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Bart De Bruyn,
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Preface

In 1980 Shult and Yanushka [82] introduced near polygons because of their connection with certain system of lines in the Euclidean space. These incidence structures can be seen as a generalization of a class of incidence structures, called generalized polygons, which themselves were introduced by Tits [92] in 1959 and are discussed in detail in [98]. In the beginning of the 80's, near polygons were studied in detail, especially by Brouwer. Several regular near polygons were characterized by their parameters (e.g. [11], [12] and [24]). Very important was the characterization of classical near polygons by Cameron [22]. Since the very beginning ([82], see also [80]) it was clear that, under some mild conditions, these incidence structures had substructures, isomorphic to generalized quadrangles, called quads. Many results about generalized quadrangles are therefore very useful for the study of near polygons. In [17], it was proved that, under some mild conditions, every two points of a near polygon are contained in a sub near polygon.

Finding new regular near polygons is an extremely interesting problem, but also a very hard one. The search for new near polygons which have quads through every two points at distance 2, is still very interesting and often more successful. This thesis contains several new classes of such near polygons. They are all in Chapter 7 and constructed in a uniform way. The whole chapter 7 is devoted to these near polygons (we call them glued near polygons) and characterizations are given.

The importance of the search for new near polygons which have quads through every two points at distance 2 is also clear from the paper [14], containing a classification of all such near hexagons with 3 points on a line. In Chapter 8, we try to do the similar classification where all near hexagons have four points on a line. There remain however four open cases. The results of Chapter 7 are extremely important during this classification.

To develop the theory of glued near polygons, we had to derive some results concerning generalized quadrangles. All these results are contained in Chapter 2, together with a brief overview of the theory of these incidence structures. For more details, we often refer to the standard work [71], see also [88].

Chapter 3 contains an overview of the basic definitions and properties (some of them being new) of near polygons. In Chapter 4, we prove that association schemes can be associated to certain near polygons. In Chapter 5, we define i-neighbourhoods of a near polygon. We also present a new construction to obtain a class of near polygons and give a property that almost characterizes these near polygons. In Chapter 6, we discuss linear representations of near polygons. We try to classify all thick near hexagons which have a linear representation and discuss the open cases. This study led to the discovery of new near polygons which in turn led us to the bigger class of glued near polygons which are studied in Chapter 7. Finally, in an appendix we give the classification of all ovoids of the near hexagon related to the Steiner system $S(5, 8, 24)$. This classification was done earlier [16], but a new proof is presented here.

Contents

Chapter 1

Basic notions and properties

1.1 The notion of incidence structure

An *incidence structure* $\mathbf{S}$ is a triple consisting of

(1) a nonempty set $\mathcal{P}$, whose elements are called *points*;

(2) a (possible) empty set $\mathcal{L}$, disjoint with $\mathcal{P}$, whose elements are called *lines*;

(3) an *incidence relation* $\mathcal{I} \subseteq (\mathcal{P} \times \mathcal{L})$.

At some other places in the literature, $\mathcal{I}$ is regarded as a subset of $(\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ and called the *symmetrized incidence relation*. One writes $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$. Unless otherwise stated, all incidence structures considered in this thesis are *finite*, i.e. $\mathcal{P}$ and $\mathcal{L}$ are finite sets. If $(p, L) \in \mathcal{I}$ (we also write $p \; \mathcal{I} \; L$), then we will say: the point p is incident with L, the point p lies on L, the line L goes through p, etc. Two different points of $\mathbf{S}$ are called *collinear* when there is a line through these points. An incidence structure is said to have *order* (s, t) if every line is incident with exactly $s + 1$ points and if every point is incident with exactly $t + 1$ lines. An incidence structure is called *thick* if every line is incident with at least three points and if every point is incident with at least three lines. With each incidence structure $\mathbf{S}$, there is associated a graph $\Gamma(\mathbf{S})$, called the *collinearity graph* or *point graph* of $\mathbf{S}$. The vertices of $\Gamma(\mathbf{S})$ are the points of $\mathbf{S}$; two different vertices are adjacent whenever they are collinear. When no confusion is possible, we will write Γ instead of $\Gamma(\mathbf{S})$. An incidence structure is called *connected* if its point graph is connected. With the graph $\Gamma(\mathbf{S})$, there corresponds a distance function, denoted by $\mathrm{d}_{\mathbf{S}}(\cdot, \cdot)$ or shortly by $\mathrm{d}(\cdot, \cdot)$ when no confusion is possible. If x is a point of $\mathbf{S}$ and if $i \in \mathbb{N}$, then $\Gamma_i(x)$ denotes the set of all points at distance i from x (in the graph Γ) and we will write $\Gamma(x) := \Gamma_1(x)$. If A and B are two sets of points, then we define $\mathrm{d}(A, B) := \min \mathrm{d}(x, y)$, where x and y range over A and B respectively. If $A = \{x\}$, we also write $\mathrm{d}(x, B)$ instead of $\mathrm{d}(\{x\}, B)$. There is another graph related to an incidence structure $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, namely the *incidence graph*: the vertices of this graph are the elements of $\mathcal{P} \cup \mathcal{L}$, two vertices being adjacent when they are incident. Two incidence structures $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ and $\mathbf{S}' = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ are called *isomorphic* when there is a bijection $\alpha : \mathcal{P} \to \mathcal{P}'$ and a bijection $\beta : \mathcal{L} \to \mathcal{L}'$ such that $p \; \mathcal{I} \; L \iff \alpha(p) \; \mathcal{I}' \; \beta(L), \forall p \in \mathcal{P}, \forall L \in \mathcal{L}$. An incidence structure is called a *partial linear space* when every line is incident with at least two points and when every two different points are incident with at most one line. If every two different points are incident with exactly one line, then the partial linear space is called a *linear space*. The partial linear space $\mathbf{S}' = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ is called a *subgeometry* of the partial linear space $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ if $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$ and if $\mathcal{I}'$ is the restriction of $\mathcal{I}$ to $\mathcal{P}' \times \mathcal{L}'$. If $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is an incidence structure without *multiple lines*, i.e. there are no two lines which are incident with the same set of points, then it is isomorphic to $\mathbf{S}' = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$, where $\mathcal{P}' = \mathcal{P}$, $\mathcal{L}' = \bigcup_{L \in \mathcal{L}} \{\{x | x \; \mathcal{I} \; L\}\}$, and where $p' \; \mathcal{I}' \; L' \iff p' \in L'$ for all $p' \in \mathcal{P}'$ and all $L' \in \mathcal{L}'$. That is the reason why, at some places in this thesis, we will consider lines as sets of points and write $p \in L$ instead of $p \; \mathcal{I} \; L$. The reader is supposed to be familiar with some incidence structures like projective and affine spaces. Important examples of incidence structures which are in general not partial linear spaces are the designs. A $t-(v, k, \lambda)$ *design* is an incidence structure $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ satisfying the following properties (for several reasons the elements of $\mathcal{L}$ will be called *block* instead of lines):

(1) there are exactly v points;

(2) each block is incident with $k \geq 2$ points;

(3) every t different points are incident with exactly λ lines.

If $\lambda = 1$, then the design is called a *Steiner system* and denoted by $S(t, k, v)$. We will meet some designs later in this thesis. In the following section, we will define the class of partial linear spaces which is the main object of study in this thesis.

1.2 Near polygons

1.2.1 Genesis

Consider the n-dimensional Euclidean space with origin O. A *line system of type* $(a_1, a_2, \ldots, a_k)$ is a set $\mathcal{A}$ of lines through O with the property that $\cos\alpha \in \{\pm a_1, \pm a_2, \ldots, \pm a_k\}$ for any two different lines of $\mathcal{A}$ at angle α. A *system of vectors of type* $(a_1, a_2, \ldots, a_k)$ is a set Σ of vectors satisfying:

(i) $\Sigma = -\Sigma := \{-x | x \in \Sigma\}$,

(ii) the norm (x, x) of x is a nonzero constant c for all $x \in \Sigma$,

(iii) if $x, y \in \Sigma$ and $x \neq \pm y$, then $\frac{(x,y)}{c} \in \{\pm a_1, \pm a_2, \ldots, \pm a_k\}$.

Line systems of type $(a_1, \ldots, a_k)$ are clearly equivalent to systems of vectors of type $(a_1, \ldots, a_k)$. In [82] E. Shult and A. Yanushka discussed line systems of type $(0, \frac{1}{3})$ and showed that they were related to a class of geometries which they called *near polygons*. The angle α for which $\cos\alpha = -\frac{1}{3}$ is the angle subtended by two chords drawn from the barycenter of a tetrahedron to two of its corner vertices. A line system of type $(0, \frac{1}{3})$ is called *tetrahedrally closed* if it has 0, 1, 2 or 4 lines in common with every set of four lines obtained by connecting the barycenter of an arbitrary tetrahedron centered at O with its four corner vertices. A system of vectors of type $(0, \frac{1}{3})$ is called a *tetrahedrally closed* $(0, \frac{1}{3})$ *system* if the corresponding line system is tetrahedrally closed.

Let Σ be a tetrahedrally closed $(0, \frac{1}{3})$ system of vectors of norm 3. Fix a vector $u \in \Sigma$ and let $\Sigma_{-1}(u)$ be the set of vectors of Σ having inner product -1 with u. If $\Sigma_{-1}(u) \neq \emptyset$, then we can define the following incidence system $\mathbf{S}$: the point set is equal to $\Sigma_{-1}(u)$; a triplet of three vectors $\{y_1, y_2, y_3\} \subseteq \Sigma_{-1}(u)$ is a line if and only if u, y_1, y_2, y_3 define the four vertices of a tetrahedron centered at O.

