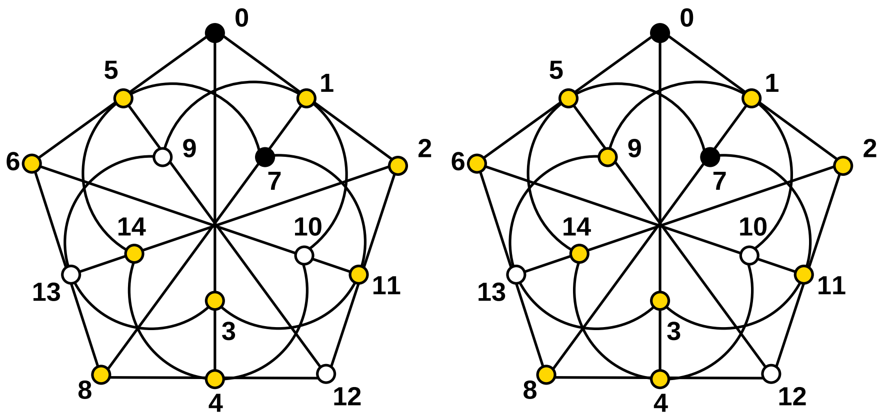
4.5 Pruning based on span and trace properties

In special situations, we can improve the pruning in a clique finding algorithm in a collinearity graph by using some properties of the generalized quadrangles and its ovoids.

We will now describe pruning techniques, which use Theorem 2.2.4 and Theorem 2.2.5.

4.5.1 Generalized quadrangles of order s

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a generalized quadrangle of order s, having a regular pair $\{x, y\}$ of non-collinear points. Let $\mathcal{O}$ be an ovoid of $\mathcal{S}$. The following observations can be used to prune a recursive process classifying all ovoids. Consider a step in the recursive process. Let A denote the set of allowed points in this step. Let y denote the point which is added to the current partial ovoid in this step. Let $\mathcal{O}'$ denote the current partial ovoid in this



(a) Partial ovoid $\{0, 7\}$ (b) Removing 9 from allowed set

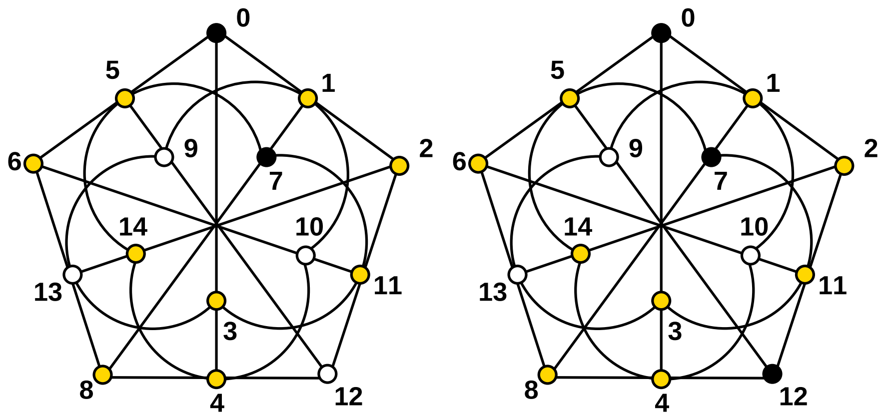
Figure 4.5: Illustrating the idea of "span" pruning in $W(2)$.

step (after adding y), which will be completed to an ovoid $\mathcal{O}$. Then:

1.ST If there is a point $x \in \mathcal{O}'$, such that (x, y) is a regular pair, then we already have $|\mathcal{O}' \cap \{x, y\}^{\perp\perp}| = 2$ (since $x, y \in \mathcal{O}'$). Hence, no other points from $\{x, y\}^{\perp\perp}$ can be used to extend $\mathcal{O}'$ to an ovoid and we can prune these possibilities from A.

Figure 4.5 illustrates this idea on the quadrangle $W(2)$. After fixing the points 0 and 7 (see Figure 4.5(a)) we get the allowed set $A = \{9, 10, 12, 13\}$. Since all points of $W(2)$ are regular (Theorem 2.3.2), $\{0, 7\}$ can be above mentioned pair $\{x, y\}$. We find the set $\{0, 7\}^{\perp\perp}$. First, $\{0, 7\}^{\perp} = \{1, 3, 5\}$, hence $\{0, 7\}^{\perp\perp} = \{0, 7, 9\}$. This means that point 9 can be removed from the allowed set.

2.ST If there is a point $x \in A$, such that (x, y) is a regular pair, we proceed as follows. Since $y \in \mathcal{O}$, then $\{x, y\}^{\perp} \notin \mathcal{O}$, i.e. $|\mathcal{O} \cap \{x, y\}^{\perp}| = 0$, implicating by Theorem 2.2.4 that $|\mathcal{O} \cap \{x, y\}^{\perp\perp}| = 2$. Since $y \in \{x, y\}^{\perp\perp}$, two possibilities remain.



(a) Partial ovoid $\{0, 7\}$ (b) Forcing 12

Figure 4.6: Illustrating the idea of "span" forcing in $W(2)$.

(a) If $|\mathcal{O}' \cap \{x, y\}^{\perp\perp}| = 2$, then it can never be extended to an ovoid by points from $\{x, y\}^{\perp\perp}$. We can prune these possibilities from A. This is similar to the situation illustrated in Figure 4.5.

(b) Suppose now that $\mathcal{O}' \cap \{x, y\}^{\perp\perp} = \{y\}$. Let $A' = A \cap \{x, y\}^{\perp\perp}$. Assume $|A'| = 1$. If this single point in A' is not added to $\mathcal{O}'$, then the resulting partial ovoid can never be extended to an ovoid, so we can force the point to be added to $\mathcal{O}'$.

Figure 4.6 illustrates this idea on the quadrangle $W(2)$. After fixing the points 0 and 7 (see Figure 4.6(a)) we get the allowed set $A = \{9, 10, 12, 13\}$. Since all points of $W(2)$ are regular (Theorem 2.3.2), $\{0, 12\}$ can be above mentioned pair $\{x, y\}$. We find the sets $\{0, 12\}^{\perp}$ and $\{0, 12\}^{\perp\perp}$. First, $\{0, 12\}^{\perp} = \{2, 5, 4\}$, hence $\{0, 12\}^{\perp\perp} = \{0, 12, 14\}$. Hence, when searching for ovoids, we can forcedly add the point 12 (see Figure 4.6(b)).

3.ST If there are two points $x, z \in A$, such that (x, z) is a regular pair of non-collinear points, then $|\mathcal{O} \cap \{x,z\}^{\perp\perp}| \in \{0,2\}$ (Theorem 2.2.4). We count now the points of $\mathcal{O}'$ contained in $\{x,z\}^{\perp\perp}$. There are three possible cases.

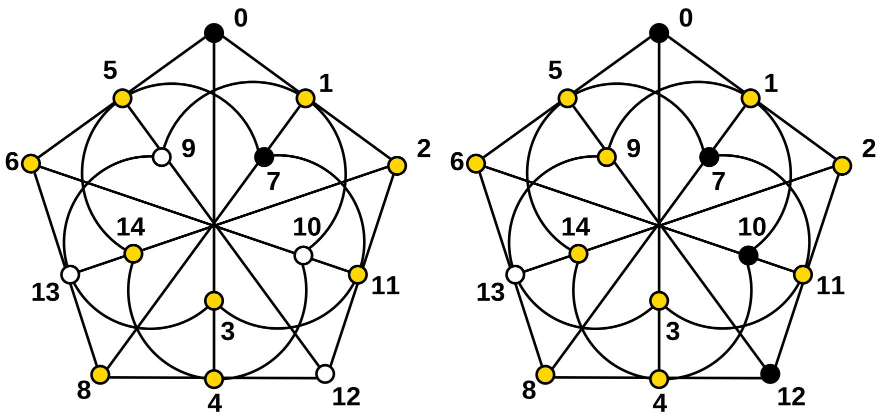
(a) If $|\mathcal{O}' \cap \{x,z\}^{\perp\perp}| = 2$, then we conclude, as above, that $\mathcal{O}'$ can never be extended by points from $\{x,z\}^{\perp\perp}$, so we can prune these possibilities from A. Figure 4.5 illustrates a similar idea as described.

(b) If $|\mathcal{O}' \cap \{x,z\}^{\perp\perp}| = 1$, then $|\mathcal{O} \cap \{x,z\}^{\perp\perp}| = 2$. Let $A' = A \cap \{x,z\}^{\perp\perp}$. As above, if $|A'| = 1$, then its single point must be forced to be added to $\mathcal{O}'$. In Figure 4.6 a similar forcing as we describe is shown.

(c) Suppose $|\mathcal{O}' \cap \{x,z\}^{\perp\perp}| = 0$. Denote $A_{\{x,z\}^\perp} = A \cap \{x,z\}^\perp$ and $A_{\{x,z\}^{\perp\perp}} = A \cap \{x,z\}^{\perp\perp}$. If $|A_{\{x,z\}^\perp}| = 1$, then $\mathcal{O}'$ can never be extended by the single point of $A_{\{x,z\}^\perp}$ and we can prune this possibility. If $|A_{\{x,z\}^\perp}| = 0$ and $|A_{\{x,z\}^{\perp\perp}}| = 2$, then the two points of $A_{\{x,z\}^{\perp\perp}}$ are forced to be added to the current partial ovoid.

Figure 4.7 illustrates this idea on the quadrangle $W(2)$. After fixing the points 0 and 7 (see Figure 4.7(a)) we get the allowed set $A = \{9, 10, 12, 13\}$. Since all points of $W(2)$ are regular (Theorem 2.3.2), $\{10, 12\}$ can be the above mentioned pair $\{x, y\}$. We find the sets $\{10, 12\}^\perp$ and $\{10, 12\}^{\perp\perp}$. First, $\{10, 12\}^\perp = \{4, 9, 11\}$, hence $\{10, 12\}^{\perp\perp} = \{3, 10, 12\}$. It means that point 9 can be removed from the allowed set and points 10, 12 can be forcedly added to the current partial ovoid (see Figure 4.7(b)).

The reader may ask why we split up this technique into partial cases instead of solving it generally in one technique. Section 4.7 will show that if we use all described partial methods, we get an effective pruning technique in terms of the number of recursive calls, which is, however, slower than the one described as the first pruning method.



(a) Partial ovoid $\{0, 7\}$

(b) Removing 9 from allowed set, forcing 10 and 12

Figure 4.7: Illustrating the idea of "span" pruning and "span" forcing in $W(2)$.

4.5.2 Generalized quadrangles of order (s, t)

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a generalized quadrangle of order (s, t) with a hyperbolic line $\{x, y\}^{\perp\perp}$ of cardinality $p+1$ with $pt = s^2$. Let $\mathcal{O}$ be an ovoid of $\mathcal{S}$. If there is a pair $\{x, y\}$, such that $\{x, y\}^{\perp\perp}$ is a required hyperbolic line, we proceed as follows. From Theorem 2.2.5 it follows that $|\mathcal{O} \cap \{x, y\}^{\perp\perp}| = 0$. This means that the current partial ovoid can never be extended by points of the allowed set, which are contained in $\{x, y\}^{\perp\perp}$. We can prune these possibilities.

4.5.3 Classical generalized quadrangles

The above techniques are described for an arbitrary generalized quadrangle satisfying the required conditions. We concentrate on the classical examples.

The only classical generalized quadrangles of order s with ovoids are $W(q)$, q even, and $Q(4,q)$. We recall that for q even, $W(q)$ is isomorphic with $Q(4,q)$. We see from Theorem 2.3.2, that all points of $W(q)$ are regular, however all points of $Q(4,q)$, q odd, are antiregular. Hence, the described technique for GQ with order s is suitable only for $W(q)$, q even.

The only classical generalized quadrangles of order $(s,t), s \neq t \neq 1$ which are known to have ovoids are $H(3,q^2)$. It is not known, whether the dual $H(4,q^2)^D$ of $H(4,q^2)$, $q > 2$ has ovoids. All points of $H(3,q^2)$ are regular (Theorem 2.3.2). It means that $|\{x,y\}^{\perp\perp}| = q+1$ for any non-collinear points x, y. However, Theorem 2.2.5 requires the existence of a hyperbolic line with cardinality q^3+1 in $H(3,q^2)$. In case of $H(4,q^2)^D$, it is known that $|\{x,y\}^{\perp\perp}| = 2$ for any non-collinear points x, y (Theorem 2.3.2). However, Theorem 2.2.5 requires the existence of a hyperbolic line with cardinality q^4+1 in $H(4,q^2)$. Hence, the described technique for GQ with order (s,t), $s \neq t$ is not suitable for any classical thick generalized quadrangle.

4.6 Heuristic algorithms

A simple greedy algorithm builds a maximal clique step by step by adding vertices from a set of allowed vertices until this set is empty.

Several strategies are possible for choosing a vertex to be added in each step. For instance, adding the vertex that leaves the largest number of vertices in the allowed set will tend to build large maximal cliques. A similar strategy, which is inspired by the pruning strategies using colorings, consists of adding the vertex that leaves the largest number of colors in the allowed set for the next step; this also results in large maximal cliques. On the other hand, choosing the vertex that leaves the least number of vertices or the least number of colors in the set of allowed vertices, is expected to result in small maximal cliques.

Starting from a maximal clique obtained by one of the above approaches, a simple restart strategy removes some of the vertices of the clique and again adds vertices until the clique is maximal. Both the removing and the adding can be done either randomly or following one of the above heuristics.

4.7 Comparison of used techniques

In what follows, we give results obtained by computer searches implementing some of the techniques described in the previous sections. The presented results show how important it is to choose an appropriate algorithm.

No further details about the Östergård algorithm and the technique of spread coloring are given. In our experiments we noticed that these techniques are not effective for our purposes. In the case of spread coloring, we noticed that the timing results depend strongly on the chosen spread. Hence, a random choice of the spread is not efficient. However, we still do not know if there is a good choice for the spread used for the spread coloring which will speed up the exhaustive algorithms.

All our programs are written in Java and call *nauty* [49] using the Java Native Interface (JNI) for the isomorph pruning. The timing results are obtained on a 1.6Ghz Pentium processor running Linux.

4.7.1 Effect of forcing vertices using look-ahead

In Tables 4.1 and 4.2 we illustrate the effect of forcing vertices when classifying all non-equivalent ovoids or spreads in a generalized quadrangle or when proving that no ovoid or spread exists. We give timings for versions of the program with and without isomorph pruning, and also versions of the program with and without a final filtering of the generated ovoids in order to obtain only the non-equivalent ovoids. For each quadrangle we also give the order $|G|$ of its collinearity graph, the running times of the different versions and the number $\#\mathcal{O}$ of ovoids obtained.

We present results for ovoids in the smallest cases of $Q(4,q)$ and $H(3,q^2)$, which are known to have ovoids, and for spreads in the case $H(4,4)$ (i.e. ovoids in $H(4,4)^D$), which are known not to exist [12].

From the Tables it is clear that the approach of forcing vertices is an effective technique. Its effect is most notable when simply generating all

GQ	$\|G\|$	No final equivalence check: Time, No forcing	No final equivalence check: Time, Forcing	No final equivalence check: $\#\mathcal{O}$	With final equivalence check: Time, No forcing	With final equivalence check: Time, Forcing	With final equivalence check: $\#\mathcal{O}$
$Q(4,7)$	400	13 s	1 s	21	16 s	4 s	1
$Q(4,8)$	585	1 722 s	93 s	532	1 865 s	275 s	2
$Q(4,9)$	820	43 355 s	529 s	14 796	58 858 s	16 390 s	2
$H(3,9)$	280	933 s	93 s	196 992	49 260 s	48 435 s	26
$H(4,4)^D$	297	75 s	4 s	0	–	–	–

Table 4.1: Effect of forcing vertices using look-ahead when searching for all ovoids. No isomorph pruning is done.

ovoids without isomorph pruning and without final equivalence check, as can be seen in the first set of timings columns in Table 4.1. Of course, when a large number of ovoids are generated (such as for $Q(4,9)$ and $H(3,9)$), the final equivalence check will account for most of the total running time, as shown in the second set of columns of Table 4.1.

When isomorph pruning is used, there is still a considerable gain in time when generating the ovoids, as can be seen in the first set of timings columns in Table 4.2. Again in some cases, e.g. $H(3,9)$, most of the total running time is spent in the final equivalence check, as can be seen in the second set of timings columns in Table 4.2.

4.7.2 Effect of span pruning and forcing

In this section we present results of the three pruning techniques based on span and trace properties from Section 4.5, for ovoids in $W(q)$, for small q even. Note that $W(q) \cong Q(4,q)$ in these cases.

In Table 4.3 we compare the running time and number of recursive calls (#c) of the different versions (1.ST, 2.ST and 3.ST) as well as a combination of all three. For each quadrangle we also list the size $|G|$ of its collinearity graph. We conclude that, although the first technique is the least effective regarding the number of recursive calls, it is the best one when comparing the running time.

GQ	$\|G\|$	No final equivalence check			With final equivalence check		
		Time			Time		
		No forcing	Forcing	$\#\mathcal{O}$	No forcing	Forcing	$\#\mathcal{O}$
$Q(4,8)$	585	13 s	3 s	12	16 s	6 s	2
$Q(4,9)$	820	47 s	5 s	59	117 s	75 s	2
$Q(4,11)$	1464	/	1 725 s	5	/	1 752 s	1
$H(3,9)$	280	6.1 s	3.6 s	783	225 s	217 s	26
$H(4,4)^D$	297	665 ms	474 ms	0	–	–	–

Table 4.2: Effect of forcing vertices using look-ahead when searching for all ovoids. For the smaller cases isomorph pruning is done on 5 levels, for $Q(4,11)$ and $H(3,9)$ isomorph pruning is done on 7 levels.

		1.ST		2.ST		3.ST		all together	
q	$\|G\|$	Time	#c	Time	#c	Time	#c	Time	#c
4	85	20 ms	7	20 ms	6	80 ms	7	80 ms	5
8	585	1.2 s	127	2.2 s	81	115s	66	80 s	47
16	4369	12 h	$2\ 10^6$	> 24 h					

Table 4.3: Effect of span and trace pruning when searching for all ovoids in $W(q)$ without final equivalence check. Isomorph pruning is done on 5 levels.

GQ	$\|G\|$	Without final equiv. check			With final equiv. check		
		Time			Time		
		No ST	With 1.ST	$\#\mathcal{O}$	No ST	With 1.ST	$\#\mathcal{O}$
$W(8)$	585	73 s	5.7 s	532	180 s	110 s	2

Table 4.4: Comparing timing results for $W(8)$ and $W(16)$. No isomorph pruning is done.

GQ	$\lvert G \rvert$	Without final equiv. check: Time, No ST	Without final equiv. check: Time, With 1.ST	Without final equiv. check: $\#\mathcal{O}$	With final equiv. check: Time, No ST	With final equiv. check: Time, With 1.ST	With final equiv. check: $\#\mathcal{O}$
$W(8)$	585	2.5 s	1.2 s	20	6.7 s	5.6 s	2
$W(16)$	4369	> 4 days	12 h	8	> 4 days	12.9 h	1

Table 4.5: Comparing timing results for $W(8)$ and $W(16)$. Isomorph pruning is done on 5 levels.

In Tables 4.4 and 4.5 we illustrate the effect of this first version (1.ST) when classifying all non-equivalent ovoids in a generalized quadrangle. We give timings for versions of the program with and without isomorph pruning, and also versions of the program with and without a final filtering of the generated ovoids in order to obtain only the non-equivalent ovoids. We also list the number $\#\mathcal{O}$ of ovoids obtained. We give comparisons only for two examples, $W(8)$ and $W(16)$. Note that the obtained results confirm the results from [58] and [52] which will be described in Section 5.1. The running time for $q < 8$ is too small to generalize. The generalized quadrangle $W(32)$ with its 33825 vertices is too large for our computer search. Nevertheless it is clear that the approach of span and trace pruning is an effective technique.

5 Maximal partial ovoids of $W(q)$ and $Q(4,q)$

Since (partial) spreads in a generalized quadrangle $\mathcal{S}$ are (partial) ovoids in $\mathcal{S}^D$, we will only discuss (partial) ovoids. We note that some of the given references deal with (partial) spreads. We found it more convenient to translate these results into corresponding dual terms.

The following chapters on partial ovoids are organized as follows. Each chapter deals with one type of generalized quadrangles and its dual. For each quadrangle we give a survey of known results: information about existence of ovoids, theoretical upper and lower bounds on the size of maximal partial ovoids, as well as earlier computer results. We compare these with the results obtained by our own computer searches. In many cases we could improve earlier results or theoretical bounds. In some cases we were able to extend the results obtained by our computer searches to a general construction.

The results collected in this chapter are to appear in *On the smallest maximal partial ovoids and spreads of the generalized quadrangles* $W(q)$ *and* $Q(4,q)$ [17].

Name	Reference	q
elliptic quadric		all
Kantor	[46]	p^h $(h > 1)$, p odd
Penttila-Williams	[60]	3^5
Ree-Tits slice	[46]	3^{2h+1}, $(h > 0)$
Thas-Payne	[56]	3^h, $(h > 2)$
Tits	[70]	$2^{2h+1}(h > 0)$

Table 5.1: The known ovoids of $Q(4,q)$.

5.1 Ovoids in $Q(4,q)$ and $W(q)$

The following result is well known. The proof can be found in e.g. [55].

Theorem 5.1.1 *The generalized quadrangle $Q(4,q)$ has an ovoid for every q.*

Proof. In $\mathrm{PG}(4,q)$ consider a hyperplane $\mathrm{PG}(3,q)$ for which $\mathrm{PG}(3,q) \cap Q(4,q)$ is an elliptic quadric Q^-. Then Q^- is an ovoid of $Q(4,q)$. □

Complete classifications of the ovoids of $Q(4,q)$ are not yet known in general. The only known ovoids in $Q(4,q)$ are listed in Table 5.1 which we found in extended form in [3].

Recently, the following result was shown by S. Ball, P. Govaerts and L. Storme [4], and confirmed by our computer searches for $q \leq 11$, q prime (see Tables 4.1 and 4.2 in Chapter 4 for $q = 7, 11$).

Theorem 5.1.2 (Ball, Govaerts and Storme [4]) *If q is prime, then every ovoid of $Q(4,q)$ is an elliptic quadric.*

T. Penttila and G. Royle [59] proved by a computer classification that $Q(4,9)$ has 2 non-equivalent ovoids. Our computer search (see Table 4.2) confirms this result.

We focus now on the dual of $Q(4,q)$, on the generalized quadrangle $W(q)$.

Theorem 5.1.3 (Payne and Thas [55]) *The generalized quadrangle $W(q)$ has ovoids if and only if q is even.*

Proof. If q is even, then $W(q)$ is isomorphic to $Q(4,q)$, and hence $W(q)$ has ovoids. If q is odd, all points of $W(q)$ are regular (Theorem 2.3.2). Theorem 2.2.4 implies that $W(q)$, q is odd, has no ovoid. □

In [66] J.A. Thas proved the following.