Theorem 1.2.1 ([82], Proposition 3.10)
If for every two points of $\Sigma_{-1}(u)$ having inner product 1, there exists at least one point of $\Sigma_{-1}(u)$ having inner product -1 with both points, then $\mathbf{S}$ is a connected partial linear space satisfying:

(a) the distance between two points is at most 3,

(b) for every point p and every line L, there exists a unique point on L nearest to p (with respect to the distance in the collinearity graph).

Two different points have distance 1, 2, respectively 3, if their inner product equals -1, $+1$, respectively 0.

Also, the converse of this theorem holds, as shown by the next theorem.

Theorem 1.2.2 ([82], Proposition 3.14)
For every connected partial linear space $\mathbf{S}$ *satisfying properties* (*a*) *and* (*b*) *of Theorem 1.2.1, there exists a tetrahedrally closed* $(0, \frac{1}{3})$ *system* Σ *of vectors of norm* 3 *and a vector* $u \in \Sigma$ *such that*

(*i*) $\Sigma = \{u, -u\} \cup (\Sigma_{-1}(u)) \cup (-\Sigma_{-1}(u))$ *and*

(*ii*) *the partial linear space defined on* $\Sigma_{-1}(u)$ *by the tetrahedra containing* u *is isomorphic to* $\mathbf{S}$.

1.2.2 Definition

A *near polygon* is a connected partial linear space satisfying the following property.

(NP) For each point p and every line L, there exists a unique point q on L nearest to p (with respect to the distance in the point graph).

This axiom is very important in this thesis, and we will often refer to it as (NP). If d is the diameter of the point graph, then the near polygon is called a *near* $2d$*-gon*. A near 0-gon has one point, but no lines. A near 2-gon consists of exactly one line with a number (≥ 2) of points on it. The near quadrangles are known as the *generalized quadrangles*. These geometries have been widely studied. Because of their importance in the theory of near polygons, we devote Chapter 2 to these geometries.

At some places in this thesis, we will also meet the *near* $(2d+1)$*-gons*, these are the connected partial linear spaces satisfying the following properties.

(1) The point graph has diameter d.

(2) For each point p and every line L with $\mathrm{d}(p, L) < d$, there exists a unique point q on L nearest to p.

(3) There exists a point p and a line L such that $\mathrm{d}(p, L) = d$.

Near 1-gons do not exist. The near 3-gons are exactly the linear spaces different from the point and the line. In this thesis we will meet a special type of near pentagon, namely the partial quadrangles. A *partial quadrangle with parameters* (s, t, μ) is a near pentagon of order (s, t) such that every two noncollinear points have exactly μ common neighbours. This thesis mainly treats the near $2d$-gons. Unless otherwise stated, we always mean a near

$2d$-gon, when we talk about a near polygon. This convention is also used in the literature.

Remark. Properties (a) and (b) of Theorem 1.2.1 are shortly written as: $\mathbf{S}$ is a near $2d$-gon with $d \leq 3$.

1.3 Generalized polygons

These incidence structures were introduced by Tits in [92]. We will only give some definitions and properties related to these incidence structures; for a detailed survey, we refer to [98].

1.3.1 Definition

Let $n \geq 3$, $n \neq 4$, be an integer. A *generalized n-gon* $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a partial linear space satisfying the following axioms:

(GP1) $\mathbf{S}$ has no subgeometry which is an ordinary k-gon, $2 < k < n$.

(GP2) Any two elements of $\mathcal{P} \cup \mathcal{L}$ are contained in some ordinary n-gon.

The above axioms also work in the case $n = 4$, but the geometries then obtained would not coincide with the generalized quadrangles which we defined above and which we will consider in the next chapter. However, only those generalized quadrangles which are degenerate (see next chapter for the definition) will not be covered by (GP1) and (GP2). Since we want that these degenerate examples are generalized polygons, we assumed $n \neq 4$ in the above definition. The generalized 3-gons are exactly the (possible degenerate) projective planes. From (GP1) and (GP2), it follows that the dual of a generalized n-gon is again a generalized n-gon (if $n \neq 4$). The reason why generalized n-gons are important in this thesis is because they are near n-gons (also for n odd).

Theorem 1.3.1
Every generalized n-gon $\mathbf{S}$, $n \geq 3$, is a near n-gon.

Proof. Suppose first that $n = 2d$, $d \geq 3$. Every two points of $\mathbf{S}$ are contained in an ordinary $2d$-gon which proves that the diameter of the point graph is at most d. Since the distance between two opposite points of such an ordinary $2d$-gon is d (otherwise (GP1) would not be satisfied), the diameter is exactly d. Take now a point a and a line L, then there exists an ordinary $2d$-gon through these elements which proves that every point of L at smallest distance from a has distance at most $d-1$ to a. If L contains two points at smallest distance from a, then (GP1) would not be satisfied. Suppose next that $n = 2d+1$, $d \geq 1$. Every two points of $\mathbf{S}$ are contained in an ordinary $(2d+1)$-gon which proves that the diameter of the point graph is at most d. Consider now an ordinary $(2d+1)$-gon as subgeometry and take a point p and a line L which are opposite in this ordinary $(2d+1)$-gon, then $\mathrm{d}(p, L) = d$ otherwise (GP1) would not be satisfied. This also proves that the diameter of the point graph is equal to d. Now, take a point p and a line L, such that $\mathrm{d}(p, L) < d$. The line L contains a unique point at distance $\mathrm{d}(p, L)$ from p, otherwise (GP1) would not be satisfied. □

Thick generalized polygons satisfy some nice properties.

Theorem 1.3.2 ([98], Theorem 1.3.2)
A generalized polygon is thick if and only if it has an ordinary $(n+1)$-gon as subgeometry.

Theorem 1.3.3 ([98], Corollary 1.5.3)
Every thick generalized n-gon has order (s,t) with $s, t \geq 2$; if n is odd, then $s = t$.

1.3.2 Projections and projectivities

Let $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a generalized n-gon and consider the incidence graph Δ with associated distance $\delta(\cdot,\cdot)$. For a vertex x of Δ, let $\Delta(x)$ denote the set of elements of $\mathcal{P} \cup \mathcal{L}$ incident with x. Two elements a and b of $\mathcal{P} \cup \mathcal{L}$ are called *opposite* if $\delta(a, b) = n$. If n is even, then two opposite elements have the same type (i.e. both lines or points); if n is odd, then two opposite elements have different types. If a and b are two different elements of $\mathcal{P} \cup \mathcal{L}$ which are not opposite, then there exists a unique element c of $\mathcal{P} \cup \mathcal{L}$ incident with b and at distance $\delta(a, b) - 1$ of a. This point c is called the *projection of a onto b*. Consider now two opposite elements a and b. Every element $c \in \Delta(a)$ has distance $n-1$ from b and hence we can project it onto b. In this way, we find a bijection

$$[a, b] : \Delta(a) \to \Delta(b),$$

called the *perspectivity from a to b*. Clearly $[a, b]$ is a bijection, with inverse $[b, a]$. For elements $a_0, \ldots, a_k$ of $\mathcal{P} \cup \mathcal{L}$ with $\delta(a_i, a_{i+1}) = n$ for $0 \leq i < k$, the bijection

$$[a_0, a_1, \ldots, a_k] := [a_{k-1}, a_k] \circ \ldots \circ [a_1, a_2] \circ [a_0, a_1]$$

is called a *projectivity from $\Delta(a_0)$ to $\Delta(a_k)$*. The set of all projectivities from $\Delta(a)$ to itself forms a permutation group $\Pi(a)$, which is called the *group* *of projectivities of a.* If A is a set of mutual opposite elements of $\mathcal{P} \cup \mathcal{L}$ containing a_0, then, in the above definition of projectivity, one can require that all elements a_i belong to A. In this way we can define a permutation group $\Pi_A(a_0)$, called the *group of projectivities of a_0 with respect to A.*

1.3.3 Classification results

In this subsection, we give some theorems which should give the reader an idea what values of n, s and t are allowed for a generalized n-gon of order (s, t).

Theorem 1.3.4 ([40])
Let $\mathbf{S}$ be a generalized n-gon of order (s, t) with $n \geq 3$, then at least one of the following holds.

- (a) *$\mathbf{S}$ is an ordinary n-gon.*
- (b) *$\mathbf{S}$ is a nondegenerate projective plane.*
- (c) *$\mathbf{S}$ is a generalized quadrangle.*
- (d) *$\mathbf{S}$ is a generalized hexagon. Moreover, if $\mathbf{S}$ is thick, then st is a square.*
- (e) *$\mathbf{S}$ is a generalized octagon. Moreover, if $\mathbf{S}$ is thick, then $2st$ is a square.*
- (f) *$\mathbf{S}$ is a generalized 12-gon which is not thick.*

Theorem 1.3.5
Let $\mathbf{S}$ be a thick generalized n-gon of order (s, t), $n \geq 4$, then one of the following holds:

- (a) *$n = 4$, $s \leq t^2$ and $t \leq s^2$ ([47]);*
- (b) *$n = 6$, $s \leq t^3$ and $t \leq s^3$ ([44]);*
- (c) *$n = 8$, $s \leq t^2$ and $t \leq s^2$ ([47]).*

For an overview of the possible orders of finite GQ we refer to Section 2.5 of Chapter 2. All known thick generalized hexagons have order (q, q), (q, q^3) or (q^3, q) with q a prime power. For $(s, t) = (2, 2)$, there are exactly two dual examples; for $(s, t) = (2, 8)$ (or $(s, t) = (8, 2)$), there is a unique example ([24]). Nothing on uniqueness is known about other orders. All known thick generalized octagons have order $(2^{2e+1}, 2^{4e+2})$ or $(2^{4e+2}, 2^{2e+1})$ with $e \in \mathbb{N}$ (the so-called *Ree-Tits octagons*, [96]). Also here nothing is known about uniqueness. We also notice that every generalized polygon which is not thick can be obtained from some ordinary polygon or some thick generalized polygon ([95],[101]).