Theorem 5.1.4 (Thas [66]) *An ovoid of $W(q)$ with ambient space $\mathrm{PG}(3,q)$, q even, is an ovoid of $\mathrm{PG}(3,q)$.*

The converse was proved earlier by B. Segre [62]. It means that the classification of ovoids in $W(q)$ is equivalent to the classification of ovoids in $\mathrm{PG}(3,q)$, q even. By the study of ovoids of $\mathrm{PG}(3,q)$, q even, the following results on ovoids of $W(q)$, q even, resp. $Q(4,q)$, q even, are known.

For $W(8)$, T. Penttila and C. Praeger [58] proved that there are 2 non-equivalent ovoids; an earlier computer classification was done by G. Fellegara [32]. All ovoids in $W(16)$ are elliptic quadrics. C.M. O'Keefe and T. Penttila proved this by a computer classification [51] and two years later without a computer [52]. Ovoids in $W(32)$ are either elliptic quadrics or Tits ovoids, a result obtained by C.M. O'Keefe, T. Penttila and G. Royle [53] with the aid of a computer.

As can be seen from the Tables 4.2 and 4.5 in Chapter 4, our computer searches confirm these results for $q = 8$ and $q = 16$.

5.2 Largest maximal partial ovoids in $W(q)$, q odd

Since the generalized quadrangle $W(q)$, q odd, has no ovoids, the question of the largest maximal partial ovoid arises.

For $W(q)$, q odd, G. Tallini obtained the following upper bound.

q	UB [65]	$|\mathcal{O}|$ found	$\#\mathcal{O}$
3	7	**7**	1
5	21	**18**	2
7	43	**33**	1
9	73	**51**	
11	111	**70**	
13	157	**92**	
17	273	**129**	
19	343	**150**	
23	507	**190**	
25	601	**203**	
27	703	**236**	

Table 5.2: Large maximal partial ovoids of $W(q)$, for small values of q, q odd, obtained by heuristic and/or exhaustive search. For $q \leq 7$, the size and classification of the largest maximal partial ovoids were determined by exhaustive search.

Theorem 5.2.1 (Tallini [65]) *Consider the generalized quadrangle $W(q)$. If q is odd and $\mathcal{O}$ is a partial ovoid, then $|\mathcal{O}| \leq q^2 + 1 - q$.*

In Table 5.2, we give the size $|\mathcal{O}|$ of the largest maximal partial ovoid found by our computer search and compare it with the value of the upper bound (UB) from Theorem 5.2.1. For $q \leq 7$, we confirmed by exhaustive search that the largest value found is indeed the size of the largest maximal partial ovoid. For these values of the parameters we classified all partial ovoids of that size. For $q = 5, 7$, the value found by our computer search improves the best known theoretical bounds. For $q \geq 9$, the largest values were obtained by heuristic search. Hence, the real sizes of the largest partial ovoids can differ from our values. However we expect that for $q = 9$ the real size is very close to the one found.

5.3 Large maximal partial ovoids in $Q(4,q)$ and $W(q)$, q even

Since $Q(4,q)$ is isomorphic to $W(q)$ for q even, all results on sizes of maximal partial ovoids of $W(q)$, q even, hold also for $Q(4,q)$, q even.

For $W(q)$, q even, G. Tallini obtained the following result, giving an upper bound for maximal partial ovoids different from ovoids.

Theorem 5.3.1 (Tallini [65]) *Consider the generalized quadrangle $W(q)$. If q is even, $q \geq 4$ and $\mathcal{O}$ is a maximal partial ovoid, then either $\mathcal{O}$ is an ovoid or $|\mathcal{O}| < q^2 - q/2$.*

Recently, M.R. Brown, J. De Beule and L. Storme [13] obtained an improvement on this result of G. Tallini.

Theorem 5.3.2 (Brown et al. [13]) *Suppose that $\mathcal{O}$ is a maximal partial ovoid of $W(q)$. If q is even, then either $\mathcal{O}$ is an ovoid or $|\mathcal{O}| \leq q^2+1-q$.*

Furthermore, this bound is sharp. They construct maximal partial ovoids of size q^2+1-q for q even.

The same result as in Theorem 5.3.2 was obtained by A. Klein and K. Metsch [47]. For q odd, the same upper bound was given by G. Tallini (see Theorem 5.2.1 in Section 5.2). A. Klein and K. Metsch [47] give a proof, which is common for even and odd q.

Theorem 5.3.3 (Klein and Metsch [47]) *Suppose that $\mathcal{O}$ is a maximal partial ovoid of $W(q)$. Then either $\mathcal{O}$ is an ovoid or $|\mathcal{O}| \leq q^2+1-q$.*

For q even, our computer searches find a maximal partial ovoid of size q^2-q+1 and no maximal partial ovoids with sizes larger than q^2-q+1 and smaller than q^2+1, thus confirming the results of [13] and [47]. We also observed the existence of a maximal partial ovoid of size $q^2-q+1-(q-2) = q^2-2q+3$, and we found no maximal partial ovoids with size larger than q^2-2q+3 and smaller than q^2-q+1 (see Table 5.3).

q	Second largest $\lvert\mathcal{O}\rvert$ found (of size q^2-2q+3)	Largest $\lvert\mathcal{O}\rvert$ found (of size q^2-q+1)
4	11	13
8	51	57
16	227	241
32	963	993

Table 5.3: Large maximal partial ovoids of $W(q)$, different from ovoids, for small values of q, q even, obtained by heuristic and/or exhaustive search. For $q = 4$, the size of the largest and second largest maximal partial ovoids, different from ovoids, were determined by exhaustive search.

We can describe in a compact way a geometric construction for maximal partial ovoids of sizes q^2-q+1 and q^2-2q+3 of $W(q)$, q even. We explain the construction on $Q(4,q)$.

Construction 5.3.4

1. First notice that $\left|C^{\perp}\right| \in \{1, q+1\}$ for any conic C in $Q(4,q)$, q is even. From this we see that if we consider a conic C in an elliptic quadric $\mathcal{O} := Q^-(3,q) \subset Q(4,q)$, then necessarily $C^{\perp}$ is a unique point c. It is easily seen that $(\mathcal{O} \cup \{c\}) \setminus C$ is a maximal partial ovoid of size q^2-q+1.

2. Now let $\mathcal{O}$ be an elliptic quadric of $Q(4,q)$ and suppose that C_1 and C_2 are two conics of $\mathcal{O}$, with $|C_1 \cap C_2| = 2$. Clearly the points $c_1 := C_1^{\perp}$ and $c_2 := C_2^{\perp}$ are not collinear (since $|C_1 \cap C_2| = 2$). If $q > 2$, it follows easily that $(\mathcal{O} \cup \{c_1, c_2\}) \setminus (C_1 \cup C_2)$ is a maximal partial ovoid of size q^2-2q+3.

q	q^2-1	Second largest $\vert\mathcal{O}\vert$ found	Largest $\vert\mathcal{O}\vert$ found	$\#\mathcal{O}$
3	8	5	**8**	1
5	24	22	**24**	1
7	48	44	**48**	1
9	80	73	**74**	
11	120	112	**120**	
13	168		**158**	
17	288		**274**	
19	360		**344**	

Table 5.4: Large maximal partial ovoids of $Q(4, q)$, for small values of q, q odd, obtained by heuristic and/or exhaustive search. For $q \leq 7$, the classification of the largest and the second largest maximal partial ovoids were determined by exhaustive search. For $q = 9$, the existence of a maximal partial ovoid of size $q^2 - 1 = 80$ was excluded by exhaustive search.

5.4 Large maximal partial ovoids in $Q(4, q),\ q$ odd

The only upper bound on a maximal partial ovoid in $Q(4, q),\ q$ odd, different from an ovoid, comes from Theorem 2.2.6. In particular, any maximal partial ovoid of size q^2 of $Q(4, q)$ is contained in a uniquely defined ovoid of $Q(4, q)$. Hence, for any maximal partial ovoid $\mathcal{O}$, different from an ovoid,

$$|\mathcal{O}| \leq q^2 - 1.$$

In Table 5.4, for each value of q, we give the size $|\mathcal{O}|$ of the largest maximal partial ovoid, different from an ovoid, found by our computer search and compare it with the value of $q^2 - 1$.

For $q = 3, 5, 7, 11$, a maximal partial ovoid of size $q^2 - 1$ is found. For $q \leq 7$ we showed by exhaustive search that the maximal partial ovoids of this size are unique up to isomorphism. For $q = 9$, it is confirmed by exhaustive search that no such maximal partial ovoid (of size $q^2-1=80$) exists. For larger values of q, no maximal partial ovoids of size q^2-1 were found by heuristic searches.

When no maximal partial ovoid of size q^2-1 is found, the largest maximal partial ovoid found has size q^2-q+2. When a maximal partial ovoid of size q^2-1 is found, the second largest maximal partial ovoid found has also size q^2-q+2. This holds for all values of q considered except for $q=3$, when the values q^2-1 and q^2-q+2 are the same (see Table 5.4).

A maximal partial ovoid of size q^2-q+2 can be constructed in the following way.

Construction 5.4.1 Let $\mathcal{O}$ be an elliptic quadric in $Q(4,q)$, q odd. Notice that $|C^\perp|\in\{0,2\}$ for any conic C in $\mathcal{O}$. Choose a conic C of $\mathcal{O}$ such that $C^\perp=\{p_1,p_2\}$. Then it is easy to see that $(\mathcal{O}\cup\{p_1,p_2\})\setminus C$ is a maximal partial ovoid of size q^2-q+2.

Inspired by our computer result for $q=9$, J. De Beule and A. Gács solved the existence problem of maximal partial ovoids of size (q^2-1) for $q=p^h$, p odd prime, $h>1$.

Theorem 5.4.2 (De Beule and Gács [22]) *Suppose that $q=p^h$, p an odd prime, $h>1$. Then $Q(4,q)$ has no complete (q^2-1)-arcs.*

For $q=5,7,11$, the maximal partial ovoids of size q^2-1 can be constructed from the blocking sets of $Q(4,q)$ of size q^2+q-2. Since an introduction to the blocking sets is needed we will discuss this construction in Section 9.2.4.

5.5 Small maximal partial ovoids in $W(q)$

5.5.1 Spectra in $W(q)$

In Tables 5.5 and 5.6 we list the spectra of sizes of maximal partial ovoids found by a computer search. For each value of q, we give the order $|G|$ of the collinearity graph of the generalized quadrangle. The last column lists

q	$\|G\|$	Spectrum found	
3	40	**4,7**	
5	156	**6,11,**	12,**14**..18
7	400	**8,15,**	17..**20**..33
9	820	**10,19,**	25..**26**..51
11	1461	**12,23,**	28..**32**..70
13	2380	**14,27,**	**38**..92
17	5220	**18,35,**	**50**..129
19	7240	**20,39,**	**56**..150
23	12720	**24,47,**	**68**..70,72..190
25	16276	**26,51,**	**74**..76,78,80..203
27	20440	**28,55,**	**80**..236

Table 5.5: Spectrum of sizes for maximal partial ovoids of $W(q)$, for small values of odd q. For $q \leq 5$, the complete spectrum was obtained by exhaustive search. For larger values of q, the results are obtained by heuristic search. For $q = 5, 7$, the size of the largest partial ovoid was determined by exhaustive search.

the sizes for which the computer search found maximal partial ovoids of that given size. The notation $a..b$ means that a maximal partial ovoid of that size has been found for all values in the interval $[a, b]$.

For $q \leq 5$, exhaustive search confirmed that the spectrum found by heuristic search is complete.

5.5.2 The smallest maximal partial ovoids

First we prove the following result on maximal partial ovoids of $W(q)$.

Theorem 5.5.1 *A maximal partial ovoid $\mathcal{O}$ of $W(q)$ is a minimal blocking set with respect to the planes of* $\mathrm{PG}(3, q)$.

q	$\|G\|$	Spectrum found		
2	15	**3,5**		
4	85	**5,9,**		**11**,13,17
8	585	**9,17,**	21..**23**..47,49,	**51**,57,65
16	4369	**17,33,**	**47**,49,51..163,165,	**227**,241,257

Table 5.6: Spectrum of sizes for maximal partial ovoids of $W(q)$, for small values of even q. For $q = 2, 4$, the complete spectrum was obtained by exhaustive search. For larger values of q, the results are obtained by heuristic search.

Proof. Consider $W(q)$ in its natural representation in $\mathrm{PG}(3,q)$ described by the symplectic polarity η. Then it follows that every maximal partial ovoid $\mathcal{O}$ of $W(q)$ must be a blocking set of $\mathrm{PG}(3,q)$ with respect to the planes of $\mathrm{PG}(3,q)$. Namely, if there is a plane π skew to $\mathcal{O}$, then the point π^η extends $\mathcal{O}$ to a larger partial ovoid, which contradicts the maximality of $\mathcal{O}$.

Now assume that it is not minimal. Suppose that the point r of $\mathcal{O}$ is not essential as point of $\mathcal{O}$, considered as blocking set with respect to the planes of $\mathrm{PG}(3,q)$. Then every plane through r contains a second point of $\mathcal{O}$. So also the plane r^η contains a second point r' of $\mathcal{O}$. Then the totally isotropic line rr' contains at least two points of $\mathcal{O}$. This is impossible. □

We now concentrate on the smallest maximal partial ovoids of $W(q)$. It can be seen from Tables 5.5 and 5.6 that the smallest maximal partial ovoids found have size $q+1$.

Construction 5.5.2 The points of a hyperbolic line L in $\mathrm{PG}(3,q)$, i.e. a line in $\mathrm{PG}(3,q)$ which is not a line of the generalized quadrangle $W(q)$, form a maximal partial ovoid of size $q+1$.

This computer result is confirmed in the following theorem.

Theorem 5.5.3 *The smallest maximal partial ovoids of $W(q)$ have size $q+1$ and consist of the point sets of the hyperbolic lines of $W(q)$.*

Proof. This is an immediate corollary of Theorem 2.2.7 and Theorem 2.2.8, since all points of $W(q)$ are regular.

There is also another proof using Theorem 5.5.1. Since, from the result of R.C. Bose and R.C. Burton (Theorem 1.6.1), the smallest blocking set with respect to the planes of PG$(3,q)$ consists of the $q+1$ points of a line, the theorem follows. □

Corollary 5.5.4

1. *The smallest maximal partial spreads of $Q(4,q)$ have size $q+1$ and consist of the lines of a regulus of* PG$(3,q)$.
2. *The smallest maximal partial spreads of $W(q)$, q even, have size $q+1$ and consist of the lines of a regulus of* PG$(3,q)$.
3. *The smallest maximal partial ovoids of $Q(4,q)$, q even, have size $q+1$ and consist of the point sets of conics having the nucleus of $Q(4,q)$ as their nucleus.*

5.5.3 The second smallest maximal partial ovoids

Now that we have classified the smallest maximal partial ovoids of $W(q)$, we focus on results on the second smallest maximal partial ovoids of $W(q)$.

For the parameters considered in our computer search, we observed that maximal partial ovoids of size $2q+1$ were always found, while no maximal partial ovoids with sizes between $q+1$ and $2q+1$ were found (see Tables 5.5 and 5.6).

Construction 5.5.5 An example of a maximal partial ovoid $\mathcal{O}$ of size $2q+1$ can be obtained by taking all points except one point p on a hyperbolic line L in PG$(3,q)$, together with one arbitrary point (not collinear with one of the remaining points of L) from each of the $q+1$ lines through p. This

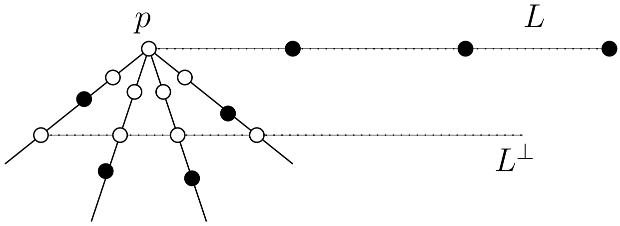


Figure 5.1: A maximal partial ovoid of size $2q+1$

example is illustrated in Figure 5.1. We see that $\mathcal{O}$ is a partial ovoid. We check the maximality. Since the hyperbolic line L is a maximal partial ovoid from Theorem 5.5.3, all points of $W(q)$ are collinear with a point of L. Hence, all points of $W(q)$ are collinear either with a point of $L \setminus \{p\}$ or with one of $q+1$ points of $\mathcal{O}$ on the lines through p.

We now present theoretical results on the second smallest maximal partial ovoids of $W(q)$. Such a maximal partial ovoid must be a blocking set with respect to the planes of $\mathrm{PG}(3,q)$ from Theorem 5.5.1. Hence, the planar non-trivial blocking sets are obvious candidates for such maximal partial ovoids. However, these are easily excluded.

Theorem 5.5.6 *A maximal partial ovoid $\mathcal{O}$ of $W(q)$, different from a hyperbolic line, cannot be a planar blocking set, i.e., the points of $\mathcal{O}$ cannot all lie in the same plane of* $\mathrm{PG}(3,q)$.

Proof. Suppose that $\mathcal{O}$ is a planar blocking set, lying in the plane π of $\mathrm{PG}(3,q)$. Let $r = \pi^\eta$. Then $r \notin \mathcal{O}$. But since $|\mathcal{O}| > q+1$, there is at least one totally isotropic line through r in π containing more than one point of $\mathcal{O}$; we have a contradiction. $\square$

We now use results on the minimal blocking sets with respect to planes of $\mathrm{PG}(3,q)$. The first result is of A.A. Bruen.

Theorem 5.5.7 (Bruen [14]) *The smallest non-trivial blocking sets with respect to planes of* $\mathrm{PG}(3, q)$ *are equal to the smallest planar non-trivial blocking sets of* $\mathrm{PG}(2, q)$.

Theorem 5.5.6 shows us that the second smallest maximal partial ovoids of $W(q)$ cannot be equal to the smallest non-trivial minimal blocking sets with respect to planes of $\mathrm{PG}(3, q)$. So for the second smallest maximal partial ovoids of $W(q)$, we need to focus on the smallest non-planar minimal blocking sets with respect to planes of $\mathrm{PG}(3, q)$. This allows us to obtain a considerably stronger result in some specific cases. We will use Theorem 1.6.4 and Theorem 1.6.5.

Recall that $s(q)$ denotes the cardinality of the second smallest non-trivial minimal blocking sets in $\mathrm{PG}(2, q)$.

Corollary 5.5.8 *The second smallest maximal partial ovoids* $\mathcal{O}$ *of* $W(q^2)$, $q = p^h$, $p > 3$ *prime,* $h \geq 1$, *contain at least* $s(q^2) + 1$ *points. If* $q = p > 2$, *then* $\mathcal{O}$ *contains at least* $3(p^2 + 1)/2 + 1$ *points.*

Proof. This follows immediately from Theorem 1.6.4 and the fact that $s(p^2) = 3(p^2 + 1)/2$ if $p > 2$ (see e.g. [64]). □

In [17], the existence of a maximal partial ovoid of $W(q^3)$ equal to a subgeometry $\mathrm{PG}(3, q)$ was posed as an open problem. Recently, S. De Winter and K. Thas proved its non-existence in [28].

Theorem 5.5.9 (De Winter and Thas [28]) *The generalized quadrangle* $W(q^3)$, $q = p^h$, $p \geq 7$ *prime, does not admit a maximal partial ovoid of size* $q^3 + q^2 + q + 1$.

This leads to the following corollary.

Corollary 5.5.10 *The second smallest maximal partial ovoids* $\mathcal{O}$ *of* $W(p^3)$, $p \geq 7$ *prime, contain at least* $3(p^3 + 1)/2$ *points.*

q	LB [17]	$\lvert\mathcal{O}\rvert$ found	$\#\mathcal{O}$
2	–	**5**	1
3	7	**7**	1
4	9	**9**	4
5	10	**11**	10

Table 5.7: The size and classification of the second smallest maximal partial ovoids of $W(q)$, $q \leq 5$ determined by exhaustive search.

Proof. The minimal blocking sets in $\mathrm{PG}(3,p^3)$, $p \geq 7$ prime, of size smaller than $3(p^3+1)/2$ have been classified in [63]. See Theorem 1.6.5 for the complete list, with exception of the Baer-subplane. The preceding results show that only a line can define a partial ovoid of $W(p^3)$. □

Finally in the case when $q = p > 2$ prime, we can use the result of Blokhuis (see Theorem 1.6.3) which states that every non-trivial planar blocking set of $\mathrm{PG}(2,p)$ contains at least $3(p+1)/2$ points.

Corollary 5.5.11 *Let $\mathcal{O}$ be a second smallest maximal partial ovoid of $W(p)$, $p > 2$ prime. Then $|\mathcal{O}| \geq 3(p+1)/2+1$.*

Remark 5.5.12 The preceding results can be translated into results on maximal partial spreads of $Q(4,q)$, on maximal partial spreads of $W(q)$, q even, and on maximal partial ovoids of $Q(4,q)$, q even.

In Table 5.7, we give the size $|\mathcal{O}|$ of the second smallest maximal partial ovoid found by our computer search. We compare it with the value of the lower bound (LB) for the second smallest maximal partial ovoid described in Theorem 5.5.8, Theorem 5.5.10 and Theorem 5.5.11. We confirmed by exhaustive search that the value found is indeed the size of the second smallest maximal partial ovoid. For these parameters we also determined all non-equivalent partial ovoids of that size.

Moreover, our computer results show that the obtained theoretical lower bounds are not sharp (e.g for $q = 5$).

5.5.4 Maximal partial ovoids of size $3q - 1$

Our computer results show the existence of a maximal partial ovoid of size $3q-1$, for all values of q considered (see Tables 5.5 and 5.6). Such a maximal partial ovoid can be constructed in the following way if $q \geq 4$.

Construction 5.5.13 Let X and Y be two skew totally isotropic lines. Choose distinct points x_1, x_2, x_3 and x on the line X and let y_i be $x_i^\perp \cap Y$, $i = 1, 2, 3$. Finally choose a point y on Y distinct from y_1, y_2, y_3 and $x^\perp \cap Y$ (we can choose y since $q \geq 4$). If we put $\mathcal{O}_1$ the set of all points of $(x_1y_2 \cup x_2y_3 \cup x_3y_1) \setminus \{x_i, y_i \mid i = 1, 2, 3\}$, then $\mathcal{O} := \mathcal{O}_1 \cup \{x, y\}$ is a maximal partial ovoid of size $3q - 1$.

To prove that $\mathcal{O}$ is indeed a partial ovoid we check here that no point of $x_1y_2 \setminus \{x_1, y_2\}$ can be collinear with a point of $x_2y_3 \setminus \{x_2, y_3\}$ (the other cases are treated analogously). By way of contradiction we assume that a point u of $x_1y_2 \setminus \{x_1, y_2\}$ is collinear with a point v of $x_2y_3 \setminus \{x_2, y_3\}$. Since u is also collinear with x_2 it follows that u is collinear with y_3. Hence, as y_3 is also collinear with y_2, we see that y_3 is collinear with x_1, a contradiction.