1.4 Projective spaces

As we already mentioned, the reader is supposed to be familiar with the (basic) theory of projective spaces. Nevertheless, we will summarize some definitions and results related to these projective spaces, especially those which will be used in this thesis and may not be known by a reader which is not too specialized in this subject.

1.4.1 Caps

A *cap* is a nonempty set of points no three of which are collinear. Caps in projective spaces of dimension $d \leq 1$ are trivial objects.

Suppose $d = 2$ and let the projective plane have order q. If q is even, then a cap has size at most $q + 2$. If this upper bound is achieved, then the cap is called a *hyperoval*. If q is odd, then a cap has size at most $q + 1$. A cap of size $q + 1$ is called an *oval* (also if q is even). A famous theorem, due to Segre, says that every oval in $\mathrm{PG}(2, q)$, q odd, is a conic ([76] and [77]). If q is even, then every oval can be extended in a unique way to some hyperoval; the point that must be added is called the *nucleus* of the oval. In $\mathrm{PG}(2, q)$, q even, there is a special type of hyperoval, which consists of a conic together with its nucleus. Such a hyperoval is called *regular*, and any other hyperoval of $\mathrm{PG}(2, q)$ is called *irregular*. Each hyperoval in $\mathrm{PG}(2, q)$ with $q \in \{2, 4, 8\}$ is regular ([78]). We refer to [89] for a survey of the known hyperovals. There is also an electronic update on the web available, see *http://www-math.cudenver.edu/~wcherowi*.

Let $\mathcal{H}$ be a hyperoval in $\mathrm{PG}(2, 2^h)$, $h \geq 1$, then we can take coordinates such that $(1, 0, 0) \in \mathcal{H}$ and $(0, 1, 0) \in \mathcal{H}$. We then can write

$$\mathcal{H} = \mathcal{H}_f := \{(1, 0, 0), (0, 1, 0)\} \cup \{(f(\lambda), \lambda, 1) | \lambda \in \mathrm{GF}(q)\},$$

where $f : \mathrm{GF}(q) \to \mathrm{GF}(q)$ is some function satisfying the following two conditions:

(a) f is a bijection,

(b) $\frac{f(\lambda_1)-f(\lambda_2)}{\lambda_1-\lambda_2} \neq \frac{f(\lambda_1)-f(\lambda_3)}{\lambda_1-\lambda_3}$ for all $\lambda_1, \lambda_2, \lambda_3 \in \mathrm{GF}(q)$ which are two by two distinct.

Using the remaining freedom in coordinatization, we even may suppose that $f(0) = 0$ and $f(1) = 1$. If we put $f(\lambda) = \lambda^2$ for all $\lambda \in \mathrm{GF}(q)$, then $\mathcal{H}_f$ is a regular hyperoval. The function $g_i(\lambda) = \lambda^{2^i}$, where $i \in \{1, \ldots, h - 1\}$ and

$(i, h) = 1$, also satisfies the above conditions (a) and (b). Every hyperoval projectively equivalent with such an $\mathcal{H}_{g_i}$ is called a *translation hyperoval.*

The maximal size of a cap in $\mathrm{PG}(3, q)$, $q \neq 2$, is $q^2 + 1$ ([8],[75]). A cap of size $q^2 + 1$ in $\mathrm{PG}(3, q)$, $q \neq 2$, is called an *ovoid.* An *ovoid* in $\mathrm{PG}(3, 2)$ is defined as an elliptic quadric. The only known examples of ovoids are the elliptic quadrics and the Tits ovoids. A *Tits ovoid* ([93]) is a set of points projectively equivalent with the following set of points in $\mathrm{PG}(3, 2^{2e+1})$:

$$\{(1, 0, 0, 0)\} \cup \{(a^{2^{e+1}} + b^{2^{e+1}+2} + ab, 1, a, b) | a, b \in \mathrm{GF}(2^{2e+1})\}.$$

Every ovoid in $\mathrm{PG}(3, q)$, q odd, is an elliptic quadric ([4]) and every ovoid in $\mathrm{PG}(3, q)$, $q \leq 32$ and even, is an elliptic quadric or a Tits ovoid ([59],[41],[55], [56],[57],[74]). If O is a nonsingular elliptic quadric or a Tits ovoid, then every other ovoid intersects it in an odd number of points (see Corollary 2 of [2]).

1.4.2 The Steiner systems $S(5, 6, 12)$ and $S(5, 8, 24)$

E. Mathieu discovered the so-called Mathieu groups ([52],[53],[54]). Witt ([99],[100]) used these groups to construct the Steiner systems $S(5, 6, 12)$ and $S(5, 8, 24)$. Lüneburg ([51]) reversed this procedure and constructed first the designs (by "extending" an affine or a projective plane) and defined then the Mathieu groups M_{12} and M_{24} as their automorphism groups. The uniqueness of both Steiner systems can be derived from this method. We will give Lüneburg's construction for $S(5, 8, 24)$ and give another construction for $S(5, 6, 12)$.

Construction of $S(5, 8, 24)$

The projective plane $\Pi = (X, B, \mathcal{I})$ of order 4 has the following properties.

(a) The set of all 168 hyperovals can be divided into three classes $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$, each of size 56. Two hyperovals are in the same class if and only if they intersect in an even number of points.

(b) The set of all 360 Baer-subplanes, i.e. subplanes of order 2, can be divided into three classes $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$, each of size 120. Two Baer-subplanes are in the same class if and only if they intersect in an odd number of points.

(c) The indices i and j can be chosen in such a way that, for $O \in \mathcal{O}_i$ and for $S \in \mathcal{B}_j$, we have $|O \cap S|$ is even if and only if $i = j$.

Construct now the following design. The point set is equal to $X \cup \{\infty_1, \infty_2, \infty_3\}$, where $\infty_1, \infty_2, \infty_3 \notin X$. There are four types of blocks:

(a) $L \cup \{\infty_1, \infty_2, \infty_3\}$ where $L \in B$;

(b) $(O \cup \{\infty_1, \infty_2, \infty_3\}) \setminus \{\infty_i\}$ for each $O \in \mathcal{O}_i$;

(c) $S \cup \{\infty_i\}$ for each $S \in \mathcal{B}_i$;

(d) $L \triangle L'$ (the symmetric difference of L and L') for all $L, L' \in B$, $L \neq L'$.

Taking the natural incidence, we find then the unique Steiner system $S(5,8,24)$. It has the following properties.

(a) Every two different blocks meet in 0, 2 or 4 points.

(b) If two blocks are disjoint, then the complement of their union is again a block.

(c) If two blocks intersect in 4 points, then their symmetric difference is again a block.

Construction of $S(5,6,12)$

Consider the following matrix over GF(3):

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & -1 \end{bmatrix}.$$

The columns of M define a set $\mathcal{K}$ of 12 points in PG(5, 3). Any set of points projectively equivalent with $\mathcal{K}$ is called a *Coxeter-cap* as it was discovered by Coxeter in [25], see also [73]. It has the following properties.

(a) No five points of $\mathcal{K}$ are contained in a threedimensional space.

(b) Every hyperplane of PG(5, 3) which contains at least 5 points of $\mathcal{K}$, contains exactly 6 points of $\mathcal{K}$.

From (a) and (b) the following construction of the unique Steiner system $S(5,6,12)$ follows. The points of the Steiner system are the points of $\mathcal{K}$, the blocks are all the sets of 6 points which arise as intersection of $\mathcal{K}$ with a hyperplane.

Remark. The rows of the matrix M generate a 6-dimensional subspace of $V(12,3)$ (i.e. the 12-dimensional vector space over $\mathrm{GF}(3)$). This subspace is called the *extended ternary Golay code.*

1.4.3 Regular spreads in $\mathrm{PG}(3,q)$

In this section, we give properties about regular spreads in $\mathrm{PG}(3,q)$. We add proofs since they are short and easy. For more details, we refer to [48].

Definitions.

(1) Let K, L, M be three lines of $\mathrm{PG}(3,q)$, two by two disjoint. There are exactly $q+1$ lines intersecting K, L and M. Every set of $q+1$ lines obtained this way is called a *regulus.*

(2) A *spread of* $\mathrm{PG}(3,q)$ is a set of q^2+1 lines partitioning the point set. A spread S of $\mathrm{PG}(3,q)$ is called *regular* if for every line $M \notin S$ the $q+1$ lines of S intersecting M define a regulus. All regular spreads of $\mathrm{PG}(3,q)$ are projectively equivalent.

Lemma 1.4.1
Let S be a spread of $\mathrm{PG}(3,q)$ and let θ be an automorphism of $\mathrm{PG}(3,q)$ which fixes each line of S. If θ fixes one point of $\mathrm{PG}(3,q)$, then θ is the identity.

Proof. Suppose that x is a fixpoint of θ. Let y be another point of $\mathrm{PG}(3,q)$ such that $xy \notin S$. Let L_1 be the line of S through y and let L_2 be a line of S meeting xy in a point different from x and y. Then $y^\theta \in L_1^\theta \cap \langle x^\theta, L_2^\theta \rangle = L_1 \cap \langle x, L_2 \rangle = \{y\}$. Hence y is fixed. Applying the same reasoning with y instead of x, we find that θ fixes each point of $\mathrm{PG}(3,q)$. $\square$

The previous lemma says that there are at most $q+1$ automorphisms of $\mathrm{PG}(3,q)$ fixing each line of a spread S. If there are exactly $q+1$ such automorphisms, then S is regular as we will show in the following lemma.

Lemma 1.4.2
If there are at least three automorphisms of $\mathrm{PG}(3,q)$ fixing each line of a spread S, then S is regular.

Proof. Let θ_1 and θ_2 be two nontrivial automorphisms of $\mathrm{PG}(3,q)$ fixing each line of S. Let M be an arbitrary line of $\mathrm{PG}(3,q)$ not belonging to S and let $L_1, L_2, \ldots, L_{q+1}$ be the $q+1$ lines of S meeting M. These $q+1$ lines determine a regulus, since they meet the three lines M, M^{θ_1} and M^{θ_2}. $\square$

Conversely, if S is a regular spread, then there are exactly $q+1$ automorphisms fixing each line of S; these automorphisms will be given now.

The case of odd q

Let m be any nonsquare of $\mathrm{GF}(q)$. For $(\lambda,\mu)\in \mathrm{GF}(q)\times \mathrm{GF}(q)$, we define

$$L_{\lambda,\mu}=\langle(\lambda,\mu,1,0),(\mu,m\lambda,0,1)\rangle,$$

and we put

$$S=\{\langle(1,0,0,0),(0,1,0,0)\rangle\}\cup\{L_{\lambda,\mu}|\lambda,\mu\in \mathrm{GF}(q)\}.$$

It is straightforward to check that S is a spread of $\mathrm{PG}(3,q)$. For $a,b\in \mathrm{GF}(q)$, let $A(a,b)$ be the following matrix:

$$A(a,b):=\begin{pmatrix} a & b & 0 & 0\\ bm & a & 0 & 0\\ 0 & 0 & a & bm\\ 0 & 0 & b & a\end{pmatrix}.$$

If $(a,b)\neq(0,0)$, then the automorphism $\bar{x}'=A(a,b)\bar{x}$ of $\mathrm{PG}(3,q)$ fixes each line of S. We obtain in this way $q+1$ (and hence all) automorphisms of $\mathrm{PG}(3,q)$ which fix each line of S. Hence S is regular. Clearly $A(a,b)\cdot A(c,d)=A(ac+bdm,ad+bc)$. Let $\sqrt{m}$ be a square root of m in $\mathrm{GF}(q^2)$. Let $\mathcal{A}:=\{A(a,b)|(a,b)\in \mathrm{GF}(q)\times \mathrm{GF}(q)\setminus\{(0,0)\}\}$, then $\mathcal{A}$ is a multiplicative group. The map $\Omega:\mathcal{A}\rightarrow \mathrm{GF}(q^2)\setminus\{0\}$, $A(a,b)\rightarrow a+b\sqrt{m}$ is a homomorphism of multiplicative groups. Let $\alpha+\beta\sqrt{m}$ be a primitive element of $\mathrm{GF}(q^2)$, then $\Omega^{-1}[(\alpha+\beta\sqrt{m})^i]$, $i\in\{0,\ldots,q\}$, are the $q+1$ matrices which define the $q+1$ automorphisms of $\mathrm{PG}(3,q)$ fixing each line of S. The group of automorphisms of $\mathrm{PG}(3,q)$ which fix each line of S is hence isomorphic to the cyclic group C_{q+1}.

The case of even q

Let l be an element of $\mathrm{GF}(q)$ for which $Tr(l)$=1. For $(\lambda,\mu)\in \mathrm{GF}(q)\times \mathrm{GF}(q)$, we define

$$L_{\lambda,\mu}=\langle(\lambda,\mu,1,0),(\lambda+l\mu,\lambda,0,1)\rangle,$$

and we put

$$S=\{\langle(1,0,0,0),(0,1,0,0)\rangle\}\cup\{L_{\lambda,\mu}|\lambda,\mu\in \mathrm{GF}(q)\}.$$

It is straightforward to check that S is a spread of $\mathrm{PG}(3,q)$. For $a,b\in \mathrm{GF}(q)$, let $A(a,b)$ be the following matrix:

$$A(a,b):=\begin{pmatrix} a+b & bl & 0 & 0\\ b & a & 0 & 0\\ 0 & 0 & a & bl\\ 0 & 0 & b & a+b\end{pmatrix}.$$

If $(a,b) \neq (0,0)$, then the automorphism $\bar{x}' = A(a,b)\bar{x}$ of $\mathrm{PG}(3,q)$ fixes each line of S. We obtain in this way $q+1$ (and hence all) automorphisms of $\mathrm{PG}(3,q)$ which fix each line of S. Hence S is regular. Clearly $A(a,b) \cdot A(c,d) = A(ac+bdl, ad+bc+bd)$. Let $i \in \mathrm{GF}(q^2)$ be such that $i^2+i=l$. Let $\mathcal{A} := \{A(a,b) | (a,b) \in \mathrm{GF}(q) \times \mathrm{GF}(q) \setminus \{(0,0)\}\}$, then $\mathcal{A}$ is a multiplicative group. The map $\Omega : \mathcal{A} \to \mathrm{GF}(q^2) \setminus \{0\}$, $A(a,b) \to a+bi$ is a homomorphism of multiplicative groups. Let $\alpha + \beta i$ be a primitive element of $\mathrm{GF}(q^2)$, then $\Omega^{-1}[(\alpha+\beta i)^j]$, $j \in \{0, \ldots, q\}$, are the $q+1$ matrices which define the $q+1$ automorphisms of $\mathrm{PG}(3,q)$ fixing each line of S. The group of automorphisms of $\mathrm{PG}(3,q)$ which fix each line of S is hence isomorphic to C_{q+1}.

1.4.4 The geometry H_q^3

Definition. Let L be a fixed line of $\mathrm{PG}(3,q)$, then the following geometry H_q^3 can be defined. The points of H_q^3 are the points of $\mathrm{PG}(3,q)$ not on L, the lines of H_q^3 are the lines of $\mathrm{PG}(3,q)$ skew to L and incidence is the one derived from $\mathrm{PG}(3,q)$.

Every automorphism of $\mathrm{PG}(3,q)$ fixing L induces an automorphism of H_q^3. Also the converse is true.

Theorem 1.4.3
Every automorphism θ of H_q^3 is induced by an automorphism ϕ of $\mathrm{PG}(3,q)$ *fixing L.*

Proof. Let α be a plane of $\mathrm{PG}(3,q)$.

(1) Suppose that α contains L and let $a \in \alpha \setminus L$. We define $\alpha^\phi = \langle a^\theta, L\rangle$. This is a good definition. If b is another point of $\alpha \setminus L$, then a^θ and b^θ are not collinear in H_q^3, since a and b are not collinear in H_q^3. Hence $\langle b^\theta, L\rangle = \langle a^\theta, L\rangle$.

(2) Suppose that $\alpha \cap L = \{r\}$ and let $a_1, a_2, a_3 \in \alpha \setminus \{r\}$ such that $\alpha = \langle a_1, a_2, a_3\rangle$ and such that no three of the points r, a_1, a_2, a_3 are collinear in $\mathrm{PG}(3,q)$. Since a_1 is not incident with a_2a_3 in H_q^3, a_1^θ is not incident with the line $a_2^\theta a_3^\theta$ in H_q^3. We put $\alpha^\phi = \langle a_1^\theta, a_2^\theta, a_3^\theta\rangle$. This is a good definition. For, we will prove now that $a_4^\theta \in \langle a_1^\theta, a_2^\theta, a_3^\theta\rangle$ for every point $a_4 \in \alpha \setminus \{r\}$. We may suppose that $a_4 \neq a_1$ and $a_4 \neq a_2$. One of the lines a_4a_1 or a_4a_2, say a_4a_1, does not meet L. The line a_4a_1 meets a_2a_3 in a point a_1'. Now, $a_4^\theta \in \langle a_1^\theta, a_1'^\theta\rangle \subseteq \langle a_1^\theta, a_2^\theta, a_3^\theta\rangle$. It is also clear that α^ϕ does not contain L.

Let M be a line of $\mathrm{PG}(3,q)$.

(1) If $M = L$, then we put $M^\phi = L$.

(2) If M does not meet L, then we put $M^\phi = M^\theta$.

(3) Suppose that M meets L in a point r. Let $\alpha_1, \ldots, \alpha_{q+1}$ be the $q+1$ planes through M, then $(M \setminus \{r\})^\theta \subseteq \alpha_1^\phi \cap \alpha_2^\phi \cap \cdots \cap \alpha_{q+1}^\phi$. Since $(M \setminus \{r\})^\theta$ is a set of q points, we have that $\alpha_1^\phi \cap \alpha_2^\phi \cap \cdots \cap \alpha_{q+1}^\phi$ is a line meeting L (since one of the planes α_i^ϕ contains L). Put $M^\phi = \alpha_1^\phi \cap \alpha_2^\phi \cap \cdots \cap \alpha_{q+1}^\phi$.

Let r be an arbitrary point of $\mathrm{PG}(3,q)$.

(1) If $r \notin L$, then we put $r^\phi = r^\theta$.