We now check the maximality. Assume that a point z would extend $\mathcal{O}$ to a larger partial ovoid. Clearly z does not belong to X or Y. The point of x_1y_2 collinear with z has to be either x_1 or y_2. Suppose without loss of generality that z is collinear with x_1. The point of x_2y_3 collinear with z has to be either x_2 or y_3, but cannot be x_2 as z is already collinear with x_1 on X. Consequently z is collinear with y_3. Finally, the point of x_3y_1 collinear with z has to be either x_3 or y_1. However it cannot be either of these points since z would then be collinear with two points on X or Y. We conclude that $\mathcal{O}$ is maximal.

5.6 Small maximal partial ovoids in $Q(4,q)$, q odd

The result in Theorem 5.6.1 can be treated as a lower bound for maximal partial ovoids of $Q(4,q)$, q odd. We will use a counting technique from [35] to prove it. The smallest integer greater than or equal to x is denoted by $\lceil x \rceil$.

Theorem 5.6.1 *Suppose that S is a maximal partial spread of $W(q)$, q odd. Then $|S| \geq \lceil 1.419q \rceil$.*

Proof. Suppose that $|S| = x$. Then there are exactly $D := q^3+q^2+q+1-x$ lines of $W(q)$ not belonging to S. Let n_i, $i = 1, \ldots, q+1$, denote the number of such lines intersecting exactly i lines of the partial spread S. Since S is a maximal partial spread, $\sum_i n_i = D$. By counting in two ways the pairs (L, M), where L is a line not belonging to S, where M is a line belonging to S, and where $L \sim M$, we obtain

$$\sum_i i n_i = x(q+1)q.$$

For the triples (L_1, L_2, M), where $L_1 \neq L_2$ are lines belonging to S, where M is a line not belonging to S and where $L_1 \sim M \sim L_2$, we obtain

$$\sum_i \binom{i}{2} n_i = \binom{x}{2}(q+1),$$

and for the quadruples (L_1, L_2, L_3, M), where L_1, L_2, L_3 are distinct lines belonging to S, where M is a line not belonging to S, and where $M \sim L_m$, $m = 1, 2, 3$, we obtain

$$\sum_i \binom{i}{3} n_i \leq \binom{x}{3} 2$$

(recall that when q is an odd prime power, $|\{L_1, L_2, L_3\}^\perp| \in \{0, 2\}$ for every triad of skew lines of $W(q)$ (see Theorem 2.1.2 and 2.3.2). Consider the polynomial $P(i) := (i - r_1)(i - r_2)(i - r_3)$ and the coefficients a_0, a_1, a_2, a_3 such that $P(i) = a_3\binom{i}{3} + a_2\binom{i}{2} + a_1 i + a_0$. We see that $a_3 = 6$, $a_2 =$

$-2(r_1+r_2+r_3)+6$, $a_1 = r_1r_2+r_1r_3+r_2r_3-(r_1+r_2+r_3)+1$, and $a_0 = -r_1r_2r_3$. Henceforth,

$$\sum_i P(i)n_i = a_3\sum_i \binom{i}{3}n_i + a_2\sum_i \binom{i}{2}n_i + a_1\sum_i in_i + a_0\sum_i n_i.$$

From this, using $a_3 > 0$, it follows that

$$\sum_i P(i)n_i \le 2a_3\binom{x}{3} + (q+1)a_2\binom{x}{2} + q(q+1)a_1x + a_0(q^3+q^2+q+1-x). \tag{5.1}$$

If we choose coefficients r_1, r_2, r_3 in such a way that $P(i)n_i \ge 0$ for every $i \in \{1,\dots,q+1\}$, then $\sum_i P(i)n_i \ge 0$ and consequently x has to be such that the right hand side of Equation (5.1) is greater than or equal to 0. The expansion of $\sum_i (i-1)(i-4)(i-5)n_i$ gives

$$0 \le 2x^3 - 13x^2 + 31x - 7x^2q + 27xq + 20xq^2 - 20q^3 - 20q^2 - 20q - 20, \tag{5.2}$$

from which we deduce that $x > 1.419q$. □

Remark 5.6.2 The result of the previous theorem can be slightly improved to $x \ge \lceil 1.419q + b \rceil$ for certain $b > 0$, by substituting $x = 1.419q + b$ and $a = 4$ in the right hand side of Equation (5.1), and by solving for the greatest b for which the obtained polynomial in q is still negative. The expression for b obtained in this way is a tedious formula in q, but it can easily be computed by computer for given q. For example, in the cases $q = 7, 9, 11$, this increases the smallest theoretical value of x by one to 11, 14 and 17, respectively. It should however be noted that b is extremely small with respect to q.

In Table 5.8, for each value of q, q odd, we give the size $|\mathcal{O}|$ of the smallest maximal partial ovoid found by our computer search and compare it with the value of the lower bound (LB) from Theorem 5.6.1 and Remark 5.6.2. For $q \le 5$ we confirmed by exhaustive search that the smallest value found is indeed the size of the smallest maximal partial ovoid.

In spite of the fact that the theoretical lower bounds are linear in q, computer results rather seem to indicate a quadratic lower bound.

q	LB [17]	$\vert\mathcal{O}\vert$ found	Non-existence
3	5	**5**	
5	8	**13**	8..12
7	11	**14**	11
9	14	**22**	
11	17	**28**	
13	19	**41**	
17	25	**67**	
19	27	**84**	

Table 5.8: Small maximal partial ovoids of $Q(4,q)$, for small values of q, q odd, obtained by heuristic and/or exhaustive search. For $q \leq 5$, the size and classification of the smallest maximal partial ovoids were determined by exhaustive search. For $q = 7$, the non-existence of maximal partial ovoids of size 11 was confirmed by exhaustive search.

5.7 Spectra of sizes for maximal partial ovoids in $Q(4,q)$, q odd

Table 5.9 gives results for maximal partial ovoids in $Q(4,q)$, q odd. We list the sizes for which our program found maximal partial ovoids of that given size.

In all cases our heuristic finds an ovoid (of size q^2+1). For $q = 3, 5$, we confirmed by exhaustive search that the spectrum found is complete. For $q = 7, 9$, we confirmed by exhaustive search for some sizes (also given in the table) that no maximal partial ovoid of that size exists.

q	$\|G\|$	Spectrum found		Non-existence
3	40	5,	**8**, 10	all other values
5	156	13..20,	**22**,**24**, 26	all other values
7	400	14,17..42,	**44**,**48**, 50	10,11,43,45,46,47,49 (open: 12,13,15,16)
9	820	22..68,70,73,	**74**,82	79,**80**
11	1461	28,30..106,109..110,	**112**,**120**, 122	
13	2380	41..42,44..136, 138,140,146,148,	**158**, 170	
17	5220	67..218,220..224,226, 228..230,232..238,240, 244,246..248,258,260,	**274**, 290	
19	7240	84..118,122..275,278,280, 282..286, 294,296,298,300, 310,312,326,328,	**344**, 362	

Table 5.9: Spectrum of sizes for maximal partial ovoids of $Q(4,q)$, for small values of odd q. For $q=3,5$, the complete spectrum was obtained by exhaustive search. For larger values of q, the results are obtained by heuristic search. For $q=7,9$, the non-existence of maximal partial ovoids of certain sizes was confirmed by exhaustive search.

6 Maximal partial ovoids of $Q^-(5, q)$ and $H(3, q^2)$

This chapter discusses the generalized quadrangle $Q^-(5, q)$ and its dual $H(3, q^2)$.

The results collected in this chapter have appeared in *Searching for maximal partial ovoids and spreads in generalized quadrangles* [19].

6.1 Large maximal partial ovoids in $Q^-(5, q)$

Since $Q^-(5, q)$ is of order (q, q^2), the following result is immediately obtained from Theorem 2.2.3.

Corollary 6.1.1 *The generalized quadrangle $Q^-(5, q)$ has no ovoids.*

The best known upper bounds on size of maximal partial ovoids are by J.A. Thas and by A. Blokhuis and G.E. Moorhouse:

Theorem 6.1.2 (Thas [67]) *If O' is a partial ovoid of $Q^-(5, q)$, then $|O'| \leq q^3 + 1 - q(q - 1)$.*

Theorem 6.1.3 (Blokhuis and Moorhouse [8]) *If K is a k-cap of a quadric in $PG(n,q)$ with $q = p^h$ and p prime, then*

$$k \leq \left[\binom{p+n-1}{n} - \binom{p+n-3}{n}\right]^h + 1.$$

R.H. Dye proved in [29] that the largest maximal partial ovoid of $Q^-(5,2)$ has size 6.

In [30], G.L. Ebert and J.W.P. Hirschfeld pose the question of the size of a complete cap. By an exhaustive computer search they show that the largest maximal partial ovoid in $Q^-(5,3)$ has size 16. All substructures of this size are isomorphic and they discuss their structure. For $q = 4$, the largest partial ovoid, which they found, has size 25. For $q = 5$, the largest partial ovoid, which they constructed, has size 42.

In [1], A. Aguglia, A. Cossidente and G.L. Ebert mention computer results on spectrum of sizes for maximal partial ovoids in $Q^-(5,q)$ (see Table 6.3). The largest partial ovoids found by their computer (non-exhaustive) search are as follows. For $q = 4$, it has size 25. For $q = 5$, the largest one, which they found, has size 39, which is smaller than the largest partial ovoid constructed by G.L. Ebert and J.W.P. Hirschfeld in [30]. For $q = 7$, the largest one has size 60 and finally for $q = 8$, the largest one has size 74.

In Table 6.1, we list the sizes $|\mathcal{O}|$ of the largest maximal partial ovoid found by our computer search. We compare it with the value of the upper bound (UB) from Theorem 6.1.2 and Theorem 6.1.3 and with the size of large maximal partial ovoids described in [1] and [30]. Note that the largest value found for $q \leq 4$ is indeed the size of the largest maximal partial ovoid – this was confirmed by exhaustive search. For small parameters we also determined all non-equivalent partial ovoids of that size.

A. Aguglia, A. Cossidente and G.L. Ebert [1] noticed that the values found by their computer search seem to indicate that upper bounds on the size should be quadratic in q, rather than the cubic upper bounds from Theorems 6.1.2 and 6.1.3. Our results also for larger values of q seem to confirm the quadratic upper bound.

q	UB		Earlier results		$\|\mathcal{O}\|$ found	$\#\mathcal{O}$
2	7	[67, 8]	6	[29]	**6**	1
3	21	[8]	16	[30]	**16**	1
4	37	[8]	25	[30, 1]	**25**	3
5	106	[67, 8]	42	[30]	**48**	
7	302	[67]	60	[1]	**98**	
8	217	[8]	74	[1]	**126**	
9	401	[8]	–		**146**	
11	1222	[67]	–		**216**	
13	2042	[67]	–		**273**	

Table 6.1: Large maximal partial ovoids of $Q^-(5,q)$, for small values of q, obtained by heuristic and/or exhaustive search. For $q \leq 4$, the size and classification of the largest maximal partial ovoids were determined by exhaustive search.

Finally, we give geometric constructions of maximal partial ovoids of size 16 in $Q^-(5,3)$ and of size 96 in $Q^-(5,7)$. The given construction of a maximal partial ovoid of size 16 in $Q^-(5,3)$ is well known. The construction of a maximal partial ovoid of size 96 in $Q^-(5,7)$ is by K. Coolsaet [20].

Construction 6.1.4 (for $Q^-(5,7)$ by K. Coolsaet) We may suppose that $Q^-(5,q)$, $q = 3,7$, has the following equation $x_0^2 + x_1^2 + \ldots + x_5^2 = 0$. Consider the set P' of points of the form $(1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$ and every cyclic permutation of the coordinates. The set P' has cardinality 32. The points of P' with an even number of minus-signs form a maximal partial ovoid of size 16 in $Q^-(5,3)$, while the points of P' with an odd number of minus-signs form another maximal partial ovoid of size 16.

A similar construction can be used in the following case. Consider the set P' of points of the form $(3, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$ and every cyclic permutation of the coordinates. The set P' has cardinality 192. The points of P' with an even number of minus-signs form a maximal partial ovoid of size 96 in $Q^-(5,7)$, while the points of P' with an odd number of minus-signs form another maximal partial ovoid of size 96.

As can be seen in Table 6.1, 16 is the size of the largest partial ovoid in $Q^-(5,3)$. However, the constructed maximal partial ovoids of size 96 in $Q^-(5,7)$ are not the largest ones. Our computer search found a maximal partial ovoid of size 98.

6.2 Small maximal partial ovoids in $Q^-(5,q)$

In [30], G.L. Ebert and J.W.P. Hirschfeld give a geometric construction for maximal partial ovoids of size q^2+1 in $Q^-(5,q)$. In further investigations, they found no maximal partial ovoids of size less than q^2+1. They mention that these q^2+1-caps can be candidates for the smallest maximal partial ovoids of $Q^-(5,q)$. They prove the following lower bound.

Theorem 6.2.1 (Ebert and Hirschfeld [30]) *Let K be a maximal partial ovoid of $Q^-(5,q)$. Then $|K| \geq 2q+1$. If $q \geq 4$, then $|K| \geq 2q+2$.*

In [1], A. Aguglia, A. Cossidente and G.L. Ebert construct other complete caps (i.e. maximal partial ovoids) of size q^2+1 in $Q^-(5,q)$. They also mention results from (extensive but not exhaustive) computer searches for possible values for the size of such complete caps. The results seem to indicate that lower bounds on the size should be quadratic in q, rather than the linear lower bound from Theorem 6.2.1. For $q = 7, 8$, they found complete caps of size less than q^2+1.

In Table 6.2, we give the size $|\mathcal{O}|$ of the smallest maximal partial ovoid found by our heuristic search and compare it with the value of the lower bound (LB) from Theorem 6.2.1, the value of q^2+1 and the size of small maximal partial ovoids from [1]. For $q \leq 4$ we confirmed by exhaustive search that the smallest value found is indeed the size of the smallest maximal partial ovoid. For $q \leq 3$ we also determined all non-equivalent partial ovoids of that size.

q	LB [30]	q^2+1	Earlier results	$\|\mathcal{O}\|$ found	$\#\mathcal{O}$
2	5	5	5 [1, 30]	**5**	1
3	7	10	10 [1, 30]	**7**	1
4	10	17	17 [1, 30]	**13**	
5	12	26	26 [1, 30]	**18**	
7	16	50	46 [1]	**32**	
8	18	65	57 [1]	**41**	
9	20	82		**52**	
11	24	122		**68**	
13	28	170		**89**	

Table 6.2: Small maximal partial ovoids of $Q^-(5,q)$, for small values of q, obtained by heuristic and/or exhaustive search. For $q \leq 4$, the size and classification of the smallest maximal partial ovoids were determined by exhaustive search.

For $q = 2,3$ the value of q^2+1 is the smallest maximal partial ovoid. For all $Q^-(5,q), q \geq 4$ considered we found maximal partial ovoids with size less than q^2+1. These results seem to indicate a subquadratic lower bound.

6.3 Spectrum of sizes for maximal partial ovoids in $Q^-(5,q)$

In [1], A. Aguglia, A. Cossidente and G.L. Ebert present (non-exhaustive) results on the spectra of sizes of maximal partial ovoids. For $q = 4$, they found maximal partial ovoids of sizes between 17 and 25, for $q = 5$ between 26 and 39. For $q = 7$, they found maximal partial ovoids of size between 46 and 60, while for $q = 8$, they found maximal partial ovoids of size between 57 and 74.

q	$\|G\|$	Earlier results [1]	Spectrum found
2	27		**5,6**
3	112		**7,11..13,16**
4	325	$\in [17,25]$	**13,15..25**
5	756	$\in [26,39]$	**18,20..44,48**
7	2752	$\in [46,60]$	**32..92,95,96,98**
8	4617	$\in [57,74]$	**41..121,123,125,126**
9	7300	–	**52..146**
11	15984	–	**68..212,214,216**
13	30772	–	**89..265, 267,268,272,273**

Table 6.3: Spectrum of sizes for maximal partial ovoids in $Q^-(5,q)$, for small values of q. For $q \leq 4$, the complete spectrum was obtained by exhaustive search. For larger values of q, the results are obtained by heuristic search.

For each $Q^-(5,q)$ considered, in Table 6.3 we list the sizes for which our program found maximal partial ovoids of that given size and compare it with the earlier results described in [1].

In all cases, we have extended the spectrum found in [1]. For $Q^-(5,4)$, we confirmed by exhaustive search that the spectrum found is complete.

6.4 Large maximal partial ovoids in $H(3,q^2)$

Theorem 6.4.1 (Payne and Thas [55]) *The generalized quadrangle $H(3,q^2)$ has ovoids.*

The following construction describes ovoids, which are sometimes called *classical ovoids.*

Let $\mathcal{H}$ be a nonsingular Hermitian variety in $\mathrm{PG}(3,q^2)$. Then any Hermitian curve, i.e. any nonsingular plane intersection of $\mathcal{H}$, is an ovoid of the generalized quadrangle $H(3, q^2)$.

Non-classical ovoids were first constructed by S.E. Payne and J.A. Thas [56] and are now known to exist in abundance. Recently T. Penttila [57] classified the ovoids in $H(3, 9)$ by a computer search and found that there are 26 non-equivalent ovoids. Our results, presented in Table 4.2 in Chapter 4, confirm these earlier results.

We focus now on the maximal strictly partial ovoids. Recently, A. Klein and K. Metsch [47] constructed maximal partial ovoids of size $q^3 - q + 1$ in $H(3, q^2)$ and showed that this example is the best possible.

Theorem 6.4.2 (Klein and Metsch [47]) *Suppose that $\mathcal{O}$ is a maximal partial ovoid of $H(3, q^2)$. Then either $\mathcal{O}$ is an ovoid or $|\mathcal{O}| \leq q^3 - q + 1$.*

In [2], A. Aguglia, G.L. Ebert and D. Luyckx produce several examples of size $q^3 - q + 1$.

In all considered cases, we have also found the largest strictly partial ovoids of size $q^3 - q + 1$ (see Table 6.5).

6.5 Small maximal partial ovoids in $H(3, q^2)$

In [43], J.W.P. Hirschfeld and G. Korchmáros present a lower bound for the size of maximal partial ovoids in Hermitian varieties $H(n, q^2)$. The given lower bound does not depend on n.

Theorem 6.5.1 (Hirschfeld and Korchmáros [43]) *The size k of a maximal partial ovoid of $H(n, q^2)$ satisfies $k \geq q^2 + 1$.*

They showed that this lower bound is sharp for $n = 3$ and even q.

A. Aguglia, G.L. Ebert and D. Luyckx [2] deal with small partial ovoids in $H(3, q^2)$. They also give a proof that this lower bound holds for $n = 3$ and show that this bound is reached if and only if q is even. This implies an increase of the lower bound if q is odd.

Corollary 6.5.2 (Aguglia, Ebert and Luyckx [2]) *If q is odd, then a maximal partial ovoid of $H(3, q^2)$ has at least $q^2 + 2$ points.*

q	LB		Earlier results [2]	$\|\mathcal{O}\|$ found
3	12	[2]	16	**16**
5	29	[28]	61	**56**
7	53	[28]	155	**142**

Table 6.4: Small maximal partial ovoids of $H(3,q^2)$, for small values of q, q odd, obtained by heuristic search.

It can be shown, that a maximal partial ovoid of $H(3,q^2)$ cannot have size exactly q^2+2. This was noticed by J.A. Thas in private communication with A. Aguglia, G.L. Ebert and D. Luyckx.

Theorem 6.5.3 (Aguglia, Ebert and Luyckx [2]) *There are no maximal partial ovoids of $H(3,q^2)$ with size q^2+2. In particular, if q is odd, then every maximal partial ovoid of $H(3,q^2)$ has size at least q^2+3.*

Recently, in [28] S. De Winter and K. Thas improved this result.

Theorem 6.5.4 (De Winter and Thas [28]) *The generalized quadrangle $H(3,q^2)$ does not admit a maximal partial ovoid of size q^2+3 if $q \geq 5$.*

Corollary 6.5.5 *Suppose that $q \geq 5$. Then the second smallest maximal partial ovoid of $H(3,q^2)$, q even, and the smallest maximal maximal partial ovoid of $H(3,q^2)$, q odd, contain at least q^2+4 points.*

In [2], the authors present constructions for a maximal partial ovoid of size 61 in $H(3,25)$ and one of size 155 in $H(3,49)$; these are the smallest maximal partial ovoids known up to now.

In Table 6.4, we give the size $|\mathcal{O}|$ of the smallest maximal partial ovoid found by our heuristic search and compare it with the value of the lower bound (LB) from Theorem 6.5.3 and Corollary 6.5.5 and the size of small maximal partial ovoids from [2].

Note, that we found smaller maximal partial ovoids than the smallest previously known one, constructed in [2].

q	$\|G\|$	Earlier results		Spectrum found
2	45	5	[43]	**5,7,9**
3	280	16..25,28	[34]	**16..25,28**
4	1105	17	[43]	**17,21,25,29..61,65**
5	3276	78..119,121,126	[34]	**56..121,126**
		61,66,71,76,81,86,91,96,101,106	[2]	
7	17200	195..337,344	[34]	**142..337,344**
		155,162,169	[2]	
8	33345	65	[43]	**121,153,166,167,174, 175,179,180,186,190, 192..505,513**

Table 6.5: Spectrum of sizes for maximal partial ovoids in $H(3,q^2)$, for small values of q, obtained by heuristic search.

6.6 Spectrum of sizes for maximal partial ovoids in $H(3,q^2)$

In [13] and [2] the authors present constructions for some maximal partial ovoids of $H(3,q^2)$ using maximal partial spreads in $\mathrm{PG}(3,q)$. In particular, if $\mathcal{S}$ is a partial spread of $\mathrm{PG}(3,q)$, then the resulting maximal partial ovoid has size $|\mathcal{O}| = (|\mathcal{S}|-1)q+1$. A. Aguglia, G.L. Ebert and D. Luyckx also summarize following references and results. In [40] maximal partial spreads of sizes $13, 14, 15 \ldots, 22$ in $\mathrm{PG}(3,5)$ are constructed by computer. Similarly, in [39] maximal partial spreads of sizes $23, 24, 25$ are constructed by computer. In general, maximal partial spreads of size n in $\mathrm{PG}(3,q)$ for odd $q \geq 7$ have been constructed for all $(q^2+1)/2+6 \leq k \leq q^2-q+2$ (see [41]). For each of these maximal partial spreads there is a corresponding maximal partial ovoid on $H(3,q^2)$ of size $(k-1)q+1$.

In [34], L. Giuzzi describes computational methods to construct maximal partial ovoids of the Hermitian surface and presents results obtained by this approach.

For each $H(3,q^2)$ considered, in Table 6.5 we list the sizes for which our program found maximal partial ovoids of that given size and compare it with the earlier results from [34] and [2].

For $q = 5, 7$, we extended the previously known spectrum as obtained in [34].