(2) If $r \in L$, let α be a plane of $\mathrm{PG}(3,q)$ for which $\alpha \cap L = \{r\}$. We put $\{r^\phi\} = \alpha^\phi \cap L$. This is a good definition. For, let β be another plane of $\mathrm{PG}(3,q)$ for which $\beta \cap L = \{r\}$. Then $(\alpha \cap \beta)^\phi \subseteq \alpha^\phi \cap \beta^\phi$. Hence $\alpha^\phi \cap L = \beta^\phi \cap L = (\alpha \cap \beta)^\phi \cap L$.

□

1.4.5 Affine representations

The incidence structure $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is called *affine* or *embedded in the finite affine space* $\mathbf{A}$ if $\mathcal{L}$ is a set of lines of $\mathbf{A}$, $\mathcal{P}$ is the union of all members of $\mathcal{L}$ and the incidence relation is the one induced by that of $\mathbf{A}$. If $\mathbf{A}'$ is the subspace of $\mathbf{A}$ generated by all points of $\mathcal{P}$, then we say that $\mathbf{A}'$ is the *ambient space* of $\mathbf{S}$. A special type of affine embedding is the so-called *linear representation*. Let Π_∞ be a projective space of dimension $n \geq 0$ embedded as a hyperplane in the projective space Π and let $\mathcal{K}$ be a nonempty subset of the point set of Π_∞. The linear representation $T_n^*(\mathcal{K})$ is the geometry with points the affine points of Π (= the points not belonging to Π_∞). The lines of $T_n^*(\mathcal{K})$ are all the lines of Π, not contained in Π_∞, which intersect $\mathcal{K}$. Incidence is the one derived from Π.

Chapter 2

Generalized quadrangles

As we already mentioned, the theory of the generalized quadrangles is extremely important in the theory of the near polygons. That is the reason why we devote a whole chapter to these geometries. The generalized quadrangles are widely studied and we will give a survey of their theory, hereby restricting ourselves to those results which we will use later in this thesis and those results which were found by the author himself. For a more complete survey, we refer to [71] and [88].

2.1 Definitions and elementary properties

A *generalized quadrangle* (GQ for short) $\mathbf{Q} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a partial linear space satisfying the following properties:

(GQ1) there exists a point a and a line L such that $(a, L) \notin \mathcal{I}$,

(GQ2) for every point a and every line L not incident with a, there exists a unique point b on L collinear with a.

Generalized quadrangles were introduced in [92]. The point graph of a generalized quadrangle determines it completely, since the lines correspond to the maximal cliques in the graph. Sometimes one requires (like in [71] and [88]) that the geometry also must have an order to be a GQ. This is not very important since, as we shall see in Theorem 2.1.1, this additional property only excludes a few trivial cases, which we will define now. A generalized quadrangle is called *trivial* when it is isomorphic to one of the three examples given below; otherwise it is called *nontrivial*.

(a) Let $t, n_1, \ldots, n_{t+1}$ be nonzero positive integers. Put $\mathcal{P} = \{(\infty)\} \cup \{x_{ij} | 1 \leq i \leq t+1, 1 \leq j \leq n_i\}, \mathcal{L} = \{L_1, \ldots, L_{t+1}\}$, let $(\infty)\ \mathcal{I}\ L_i, \forall i \in$

$\{1,\ldots,t+1\}$, and let $x_{ij}\ \mathcal{I}\ L_k$ if and only if $i=k$. The incidence structure $\mathbf{Q}=(\mathcal{P},\mathcal{L},\mathcal{I})$ is then a GQ. All GQ's which can be constructed this way are called *degenerate*.

(b) Let s_1 and s_2 be nonzero positive integers. Put $\mathcal{P}=\{x_{ij}|1\leq i\leq s_1+1, 1\leq j\leq s_2+1\}, \mathcal{L}=\{L_1,\ldots,L_{s_1+1},M_1,\ldots,M_{s_2+1}\}$, let $x_{ij}\ \mathcal{I}\ L_k$ if and only if $i=k$ and let $x_{ij}\ \mathcal{I}\ M_k$ if and only if $j=k$. The incidence structure $\mathbf{Q}=(\mathcal{P},\mathcal{L},\mathcal{I})$ is then a GQ. The GQ's arising this way are called *grids*, more precisely the GQ defined above is an $(s_1+1)\times(s_2+1)$-grid. If $s_1=s_2$, then the grid is called *symmetrical*; otherwise it is called *nonsymetrical*.

(b) The *dual grids* are the point-line duals of the grids. A description follows by interchanging the roles of the points and lines in the above description of the grids.

We prove now the following known result about GQ's.

Theorem 2.1.1
Every nontrivial GQ *has an order* (s,t) *with* $s,t\geq 2$.

Proof. For a point x of a generalized quadrangle, let t_x+1 denote the number of lines through it. The proof goes in several steps.
Step 1. We prove that two noncollinear points x and y are incident with the same number of lines. To do this, we count in two different ways the pairs (L,M), where L and M are two intersecting lines incident with x, respectively y. By (GQ2), we have $t_x+1=t_y+1$ or $t_x=t_y$.
Step 2. We prove that a GQ is degenerate if there exists a point x which is incident with only one line L_x. For a point y not on L_x, let L_y be the unique line through it which intersects L_x, denote the intersection point by y'. Now x and y are not collinear, otherwise L_x and L_y are two different lines incident with x. By Step 1, L_y is the unique line through y. It suffices to prove now that $y_1'=y_2'$ for every two points y_1 and y_2 not on L_x. If $y_2\ \mathcal{I}\ L_{y_1}$, then $L_{y_1}=L_{y_2}$ and $y_1'=y_2'$. Suppose therefore that y_2 is not a point of L_{y_1}. The line L_{y_2} is the unique line through y_2 which intersects L_{y_1} in a point z. The point z is incident with at least two lines, hence $z\ \mathcal{I}\ L_x$ and $y_1'=y_2'=z$.
Step 3. We prove that all points are incident with the same number of lines if there exists a line which is incident with at least three points. Dually, all lines are incident with the same number of points if there exists a point which is incident with at least three lines. Let L be a line incident with at least three points and let x and y be two arbitrary points; we will prove that $t_x=t_y$. By Step 1, we may suppose that x and y are collinear. If x and y are incident with L, then there exists a point z not collinear with x and y;

hence $t_x = t_z = t_y$. If exactly one of the two points, say x, is incident with L, let $z \neq x$ be another point of L. Then $t_x = t_z = t_y$. Finally, suppose that x and y are not incident with L, then there exists a point z on L not collinear with x and y. Once again $t_x = t_z = t_y$.
Step 4. We prove that a GQ is a dual grid if all the lines are incident with exactly two points. Dually, a GQ is a grid if all the points are incident with exactly two lines. Let p be a fixed point of the GQ and partition the point set $\mathcal{P}$ in two sets $X = \{x_1, \ldots, x_{s_1+1}\}$ and $Y = \{y_1, \ldots, y_{s_2+1}\}$. The elements of X have even distance to p and the elements of Y have odd distance to p. Let $\{z_1, z_2\}$ be a line of the GQ; this line contains a unique point nearest to p. Hence $z_1 \in X, z_2 \in Y$ or $z_1 \in Y, z_2 \in X$. This implies that the point graph Γ of the GQ is a bipartite graph. Now, let $x_i \in X$ and $y_j \in Y$ be arbitrary. If x_i and y_j are not collinear, then $\mathrm{d}(x_i, y_j) \geq 3$, a contradiction. Hence Γ is a complete bipartite graph. Putting L_{ij} the line through x_i and y_j, we find the dual of the description given above for the grids. $\square$

Remark. The point-line dual of a nondegenerate GQ is again a GQ.

A GQ of order (s, t) is sometimes denoted by $\mathrm{GQ}(s, t)$. If $s = t$ then we say that the GQ has order s; in this case the GQ is also denoted by $\mathrm{GQ}(s)$. A GQ of order $(s, 1)$ is a symmetrical grid, dually a GQ of order $(1, t)$ is a symmetrical dual grid. We have the following restrictions on the parameters (s, t).

Theorem 2.1.2 ([71])
Let $\mathbf{Q} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a $\mathrm{GQ}(s, t)$, then

(1) $|\mathcal{P}| = (s+1)(st+1)$ *and* $|\mathcal{L}| = (t+1)(st+1)$,

(2) $s + t$ *divides* $st(s+1)(t+1)$.

Let $\mathbf{Q} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a $\mathrm{GQ}(s, t)$. For $x \in \mathcal{P}$, let $x^\perp$ denote the set of all points collinear with x (hence $x \in x^\perp$). If $A \subseteq P$, then we define $A^\perp = \bigcap_{x \in A} x^\perp$. If x and y are two different points, then the *span* of the pair $\{x, y\}$ is defined as $sp(x, y) := \{x, y\}^{\perp\perp} = (\{x, y\}^\perp)^\perp$. If x and y are not collinear, then $sp(x, y)$ is also called the *hyperbolic line* through x and y. Each hyperbolic line contains at most $t + 1$ points. A pair $\{x, y\}$ is called *regular* when

(i) x is collinear with y, or

(ii) x is not collinear with y and $|sp(x, y)| = t + 1$.

A point x is called *regular* if $\{x, y\}$ is regular for each $y \neq x$. Dually, the notion of regularity can be defined for lines of a GQ. A *triad* of points is a set of three pairwise noncollinear points. A *center of a triad* T is just a point of $T^{\perp}$.

Theorem 2.1.3 ([9],[21],[46],[47])
Let $\mathbf{Q} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a $\mathrm{GQ}(s,t)$. If $s > 1$ then $t \leq s^2$ and equality holds if and only if each triad of points has the same number of centers. This number of centers is then equal to $s+1$. Dually, $s \leq t^2$ if $t > 1$.

2.2 A description of some classes of GQ's

This section does not give a survey of all known GQ's, just an overview of these GQ's which we will meet at least once in this thesis.

2.2.1 The classical GQ's

The examples given here were first recognized as GQ's by Tits ([92]).

(a) Consider in $\mathrm{PG}(d,q)$, $d \in \{3,4,5\}$, a nonsingular quadric with projective index 1. The points of Q together with the lines of Q form then a GQ, denoted by $Q(d,q)$. It has the following parameters:

$Q(3,q)$: $s = q, t = 1$;

$Q(4,q)$: $s = t = q$;

$Q(5,q)$: $s = q, t = q^2$.

(b) Consider in $\mathrm{PG}(d,q^2)$, $d \in \{3,4\}$, a nonsingular Hermitian variety H. The points of H together with the lines of H form then a GQ denoted by $H(d,q^2)$. It has the following parameters:

$H(3,q^2)$: $s = q^2, t = q$;

$H(4,q^2)$: $s = q^2, t = q^3$.

(c) Consider in $\mathrm{PG}(3,q)$ a symplectic polarity ζ. The incidence structure with points the points of $\mathrm{PG}(3,q)$, with lines the totally isotropic lines of $\mathrm{PG}(3,q)$ with respect to ζ and with natural incidence, is then a $\mathrm{GQ}(q)$ denoted by $W(q)$.

2.2.2 The GQ $T_2^*(H)$

Let $\mathrm{PG}(2, 2^h)$, $h \geq 1$, be embedded as a hyperplane Π_∞ in $\mathrm{PG}(3, 2^h)$. For each hyperoval in Π_∞, the linear representation $T_2^*(H)$ is a GQ of order $(2^h - 1, 2^h + 1)$, see [1] and [45].

2.2.3 The GQ $AS(q)$, q odd prime power

The GQ $AS(q)$, q odd prime power, is constructed as follows (see [1]). The points of $AS(q)$ are the points of $\mathrm{AG}(3, q)$. The lines are the following curves of $\mathrm{AG}(3, q)$:

(i) $x = \sigma, y = a, z = b$;

(ii) $x = a, y = \sigma, z = b$;

(iii) $x = c\sigma^2 - b\sigma + a, y = -2c\sigma + b, z = \sigma$.

Here the parameter σ ranges over $\mathrm{GF}(q)$ and a, b, c are arbitrary elements of $\mathrm{GF}(q)$. The incidence is the natural one. The GQ has order $(q - 1, q + 1)$.

2.2.4 The GQ $T_d(O), d \in \{2, 3\}$

The examples of GQ's which we will describe now were discovered by Tits and first appeared in Dembowski [37]. Let the projective space $\mathrm{PG}(d, q)$, $d \in \{2, 3\}$, be embedded as a hyperplane Π_∞ in $\mathrm{PG}(d + 1, q)$ and consider an oval (in the case $d = 2$) or an ovoid (in the case $d = 3$) in Π_∞. We then can construct a GQ $T_d(O)$ as follows. The points are of the following types:

(i) the points of $\mathrm{PG}(d + 1, q) \setminus \Pi_\infty$,

(ii) the hyperplanes X of $\mathrm{PG}(d + 1, q)$ for which $|X \cap O| = 1$;

(iii) one new symbol (∞).

The lines are of the following types:

(a) the lines of $\mathrm{PG}(d+1, q)$ which are not contained in Π_∞ and which meet O;

(b) the points of O.

Incidence is defined as follows. A point of type (i) is incident only with lines of type (a); here incidence is the one from $\mathrm{PG}(d+1,q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of O contained in it. The point (∞) is incident with no line of type (a) and all lines of type (b). The just defined incidence structure is a GQ with the following parameters:

$T_2(O)$: $s = t = q$;

$T_3(O)$: $s = q, t = q^2$.

2.2.5 The GQ S_{xy}^-

The generalized quadrangle S_{xy}^- first appeared in [60], but we give the description taken from [66]. Let $\mathcal{H}$ be a hyperoval in $\mathrm{PG}(2,2^h)$, $h \geq 1$, which is embedded as a hyperplane π in $\mathrm{PG}(3,2^h)$. The following GQ of order $(2^h+1, 2^h-1)$ can then be constructed. The points of S_{xy}^- are of three types:

(i) points of $\mathrm{PG}(3,2^h)$ not contained in π;

(ii) planes through x not containing y;

(iii) planes through y not containing x.

The lines of S_{xy}^- are those lines of $\mathrm{PG}(3,2^h)$ which are not contained in π and which intersect $\mathcal{H} \setminus \{x,y\}$. A point of S_{xy}^- and a line of S_{xy}^- are incident if and only if they are incident as objects of $\mathrm{PG}(3,2^h)$.

2.3 Classification and isomorphisms

The following theorem gives the (anti-)isomorphisms between some classes of GQ's which we just defined. For proofs, we refer to [71].

Theorem 2.3.1

(1) *$Q(3,q)$ is a grid.*

(2) *$Q(4,q)$ is isomorphic to the dual of $W(q)$. Moreover, $Q(4,q)$ (or $W(q)$) is self-dual if and only if q is even.*

(3) *$Q(5,q)$ is isomorphic to the dual of $H(3,q^2)$.*

(4) *$T_2(O)$ is isomorphic to $Q(4,q)$ if and only if O is a conic, it is isomorphic to $W(q)$ if and only if q is even and O is a conic.*

(5) $T_3(O)$ *is isomorphic to* $Q(5,q)$ *if and only if* O *is an elliptic quadric in* $\mathrm{PG}(3,q)$.

(6) $AS(3)$ *is isomorphic to* $Q(5,2)$.

For (anti-)isomorphisms between two GQ's of type $T_2(O)$, $T_2^*(O)$ or S_{xy}^-, we refer to [67]. Concerning isomorphisms between $\mathrm{GQ}'s$ of type $T_2^*(O)$, one can say the following.

Theorem 2.3.2 ([7])
Let O_i, $i \in \{1,2\}$, *be a hyperoval of* π_i. *Then* $T_2^*(O_1) \simeq T_2^*(O_2)$ *if and only if there is an isomorphism* θ *of the plane* π_1 *containing* O_1 *onto the plane* π_2 *containing* O_2 *for which* $O_1^\theta = O_2$.

We give now classification results.

Let $\mathbf{Q}$ be a GQ of order $(2,t)$. By Theorems 2.1.2 and 2.1.3, it follows that $t=1$, $t=2$ or $t=4$. Clearly, there is a unique GQ of order $(2,1)$, namely the (3×3)-grid. It is also not so hard to prove that there is a unique GQ of order 2, see e.g. Theorem 5.2.3. of [71]. The uniqueness of the GQ of order $(2,4)$ was proved independently at least 5 times ([38], [42], [79], [81] and [85]). Hence summarizing we have the following theorem.

Theorem 2.3.3
There are three GQ*'s with order* $(s,t)=(2,t)$, *namely the following ones:*

(1) *the* (3×3)*-grid;*

(2) $W(2)$*;*

(3) $Q(5,2)$.

There is another interesting model for $W(2)$ which goes back to Sylvester ([83]). A *duad* is an unordered pair $ij = ji$ of distinct elements from the set $\{1,2,3,4,5,6\}$. A *syntheme* is a set $\{ij, kl, mn\}$ of three duads for which i,j,k,l,m,n are distinct. The incidence structure with points the duads, with lines the synthemes and with containment as incidence relation, is then the unique GQ of order 2.

Let $\mathbf{Q}$ be a GQ of order $(3,t)$. By Theorems 2.1.2 and 2.1.3, we have that $t \in \{1,3,5,6,9\}$. In [38], it is proved that there is no GQ of order $(3,6)$. The determination of all GQ's of order 3 was achieved in [38] and [62]. The uniqueness of the GQ of order $(3,5)$ was proved in [38]. The uniqueness of the

GQ of order $(3,9)$ was independently proved by Dixmier and Zara [38] and Cameron (see [91]). All these proofs, some of them simplified or streamlined, are also contained in [71]. Hence summarizing we have the following theorem.

Theorem 2.3.4
There are five GQ*'s with order* $(s,t) = (3,t)$*, namely the following ones:*

(1) *the* (4×4)*-grid;*

(2) $W(3)$*;*

(3) $Q(4,3)$*;*

(4) $T_2^*(H)$ *with* H *the, up to projective equivalence, unique hyperoval of* $\mathrm{PG}(2,4)$*;*

(5) $Q(5,3)$*.*

There is no complete classification for $s \geq 4$. It is known that a GQ of order 4 must be isomorphic to $W(4)$, see [63].

2.4 Some general constructions for generalized quadrangles

In Section 2.2, we defined several classes of generalized quadrangles. Sometimes, two or more of these classes can be obtained with the same construction. In this section, we will describe only two of these constructions. The AT-construction was found by the author himself and yields new models for several generalized quadrangles, being natural models for further applications (see Chapter 7).

2.4.1 The Payne-construction

If x is a regular point of a GQ $\mathbf{Q}$ of order s, then a GQ $P(\mathbf{Q},x)$ of order $(s-1,s+1)$ can be constructed as follows (see [60] and [71]). The points of $P(\mathbf{Q},x)$ are the points of $\mathbf{Q}$ not collinear with x; the lines of $P(\mathbf{Q},x)$ are the lines of $\mathbf{Q}$ not incident with x together with all the hyperbolic lines through x; incidence is the natural one. We give now some examples, see [71] for more details.