For $H(3,16)$, we found a maximal partial ovoid whose size meets the lower bound of $q^2 + 1 = 17$. We also found maximal partial ovoids of size $21 = q^2 + q + 1$, $25 = q^2 + 2q + 1$, $29 = q^2 + (q-1)q + 1$, and then all sizes until $61 = q^3 - q + 1$.

7 Maximal partial ovoids and spreads of $H(4, q^2)$

In this chapter we present some new results, found by exhaustive and heuristic searches, for maximal partial ovoids and spreads in the Hermitian variety $H(4, q^2)$.

The results collected in this chapter are to appear in *Clique algorithms for finding substructures in generalized quadrangles* [18].

7.1 Large maximal partial ovoids and spreads in $H(4, q^2)$

Theorem 7.1.1 (Payne and Thas [55]) *The generalized quadrangle $H(4, q^2)$ has no ovoids.*

Proof. Suppose $H(4, q^2)$ has an ovoid $\mathcal{O}$, and let $\{x, y\} \subset \mathcal{O}$. Then $|\{x, y\}^{\perp\perp}| = q + 1$, see Theorem 2.3.2. Since $qt = s^2$, by Theorem 2.2.5 $\mathcal{O}$ has an empty intersection with $\{x, y\}^{\perp\perp}$, a contradiction. □

By a computer search A. Brouwer [12] proved that $H(4, 4)$ has no spread. For $q > 2$, the existence of spreads is an open problem.

q	Partial ovoids			Partial spreads		
	UB [50]	$\|\mathcal{O}\|$ found	$\#\mathcal{O}$	$st+1$	$\|\mathcal{S}\|$ found	$\#\mathcal{S}$
2	25	**21**	1	33	**29**	6
3	201	**105**		244	**162**	
4	577	**289**		1025	**494**	

Table 7.1: Large maximal partial ovoids and spreads of $H(4, q^2)$, for small values of q, obtained by heuristic and/or exhaustive search. For $q = 2$, the size and classification of the largest maximal partial ovoids and spreads were determined by exhaustive search.

For the size of a maximal partial ovoid in $H(4, q^2)$ there are theoretical upper bounds by G.E. Moorhouse and by P. Govaerts:

Theorem 7.1.2 (Moorhouse [50]) *If K is a k-cap of a Hermitian variety in $\mathrm{PG}(4, q^2)$, with $q = p^h$ and p prime, then*

$$k \leq \left[\binom{p+3}{4}^2 - \binom{p+2}{4}^2 \right]^h + 1.$$

Theorem 7.1.3 (Govaerts [36]) *If O' is a partial ovoid of $H(4, q^2)$, then $|O'| < q^5 - (4q-1)/3$.*

In Table 7.1, we list the sizes $|\mathcal{O}|$ and $|\mathcal{S}|$ of the largest maximal partial ovoid and spread found by our computer search. We compare it with the value of the upper bound (UB) from Theorem 7.1.2 or with the value of $st+1$ which would be the size of a spread in $H(4, q^2)$. Note that the largest values found for $q = 2$ are indeed the size of the largest maximal partial ovoid, resp. spread; this was confirmed by exhaustive search.

Exhaustive search also shows that the maximal partial ovoid of size 21 in $H(4, 4)$ is unique up to equivalence, while $H(4, 4)$ has 6 non-equivalent maximal partial spreads of size 29.

7.2 Small maximal partial ovoids and spreads in $H(4,q^2)$

In Section 6.5 we mentioned a lower bound for the size of maximal partial ovoids in Hermitian varieties $H(n,q^2)$. This is from J.W.P. Hirschfeld and G. Korchmáros [43] and does not depend on n.

Theorem 7.2.1 (Hirschfeld and Korchmáros [43]) *The size k of a complete cap of a Hermitian variety $\mathcal{U}_n$ in $\mathrm{PG}(n,q^2)$ satisfies $k \geq q^2+1$.*

This bound was sharp for $n=3$ and q even. It is not known yet whether there exist also examples of maximal partial ovoids of size q^2+1 for $n>3$.

J.W.P. Hirschfeld and G. Korchmáros also describe a complete cap of size q^3+1 for any q, which is currently the smallest known complete cap of the Hermitian variety:

Theorem 7.2.2 (Hirschfeld and Korchmáros [43]) *Let α be a plane of $\mathrm{PG}(n,q^2)$ which meets the Hermitian variety $\mathcal{U}_n$ in an non-degenerate Hermitian curve $\mathcal{U}_2$. Then $\mathcal{U}_2$ is a complete cap of $\mathcal{U}_n$ of size q^3+1.*

A lower bound for the size of maximal partial spreads in $H(4,q^2)$ is due to S. De Winter and K. Thas [28].

Theorem 7.2.3 (De Winter and Thas [28]) *The smallest maximal partial spread of $H(4,q^2)$ contains at least q^3+3 lines.*

The authors conjectured that maximal partial spreads of size q^3+3 do probably not exist. However, as will be shown in Table 7.2, our searches found a maximal partial spread of size $q^3+3=11$ for $q=2$.

In Table 7.2, we list the sizes $|\mathcal{O}|$ and $|\mathcal{S}|$ of the smallest maximal partial ovoid and spread found by our computer search. We compare them with the values of the lower bounds (LB) from Theorem 7.2.1 and Theorem 7.2.3. Note that the smallest values found for $q=2$ are indeed the size of the smallest maximal partial ovoid, resp. spread; this was confirmed by exhaustive search.

q	Partial ovoids		Partial spreads	
	LB [43]	$\lvert\mathcal{O}\rvert$ found	LB [28]	$\lvert\mathcal{S}\rvert$ found
2	5	**9**	11	**11**
3	10	**28**	30	**86**
4	17	**65**	67	**303**

Table 7.2: Small maximal partial ovoids and spreads of $H(4, q^2)$, for small values of q, obtained by heuristic and/or exhaustive search. For $q = 2$, the size and classification of the smallest maximal partial ovoids and spreads were determined by exhaustive search.

Our searches confirm the existence of maximal partial ovoids of size $q^3 + 1$ in $H(4, q^2)$, for small q. No maximal partial ovoids with size smaller than $q^3 + 1$ were found, and for $q = 2$ exhaustive search excludes the existence of maximal partial ovoids with size smaller than $q^3 + 1 = 9$. We also observe the existence of maximal partial ovoids of size $q^3 + 1 + iq$ for small values of $i \geq 1$.

7.3 Spectrum of sizes for maximal partial ovoids and spreads in $H(4, q^2)$

In Table 7.3 we give computer results for maximal partial ovoids in $H(4, q^2)$, while Table 7.4 gives results for maximal partial spreads in $H(4, q^2)$. Recall that a (partial) spread in $H(4, q^2)$ is a (partial) ovoid in $H(4, q^2)^D$.

For maximal partial ovoids in $H(4, 4)$, we confirmed by exhaustive search that the spectrum found is complete, i.e. exhaustive search confirmed that no maximal partial ovoids with size less than 9, or with sizes 10, 18 or 20 exist.

For maximal partial spreads in $H(4, 4)$, we confirmed by exhaustive search that no maximal partial spreads with size less than 11 exist; for sizes 12, 13, 14 it remains open whether such maximal partial spreads exist.

q	$\|G\|$	Spectrum found
2	165	**9,11..17,19,21**
3	2440	**28,31,34..97,99..100,105**
4	17425	**65,69,73,77,81,85..287,289**

Table 7.3: Spectrum of sizes for maximal partial ovoids of $H(4,q^2)$, for small values of q, obtained by exhaustive and/or heuristic search.

q	$\|G\|$	Spectrum found	Non-existence
2	297	**11,15..29**	< 11 and > 29
3	6832	**86,88..162**	
4	66625	**303,307..494**	

Table 7.4: Spectrum of sizes for maximal partial spreads of $H(4,q^2)$, for small values of q, obtained by exhaustive and/or heuristic search.

8 Minimal blocking sets in generalized quadrangles

In the previous chapters we dealt with the following problem. What is the largest set of points $\mathcal{O}$, such that each line of the generalized quadrangle $\mathcal{S}$ is incident with at most one point of $\mathcal{O}$? Now, the following question naturally arises. What is the smallest set of points $\mathcal{B}$, such that each line of the generalized quadrangle $\mathcal{S}$ is incident with at least one point of $\mathcal{B}$?

In this chapter we introduce minimal blocking sets and we give an overview of necessary definitions. We describe our search algorithms and finally we present some known results on the small minimal blocking sets.

8.1 Introduction and preliminaries

Consider a generalized quadrangle $\mathcal{S} = (P, B, \mathrm{I})$ of order (s,t), $s > 1, t > 1$. A *blocking set* of $\mathcal{S}$ is a set $\mathcal{B}$ of points of $\mathcal{S}$ such that each line of $\mathcal{S}$ is incident with at least one point of $\mathcal{B}$. A *cover* of $\mathcal{S}$ is a set $\mathcal{C}$ of lines of $\mathcal{S}$ such that each point of $\mathcal{S}$ is incident with at least one line of $\mathcal{C}$. A blocking set $\mathcal{B}$ is *minimal* if $\mathcal{B} \setminus p$ is not a blocking set for any point $p \in \mathcal{B}$. A cover $\mathcal{C}$ is *minimal* if $\mathcal{C} \setminus L$ is not a cover for any line $L \in \mathcal{C}$. It is clear that a blocking set in the generalized quadrangle $\mathcal{S}$ is a cover in the dual $\mathcal{S}^D$.

Necessarily $|\mathcal{B}| \geq st+1$ for a generalized quadrangle $\mathcal{S} = (P, B, \mathrm{I})$ with equality if and only if $\mathcal{B}$ is an ovoid. Necessarily $|\mathcal{C}| \geq st+1$ for a generalized quadrangle $\mathcal{S} = (P, B, \mathrm{I})$ with equality if and only if $\mathcal{C}$ is a spread.

Let $\mathcal{B}$ be a blocking set of $\mathcal{S}$ of size $st+1+r$. We call r the *excess* of the blocking set. A line of $\mathcal{S}$ is called a *multiple line* if it contains at least two points of $\mathcal{B}$. The *excess of a line* is the number of points of $\mathcal{B}$ it contains, minus one. The *weight of a point* of $\mathcal{S}$ with respect to $\mathcal{B}$ is the minimum of the excesses of the lines of $\mathcal{S}$ passing through this point.

Let $\mathcal{C}$ be a cover of $\mathcal{S}$ of size $st+1+r$. We call r the *excess* of the cover. A point of $\mathcal{S}$ is called a *multiple point* if it lies on at least two lines of $\mathcal{C}$. The *excess of a point* is the number of lines of $\mathcal{C}$ passing through this point, minus one. The *weight of a line* of $\mathcal{S}$ with respect to $\mathcal{C}$ is the minimum of the excesses of the points of $\mathcal{S}$ belonging to this line.

A *sum of lines* $\mathcal{L}$ of $\mathrm{PG}(n, q)$ is a collection of lines of $\mathrm{PG}(n, q)$, where each line is accorded a positive integer, called its weight. Furthermore, the *weight* of a point with respect to $\mathcal{L}$ is the sum of the weights of the lines of $\mathcal{L}$ through this point.

A *pencil* of generalized quadrangle $\mathcal{S}$ with parameters (s, t) is the set of $t+1$ lines on a point of $\mathcal{S}$.

Let $\mathcal{C}$ be a cover and let $a_i = |\{x \in P \mid x$ is incident with exactly i lines of $\mathcal{C}\}|$, $1 \leq i \leq t+1$. We count the numbers of points in P

$$\sum_{j=1}^{t+1} a_j = (1+s)(1+st).$$

Next, we count in two ways the number of ordered pairs (x, L), where $L \in \mathcal{C}$ and $x \in L$ is a point of $\mathcal{S}$.

$$\sum_{j=1}^{t+1} j a_j = (1+s)(1+st+r).$$

Together we obtain

$$\sum_{j=2}^{t+1}(j-1)a_j = (1+s)r.$$

This result is stated in the following lemma.

Lemma 8.1.1 *Suppose that $\mathcal{C}$ is a cover of a generalized quadrangle of order (s,t), of size $st+1+r$. The sum of excesses of the points of the generalized quadrangle is $r(s+1)$.*

The next results give information about the structure of multiple points of a cover in a generalized quadrangle. We state the dual version as well.

Lemma 8.1.2 (Blokhuis et al. [9]) *Let $\mathcal{C}$ be a minimal cover of a generalized quadrangle $\mathcal{S}$ with parameters (s,t), having excess satisfying $r \leq 2s$. Then every point of $\mathcal{S}$ lies on at most $r+1$ lines of $\mathcal{C}$.*

Let $\mathcal{B}$ be a minimal blocking set of a generalized quadrangle $\mathcal{S}$ with parameters (s,t), having excess satisfying $r \leq 2t$. Then every line of $\mathcal{S}$ contains at most $r+1$ points of $\mathcal{B}$.

Theorem 8.1.3 (Eisfeld et al. [31]) *Let $\mathcal{C}$ be a cover of a classical generalized quadrangle $\mathcal{S}$ of order (q,t) embedded in $PG(n,q)$. Let $|\mathcal{C}| = qt+1+r$, with $q+r$ smaller than the cardinality of the smallest non-trivial blocking sets in* $\mathrm{PG}(2,q)$. *Then the multiple points of $\mathcal{C}$ form a sum of lines of $PG(n,q)$, where the weight of a line in this sum is equal to the weight of this line with respect to the cover, and with the sum of the weights of the lines equal to r.*

The formulation of Theorem 8.1.3 is given in [26]. The authors reformulated the original theorem from [31] by replacing "contained in Q" by "of $\mathrm{PG}(n,q)$". They noticed that one has to be careful concerning the interpretation of the original theorem. In general, the lines of the sum of the lines are not necessarily the lines of the generalized quadrangle $\mathcal{S}$. Hence, the interpretation is that the sum of the lines is a sum of lines of $\mathrm{PG}(n,q)$.

We will concentrate on the small blocking sets in the classical generalized quadrangles. It is easy to see that the following construction gives a minimal blocking set of size $st + s$.

Construction 8.1.4 Take a point x of $\mathcal{S}$ and all lines of $\mathcal{S}$ through this point. Then all $s(t + 1)$ points on these lines (except x) form a minimal blocking set of size $st + s$.

Our aim is to find a minimal blocking set of size smaller than $st + s$, which is not an ovoid (in case the generalized quadrangle contains an ovoid).

The following sections describe exhaustive and non-exhaustive search algorithms, similar to those for searching for maximal partial ovoids (Chapter 4). We remark that the problem of searching for minimal blocking sets of a generalized quadrangle cannot be translated to a problem on the collinearity graph. Hence, many pruning techniques described in Chapter 4 cannot be used.

8.2 Exhaustive search by removing points

A first class of algorithms are exhaustive algorithms, where we use a standard backtracking algorithm which tries in every recursion step to reduce a blocking set (which is not minimal yet) by removing the points of a set A_N of allowed points in a systematic way. When reaching a point where the set A_N is empty, a new minimal blocking set has been found.

Now we use a similar reasoning as in Subsection 4.2.1. To prevent the duplication of work, we define a set A of allowed remaining points, i.e., in each recursive call the removed point is taken from this set.

Algorithm 8.1 shows the pseudo-code of a backtracking search by removing points for the minimal blocking set of the given size *desired* in a given quadrangle with points $p_1, \ldots, p_n$. The function *bound* will be explained in more details in Section 8.4. The recursive function *blockset* takes as parameters the sets A and A_N of allowed points and the partial blocking set B constructed at the current step in the recursive process. It takes also as parameter the desired size of a minimal blocking set (*desired*). In the original call, all sets A, A_N and B are the full point set P of the generalized quadrangle. When the allowed set A_N is empty, a new minimal blocking set has been found and it is checked whether this blocking set is of the desired size. If the set A_N is not empty, all points of A are used in a systematic way to reduce the blocking set B and for each case a recursive call is performed.

We create a set $X(p_i)$ by considering all lines L through p_i. If L contains only one point p_L, which still belongs to the allowed set A_N, we add p_L to $X(p_i)$. The new allowed sets A_N and A in the recursive call are restricted to points from the set $A \setminus X(p_i)$ and $A_N \setminus X(p_i)$, respectively, of the point p_i which is currently removed from B.

Note that before calling the recursive function *blockset*, the currently removed point p_i is removed from A.

8.3 Exhaustive search by adding points

Note that Algorithm 8.1 is a backtracking search which tries in every recursion step to reduce a blocking set (which is not yet minimal) by removing the points of a set A of remaining allowed points in a systematic way.

In some cases it may be more suitable to use another form of backtracking search, namely a backtracking search which tries in every recursion step to extend a "partial blocking set" (which is not a blocking set yet) by adding the points of a set A of allowed remaining points in a systematic way. When reaching a point where the set A is empty, a new minimal blocking set has been found.

8.3.1 Allowed set

Now the question of determining the allowed set arises. To obtain a blocking set, which is minimal, we need to guarantee that the situation from Figure 8.1 does not occur.

There are two possible cases when points are removed from the allowed set in a recursive step and are not the point, which was actually added to the current partial blocking set.

Algorithm 8.1 Backtracking search for a minimal blocking set by removing points

function *blockset* (*desired*)
  blockset (P, P, P, *desired*)
function *blockset* (A, A_N, B, *desired*)
  if $size(B) < desired$ **then**
    return
  if $A_N = \emptyset$ **then**
    {Found a new minimal blocking set}
    if $size(B) = desired$ **then**
      {Found a minimal blocking set of the desired size}
      return
  else
    while $A \neq \emptyset$ **do**
      if $size(B) - bound(A) \geq desired$ **then**
        return
      $i \leftarrow \min\{j \mid p_j \in A\}$
      $A \leftarrow A \setminus \{p_i\}$
      blockset ($A \setminus X(p_i)$, $A_N \setminus X(p_i)$, $B \setminus \{p_i\}$, *desired*)
function *bound* (A)
  {Determines a bound on the number of points from A that might be removed from the blocking set}

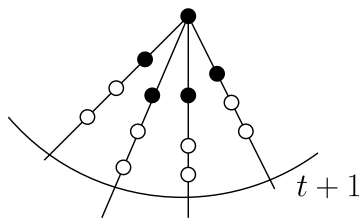
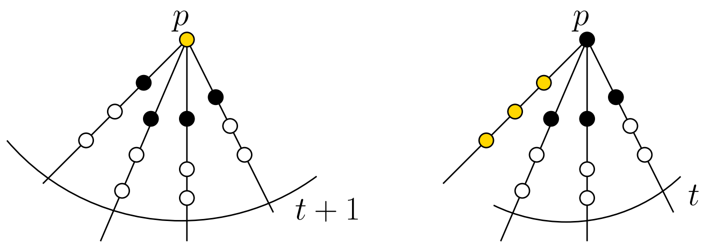


Figure 8.1: Resulting blocking set would not be minimal



(a) (b)

Figure 8.2: Removing points from the allowed set

(A_I) There is a point p, such that all $t+1$ lines through this point contain a point from the current partial blocking set. Then, the point p must be removed from the allowed set (Figure 8.2(a)).

(A_{II}) There is a point p, which already belongs to the current partial blocking set and t lines through this point contain a point (different from p) from the current partial blocking set. Then the points on the line, which is not among the above t lines, must be removed from the allowed set (Figure 8.2(b)).

The above algorithm can still result in a structure which is not a blocking set. We illustrate this on a simple example, see Figure 8.3. Let us search for a minimal blocking set in the smallest thick generalized quadrangle from Figure 2.2. For example, after adding the points 0,1,2, 3 we have to remove the points 5 and 6, as described in (A_{II}) and illustrated in Figure 8.2(b). After adding the point 8 we have to remove the point 13, as described in (A_I) and illustrated in Figure 8.2(a). We continue in the same way. The algorithm creates two not covered lines, which contain no allowed points anymore. So the resulting set cannot be a blocking set.

To avoid this situation, we combine the algorithm with the forcing technique described in the following subsection.

partial blocking set	allowed set
$\emptyset$	P
$\{0\}$	$P \setminus \{0\}$
$\{0,1\}$	$P \setminus \{0,1\}$
$\{0,1,2\}$	$P \setminus \{0,1,2\}$
$\{0,1,2,3\}$	$\{4,7,8,9,10,11,12,13,14\}$
$\{0,1,2,3,8\}$	$\{4,7,11,12,14\}$
$\{0,1,2,3,8,14\}$	$\emptyset$
Not a blocking set!	

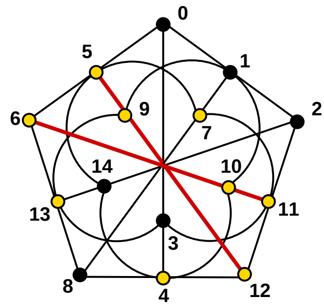


Figure 8.3: The algorithm resulting in non-blocking set

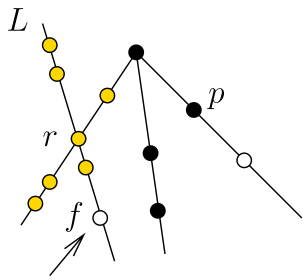


Figure 8.4: Illustrating the idea of forcing points

8.3.2 Forcing points

Consider a step in the recursive process where the current partial blocking set gives rise to a line for which only one point still belongs to the allowed set A. If that point is not added to the current partial blocking set, then the resulting partial blocking set can never be extended to a blocking set, so we can prune these possibilities and force the point to be added to the current partial blocking set.

Figure 8.4 illustrates this idea. Imagine that after adding a point p in the recursive step, a point r was removed from the current set of allowed points. Suppose that there is only one point f on a line L left, which can be added to the partial blocking set. Hence, when searching for minimal blocking set, we can forcedly add this point to the partial blocking set, thus pruning the possible extensions that do not contain this point.

In Figure 8.5 we illustrate again the algorithm with forcing points, following the example from Figure 8.3. The resulting structure is a blocking set.

However, the adding algorithm combined with the forcing technique is probably still not sufficient for all general cases. An example of such a situation is given in Figure 8.6. Suppose that after adding a point p in the recursive step, more points (here r_1 and r_2) on a still not covered line L were removed from the set of allowed points together. If r_1 and r_2 were the last allowed points on the line L, L becomes a not covered line with no allowed points on it, i.e., the resulting structure would not be a blocking set. Hence, in each recursive call we must add a check to see whether such a line occurs. If so, we return.

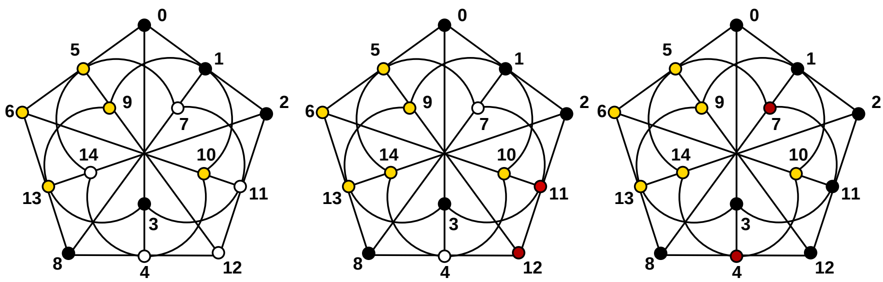
This adding technique gets complicated. However, it turns out to be effective in some specific cases, as will be described in Subsection 9.2.2.

8.3.3 Pseudo-code

Algorithm 8.2 shows the pseudo-code of a backtracking search by adding points for the minimal blocking set size *desired* in a given generalized quadrangle with points $p_1, \ldots, p_n$. The recursive function *blockset* takes as parameters the set A and A_N of allowed points and the partial blocking set B constructed at the current step in the recursive process. It takes also as parameter the desired size of a minimal blocking set (*desired*). In the original call the sets A and A_N are the full point set P of the generalized quadrangle and B is the empty set. When the allowed set A_N is empty, a new minimal blocking set has been found and it is checked whether this minimal blocking set is of the desired size. If the set A_N is not empty, all points of A are used in a systematic way to extend the partial blocking set B and for each case a recursive call is performed.

Note that before calling the recursive function *blockset*, the currently added point p_i is removed from A. The details of determining the new

partial blocking set	allowed set	forced points
$\emptyset$	P	
$\{0\}$	$P \setminus \{0\}$	
$\{0,1\}$	$P \setminus \{0,1\}$	
$\{0,1,2\}$	$P \setminus \{0,1,2\}$	
$\{0,1,2,3\}$	$\{4,7,8,9,10,11,12,13,14\}$	
$\{0,1,2,3,8\}$	$\{4,7,11,12,14\}$	$\{11,12\}$
$\{0,1,2,3,8,11,12\}$	$\{4,7\}$	$\{4,7\}$
$\{0,1,2,3,4,7,8,11,12\}$	$\emptyset$	
Blocking set found!		



(a) Partial blocking set $\{0,1,2,3,8\}$

(b) Forcing $\{11,12\}$

(c) Forcing $\{4,7\}$ which results in the blocking set

Figure 8.5: The algorithm with forcing points resulting in a blocking set

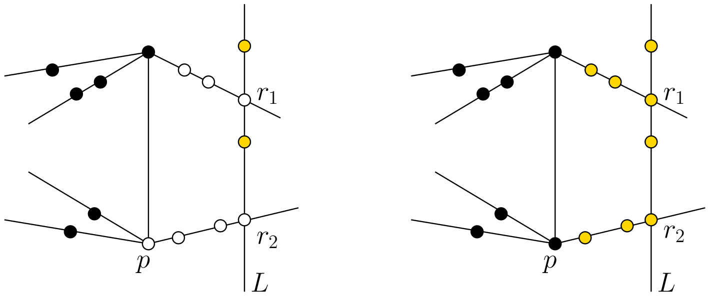


Figure 8.6: A situation resulting in a non-blocking set

allowed set will be discussed for specific cases.

8.4 Pruning strategies

Pruning strategies are used to avoid going through every single "partial blocking set" of the generalized quadrangle. Typically this consists in a bounding function (called *bound* in Algorithm 8.1) which gives a lower bound on the number of points that can still be removed from the current blocking set. E.g. when searching for minimal blocking sets by removing points, a straightforward idea is to backtrack when the set A becomes so small that even if all its points could be removed to form a minimal blocking set, the size of that minimal blocking set would not deceed the size of the smallest blocking set found so far; in that case $bound(A)$ is simply $|A|$.

We have already dealt with isomorph rejection in algorithms for searching maximal partial ovoids and spreads. The same technique can be used to search for minimal blocking sets. Having determined in a step of the search process the set stabilizer of the current blocking set in the automorphism group of the quadrangle, it suffices to try only one point of each orbit of the stabilizer for reducing the current blocking set in the next recursive steps instead of trying to remove all points of the allowed set. We use *nauty* [49] to compute the set stabilizer and its orbits.

Algorithm 8.2 Backtracking search for a minimal blocking set by adding and forcing points

function *blockset* (*desired*)
  blockset (P, $\emptyset$, $\emptyset$, *desired*)
function *blockset* (A_{old}, A_N, old, B, *desired*)
  $A \leftarrow newAllowed(A_{\text{old}})$
  $A_N \leftarrow newAllowed(A_{N,\text{old}})$
  if $size(B) > desired$ **then**
    return
  if $A_N = \emptyset$ **then**
    {Found a new minimal blocking set}
    if $size(B) = desired$ **then**
      {Found a minimal blocking set of the desired size}
      return
  else
    while $A \neq \emptyset$ **do**
      if $size(B) - bound(A) \geq desired$ **then**
        return
      $i \leftarrow \min\{j \mid p_j \in A\}$
      $A \leftarrow A \setminus \{p_i\}$
      {Forcing of points}
      $F \leftarrow \emptyset$
      addForcedPoints(A, B, F)
      if $size(F) = 0$ **then**
        {No points need to be forced}
        blockset (A, A_N, $B \cup \{p_i\}$, *desired*)
      else
        if $size(B) + 1 + size(F) \leq desired$ **then**
          {some points need to be forced}
          $A_{temp} \leftarrow forcePoints$ (A, A_N, B, F)
          blockset (A, A_N, $B \cup \{p_i\} \cup F$, *desired*)
          unforcePoints (B, F)

Algorithm 8.2 (continued) Backtracking search for a minimal blocking set by adding and forcing points

function *bound* (A)

{Determines a bound on the number of points from A that might be added to the partial blocking set}

function *newAllowed* (A_{old})

{Determines a new allowed set from the old one}

function *addForcedPoints* (A,B,F)

{Determines those points that are unique allowed points on a non-covered line and adds them to the given set}

function *forcePoints* (A, A_N, B, F)

{Forces the given set of points in a current "partial blocking set". Returns the remaining sets of allowed points}

function *unforcePoints* (B, F)

{Undoes the action of forcePoints}

Algorithm 8.3 gives the pseudo-code for a minimal blocking set search algorithm by removing points using isomorph pruning. The function *block-set* now takes as an extra parameter the current level in the recursion tree. In case this level is smaller than a predefined maximum number of levels (*maxlevel*), the set stabilizer for the current blocking set B is determined and B is reduced by one representative of every orbit in a systematic way: the first point p of the orbit tried by the recursive process is added effectively, after handling p all points in the same orbit as p are removed from the set A of allowed points. At deeper levels the recursive process reduces the current blocking set by every point from the allowed set, as described earlier in Algorithm 8.1.

Algorithm 8.3 Minimal blocking set searching by removing points with isomorph pruning

```
function blockset (A, A_N, B, desired, level)
  if size(B) < desired then
    return
  if A_N = ∅ then
    {Found a new minimal blocking set}
    if size(B) = desired then
      {Found a minimal blocking set of the desired size}
      return
  else if level < maxlevel then
    {Remove only one point per orbit of the set stabilizer of B}
    compute set stabilizer of B and determine its orbits
    while A ≠ ∅ do
      if size(B) − bound(A) ≥ desired then
        return
      i ← min{j | p_j ∈ A}
      A ← A \ {p_i}
      blockset (A \ X(p_i), A_N \ X(p_i), B \ {p_i}, desired, level + 1 )
      for all p_j in orbit of p_i do
        A ← A \ {p_j}
  else
    {Remove all points in allowed set A}
    while A ≠ ∅ do
      if size(B) − bound(A) ≥ desired then
        return
      i ← min{j | p_j ∈ A}
      A ← A \ {p_i}
      blockset (A \ X(p_i), A_N \ X(p_i), B \ {p_i}, desired, level + 1)
```

8.5 Non-exhaustive search algorithms

Heuristic algorithms were already described as an efficient tool for exploring the spectrum of sizes for which maximal partial ovoids exist. This class of algorithms also gives new results for minimal blocking sets.

A simple greedy algorithm builds a minimal blocking set step by step, by removing points from a set of allowed points, until this set is a minimal blocking set. Removing a point that leaves the largest number of points in the allowed set, will tend to build small minimal blocking sets. Starting from a minimal blocking set obtained by this approach, a simple restart strategy adds some of the points and again removes points until the blocking set is minimal. Both the adding and the removing can be done either randomly or following the above heuristic.

8.6 Known results on minimal blocking sets of some classical generalized quadrangles

In what follows we collect the known results on the smallest minimal blocking sets of the classical generalized quadrangles. In Subsection 8.6.3 we present some new results on the small minimal blocking sets of $H(3,q^2)$ obtained by computer searches. In this work we focused on the generalized quadrangle $Q(4,q)$. Our results on the small minimal blocking sets of $Q(4,q)$ will be treated separately in the following chapter.

8.6.1 Blocking sets of $W(q)$, q odd

In [31] Eisfeld, Storme, Szőnyi and Sziklai proved the following lower bound for blocking sets of $W(q)$, q odd.

Theorem 8.6.1 (Eisfeld et al. [31]) *Let $\mathcal{B}$ be a blocking set of $W(q)$, q odd, then $|\mathcal{B}| > q^2+1+(q-1)/3$.*

This result was improved by A. Klein and K. Metsch in [47].

Theorem 8.6.2 (Klein and Metsch [47]) *Let q be odd. Then a blocking set of $W(q)$ contains at least*

$$q^2 - q - \frac{3}{2} + \frac{\sqrt{8q^2 + 20q + 25}}{2}$$

points.

Remark 8.6.3 (Eisfeld et al. [31]) It is possible to construct a minimal blocking set of $W(q)$ of size $q^2 + 1 + (q - 2)$ in the following way. Consider a fixed line L of $Q(4, q)$ and consider the set S of all $q^2 + q$ lines of $Q(4, q)$ intersecting L in one point. Replace in a fixed three-dimensional space through L intersecting $Q(4, q)$ in a hyperbolic quadric $Q^+(3, q)$, the $q + 1$ lines of this hyperbolic quadric $Q^+(3, q)$ in S by the q other lines of $Q^+(3, q)$ in the regulus containing L. Then a minimal cover of size $q^2 + 1 + (q - 2)$ is obtained. For $q = 3, 4, 5$ it can be shown that the smallest minimal blocking sets have this size $q^2 + 1 + (q - 2)$.

The formulation of the previous remark is as used in [31], also includes q even. In the next chapter we deal with small minimal blocking sets of $Q(4, q)$ and we present new results for both q even and odd. Since $Q(4, q)$ is isomorphic to $W(q)$, for q even, we compare results from Remark 8.6.3 with our results in Section 9.1.

8.6.2 Blocking sets of $H(4, q^2)$

In [31] we found the following result on $H(4, q^2)$.

Theorem 8.6.4 (Eisfeld et al. [31]) *The generalized quadrangle $H(4, q^2)$ does not contain minimal blocking sets of size $q^5 + 2$ when $q > 2$, of size $q^5 + 3$ when $q > 3$, of size $q^5 + 4$ when $q > 4$.*

Recently, in [25] the following result was proved, giving a lower bound for minimal blocking sets of $H(4, q^2)$.

q	# points	q^3+1	q^3+q^2	Spectrum found
2	45	9	12	**10,11**
3	280	28	36	**29..35**
4	1105	65	80	**72..79**

Table 8.1: Spectrum of sizes for minimal blocking sets of $H(3,q^2)$ smaller than q^3+q^2, for small values of q, obtained by heuristic search.

Theorem 8.6.5 (De Beule and Metsch [25]) *Suppose that $\mathcal{B}$ is a minimal set of points of $H(4,q^2)$ with the property that $\mathcal{B}$ meets every line of $H(4,q^2)$. Then $\mathcal{B} \geq q^5+q^2$ and this bound is sharp.*

In [25] the authors construct a minimal blocking set of size q^5+q^2 and show its uniqueness.

8.6.3 Blocking sets of $H(3,q^2)$

Theorem 8.1.3 is valid, in dual form, for the blocking sets of the classical generalized quadrangle $H(3,q^2)$. This follows from the fact that $H(3,q^2)$ is the dual of $Q^-(5,q)$.

To our knowledge there is nothing known about small minimal blocking sets in $H(3,q^2)$. We present some new results found by our heuristic searches in Table 8.1. For small values of q we give the number of points of the generalized quadrangle, the size q^3+1 of an ovoid and the size q^3+q^2 of a blocking set described in Construction 8.1.4. Finally the last column lists the sizes smaller than q^3+q^2 for which our program found minimal blocking sets of that given size. The notation $a..b$ means that for all values in the interval $[a,b]$ a minimal blocking set of that size has been found.

For $q=2$ we classified all minimal blocking sets of small size. There are two minimal blocking sets of size 10, twelve minimal blocking sets of size 11 and thirty minimal blocking sets of size 12.

Finally, we can construct a spectrum of minimal blocking sets for each value from $q^3 + r$, $r = 1, \ldots, q^2$.

Construction 8.6.6 Consider a Hermitian curve $\mathcal{H}$ contained in $H(3, q^2)$. This curve is contained in a plane, denoted by π. Consider any point $p \in \mathcal{H}$. Consider the $q^2 + 1$ lines of π on p. Exactly q^2 of them intersect $\mathcal{H}$ in a Hermitian variety H on a line (this is a $H(1, q^2)$). It is clear that $H^\perp$ is again a Hermitian variety on a line. Consider r different lines L_i on p and there corresponding Hermitian varieties H_i. Consider the r Hermitian varieties $H_i^\perp$, they are mutually skew and lie all in $p^\perp$. It is clear that the set $\mathcal{H} \setminus (\cup_{i=1}^r H_i) \cup_{i=1}^r H_i^\perp$ is a minimal blocking set of size $q^3 + r$ of $H(3, q^2)$. Since we have exactly q^2 suitable lines on p, we can construct a spectrum of minimal blocking sets of size $q^3 + r$, $r = 1, \ldots, q^2$.

9 Minimal blocking sets in $Q(4,q)$

This chapter deals with small minimal blocking sets of the generalized quadrangle $Q(4,q)$.

9.1 Minimal blocking sets in $Q(4,q)$, q even

It is known that $Q(4,q)$ always has an ovoid. Considering minimal blocking sets of $Q(4,q)$, q even, the following result is known.

Theorem 9.1.1 (Eisfeld et al. [31]) *Let $\mathcal{B}$ be a blocking set of the quadric $Q(4,q)$, q even. If $q \geq 32$ and $|\mathcal{B}| \leq q^2+1+\sqrt{q}$, then $\mathcal{B}$ contains an ovoid of $Q(4,q)$.*

If $q = 4, 8, 16$ and $|\mathcal{B}| \leq q^2+1+\frac{q+4}{6}$, then $\mathcal{B}$ contains an ovoid of $Q(4,q)$.

We focus on the small minimal blocking sets. The following results were obtained by heuristic computer search, described in 8.5.

For each q considered, in Table 9.1 we list the sizes for which our program found minimal blocking set of that given size smaller than q^2+q. We also compare them with the sizes of the known small minimal blocking set from Remark 8.6.3. For each value of q, the number of points $|P|$ of the corresponding generalized quadrangle is also given.

q	$\|P\|$	Earlier results [31]	Spectrum found
4	85	19	19
8	585	71	69,71
16	4369	271	269,271

Table 9.1: Spectrum of sizes for minimal blocking sets in $Q(4,q)$, for small values of q, q even. All results are obtained by heuristic search.

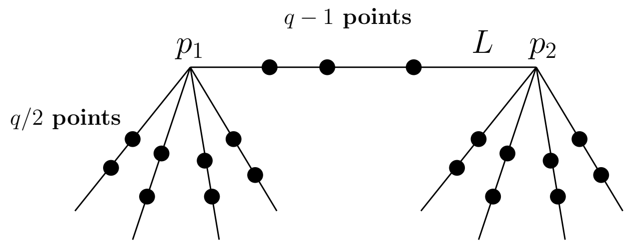


Figure 9.1: Blocking set of $Q(4,q)$ of size q^2+q-1

For each q even considered, our computer searches find a minimal blocking set of size q^2+q-1. We also observed the existence of a minimal blocking set of size q^2+q-3 for $q=4,8,16$.

We now investigate the structure of these blocking sets. This is also done by computer.

Blocking set $\mathcal{B}$ of size q^2+q-1 for $q=2,4,8,16$.
There is a line L and two points p_0 and p_1 on this line not contained in the blocking set $\mathcal{B}$. All $q-1$ points from L, except for the points p_1 and p_2, belong to $\mathcal{B}$. Through the points p_1 and p_2 there are $2q$ lines (excluding the line L) and on each line there are $q/2$ points of $\mathcal{B}$. These $q-1+(2q)(q/2)$ points form a minimal blocking set of size q^2+q-1. This is illustrated in Figure 9.1. We note that for $q=2$ we get an ovoid.

Blocking set $\mathcal{B}$ of size q^2+q-3 for $q=4,8,16$.
There is a line L and four points p_0, p_1, p_2 and p_3 on this line not contained in the blocking set $\mathcal{B}$. All $q-3$ points from L, except for the points p_i, $i=0,1,2,3$, belong to $\mathcal{B}$. Through the points p_i, $i=0,1,2,3$ there are $4q$ lines (excluding the line L) and on each line there are $q/4$ points of $\mathcal{B}$. These $q-3+(4q)(q/4)$ points form a minimal blocking set of size q^2+q-3. We note that for $q=4$ we get an ovoid.

Blocking set $\mathcal{B}$ of size $q^2+q-(2^k-1)$ for $q\geq 2^k$, $k>0$.
In the previous cases a minimal blocking set of the smallest possible parameter was an ovoid. If $q=8$, a minimal blocking set of size q^2+q-7 is an ovoid. However, our heuristic searches did not find a minimal blocking set of that size for $q>8$. Now the following question naturally arises. Is it possible to construct a blocking set of size q^2+q-7 for $q>8$ using the previous ideas? Even more generally, are there any blocking sets of size $q^2+q-(2^k-1)$, for $q\geq 2^k$, $k>0$ with the following structure?

There is a line L and 2^k points p_i, $0\leq i\leq 2^k-1$, on this line not contained in the blocking set $\mathcal{B}$. All $q+1-2^k$ points from L, except for the points p_i, belong to $\mathcal{B}$. Through all points p_i there are 2^kq lines (excluding the line L) and on each line there are $q/2^k$ points of $\mathcal{B}$. Is it possible that these $(q+1-2^k)+(2^kq)(q/2^k)=q^2+q-(2^k-1)$ points form a minimal blocking set?

Remark 8.6.3 gives a construction of minimal blocking sets of size q^2+q-1 of $W(q)$. We give a construction of minimal blocking sets of size q^2+q-1 corresponding to our computer results.

Construction 9.1.2 Let $\mathcal{B}$ be the blocking set from Construction 8.1.4 of size q^2+q. It is a pencil on a point p_1, where all points of the lines of the pencil belong to $\mathcal{B}$ except p_1. Let L be a line of this pencil and let p_2 be a point on L different from p_1. We consider a conic C_1, containing p_2, such that its nucleus is the nucleus of $Q(4,q)$. This conic intersects the pencil through the point p_1 in $q+1$ mutually non-collinear points $c_1^1, c_2^1, \dots c_{q+1}^1$, where $c_1^1=p_2$. The conic $C_1^\perp$ intersects the pencil on p_2 in $q+1$ non-collinear points $c_1^2, c_2^2, \dots c_{q+1}^2$, where $c_1^2=p_1$. Now, we remove the $q+1$

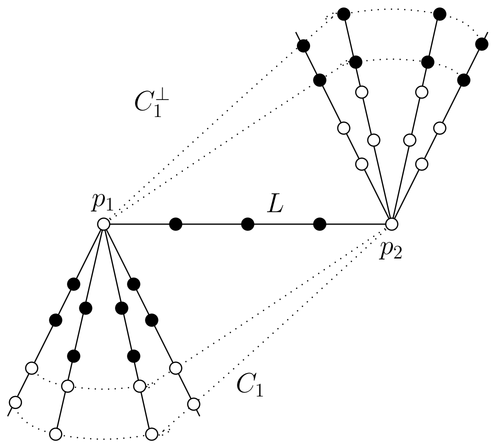


Figure 9.2: Construction of blocking sets of size q^2+q-1

points $c_1^1, c_2^1, \ldots c_{q+1}^1$ from the blocking set and add the q points $c_2^2, \ldots c_{q+1}^2$. The obtained set of size q^2+q-1 is indeed a blocking set, since we substitute the points of C_1 by the points of $C_1^\perp$. Checking the minimality is easy, since through each point of the new set there is at least one line containing only one point of the set. Notice that the points of conics can be interchanged $q-1$ times. This construction is illustrated in Figure 9.2.

Finally we note that we are currently working on a geometrical construction for minimal blocking sets of size smaller than q^2+q-1.

9.2 Minimal blocking sets in $Q(4,q)$, q odd

Theorem 9.1.1 in Section 9.1 gives a lower bound for minimal blocking sets of $Q(4,q)$, q even. No similar result is known for q odd. It is even not known, whether or not $Q(4,q)$, q odd, has a minimal blocking set of cardinality q^2+2. In [24], the authors were able to solve this problem when q is an odd prime. They proved that for q an odd prime no minimal blocking set of size q^2+2 exist.

We will focus on small minimal blocking sets of $Q(4, q)$, q odd, of size smaller than $q^2 + q$. Before mentioning our results, we give more details on the structure of multiple lines.

9.2.1 Structure of multiple lines

First, we define a regulus and its opposite regulus in $Q(4, q)$. Consider a grid $Q^+(3, q) \subset Q(4, q)$. A *regulus* of $Q(4, q)$ is a set of $q + 1$ pairwise disjoint lines in $Q^+(3, q)$. The set of the remaining $q + 1$ pairwise disjoint lines in $Q^+(3, q)$ is called the *opposite regulus.*

Consider a cover of the generalized quadrangle $\mathcal{S} = W(q)$, satisfying the conditions of Theorem 8.1.3. This cover dualizes to a blocking set $\mathcal{B}$ of the generalized quadrangle $\mathcal{S}' = Q(4, q)$. The sum of multiple lines can now be described by pencils, i.e. $q + 1$ lines of $\mathcal{S}'$ on a point, the dual of a line of $PG(3, q)$ which is also a line of $\mathcal{S}$, and reguli corresponding to the $q+1$ points on a line of $PG(3, q)$ which is not a line of $\mathcal{S}$. This is illustrated in Figure 9.3.

The following lemma is given in [26].

Lemma 9.2.1 (De Beule and Storme [26]) *Suppose that $\mathcal{C}$ is a cover of $\mathcal{S} = W(q)$, of size $q^2 + 1 + r$, with $q + r$ smaller than the cardinality of the smallest non-trivial blocking sets in* PG$(2, q)$*, such that the multiple points of $\mathcal{C}$ are a sum $\mathcal{A}$ of lines of* PG$(3, q)$*. If L is a line of $\mathcal{A}$, L not a line of $W(q)$, then $L^\perp \in \mathcal{A}$, with $\perp$ the symplectic polarity corresponding to $W(q)$.*

Suppose that $\mathcal{B}$ is a blocking set of $Q(4, q)$, of size $q^2 + 1 + r$, with $q + r$ smaller than the cardinality of the smallest non-trivial blocking set in PG$(2, q)$. For lower bounds of minimal blocking sets of PG$(2, q)$, we refer to Section 1.6. Since $q + r$ should be smaller than the value of this lower bound, we get an upper bound for r.