(1) Consider the GQ $W(q)$ related to the symplectic polarity ζ of $\mathrm{PG}(3,q)$. Each point x of $W(q)$ is regular and we find the following description for $P(W(q),x)$. The points are the points of $\mathrm{PG}(3,q)$ not in the plane x^ζ; the lines are the lines of $W(q)$ not in x^ζ together with those lines of $\mathrm{PG}(3,q)$ through x but not contained x^ζ; incidence is the natural one. If q is even then $P(\mathbf{Q},x)$ is isomorphic to $T_2^*(O)$ with O a regular hyperoval; if q is odd then $P(\mathbf{Q},x)$ is isomorphic to $AS(q)$. If q is odd, then the hyperbolic lines of $W(q)$ through x correspond with the lines of type (i) (see model given in Section 2.2.3).

(2) Consider the GQ $T_2(O)$ with O an oval of $\mathrm{PG}(2,2^h)$, $h \geq 1$. The point (∞) is regular and $P(T_2(O),(\infty))$ is isomorphic to $T_2^*(O \cup \{x\})$, where x is the nucleus of the oval O.

(3) Let $\mathcal{H}$ be a hyperoval of $\mathrm{PG}(2,2^h)$ and let $x,y \in \mathcal{H}$. The line y of $T_2(\mathcal{H} \setminus \{x\})$ (i.e. a line of type (b) in the model given in Section 2.2.4) is regular and $P(T_2(\mathcal{H} \setminus \{x\}),y) \simeq S^-_{xy}$, see [66].

2.4.2 The AT-construction

We describe now a procedure which will give generalized quadrangles. It was found by the author, see [27].

(1) Let $\mathbf{D}$ be a Steiner system $S(2,k,v)$ with $1 < k \leq v$ and let $\mathcal{P}$ be the set of its points. The number of blocks through a point is $\frac{v-1}{k-1}$. Hence, we can put $k = s+1$ and $v = st+1$ with s and t strict positive integers. The total number of blocks is equal to $\frac{v(v-1)}{k(k-1)}$, hence $(s+1)|t(t-1)$.

(2) Find a group K of order $s+1$ (with a multiplication operation, and identity e) and a map $\Delta : \mathcal{P} \times \mathcal{P} \mapsto K, (x,y) \mapsto \delta_{xy}$ such that the points x,y,z are collinear if and only if $\delta_{xy}\delta_{yz} = \delta_{xz}$. If this holds, then we say that the triple $(\mathbf{D},K,\Delta)$ is *admissible*. Note that $\delta_{xx} = e$ and $\delta_{yx} = \delta_{xy}^{-1}$ for all $x,y \in \mathcal{P}$. Hence $\delta_{xy}\delta_{yz}\delta_{zx} = e$ for collinear points $x,y,z \in \mathcal{P}$.

The following theorem says now that every admissible triple (AT for short) yields a generalized quadrangle.

Theorem 2.4.1
Suppose that $(\mathbf{D},K,\Delta)$ is an admissible triple. Let Γ be the graph with vertex set $K \times \mathcal{P}$; two different vertices (k_1,x) and (k_2,y) are adjacent whenever

(a) $x = y$ and $k_1 \neq k_2$, or

(b) $x \neq y$ *and* $k_2 = k_1\delta_{xy}$.

Then Γ *is the collinearity graph of a* $\mathrm{GQ}(s,t)$.

Proof. The graph Γ contains $(1+s)(1+st)$ vertices and every vertex is adjacent to $s(t+1)$ others. We will prove that every two adjacent vertices are contained in a unique maximal clique and that this clique contains exactly $s+1$ elements. If we consider these cliques as the lines of a geometry $\mathbf{Q}$ with the vertices of Γ as points, then every point of $\mathbf{Q}$ is incident with $t+1$ lines and the number of points at distance at most one to a fixed line is $(s+1)+(s+1)ts = (s+1)(1+st)$. Hence $\mathbf{Q}$ is a $\mathrm{GQ}(s,t)$.

Now, suppose that $p_1 = (k_1, x)$ and $p_2 = (k_2, y)$ are two adjacent vertices of Γ; we determine how the common neighbours (k_3, z) look like. If $x = y \neq z$, then $k_3 = k_1\delta_{xz} = k_2\delta_{xz}$, implying that $k_1 = k_2$, a contradiction. Hence if $x = y$, then p_1 and p_2 are in a unique maximal clique containing all the points (k, x) with $k \in K$. If $x \neq y$, then also $x \neq z \neq y$ and $k_3 = k_1\delta_{xz} = k_2\delta_{yz} = k_1\delta_{xy}\delta_{yz}$. This implies that $\delta_{xz} = \delta_{xy}\delta_{yz}$ or that $z \in xy$. It follows now easily that p_1 and p_2 are contained in a unique maximal clique, namely $\{(k_1\delta_{xz}, z) | z \mathrm{\ I\ } xy\}$. $\square$

If there is an admissible triple with $\mathbf{D}$ and K as components, then there are a lot of admissible triples with $\mathbf{D}$ and K as components, as we will show now; however all the corresponding GQ's turn out to be isomorphic.

Theorem 2.4.2
Let $(\mathbf{D}, K, \Delta)$ *be an admissible triple. With every point* x *of* $\mathbf{D}$, *we associate an element* $\delta_x \in K$. *Put* $\Delta'(x,y) = \delta'_{xy} = \delta_x^{-1}\delta_{xy}\delta_y$ *for every two points* x, y *of* $\mathbf{D}$. *Then*

(1) $(\mathbf{D}, K, \Delta')$ *is an admissible triple,*

(2) *the two corresponding* GQ*'s are isomorphic.*

Proof. Part (1) is easily verified. Now, let $\mathbf{Q}_1$ and $\mathbf{Q}_2$ be the generalized quadrangles corresponding to $(\mathbf{D}, K, \Delta)$ and $(\mathbf{D}, K, \Delta')$ respectively. The map $(k, x) \mapsto (k\delta_x, x)$ defines then an isomorphism between $\mathbf{Q}_1$ and $\mathbf{Q}_2$. $\square$

Theorem 2.4.3
Let $(\mathbf{D}, K, \Delta)$ *be an admissible triple and let* θ *be an automorphism of the group* K. *Put* $\Delta''(x,y) = \delta''_{xy} = \delta_{xy}^{\theta}$ *for every two points* x, y *of* $\mathbf{D}$. *Then*

(1) $(\mathbf{D}, K, \Delta'')$ *is an admissible triple,*

(2) *the two corresponding* GQ*'s are isomorphic.*

Proof. Part (1) is easily verified. Now, let $\mathbf{Q}_1$ and $\mathbf{Q}_2$ be the generalized quadrangles corresponding to $(\mathbf{D}, K, \Delta)$ and $(\mathbf{D}, K, \Delta'')$ respectively. The map $(k, x) \mapsto (k^\theta, x)$ defines then an isomorphism between $\mathbf{Q}_1$ and $\mathbf{Q}_2$. $\square$

Suppose that $(\mathbf{D}, K, \Delta)$ is an admissible triple. If $\mathbf{D}$ is not a line ($t > 1$), then the fact that there are at least as many blocks as points (Fundamental Theorem of linear spaces, [26]) implies that $t \geq s + 1$. If $t = s + 1$, then $\mathbf{D}$ is an $S(2, s + 1, s^2 + s + 1)$ and hence a projective plane, but we will prove that this is impossible. Hence $s = 1$, $t = 1$ or $s + 2 \leq t \leq s^2$.

Theorem 2.4.4
There are no admissible triples where the Steiner system is a projective plane.

Proof. If there was such an admissible triple, then there would exist a GQ of order $(s, s + 1)$ with $s \geq 2$, but this contradicts the condition $(s + t) \mid st(s + 1)(t + 1)$ holding for a GQ(s, t), see Theorem 2.1.2. $\square$

2.5 The known GQ's with an AT-model

Every known GQ has one of the following parameters:

(I) $(s, 1)$ or $(1, t)$ with $s, t \geq 1$;

(II) $(q - 1, q + 1)$ or $(q + 1, q - 1)$ with q a prime power;

(III) (q, q) with q a prime power;

(IV) (q^2, q^3) or (q^3, q^2) with q a prime power;

(V) (q, q^2) or (q^2, q) with q a prime power.

Only the following cases remain after checking the conditions (i) $(s + 1) \mid t(t - 1)$ and (ii) $s + 2 \leq t \leq s^2$ if $s, t \neq 1$:

(a) $(s, 1)$ or $(1, t)$ with $s, t \geq 1$;

(b) $(q - 1, q + 1)$ with q a prime power;

(c) (q, q^2) with q a prime power.

There are examples of GQ's with an AT-model for each of these parameters.

2.5.1 The trivial GQ's

(1) Let $\mathbf{D}$ be a line of length $s+1$ $(t=1)$ and let K be any group of order $s+1$. For $x, y \in \mathcal{P}$ put $\delta_{xy} = e$. The corresponding generalized quadrangle is an $(s+1) \times (s+1)$-grid.

(2) Let $\mathbf{D}$ be an $S(2,2,t+1)$ (i.e. the complete graph K_{t+1}), let $K = \{e, a\}$ be the group of order 2. Put $\delta_{xy} = e$ if $x = y$ and $\delta_{xy} = a$ if $x \neq y$. The corresponding generalized quadrangle is a dual grid.