For small values of q we obtained:

- $q = 5$, $r < 4$

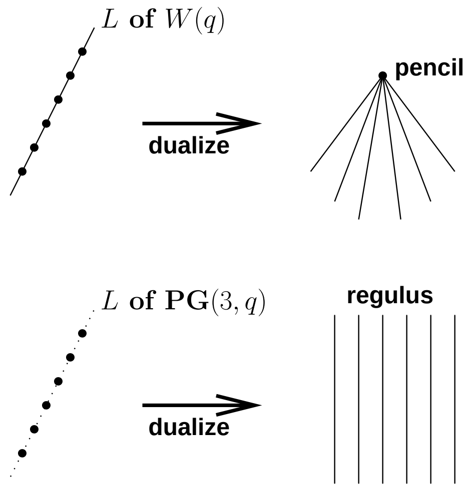


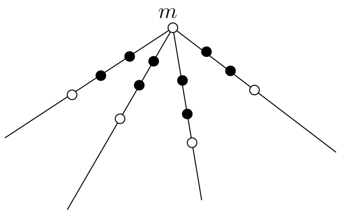
Figure 9.3: Pencil and regulus

- $q = 7,\ r < 5$
- $q = 9,\ r < 4$
- $q = 11,\ r < 7$

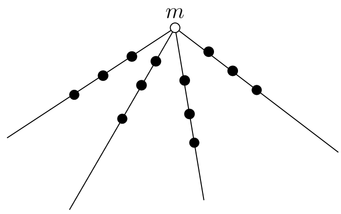
Now, more details for $r = 1$ and $r = 2$ are given. Recall that r is the sum of the weights of the lines in the sum of lines, corresponding to the multiple points of the cover. For every case we describe all possible subcases, which are mutually exclusive. We always start by applying Theorem 8.1.3 to the cover $\mathcal{C}$ of $W(q)$ and then we give the dual information (see also [21]).

The case $r = 1$ for all q odd

Theorem 8.1.3 implies that all multiple points of $\mathcal{C}$ lie on the unique line $M \in \mathrm{PG}(3, q)$ with weight 1. Hence necessarily $M = M^{\perp}$, otherwise the sum of lines would contain two lines with weight 1 (see Lemma 9.2.1), a contradiction since the sum of the weight of the lines equals to $r = 1$.



(a) The case $r = 1$ for all q odd, one pencil



(b) The case $r = 2$ for all $q > 3$ odd, one pencil

Figure 9.4: One pencil

The latter implies $M \in W(q)$. All points $p \in M$ have weight 1 (see Lemma 8.1.1), hence they are covered by two lines of $\mathcal{C}$.

For the blocking set $\mathcal{B}$ of $Q(4,q)$: there is exactly one point $m \in Q(4,q) \setminus \mathcal{B}$, such that all $q+1$ lines of $Q(4,q)$ on m are the multiple lines and meet $\mathcal{B}$ in exactly two points. It is illustrated for $q = 3$ in Figure 9.4(a). We will denote this case by $\mathrm{C_I}$.

The case $r = 2$ for all q odd

1. All multiple points of $\mathcal{C}$ lie on the unique line $M \in \mathrm{PG}(3,q)$ with weight 2. Hence necessarily $M = M^\perp$, hence $M \in W(q)$. All points $p \in M$ have weight 2, hence they are covered by three lines of $\mathcal{C}$.

   For the blocking set $\mathcal{B}$ of $Q(4,q)$: there is exactly one point $m \in Q(4,q) \setminus \mathcal{B}$, such that all $q+1$ lines of $Q(4,q)$ on m are the multiple lines and meet $\mathcal{B}$ in exactly three points, see Figure 9.4(b). We will denote this case by $\mathrm{C_{II}}$.

2. All multiple points of $\mathcal{C}$ lie on two lines $M, N \in \mathrm{PG}(3,q)$ of weight 1. Either $M = M^\perp$ and $N = N^\perp$, and hence $M, N \in W(q)$, or $M^\perp = N$ and hence $M, N \in \mathrm{PG}(3,q) \setminus W(q)$.

   (a) $M = M^\perp$ and $N = N^\perp$:

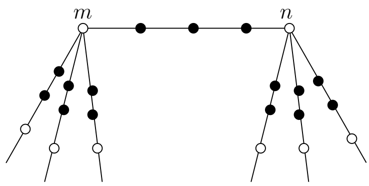


Figure 9.5: The case $r = 2$ for all q odd, two collinear pencils

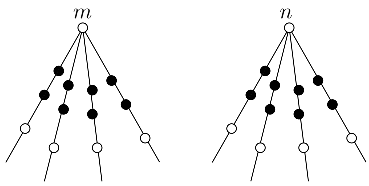


Figure 9.6: The case $r = 2$ for all q odd, two non-collinear pencils

i. M and N intersect in a point p. Necessarily, all points on M and N have weight 1, except for exactly one point having weight 2.
For the blocking set $\mathcal{B}$ of $Q(4, q)$: there are two points $m, n \in Q(4, q) \setminus \mathcal{B}$, the line $\langle m, n \rangle$ is a line of $Q(4, q)$, and all the $q + 1$ lines on m or n meet $\mathcal{B}$ in exactly 2 points, except for exactly one line on m or n, meeting $\mathcal{B}$ in exactly 3 points. It is shown for $q = 3$ in Figure 9.5. We will denote this case by $\mathrm{C_{III}}$.

ii. M and N are skew. Necessarily, all points on M and N have weight 1.
For the blocking set $\mathcal{B}$ of $Q(4, q)$: there are two points $m, n \in Q(4, q) \setminus \mathcal{B}$, $\langle m, n \rangle$ is not a line of $Q(4, q)$, such that all the $q+1$ lines on m or n meet $\mathcal{B}$ in exactly 2 points. We illustrate it for $q = 3$ in Figure 9.6. We will denote this case by $\mathrm{C_{IV}}$.

(b) $M^{\perp} = N$:
All points on the lines M and N have weight exactly 1. The $q+1$

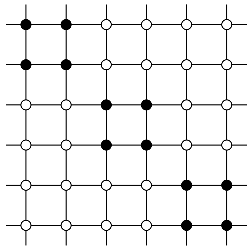


Figure 9.7: The case $r = 2$ for all q odd, regulus and opposite regulus

points on a line M of $\mathrm{PG}(3,q) \setminus W(q)$ dualise to $q+1$ skew lines of a regulus $\mathcal{R} \subset Q(4,q)$. Such a regulus intersects $Q(4,q)$ in a hyperbolic quadric $Q^+(3,q)$. The opposite regulus corresponds with $M^\perp$, and intersects $Q(4,q)$ in the same hyperbolic quadric. For the blocking set $\mathcal{B}$ of $Q(4,q)$: there is regulus $\mathcal{R}_M$ and opposite regulus $\mathcal{R}_N$ such that every line of $\mathcal{R}_M$ and every line of $\mathcal{R}_N$ meet $\mathcal{B}$ in exactly 2 points, see Figure 9.7. We will denote this case by $\mathrm{C_V}$.

Remark 9.2.2 S. De Winter [27] constructed a minimal blocking set $\mathcal{B}$ of size q^2+1+2 of $Q(4,5)$, which contains exactly 12 points on a hyperbolic quadric as described in the last case.

Using information about the structure of multiple lines, the authors of [26] and [23] were able to prove the following results for $q = 3$, and for $q = 5,7$ with the aid of the computer.

Lemma 9.2.3 (De Beule, Storme [26]) *If $\mathcal{B}$ is a minimal blocking set of $Q(4,3)$ different from an ovoid, then $|\mathcal{B}| > 11$.*

Lemma 9.2.4 (De Beule, Hoogewijs, Storme [23]) *If $\mathcal{B}$ is a minimal blocking set of $Q(4,q)$, $q = 5,7$ different from an ovoid of $Q(4,q)$, then $|\mathcal{B}| > q^2+2$.*

Later, J. De Beule and K. Metsch [24] proved the already mentioned result.

Theorem 9.2.5 (De Beule, Metsch [24]) *If q is an odd prime, then $Q(4, q)$ does not have a minimal blocking set of size $q^2 + 2$.*

In [23] the authors performed a computer search to exclude the existence of a minimal blocking set of $Q(4, 7)$ of size $q^2 + 3$ satisfying a special property:

Lemma 9.2.6 (De Beule, Hoogewijs, Storme [23]) *There is no minimal blocking set $\mathcal{B}$ of size $q^2 + 3$ on $Q(4, 7)$ such that there is one point of $Q(4, 7)$ with $q + 1$ lines on it being blocked by exactly three points of $\mathcal{B}$.*

Now the following questions occur:

- Are there other minimal blocking sets of size $q^2 + 3$ in $Q(4, 5)$?
- Is there any blocking set of size $q^2 + 3$ in $Q(4, 7)$?
- Is there any minimal blocking set of size at least $q^2 + 2$ and smaller than $q^2 + q$ in $Q(4, q)$, q odd?

9.2.2 Pruning based on structure of multiple lines

In Section 8.3, an exhaustive algorithm for searching minimal blocking sets by adding points was described. In what follows, we explain a more efficient way of determining the allowed set and of forcing of points, when searching in $Q(4, q)$, q odd and $r = 1$, or $q > 3$ odd and $r = 2$.

Allowed set

In Subsection 9.2.1 we described the way in which the multiple lines are structured. This will allow us to remove more points from the allowed set than in the general case. From now on, we consider only the values of parameters q and r for which the structure of the multiple lines was described. We recall that for $r = 1$, there was just one case, while for $r = 2$, there were four possible ways in which the multiple lines can be structured. We denoted them C_{I}, C_{II}, C_{III}, C_{IV} and C_{V} in the order as they were explained.

The first step in the algorithm is to determine the excess of the lines. Since a generalized quadrangle is transitive on the pairs of points and on the pairs of lines, we can fix the structure, i.e. pencils or reguli in all considered cases. In particular, we determine the lines, which are multiple with the considered excess. Let *midpoint* be a point, where the lines of the pencil meet. In cases C_{I}, C_{II}, C_{III}, C_{IV} (pencils), we can also directly remove the midpoint(s) of pencil(s) from the allowed set.

In each recursive step we determine the new allowed set. Let p be a point currently added to B in this recursive step. If adding the point p gives rise to a line, which attains the expected excess, we can remove all still allowed points on this line from the allowed set.

In Subsection 8.3.1 we explained which points need to be removed from the allowed set in a general case. Now the following question arises. Is it necessary to check, if the situations from Figure 8.2(shown again in Figure 9.8) can occur here? Do we need to remove some extra points from the allowed set? In what follows, we explain that removing points from the lines attaining the excess will be sufficient.

Looking back at the possible structures, it follows immediately that:

1. The situation from Figure 9.8(b) will never occur, since there are never t multiple lines through one point.

2. We do not need to check whether there is such a situation as shown in Figure 9.8(a). To explain this, we consider two cases:

    (a) All lines meeting in a point are multiple. The only possible cases for $r = 1$, or $q > 3$ odd and $r = 2$, are C_{I}, C_{II}, C_{III}, C_{IV}. However, the midpoints of such pencils were already removed in the start of the algorithm and are not in the allowed set anymore.

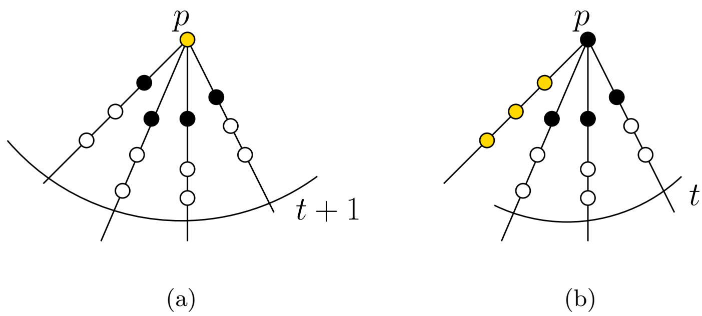


Figure 9.8: Removing points from the allowed set

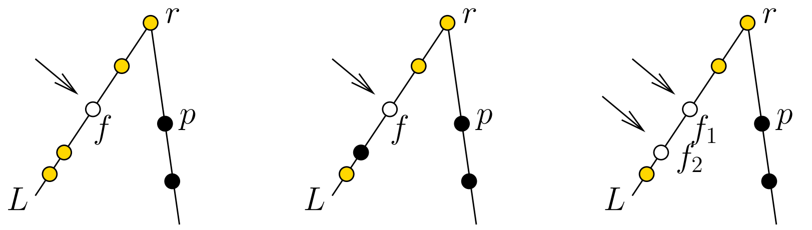
(b) There is a line among the lines from Figure 9.8(a), which is not multiple. Immediately after adding a point on this line, the excess of the line was attained and all points from this line were removed from the allowed set, i.e., the midpoint is not in the allowed set anymore.

Forcing points

Using information about the structure of multiple lines, we can also improve the forcing method.

Let e_L be the excess of a line L and let b_L be the number of points of the current "partial blocking set" on the line L. Let A_L be the set of still allowed points on the line L. Consider a step in the recursive process where the current "partial blocking set" gives rise to a line for which $e_L+1 = |A_L|+b_L$ holds.

If points of A_L are not added to the current "partial blocking set", then the resulting "partial blocking set" can never be extended to a minimal blocking set, so we can prune these possibilities and force the points to be added to the current "partial blocking set".



(a) Forcing a point on a non-multiple line

(b) Forcing a point on a multiple line

(c) Forcing two points on a multiple line

Figure 9.9: Illustrating the idea of forcing points

Figure 9.9 illustrates this idea. Suppose that after adding a point p in the recursive step, a point r was removed from the current set of allowed points. In Figure 9.9(a), there is only one point f on the non-multiple line L left which can be added to the "partial blocking set". Similarly, let L be a multiple line with excess 1, as shown in Figures 9.9(b) and 9.9(c). There is only one point (resp. two points) left on the multiple line L which can be added to the "partial blocking set", see Figure 9.9(b)(resp. 9.9(c)). Hence, when searching for a minimal blocking set, we can forcedly add these points to the "partial blocking set", thus pruning the possible extensions that do not contain these points.

9.2.3 Results of the exhaustive search

We already mentioned that in [23] a computer search was performed to obtain new results. Using the exhaustive algorithm described in Section 8.3, we obtain the following results in less search time than in [23].

In Table 9.2, we compare our search times with the timings found in [23]. We give also the number of points of the generalized quadrangles, the excess r of the blocking set $\mathcal{B}$ and the size $|\mathcal{B}|$ of the blocking set. Because of the evident improvement in running time we tried to get more results.

	# points	r	$\|\mathcal{B}\|$	# $\mathcal{B}$	Time	
					Earlier results [23]	Our results
$Q(4,5)$	156	1	27	0	10 s	< 1 s
$Q(4,7)$	400	1	51	0	9 days	7 min
$Q(4,7)$	400	$2(\mathrm{C_{II}})$	52	0	7 hours	40 s

Table 9.2: Effect of our algorithm when comparing with the earlier results in [23].

	# points	$\|\mathcal{B}\|$	# $\mathcal{B}$			
			$\mathrm{C_{II}}$	$\mathrm{C_{III}}$	$\mathrm{C_{IV}}$	$\mathrm{C_{V}}$
$Q(4,5)$	156	28	0	0	0	1
$Q(4,7)$	400	52	0	0	0	0

Table 9.3: Classification of minimal blocking sets of size q^2+3 in $Q(4,5)$ and $Q(4,7)$.

By exhaustive search we determined the classification of minimal blocking sets of size q^2+3 for $Q(4,5)$ and $Q(4,7)$. The results are shown in Table 9.3. For both values of q and for each of the cases $\mathrm{C_{II}}$, $\mathrm{C_{III}}$, $\mathrm{C_{IV}}$, $\mathrm{C_{V}}$, we classified all minimal blocking sets of size q^2+3. We summarize these results in the following lemma.

Lemma 9.2.7 *There is unique minimal blocking set of size $q^2+3=28$ in $Q(4,5)$ and there is no minimal blocking set of size q^2+3 in $Q(4,7)$.*

9.2.4 Minimal blocking sets of size q^2+q-2

Using the heuristic strategies from Section 8.5 we obtain minimal blocking sets of size q^2+q-2 (i.e. with excess $r=q-3$) for $q=5,7,9,11$. Hence, for small values of q, minimal blocking sets (which are not ovoids) of size smaller than q^2+q do exist.

We now investigate the structure of these blocking sets. This is also

	# all multiple lines	r	# grids
$Q(4,5)$	12	2	1
$Q(4,7)$	32	4	2
$Q(4,11)$	96	8	4
$\boldsymbol{Q(4,q)}$ if $\boldsymbol{q \in \{5,7,11\}}$	$\boldsymbol{(q+1)(q-3)}$	$\boldsymbol{q-3}$	$\boldsymbol{(q-3)/2}$

Table 9.4: Structure of the blocking sets of $Q(4, q)$, $q = 5, 7, 11$ of size $q^2 + q - 2$.

done by a computer search. We focus on the multiple lines of the blocking sets and the way, in which they are structured. The case $q = 9$ is different and we will mention it separately.

For $q = 5, 7, 11$ we observed that, for the blocking sets found, all multiple lines have excess one, which means that the multiple lines contain two points of the blocking set.

Already S. De Winter noticed that for $q = 5$ all multiple lines form a grid. We observed that also for $q = 7, 11$ grids are formed by the multiple lines; moreover, more than one grid appears, see Table 9.4. For these cases the number of multiple lines is $2(q + 1)(q - 3)/2$ and the number of grids formed by multiple lines is $(q - 3)/2$. Further, for $q = 7, 11$ there are $q + 1$ points of $\mathcal{B}$ common to all grids.

Finally we observed that it is possible to obtain a maximal partial ovoid of size $q^2 - 1$ by removing these $q+1$ common points and adding two points of the "perp" of this set. In $Q(4, 5)$ are there 4 possibilities of obtaining such a maximal partial ovoid by removing $q + 1$ points.

For $q = 9$ we observed that, for the maximal blocking sets found, all multiple lines have excess three. The blocking set found has 20 multiple lines and we checked that they form a grid.

As already mentioned in Section 5.4, we found by exhaustive search and J. De Beule and A. Gács recently proved that no maximal partial ovoids of size $q^2 - 1$ exist in $Q(4, 9)$.

To finish this section we want to mention our effort, which was not successful yet. We tried to find a blocking set of size q^2+q-2 for larger q, especially for $q=13$. If we suppose that this blocking set would be of the same structure, it should contain 5 grids with 14 points in the intersection. So far we have found no blocking set of the given size by our (not exhaustive!) computer search. We expect that no blocking set of $Q(4,13)$ of size q^2+q-2 exists.

A Nederlandstalige samenvatting

In deze Nederlandstalige appendix geven we een kort overzicht van de belangrijkste technieken en resultaten die in deze scriptie gepresenteerd worden. We volgen de structuur van de Engelstalige tekst.

A.1 Inleiding

In dit werk ontwikkelen we hoofdzakelijk algoritmen voor het bestuderen van speciale deelstructuren in combinatorische objecten uit de eindige meetkunde en de grafentheorie. In hoofdstuk 1 beschrijven we de verschillende objecten en aanverwante terminologie, die voor de scriptie relevant zijn. Dit omvat enkele basisconcepten van backtracking en heuristische algoritmen, elementaire definities uit de grafentheorie, en tenslotte definities van begrippen uit de projectieve meetkunde (zoals incidentiestructuren, projectieve ruimten, kwadrieken en Hermitische variëteiten, en polaire ruimten).

A.2 Veralgemeende vierhoeken

In hoofdstuk 2 bespreken we eindige veralgemeende veelhoeken, waarbij we de aanpak van S.E. Payne en J.A. Thas [55] volgen. We beperken ons hierbij tot definities en eigenschappen die relevant zijn onze doeleinden.

Eindige veralgemeende vierhoeken

Een *eindige veralgemeende vierhoek* is een incidentiestructuur $\mathcal{S} = (P, B, \mathrm{I})$, met I een symmetrische incidentierelatie tussen punten en rechten, die aan de volgende axioma's voldoet:

(i) Elk punt is incident met $1 + t$ rechten ($t \geq 1$) en twee verschillende punten zijn incident met ten hoogste één rechte;

(ii) Elke rechte is incident met $1 + s$ punten ($s \geq 1$) en twee verschillende rechten zijn incident met ten hoogste één punt;

(iii) Zij x een punt en L een rechte die niet incident is met x, dan bestaat er een uniek paar $(y, M) \in P \times B$ waarvoor $x \mathrm{\ I\ } M \mathrm{\ I\ } y \mathrm{\ I\ } L$.

De gehele getallen s en t worden de *parameters* van de veralgemeende vierhoek genoemd. Men zegt dat $\mathcal{S}$ *orde* (s, t) heeft; wanneer $s = t$ is, dan zegt men dat $\mathcal{S}$ orde s heeft. Door de rollen van punten en rechten om te wisselen, bekomt men de *duale* van een veralgemeende vierhoek. Een veralgemeende vierhoek van orde $(s, 1)$ wordt een *rooster* genoemd, terwijl een veralgemeende vierhoek van orde $(1, t)$ een *duaal rooster* genoemd wordt.

Ovoïdes en spreads

Een *ovoïde* van $\mathcal{S}$ is een verzameling $\mathcal{O}$ van punten van $\mathcal{S}$, zodanig dat elke rechte van $\mathcal{S}$ incident is met een uniek punt van $\mathcal{O}$. Een *partiële ovoïde* van $\mathcal{S}$ is een verzameling $\mathcal{O}$ van punten van $\mathcal{S}$, zodanig dat elke rechte van $\mathcal{S}$ incident is met ten hoogste één punt van $\mathcal{O}$. Een *spread* van $\mathcal{S}$ is een verzameling $\mathcal{R}$ van rechten van $\mathcal{S}$, zodanig dat elk punt van $\mathcal{S}$ incident is met een unieke rechte van $\mathcal{R}$. Een *partiële spread* van $\mathcal{S}$ is een verzameling $\mathcal{R}$

van rechten van $\mathcal{S}$, zodanig dat elk punt van $\mathcal{S}$ incident is met ten hoogste één rechte van $\mathcal{R}$. Een (partiële) ovoïde in $\mathcal{S}$ is een (partiële) spread in zijn duale $\mathcal{S}^D$. Een partiële ovoïde (of spread) wordt *maximaal* of *compleet* genoemd als ze niet bevat is in een grotere partiële ovoïde (of spread).

A.3 Computervoorstelling

Om een computerzoektocht te kunnen uitvoeren, is een geschikte voorstelling van een veralgemeende vierhoek nodig. We gebruiken hiervoor een incidentiematrix.

In hoofdstuk 3 bespreken we een *coördinatisering*, d.i. een methode om de elementen van een veralgemeende veelhoek, i.h.b. een veralgemeende vierhoek, te labelen. We baseren ons hierbij hoofdzakelijk op de labeling, ingevoerd door G. Hanssens en H. Van Maldeghem in [38].

Verder bespreken we kort een methode om de punten van een klassieke veralgemeende vierhoek te indiceren, hetgeen leidt tot een constructie van een incidentiematrix voor de veralgemeende vierhoek.

Ten slotte associëren we met een veralgemeende vierhoek $\mathcal{S}$ een zogenaamde *collineariteitsgraaf* of *puntengraaf* $G_\mathcal{S}$. Hierin corresponderen de punten van P met de toppen van $G_\mathcal{S}$, en twee toppen zijn adjacent als en slechts als de corresponderende punten collineair zijn. De graaf $G_\mathcal{S}$ is sterk regulier [55] met parameters

$$v = (s+1)(st+1), \quad k = s(t+1), \quad \lambda = s-1, \quad \mu = t+1\,.$$

Een ovoïde van een veralgemeende vierhoek $\mathcal{S}$ correspondeert met een maximum onafhankelijke verzameling van grootte $st+1$ in $G_\mathcal{S}$, of equivalent, met een maximum kliek in zijn complement $\overline{G_\mathcal{S}}$. Een spread van $\mathcal{S}$ is een maximum onafhankelijke verzameling van grootte $st+1$ in de collineariteitsgraaf $G_{\mathcal{S}^D}$ van de duale veralgemeende vierhoek $\mathcal{S}^D$. Maximale partiële ovoïden en spreads zijn maximale klieken in $\overline{G_\mathcal{S}}$ of $\overline{G_{\mathcal{S}^D}}$.

A.4 Algoritmen voor het zoeken naar maximale partiële ovoïdes

Het al dan niet bestaan van ovoïden en spreads in eindige veralgemeende vierhoeken is reeds uitgebreid bestudeerd [67, 68]. Wanneer van een veralgemeende vierhoek $\mathcal{S}$ geweten is dat hij geen ovoïde heeft, dan ligt het voor de hand om te vragen naar de grootste verzameling $\mathcal{O}$ van punten, zodanig dat elke rechte van $\mathcal{S}$ met hoogstens één punt van $\mathcal{O}$ incident is. In een veralgemeende vierhoek die wel ovoïden heeft, kunnen we ook zoeken naar de grootste partiële ovoïde die geen ovoïde is. Duaal kunnen dezelfde vragen voor spreads gesteld worden.

Heel wat onderzoek is reeds gebeurd voor partiële ovoïden en spreads van grootte $st + 1 - d$, met kleine deficiëntie d. Speciale aandacht gaat hierbij naar de uitbreidbaarheid van dergelijke partiële ovoïden en spreads tot ovoïden en spreads [13, 37]. Enkele theoretische bovengrenzen voor de grootte van een maximale partiële ovoïde of spread zijn gekend.

Recent werd ook aandacht besteed aan de kleinste maximale partiële ovoïden en spreads in veralgemeende vierhoeken.

Onze resultaten voor (partiële) ovoïden en spreads zijn hoofdzakelijk door computerzoektochten bekomen. Paragraaf 4.2 beschrijft exhaustieve zoekalgoritmen, waarbij we standaardalgoritmen voor het zoeken naar klieken gebruiken, aangevuld met standaard snoeitechnieken.

In paragrafen 4.3, 4.4 en 4.5, beschrijven we nieuwe snoeitechnieken die steunen op specifieke eigenschappen van veralgemeende vierhoeken. Deze benadering levert exacte antwoorden omtrent bv. de grootte van de grootste/kleinste maximale partiële ovoïde of spread, of de classificatie van alle maximale partiële ovoïden of spreads van een gegeven grootte. Deze resultaten verbeteren de beste theoretische grenzen die tot nu toe gekend zijn.

Men kan zich ook afvragen of, voor een gegeven grootte, er maximale partiële ovoïden of spreads van die grootte bestaan. In het bijzonder zijn we geïnteresseerd in het spectrum van waarden waarvoor maximale partiële ovoïden of spreads van die grootte bestaan. De klasse algoritmen die beschreven wordt in paragraaf 4.6, is gebaseerd op heuristische technieken en blijkt in de praktijk zeer geschikt te zijn voor het verkennen van deze spectra.

De technieken beschreven in hoofdstuk 4 zullen gepubliceerd worden in het artikel *Clique algorithms for finding substructures in generalized quadrangles* [18].

A.5 Maximale partiële ovoïden van $W(q)$ en $Q(4,q)$

De hoofdstukken 5, 6 en 7 behandelen partiële ovoïden en spreads in klassieke veralgemeende vierhoeken, en zijn als volgt georganiseerd. Elk hoofdstuk behandelt een type veralgemeende vierhoek en zijn duale. Eerst en vooral geven we een overzicht van de gekende resultaten omtrent het al dan niet bestaan van ovoïden en spreads, evenals de gekende theoretische boven- en ondergrenzen voor de grootte van maximale partiële ovoïden of spreads, en – waar beschikbaar – resultaten van eerdere computerzoektochten. Deze resultaten dienen als vergelijkingsmateriaal voor de resultaten bekomen door onze eigen computerzoektochten.

In deze samenvatting beperken we ons tot het vermelden van de meest interessante resultaten. We geven geen resultaten voor de spectra, en bij het vergelijken met eerdere resultaten vermelden we enkel de referentie.

De resultaten verzameld in hoofdstuk 5 zullen gepubliceerd worden in het artikel *On the smallest maximal partial ovoids and spreads of the generalized quadrangles $W(q)$ and $Q(4,q)$* [17].

A.5.1 Ovoïden in $Q(4,q)$ en $W(q)$

De volgende resultaten zijn welbekend.

Stelling A.5.1 *De veralgemeende veelhoek $Q(4,q)$ heeft steeds ovoïden.*

Stelling A.5.2 *De veralgemeende veelhoek $W(q)$ heeft ovoïden als en slechts als q even is.*

q	BG [65]	$\lvert\mathcal{O}\rvert$ gevonden	$\#\mathcal{O}$
3	7	**7**	1
5	21	**18**	2
7	43	**33**	1
9	73	**51**	
11	111	**70**	
13	157	**92**	
17	273	**129**	
19	343	**150**	
23	507	**190**	
25	601	**203**	
27	703	**236**	

Tabel A.1: Grote maximale partiële ovoïden van $W(q)$, voor kleine q-waarden, q oneven, bekomen door heuristische en/of exhaustieve zoektochten. Voor $q \leq 7$ is de grootte en de classificatie van de grootste maximale partiële ovoïden door een exhaustieve zoektocht bepaald.

A.5.2 De grootste maximale partiële ovoïden in $W(q)$, q oneven

Aangezien de veralgemeende vierhoek $W(q)$, voor q oneven, geen ovoïden heeft, zoeken we naar de grootste maximale partiële ovoïde.

In tabel A.1 geven we de grootte $|\mathcal{O}|$ van de grootste maximal partiële ovoïde gevonden door onze computerzoektocht, en vergelijken deze met de beste gekende bovengrens (BG) uit [65]. Voor $q \leq 7$, hebben we door een exhaustieve zoektocht bevestigd dat de grootste waarde die we vinden, ook effectief de grootte van de grootste maximale partiële ovoïde is. Voor deze q-waarden classificeren we ook alle partiële ovoïden van deze grootte. Voor $q = 5, 7$ vormt de waarde gevonden door onze computerzoektocht een verbetering op de gekende theoretische grens. Voor $q \geq 9$ geven we de

q	Tweede-grootste $\lvert\mathcal{O}\rvert$ gevonden (van grootte q^2-2q+3)	Grootste $\lvert\mathcal{O}\rvert$ gevonden (van grootte q^2-q+1)
4	11	13
8	51	57
16	227	241
32	963	993

Tabel A.2: Grote maximale strikt partiële ovoïden van $W(q)$, voor kleine q even, bekomen door heuristische en/of exhaustieve zoektochten. Voor $q=4$ werd de grootte van de grootste en de tweede-grootste maximale strikt partiële ovoïde bepaald door een exhaustieve zoektocht.

waarden die bekomen werden door een heuristische zoektocht. De effectieve waarde van de grootste maximale partiële ovoïde kan dus nog groter zijn dan de hier gepresenteerde waarde. Voor $q=9$ verwachten we echter dat de effectieve waarde zeer dicht bij onze gevonden waarde (51) ligt.

A.5.3 Grote maximale partiële ovoïden in $Q(4,q)$ en $W(q)$, q even

Voor q even, merkten we op dat er maximale partiële ovoïden van grootte $q^2-q+1-(q-2)=q^2-2q+3$ bestaan, en we vonden geen maximale partiële ovoïden met grootte groter dan q^2-2q+3 en kleiner dan q^2-q+1 (zie tabel A.2).

Een meetkundige constructie voor maximale partiële ovoïden van grootte q^2-q+1 and q^2-2q+3 of $W(q)$, q even, kan als volgt beschreven worden. We beschrijven de constructie in $Q(4,q)$, die isomorf is met $W(q)$ in dit geval (q even).

Constructie A.5.3

1. Merk op dat $\left|C^{\perp}\right| \in \{1, q+1\}$ voor elke kegelsnede C in $Q(4,q)$, q even. Wanneer we een kegelsnede C in een elliptische kwadriek $\mathcal{O} := Q^{-}(3,q) \subset Q(4,q)$ beschouwen, dan is $C^{\perp}$ noodzakelijkerwijze een uniek punt c. Het is gemakkelijk in te zien dat $(\mathcal{O} \cup \{c\}) \setminus C$ een maximale partiële ovoïde van grootte $q^2 - q + 1$ is.

2. Zij $\mathcal{O}$ een elliptische kwadriek van $Q(4,q)$ en zij C_1 en C_2 twee kegelsneden van $\mathcal{O}$, met $|C_1 \cap C_2| = 2$. Het is duidelijk dat de punten $c_1 := C_1^{\perp}$ en $c_2 := C_2^{\perp}$ niet collineair zijn (aangezien $|C_1 \cap C_2| = 2$). Voor $q > 2$ volgt gemakkelijk dat $(\mathcal{O} \cup \{c_1, c_2\}) \setminus (C_1 \cup C_2)$ een maximale partiële ovoïde van grootte $q^2 - 2q + 3$ is.

A.5.4 Grote maximale partiële ovoïden in $Q(4,q)$, q oneven

Voor elke maximale (strikt) partiële ovoïde $\mathcal{O}$ in $Q(4,q)$, q oneven, geldt dat

$$|\mathcal{O}| \leq q^2 - 1.$$

In tabel A.3 geven we voor elke waarde van q de grootte $|\mathcal{O}|$ van de grootste maximale strikt partiële ovoïde gevonden door onze computerzoektochten en vergelijken deze met de waarde van $q^2 - 1$.

Voor $q = 3, 5, 7, 11$ vonden we een maximale partiële ovoïde van grootte $q^2 - 1$. Voor $q \leq 7$ wijst een exhaustieve zoektocht uit dat de maximale partiële ovoïden van deze grootte uniek zijn, op een isomorfisme na. Voor $q = 9$ bevestigen we door een exhaustieve zoektocht dat geen maximale partiële ovoïde van grootte $q^2 - 1 = 80$ bestaat. Voor grotere q-waarden vinden onze heuristische zoektochten geen maximale partiële ovoïde van grootte $q^2 - 1$.

In de gevallen waar we geen maximale partiële ovoïde van grootte $q^2 - 1$ vinden, heeft de grootste maximale partiële ovoïde die we vinden, grootte $q^2 - q + 2$. In de gevallen waar we wel een maximale partiële ovoïde van grootte $q^2 - 1$ vinden, heeft de tweede-grootste maximale partiële ovoïde die we vinden, grootte $q^2 - q + 2$. Dit geldt voor elke beschouwde q-waarde, behalve in zekere zin voor $q = 3$, waarvoor de waarden van $q^2 - 1$ en $q^2 - q + 2$ samenvallen (zie tabel A.3).

q	$q^2 - 1$	Tweede-grootste $\|\mathcal{O}\|$ gevonden	Grootste $\|\mathcal{O}\|$ gevonden	$\#\mathcal{O}$
3	8	5	**8**	1
5	24	22	**24**	1
7	48	44	**48**	1
9	80	73	**74**	
11	120	112	**120**	
13	168		**158**	
17	288		**274**	
19	360		**344**	

Tabel A.3: Grote maximale partiële ovoïden van $Q(4, q)$, voor kleine q oneven, bekomen door heuristische en/of exhaustieve zoekalgoritmen. Voor $q \leq 7$ werd de classificatie van de grootste en de tweede-grootste maximale partiële ovoïden gedaan door een exhaustief zoekalgoritme. Voor $q = 9$ werd het bestaan van een maximale partiële ovoïde van grootte $q^2 - 1 = 80$ uitgesloten door een exhaustieve zoektocht.

Geïnspireerd door onze computerresultaten voor $q = 9$ hebben J. De Beule en A. Gács recent het probleem van het al dan niet bestaan van maximale partiële ovoïden van grootte $q^2 - 1$ opgelost voor $q = p^h$ met p een oneven priemgetal, $h > 1$.

Stelling A.5.4 (De Beule and Gács [22]) *Zij $q = p^h$, p een oneven priemgetal, en $h > 1$. Dan heeft $Q(4, q)$ geen complete $(q^2 - 1)$-bogen.*

Voor $q = 5, 7, 11$ kunnen de maximale partiële ovoïden van grootte $q^2 - 1$ geconstrueerd worden uit de blokkerende verzamelingen van $Q(4, q)$ van grootte $q^2 + q - 2$. We bespreken deze constructie in paragraaf 9.2.4.

A.5.5 Kleine maximale partiële ovoïden in $W(q)$

De kleinste maximale partiële ovoïden

We beschouwen eerst de kleinste maximale partiële ovoïden van $W(q)$. Uit tabellen 5.5 en 5.6 met de spectra blijkt dat de kleinste maximale partiële ovoïden die we vinden, grootte $q + 1$ hebben.

Constructie A.5.5 De punten van een hyperbolische rechte L in $\mathrm{PG}(3, q)$, m.a.w. een rechte in $\mathrm{PG}(3, q)$ die geen rechte van de veralgemeende vierhoek $W(q)$ is, vormen een maximale partiële ovoïde van grootte $q + 1$.

Dit computerresultaat wordt bevestigd in de volgende stelling.

Stelling A.5.6 *De kleinste maximale partiële ovoïden van $W(q)$ hebben grootte $q + 1$ en bestaan uit de puntenverzamelingen van de hyperbolische rechten van $W(q)$.*

Gevolg A.5.7 *Een maximale partiële ovoïde van $W(q)$ is een blokkerende verzameling met betrekking tot de vlakken van* $\mathrm{PG}(3, q)$.

Gevolg A.5.8

1. *De kleinste maximale partiële spreads van $Q(4, q)$ hebben grootte $q+1$ en bestaan uit de rechten van een regulus van* $\mathrm{PG}(3, q)$.

2. *De kleinste maximale partiële spreads van $W(q)$, q even, hebben grootte $q+1$ en bestaan uit de rechten van een regulus van* $\mathrm{PG}(3,q)$.

3. *De kleinste maximale partiële ovoïden van $Q(4,q)$, q even, hebben grootte $q+1$ en bestaan uit de puntenverzamelingen van kegelsneden die de kern van $Q(4,q)$ als hun kern hebben.*

De tweede-kleinste maximale partiële ovoïden

Nu we de kleinste maximale partiële ovoïden van $W(q)$ geclassificeerd hebben, bekijken we de tweede-kleinste maximale partiële ovoïden van $W(q)$.

Voor de parameters waarvoor we een computerzoektocht doorvoerden, merken we op dat maximale partiële ovoïden van grootte $2q+1$ steeds gevonden werden, terwijl we geen maximale partiële ovoïden met groottes tussen $q+1$ en $2q+1$ vonden (zie tabellen 5.5 en 5.6).

Constructie A.5.9 Een voorbeeld van een maximale partiële ovoïde van grootte $2q+1$ wordt bekomen door alle punten, behalve één willekeurig punt p, van een hyperbolische rechte L in $\mathrm{PG}(3,q)$ te nemen, samen met één willekeurig punt (dat niet collineair is met de overblijvende punten van L) van elk van de $q+1$ rechten door p.

Over de tweede-kleinste maximale partiële ovoïden van $W(q)$ hebben we ook theoretische resultaten bekomen, waarvan we belangrijkste hier vermelden.

Gevolg A.5.10 *De tweede-kleinste maximale partiële ovoïden $\mathcal{O}$ van $W(q^2)$, $q=p^h$, $p>3$ priem, $h\geq 1$, bevatten ten minste $s(q^2)+1$ punten, waarbij $s(q^2)$ de grootte van de tweede-kleinste minimale blokkerende verzameling in* $\mathrm{PG}(2,q^2)$ *voorstelt. Als $q=p>2$, dan bevat $\mathcal{O}$ ten minste* $3(p^2+1)/2+1$ *punten.*

Deze grens kan verbeterd worden door gebruik te maken van een recent resultaat van S. De Winter and K. Thas [28].

q	OG [17]	$\lvert\mathcal{O}\rvert$ gevonden	$\#\mathcal{O}$
2	–	**5**	1
3	7	**7**	1
4	9	**9**	4
5	10	**11**	10

Tabel A.4: Grootte en classificatie van de tweede-kleinste maximale partiële ovoïden van $W(q)$, $q \leq 5$, bepaald door een exhaustieve computerzoektocht.

Gevolg A.5.11 *De tweede-kleinste maximale partiële ovoïden $\mathcal{O}$ van $W(p^3)$, $p \geq 7$ priem, bevatten ten minste $3(p^3+1)/2$ punten.*

Gevolg A.5.12 *Zij $\mathcal{O}$ een tweede-kleinste maximale partiële ovoïde van $W(p)$, $p > 2$ priem. Dan $\lvert\mathcal{O}\rvert \geq 3(p+1)/2+1$.*

Opmerking A.5.13 De voorgaande resultaten kunnen vertaald worden in resultaten voor maximale partiële spreads van $Q(4,q)$, maximale partiële spreads van $W(q)$, q even, en maximale partiële ovoïden van $Q(4,q)$, q even.

In tabel A.4 geven we, voor $q \leq 5$, de grootte van de tweede-kleinste maximale partiële ovoïde die gevonden werd door onze computerzoektochten. We vergelijken ze met de waarde van de ondergrens (OG) voor de tweede-kleinste maximale partiële ovoïde, beschreven in de stellingen A.5.10, A.5.11 en A.5.12. We bevestigden door een exhaustieve computerzoektocht dat de gevonden waarde effectief de grootte van de tweede-kleinste maximale partiële ovoïde is. Voor deze q-waarden bepaalden we ook alle niet-equivalente maximale partiële ovoïden van die grootte.

Merk op dat onze computerresultaten aangeven dat de bekomen theoretische ondergrenzen niet scherp zijn (bv. voor $q = 5$).

Maximale partiële ovoïden van groote $3q - 1$

Onze computerresultaten geven ook het bestaan aan van maximale partiële ovoïden van grootte $3q - 1$, voor alle beschouwde waarden van q (zie tabellen 5.5 en 5.6). Een dergelijke maximale partiële ovoïde kan op de volgende manier geconstrueerd worden voor $q \geq 4$.

Constructie A.5.14 Zij X en Y twee scheve totaal isotrope rechten. Kies verschillende punten x_1, x_2, x_3 en x op de rechte X en zij $y_i = x_i^\perp \cap Y$, $i = 1, 2, 3$. Tenslotte kiezen we een punt y op Y verschillend van y_1, y_2, y_3 en $x^\perp \cap Y$ (we kunnen y kiezen, omdat $q \geq 4$). Noemen we $\mathcal{O}_1$ de verzameling van alle punten van $(x_1y_2 \cup x_2y_3 \cup x_3y_1) \setminus \{x_i, y_i \mid i = 1, 2, 3\}$, dan is $\mathcal{O} := \mathcal{O}_1 \cup \{x, y\}$ een maximale partiële ovoïde van grootte $3q - 1$.

A.5.6 Kleine maximale partiële ovoïden in $Q(4, q)$, q oneven

Vervolgens bestuderen we de kleinste maximale partiële ovoïden in $Q(4, q)$, q oneven. Aangezien $W(q)$ de duale is van $Q(4, q)$, kan het volgende resultaat voor maximale partiële ovoïdes in $W(q)$ beschouwd worden als een ondergrens voor de grootte van maximale partiële ovoïden in $Q(4, q)$, q oneven.

Stelling A.5.15 *Zij S een maximale partiële spread in $W(q)$, q oneven. Dan $|S| \geq \lceil 1.419q \rceil$.*

In tabel A.5 geven we, voor elke waarde van q, de grootte $|\mathcal{O}|$ van de kleinste maximale partiële ovoïde die gevonden werd door onze computerzoektocht. We vergelijken deze waarde met de waarde van de ondergrens (OG) beschreven in stelling A.5.15. Voor $q \leq 5$ bevestigden we door een exhaustieve zoektocht dat de kleinste waarde die we vinden, ook effectief de grootte van de kleinste maximale partiële ovoïde is. Voor deze q-waarden bepaalden we ook alle niet-equivalente maximale partiële ovoïden van deze grootte. Merk op dat, niettegenstaande het feit dat de theoretische grenzen lineair in q zijn, de computerresultaten eerder wijzen op een kwadratische ondergrens.

q	OG [17]	$\|\mathcal{O}\|$ gevonden	$\#\mathcal{O}$	Niet-bestaan
3	5	**5**	1	
5	8	**13**		8..12
7	11	**14**		11
9	14	**22**		
11	17	**28**		
13	19	**41**		
17	25	**67**		
19	27	**84**		

Tabel A.5: Kleine maximale partiële ovoïden in $Q(4, q)$, voor kleine q oneven, bekomen door heuristische en/of exhaustieve zoektochten. Voor $q \leq 5$ werd de grootte en de classificatie van de kleinste maximale partiële ovoïden bekomen door een exhaustieve zoektocht. Voor $q = 7$ werd het bestaan van maximale partiële ovoïden van grootte 11 weerlegd door een exhaustieve zoektocht.

q	BG		Eerdere resultaten		$\|\mathcal{O}\|$ gevonden	$\#\mathcal{O}$
2	7	[67, 8]	6	[29]	**6**	1
3	21	[8]	16	[30]	**16**	1
4	37	[8]	25	[30, 1]	**25**	3
5	106	[67, 8]	42	[30]	**48**	
7	302	[67]	60	[1]	**98**	
8	217	[8]	74	[1]	**126**	
9	401	[8]	–		**146**	
11	1222	[67]	–		**216**	
13	2042	[67]	–		**273**	

Tabel A.6: Grote maximale partiële ovoïden in $Q^-(5,q)$, voor kleine q, bekomen door heuristische en/of exhaustieve computerzoektochten. Voor $q \leq 4$ werd de grootte en de classificatie van de grootste maximale partiële ovoïden bekomen door exhaustieve zoektochten.

A.6 Maximale partiële ovoïden in $Q^-(5,q)$ en $H(3,q^2)$

Hoofdstuk 6 behandelt de veralgemeende vierhoek $Q^-(5,q)$ en zijn duale $H(3,q^2)$.

De resultaten besproken in dit hoofdstuk zijn verschenen in het artikel *Searching for maximal partial ovoids and spreads in generalized quadrangles* [19].

A.6.1 Grote maximale partiële ovoïden in $Q^-(5,q)$

In tabel A.6 geven we de grootte $|\mathcal{O}|$ van de grootste maximale partiële ovoïde die gevonden werd door onze computerzoektochten. We vergelijken ze met de waarde van de beste gekende bovengrenzen (BG) uit [67] en [8], en met de grootte van de grote maximale partiële ovoïden beschreven in [1] en [30]. Merk op dat de grootste bekomen waarden voor $q \leq 4$ effectief de grootte van de grootste maximale partiële ovoïde is – dit werd door een exhaustieve computerzoektocht bevestigd. Voor kleine q bepaalden we ook alle niet-equivalente maximale partiële ovoïden van die grootte.

De resultaten lijken te wijzen op een kwadratische bovengrens.

We geven een meetkundige constructie van een maximale partiële ovoïde van grootte 96 in $Q^-(5,7)$. Eerst construeren we een maximale partiële ovoïde van grootte 12 in $Q^-(5,3)$. Deze zijn reeds gekend (bv. [30]). We geven een andere beschrijving, die kan aangepast worden om een maximale partiële ovoïde van grootte 96 in $Q^-(5,7)$ te construeren.

Constructie A.6.1 (door K. Coolsaet) Beschouw de kwadriek $x_0^2 + x_1^2 + \ldots + x_5^2 = 0$. Zij P' de verzameling punten van de gedaante $(1, \pm1, \pm1, \pm1, \pm1, \pm1)$ en elke cyclische permutatie van de coördinaten. De verzameling P' heeft kardinaliteit 32. De punten van P' met een even aantal mintekens vormen een maximale partiële ovoïde van grootte 16 in $Q^-(5,3)$, terwijl de punten van P' met een oneven aantal mintekens een andere maximale partiële ovoïde van grootte 16 vormen.

Op analoge manier beschouwen we de verzameling P' van punten van de gedaante $(3, \pm1, \pm1, \pm1, \pm1, \pm1)$, en elke cyclische permutatie van de coördinaten. De verzameling P' heeft kardinaliteit 192. De punten van P' met een even aantal mintekens vormen een maximale partiële ovoïde van grootte 96 in $Q^-(5,7)$, terwijl de punten van P' met een oneven aantal mintekens een andere maximale partiële ovoïde van grootte 96 vormen.

Zoals blijkt uit tabel A.6, is 16 de grootte van de grootste maximale partiële ovoïde in $Q^-(5,3)$. Daarentegen is de geconstrueerde maximale partiële ovoïde van grootte 96 in $Q^-(5,7)$ niet de grootste. Onze computerzoektochten vonden immers een maximale partiële ovoïde van grootte 98.

A.6.2 Kleine maximale partiële ovoïden in $Q^-(5,q)$

In tabel A.7 geven we de grootte $|\mathcal{O}|$ van de kleinste maximale partiële ovoïde die gevonden werd door onze computerzoektochten. We vergelijken ze met de waarde van de beste gekende ondergrens (OG) uit [30], de waarde van q^2+1 en de grootte van de kleine maximale partiële ovoïden uit [1].

q	OG [30]	q^2+1	Eerdere resultaten	$\|\mathcal{O}\|$ gevonden	$\#\mathcal{O}$
2	5	5	5 [1, 30]	**5**	1
3	7	10	10 [1, 30]	**7**	1
4	10	17	17 [1, 30]	**13**	
5	12	26	26 [1, 30]	**18**	
7	16	50	46 [1]	**32**	
8	18	65	57 [1]	**41**	
9	20	82		**52**	
11	24	122		**68**	
13	28	170		**89**	

Tabel A.7: Kleine maximale partiële ovoïden in $Q^-(5,q)$, voor kleine q, bekomen door heuristische en/of exhaustieve zoektochten. Voor $q \leq 4$ werd de grootte en de classificatie van de kleinste maximale partiële ovoïde bekomen door exhaustieve zoektochten.

q	LB		Eerdere resultaten [2]	$\lvert\mathcal{O}\rvert$ gevonden
3	12	[2]	16	**16**
5	29	[28]	61	**56**
7	53	[28]	155	**142**

Tabel A.8: Kleine maximale partiële ovoïden in $H(3, q^2)$, voor kleine q, bekomen door heuristische computerzoektochten.

Voor $q \leq 4$ bevestigden we door exhaustieve zoektochten dat de kleinste waarde die we vinden, ook effectief de grootte van de kleinste maximale partiële ovoïde is. Voor $q \leq 3$ bepaalden we ook alle niet-equivalente maximale partiële ovoïden van die grootte.

Voor $q = 2, 3$ is $q^2 + 1$ de grootte van de kleinste maximale partiële ovoïde. Voor alle beschouwde $Q^-(5, q), q \geq 4$ vonden we maximale partiële ovoïden met grootte kleiner dan $q^2 + 1$. Deze resultaten lijken te wijzen op een subkwadratische ondergrens.

A.6.3 Kleine maximale partiële ovoïden in $H(3, q^2)$

In tabel A.8 geven we de grootte $|\mathcal{O}|$ van de kleinste maximale partiële ovoïde gevonden door onze computerzoektochten. We vergelijken ze met de waarde van de beste gekende ondergrens (OG) uit [2] en [28], en met de grootte van de kleine maximale partiële ovoïden uit [2].

Merk op dat we kleinere maximale partiële ovoïden gevonden hebben dan de kleinste tot nu toe gekende, geconstrueerd in [1].

A.7 Maximale partiële ovoïden en spreads in $H(4, q^2)$

In hoofdstuk 7 presenteren we resultaten, bekomen door exhaustieve en heuristische zoektochten, voor maximale partiële ovoïden en spreads in de Hermitische variëteit $H(4, q^2)$.

q	maximale partiële ovoïden			maximale partiële spreads		
	BG [50]	$\lvert\mathcal{O}\rvert$ gevonden	$\#\mathcal{O}$	$st+1$	$\lvert\mathcal{S}\rvert$ gevonden	$\#\mathcal{S}$
2	25	**21**	1	33	**29**	6
3	201	**105**		244	**162**	
4	577	**289**		1025	**494**	

Tabel A.9: Grote maximale partiële ovoïden en spreads in $H(4,q^2)$, voor kleine q, bekomen door heuristische en/of exhaustieve zoektochten. Voor $q = 2$ werd de grootte en de classificatie van de grootste maximale partiële ovoïden en spreads bepaald door exhaustieve zoektochten.

De resultaten in dit hoofdstuk zullen gepubliceerd worden in het artikel *Clique algorithms for finding substructures in generalized quadrangles* [18].

A.7.1 Grote maximale partiële ovoïden en spreads in $H(4,q^2)$

Stelling A.7.1 (Payne and Thas [55]) *$H(4,q^2)$ heeft geen ovoïden. Voor $q = 2$ heeft hij geen spread, zoals aangetoond door een niet-gepubliceerd computerresultaat van Brouwer [12]. Voor $q > 2$ is het al dan niet bestaan van een spread een open probleem.*

In tabel A.9 geven we de groottes $|\mathcal{O}|$ en $|\mathcal{S}|$ van de grootste maximale partiële ovoïde en spread die door onze computerzoektochten gevonden werden. We vergelijken deze met de waarde van de beste gekende bovengrens (BG) uit [50], of met de waarde van $st+1$ (hetgeen de grootte van een spread in $H(4,q^2)$ zou zijn). Merk op dat de grootste waarden die gevonden werden voor $q = 2$, effectief de grootte van de grootste maximale partiële ovoïde, resp. spread, zijn; dit werd door een exhaustieve computerzoektocht bevestigd.

Exhaustieve zoektochten tonen ook aan dat de maximale partiële ovoïde van grootte 21 in $H(4,4)$ uniek is op een isomorfisme na, terwijl $H(4,4)$ 6 niet-equivalente maximale partiële spreads van grootte 29 heeft.

q	maximale partiële ovoïden		maximale partiële spreads	
	OG [43]	$\|\mathcal{O}\|$ gevonden	OG [28]	$\|\mathcal{S}\|$ gevonden
2	5	**9**	11	**11**
3	10	**28**	30	**86**
4	17	**65**	67	**303**

Tabel A.10: Kleine maximale partiële ovoïden en spreads in $H(4, q^2)$, voor kleine q, bekomen door heuristische en/of exhaustieve zoektochten. Voor $q = 2$ werd de grootte en de classificatie van de kleinste maximale partiële ovoïden en spreads bekomen door exhaustieve zoektochten.

A.7.2 Kleine maximale partiële ovoïden en spreads in $H(4, q^2)$

In tabel A.10 geven we groottes $|\mathcal{O}|$ en $|\mathcal{S}|$ van de kleinste maximale partiële ovoïde en spread die gevonden werden door onze computerzoektocht. We vergelijken deze met de waarden van de beste gekende ondergrenzen (OG) uit [43] en [28]. Merk op dat de kleinste waarden die voor $q = 2$ gevonden werden, effectief de grootte van de kleinste maximale partiële ovoïde, resp. spread, zijn; dit werd door exhaustieve zoektochten bevestigd.

Onze computerzoektochten geven ook het bestaan aan van maximale partiële ovoïden van grootte $q^3 + 1$ in $H(4, q^2)$, voor kleine q. Geen maximale partiële ovoïden met grootte kleiner dan $q^3 + 1$ werden gevonden, en voor $q = 2$ bevestigt een exhaustieve zoektocht dat er geen maximale partiële ovoïden met grootte kleiner dan $q^3 + 1 = 9$ bestaan.

We merken ook op dat er maximale partiële ovoïden bestaan met groottes $q^3 + 1 + iq$ voor kleine waarden van $i \geq 1$.

Recent stelden S. De Winter en K. Thas [28] als conjectuur dat maximale partiële spreads van grootte $q^3 + 3$ waarschijnlijk niet bestaan. Echter, zoals blijkt uit tabel A.10, vinden onze computerzoektochten voor $q = 2$ een maximale partiële spread met grootte $q^3 + 3 = 11$.

A.8 Minimale blokkerende verzamelingen in veralgemeende vierhoeken

In de voorgaande hoofdstukken behandelden we het probleem van het bepalen van de grootste verzameling $\mathcal{O}$ van punten, zodanig dat elke rechte van de veralgemeende vierhoek $\mathcal{S}$ met ten hoogste één punt van $\mathcal{O}$ incident is. Op natuurlijke manier rijst de volgende vraag: Wat is de kleinste verzameling $\mathcal{B}$ van punten, zodanig dat elke rechte van de veralgemeende vierhoek $\mathcal{S}$ met ten minste één punt van $\mathcal{B}$ incident is?

In hoofdstuk 8 definiëren we minimale blokkerende verzamelingen en enkele relevante aanverwante begrippen. We beschrijven onze zoekalgoritmen en geven een overzicht van de eerder gekende resultaten voor kleine minimale blokkerende verzamelingen.

A.8.1 Inleiding

Zij $\mathcal{S} = (P, B, \mathrm{I})$ een veralgemeende vierhoek van orde (s, t), $s > 1, t > 1$. Een *blokkerende verzameling* van $\mathcal{S}$ is een verzameling $\mathcal{B}$ van punten van $\mathcal{S}$, zodanig dat elke rechte van $\mathcal{S}$ met minstens één punt van $\mathcal{B}$ incident is. Een *bedekking* van $\mathcal{S}$ is een verzameling $\mathcal{C}$ van rechten van $\mathcal{S}$, zodanig dat elk punt van $\mathcal{S}$ met minstens één rechte van $\mathcal{C}$ incident is. Een blokkerende verzameling $\mathcal{B}$ is *minimaal* als $\mathcal{B} \setminus \{p\}$ geen blokkerende verzameling is, voor elk punt $p \in \mathcal{B}$. Een bedekking $\mathcal{C}$ is *minimaal* als $\mathcal{C} \setminus \{L\}$ geen bedekking is, voor elke rechte $L \in \mathcal{C}$. Een blokkerende verzameling in een veralgemeende vierhoek $\mathcal{S}$ is een bedekking in zijn duale S^D.

Het is onmiddellijk duidelijk dat $|\mathcal{B}| \geq st + 1$ voor een veralgemeende vierhoek $\mathcal{S} = (P, B, \mathrm{I})$, waarbij de gelijkheid optreedt als en slechts als $\mathcal{B}$ een ovoïde is. Analoog is $|\mathcal{C}| \geq st + 1$ voor een veralgemeende vierhoek $\mathcal{S} = (P, B, \mathrm{I})$, met gelijkheid als en slechts als $\mathcal{C}$ een spread is.

Zij $\mathcal{B}$ een blokkerende verzameling van $\mathcal{S}$ van grootte $st + 1 + r$. We noemen r het *surplus* van de blokkerende verzameling. Een rechte van $\mathcal{S}$ wordt een *meervoudig geblokkeerde rechte* (in het vervolg kort *meervoudige* *rechte*) genoemd als ze minstens twee punten van $\mathcal{B}$ bevat. Het *surplus van een rechte* is het aantal punten van $\mathcal{B}$ dat ze bevat, min één.

A.8.2 Algoritmen voor kleine minimale blokkerende verzamelingen

We concentreren ons op de kleine minimale blokkerende verzamelingen in de klassieke veralgemeende vierhoeken. Het is gemakkelijk in te zien dat de volgende constructie een minimale blokkerende verzameling van grootte $st + s$ oplevert.

Constructie A.8.1 Zij x een punt van $\mathcal{S}$ en beschouw alle rechten van $\mathcal{S}$ door dit punt. Dan vormen alle $s(t+1)$ punten op deze rechten (behalve x) een minimale blokkerende verzameling van grootte $st + s$.

Ons doel is het vinden van een minimale blokkerende verzameling met grootte kleiner dan $st + s$, die geen ovoïde is (in het geval dat de veralgemeende vierhoek een ovoïde bevat).

De paragrafen 8.2 t.e.m. 8.5 beschrijven exhaustieve en niet-exhaustieve zoekalgoritmen, analoog aan de zoekalgoritmen voor maximale partiële ovoïden (zie hoofdstuk 4). Merk op dat het probleem van het bepalen van minimale blokkerende verzamelingen niet kan vertaald worden naar een probleem in de collineariteitsgraaf, zodat verscheidene van de snoeitechnieken besproken in hoofdstuk 4 hier niet kunnen gebruikt worden.

A.8.3 Gekende resultaten voor minimale blokkerende verzamelingen in enkele klassieke veralgemeende vierhoeken

In paragraaf 8.6 bundelen we de gekende resultaten voor minimale blokkerende verzamelingen in enkele klassieke veralgemeende vierhoeken. We geven ook enkele nieuwe resultaten, bekomen door onze computerzoektochten, voor kleine minimale blokkerende verzamelingen in $H(3, q^2)$, waarover tot nu toe nog niets gekend is.

q	# punten	q^3+1	q^3+q^2	Gevonden spectrum
2	45	9	12	**10,11**
3	280	28	36	**29..35**
4	1105	65	80	**72..79**

Tabel A.11: Spectrum van groottes voor minimale blokkerende verzamelingen in $H(3,q^2)$ kleiner dan q^3+q^2, voor kleine q, bekomen door heuristische zoektochten.

In tabel A.11 geven we, voor kleine q, het aantal punten van de veralgemeende vierhoek, de grootte q^3+1 van een ovoïde en de grootte q^3+q^2 van een blokkerende verzameling beschreven in constructie A.8.1. De laatste kolom geeft de groottes kleiner dan q^3+q^2 waarvoor onze computerzoektochten minimale blokkerende verzamelingen met die grootte gevonden hebben. De notatie $a..b$ geeft aan dat voor alle waarden in het interval $[a,b]$ een minimale blokkerende verzameling met die grootte gevonden werd.

Voor $q=2$ classificeerden we ook alle minimale blokkerende verzamelingen van kleine grootte. Er zijn twee minimale blokkerende verzamelingen van grootte 10, twaalf van grootte 11 en dertig van grootte 12.

A.9 Minimale blokkerende verzamelingen in $Q(4,q)$

In dit werk hebben we vooral gezocht naar kleine minimale blokkerende verzamelingen in de veralgemeende vierhoek $Q(4,q)$. De resultaten hiervan worden apart behandeld in hoofdstuk 9.

A.9.1 Minimale blokkerende verzamelingen in $Q(4,q)$, q even

Het is gekend dat $Q(4,q)$ steeds een ovoïde heeft. Betreffende minimale blokkerende verzamelingen in $Q(4,q)$, q even, is het volgende resultaat gekend.

q	$\|P\|$	Eerdere resultaten [31]	Gevonden spectrum
4	85	19	19
8	585	71	69,71
16	4369	271	269,271

Tabel A.12: Spectrum van groottes voor minimale blokkerende verzamelingen $Q(4, q)$, voor kleine q, q even. Alle resultaten werden bekomen door heuristische zoektochten.

Stelling A.9.1 (Eisfeld et al. [31]) *Zij $\mathcal{B}$ een blokkerende verzameling van de kwadriek $Q(4, q)$, q even. Als $q \geq 32$ en $|\mathcal{B}| \leq q^2 + 1 + \sqrt{q}$, dan bevat $\mathcal{B}$ een ovoïde van $Q(4, q)$. Als $q = 4, 8, 16$ en $|\mathcal{B}| \leq q^2 + 1 + \frac{q+4}{6}$, dan bevat $\mathcal{B}$ een ovoïde van $Q(4, q)$.*

D.m.v. heuristische zoekalgoritmen, zoals beschreven in paragraaf 8.5, bekwamen we de volgende resultaten.

Voor elke beschouwde q geven we in tabel A.12 de groottes kleiner dan $q^2 + q$ waarvoor onze computerzoektochten minimale blokkerende verzamelingen gevonden hebben. We vergelijken deze resultaten met de resultaten uit [31].

Voor q even vinden onze computerzoektochten een minimale blokkerende verzameling met grootte $q^2 + q - 1$. We vinden ook minimale blokkerende verzamelingen met grootte $q^2 + q - 3$ voor $q = 4, 8, 16$.

A.9.2 Minimale blokkerende verzameling in $Q(4, q)$, q oneven

Stelling A.9.1 geeft een ondergrens voor de grootte van minimale blokkerende verzamelingen van $Q(4, q)$, q even. Voor q oneven is geen dergelijk resultaat gekend. In het algemeen is zelfs niet geweten of $Q(4, q)$, q oneven, een minimale blokkerende verzameling van grootte $q^2 + 2$ heeft. In [24] lossen de auteurs dit probleem op voor q een oneven priemgetal.

Stelling A.9.2 (De Beule, Metsch [24]) *Als q een oneven priemgetal is, dan heeft $Q(4, q)$ geen minimale blokkerende verzameling van grootte $q^2 + 2$.*

In [23] voeren de auteurs een computerzoektocht uit om het bestaan van een minimale blokkerende verzameling van grootte q^2+3 met een specifieke eigenschap in $Q(4, 7)$ uit te sluiten.

Lemma A.9.1 (De Beule, Hoogewijs, Storme [23]) *Er bestaat geen minimale blokkerende verzameling $\mathcal{B}$ van grootte $q^2 + 3$ in $Q(4, 7)$ zodanig dat er één enkel punt van $Q(4, 7)$ bestaat met $q+1$ rechten erdoor die door precies drie punten van $\mathcal{B}$ geblokkeerd worden.*

De volgende vragen komen naar voren:

- Zijn er andere minimale blokkerende verzamelingen van grootte q^2+3 in $Q(4, 5)$?
- Zijn er minimale blokkerende verzamelingen van grootte $q^2 + 3$ in $Q(4, 7)$?
- Zijn er minimale blokkerende verzamelingen van grootte ten minste $q^2 + 2$ en kleiner dan $q^2 + q$ in $Q(4, q)$, q oneven?

Met een verbetering van onze zoektechnieken uit paragraaf 9.2.2 bekomen we de volgende resultaten.

Lemma A.9.2 *Er is een unieke minimale blokkerende verzameling van grootte $q^2 + 3 = 28$ in $Q(4, 5)$ en er is geen minimale blokkerende verzameling van grootte $q^2 + 3$ in $Q(4, 7)$.*

Minimale blokkerende verzamelingen van grootte $q^2 + q - 2$

Gebruik makend van heuristische zoektochten bekomen we minimale blokkerende verzamelingen van grootte q^2+q-2 (m.a.w. met surplus $r = q-3$) voor $q = 5, 7, 9, 11$. Dus, voor kleine q, bestaan er minimale blokkerende verzamelingen (die geen ovoïde zijn) met grootte kleiner dan $q^2 + q$.

We bestuderen nu de structuur van deze minimale blokkerende verzamelingen. Dit gebeurde ook via een computerzoektocht. We concentreren ons op de manier waarop de meervoudige rechten van de minimale blokkerende verzamelingen gestructureerd zijn. Het geval $q = 9$ gedraagt zich anders dan de andere, en wordt dan ook apart behandeld.

Voor $q = 5, 7, 11$ merken we op dat, voor de gevonden minimale blokkerende verzamelingen, alle meervoudige rechten surplus één hebben, hetgeen betekent dat de meervoudige rechten twee punten van de minimale blokkerende verzameling bevatten.

Eerder merkte S. De Winter reeds op dat voor $q = 5$ alle meervoudige rechten een rooster vormen. We merkten op dat ook voor $q = 7, 11$ roosters gevormd worden door de meervoudige rechten; bovendien treedt meer dan één rooster op (zie tabel A.13). Voor deze gevallen is het aantal meervoudige rechten gegeven door $2(q+1)(q-3)/2$ en het aantal roosters gevormd door de meervoudige rechten door $(q-3)/2$. Verder, voor $q = 7, 11$, zijn er $q+1$ punten van $\mathcal{B}$ gemeenschappelijk aan alle roosters.

	# meervoudige rechten	r	# roosters
$Q(4,5)$	12	2	1
$Q(4,7)$	32	4	2
$Q(4,11)$	96	8	4
$\boldsymbol{Q(4,q)}$ als $\boldsymbol{q \in \{5,7,11\}}$	$\boldsymbol{(q+1)(q-3)}$	$\boldsymbol{q-3}$	$\boldsymbol{(q-3)/2}$

Tabel A.13: Structuur van de minimale blokkerende verzamelingen in $Q(4,q)$, $q = 5, 7, 11$, van grootte $q^2 + q - 2$.

Tenslotte merken we op dat het mogelijk is om een maximale partiële ovoïde van grootte $q^2 - 1$ te bekomen door deze $q + 1$ gemeenschappelijke punten te verwijderen en twee punten van de "perp" van deze verzameling toe te voegen. In $Q(4,5)$ zijn er 4 mogelijkheden om dergelijke maximale partiële ovoïde te bekomen.

Voor $q = 9$ merkten we op dat, voor de gevonden minimale blokkerende verzamelingen, alle meervoudige rechten surplus drie hebben. De gevonden minimale blokkerende verzameling heeft 20 meervoudige rechten en we controleerden dat deze een rooster vormen.

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