2.5.2 The GQ $P(W(q), x)$

Let $\mathbf{D}$ be the Desarguesian affine plane $\mathrm{AG}(2,q)$ with point set $\mathcal{P} = \{(x_1, x_2) | x_1, x_2 \in \mathrm{GF}(q)\}$. Three points $x = (x_1, x_2), y = (y_1, y_2)$ and $z = (z_1, z_2)$ are collinear if and only if

$$\begin{vmatrix} x_1 & x_2 & 1 \\ y_1 & y_2 & 1 \\ z_1 & z_2 & 1 \end{vmatrix} = 0.$$

This condition can be rewritten as

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix}.$$

If we take K equal to the additive group of $\mathrm{GF}(q)$ and $\delta_{xy} = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$, then $(\mathbf{D}, K, \Delta)$ is an admissible triple, which gives rise to a generalized quadrangle $\mathbf{Q}$ of order $(q-1, q+1)$. The generalized quadrangle $\mathbf{Q}$ has points (X_0, X_1, X_2) with $X_0, X_1, X_2 \in \mathrm{GF}(q)$; these are the points of $\mathrm{AG}(3,q)$. Two different points (X_0, X_1, X_2) and (Y_0, Y_1, Y_2) are collinear if and only if

(a) $X_0 \neq Y_0, X_1 = Y_1, X_2 = Y_2$, or

(b) $Y_0 = X_0 + X_1 Y_2 - X_2 Y_1$ and $(X_1, X_2) \neq (Y_1, Y_2)$.

This enables us to calculate the lines (see proof of Theorem 2.4.1).

(I) The above case (a) yields the lines $\{(\sigma, a, b) | \sigma \in \mathrm{GF}(q)\}$, $a, b \in \mathrm{GF}(q)$.

(II) Consider the line $\{(a, \sigma) | \sigma \in \mathrm{GF}(q)\}$ of $\mathrm{AG}(2,q)$. Then the corresponding lines of $\mathbf{Q}$ are the sets $\{(b' + \begin{vmatrix} a & \sigma' \\ a & \sigma \end{vmatrix}, a, \sigma) | \sigma \in \mathrm{GF}(q)\}$, $b', \sigma' \in \mathrm{GF}(q)$, that is, the sets $\{(b + a\sigma, a, \sigma) | \sigma \in \mathrm{GF}(q)\}$, $b \in \mathrm{GF}(q)$.

(III) Consider the line $\{(\sigma, a\sigma + b)|\sigma \in \mathrm{GF}(q)\}$ of $\mathrm{AG}(2,q)$. Then the corresponding lines of $\mathbf{Q}$ are the sets $\{(b' + \begin{vmatrix} \sigma' & a\sigma' + b \\ \sigma & a\sigma + b \end{vmatrix}, \sigma, a\sigma + b)|\sigma \in \mathrm{GF}(q)\}$, $b', \sigma' \in \mathrm{GF}(q)$, that is, the sets $\{(c - b\sigma, \sigma, a\sigma + b)|\sigma \in \mathrm{GF}(q)\}$, $c \in \mathrm{GF}(q)$.

Now, embed $\mathrm{AG}(3,q)$ in $\mathrm{PG}(3,q)$ (by adding a new coordinate X_3 and identification of the points (X_0, X_1, X_2) of $\mathrm{AG}(3,q)$ and $(X_0, X_1, X_2, 1)$ of $\mathrm{PG}(3,q)$). Consider in $\mathrm{PG}(3,q)$ the generalized quadrangle $W(q)$ arising from the symplectic polarity ζ determined by the equation $X_0Y_3 + X_1Y_2 - X_2Y_1 - X_3Y_0 = 0$. The point $x = (1,0,0,0)$ is a regular point of $W(q)$ and we have $\mathbf{Q} \simeq P(W(q), x)$. Indeed, the lines of type (I) are the lines through x not contained in x^ζ and the lines of type (II) and (III) are the totally isotropic lines of ζ not contained in x^ζ. Hence we have the following conclusions.

(1) If q is odd, then $\mathbf{Q} \simeq AS(q)$.

(2) If q is even, then $\mathbf{Q} \simeq T_2^*(O)$, with O a regular hyperoval.

2.5.3 The GQ $T_2^*(O)$

There are other admissible triples related to the affine plane $\mathrm{AG}(2,q)$ and the additive group of $\mathrm{GF}(q)$. The affine plane $\mathrm{AG}(2,q)$ has $q+1$ directions determined by the vectors $\bar{e}_1, \ldots, \bar{e}_{q+1}$. For two different points x and y of $\mathrm{AG}(2,q)$, one has that $\overline{xy} = \delta_{xy}\bar{e}_{xy}$ with $\bar{e}_{xy} \in \{\bar{e}_1, \ldots, \bar{e}_{q+1}\}$ the vector corresponding to the direction of the line xy. This defines δ_{xy} when $x \neq y$. If $x = y$, put $\delta_{xy} = 0$. We determine now under which conditions we get an admissible triple. If x, y, z are three collinear points, then clearly $\delta_{xy} + \delta_{yz} = \delta_{xz}$. Conversely, suppose that $\delta_{xy} + \delta_{yz} = \delta_{xz}$ with x, y, z three noncollinear points. Then $\bar{e}_{xy}, \bar{e}_{yz}, \bar{e}_{xz}$ are three mutual different vectors which satisfy $\delta_{xy}\bar{e}_{xy} + \delta_{yz}\bar{e}_{yz} = \delta_{xz}\bar{e}_{xz}$ or $\bar{e}_{xz} = \lambda\bar{e}_{xy} + (1-\lambda)\bar{e}_{yz}$ for some $\lambda \in \mathrm{GF}(q)$. We hence obtain an admissible triple if and only if

$$\bar{e}_k - \lambda\bar{e}_i - (1-\lambda)\bar{e}_j \neq 0 \tag{2.1}$$

holds for all $\lambda \in \mathrm{GF}(q)$ and any three mutual distinct vectors $\bar{e}_i, \bar{e}_j, \bar{e}_k$. Now, if a is a fixed point of $\mathrm{AG}(2,q)$, then $H = \{a, a+\bar{e}_1, \ldots, a+\bar{e}_{q+1}\}$ is a set of $q+2$ points. If $\mathrm{PG}(2,q)$ is the projective completion of $\mathrm{AG}(2,q)$, then condition (2.1) holds if and only if H is a hyperoval of $\mathrm{PG}(2,q)$.

Theorem 2.5.1
If condition (2.1) holds, then the generalized quadrangle arising from the corresponding admissible triple is isomorphic to $T_2^*(H)$.

Proof. Let $\mathbf{Q}$ be the generalized quadrangle arising from the admissible triple. The points of $\mathbf{Q}$ are the elements of $\mathrm{GF}(q) \times \mathrm{GF}(q) \times \mathrm{GF}(q)$. A line of $\mathbf{Q}$ is of one of the following types.

(i) $\{(\lambda, x, y) | \lambda \in \mathrm{GF}(q)\}$ for fixed $x, y \in \mathrm{GF}(q)$.

(ii) $\{(a+\lambda, b+\lambda e_{i1}, c+\lambda e_{i2}) | \lambda \in \mathrm{GF}(q)\}$ for fixed $a, b, c \in \mathrm{GF}(q)$ and fixed $\bar{e}_i = (e_{i1}, e_{i2})$.

Now, embed the points of $\mathbf{Q}$ in $\mathrm{PG}(3, q)$ as follows. Let the point (x, y, z) of $\mathbf{Q}$ correspond to the point $(x, y, z, 1)$ of $\mathrm{PG}(3, q)$. The lines of $\mathbf{Q}$ intersect the plane at infinity of $\mathrm{PG}(3, q)$ (this is the plane with points $(x, y, z, 0)$) in one of the $q+2$ points of the set $\{(1, 0, 0, 0)\} \cup \{(1, e_{i1}, e_{i2}, 0) |\ i \in \{1, \ldots, q+1\}\}$ and this latter set defines a hyperoval projectively equivalent to H. □

2.5.4 The GQ $Q(5, q)$

Let the vector space $V(3, q^2)$ be equiped with a nonsingular Hermitian form $(\cdot, \cdot)$, i.e. $(\sum \mu_i v_i, \sum \lambda_j w_j) = \sum \sum \mu_i \lambda_j^q (v_i, w_j)$, and let U be the corresponding unital of $\mathrm{PG}(2, q^2)$. There is a Steiner system $\mathbf{D} = S(2, q+1, q^3+1)$ related to U (the blocks are the intersections of U with nontangent lines). Let $K = \{\alpha \in \mathrm{GF}(q^2) | \alpha^{q+1} = 1\}$ with multiplication inherited from $\mathrm{GF}(q^2)$. Let $z = \langle \bar{a} \rangle$ be a fixed point of U. For two points $x = \langle \bar{b} \rangle$ and $y = \langle \bar{c} \rangle$ of U, we define

$$\begin{aligned} \Delta(x, y) &= -(\bar{a}, \bar{b})^{q-1} (\bar{b}, \bar{c})^{q-1} (\bar{c}, \bar{a})^{q-1} \in K \text{ if } x, y, z \text{ are mutually different;} \\ &= 1 \text{ otherwise.} \end{aligned}$$

This is a good definition. For, if we replace $\bar{b}$ by $\mu \bar{b}$ and $\bar{c}$ by $\lambda \bar{c}$ with $\mu, \lambda \in \mathrm{GF}(q^2) \setminus \{0\}$, then the above value for $\Delta(x, y)$ is unaltered. We will prove in Section 2.7.1 that $(\mathbf{D}, K, \Delta)$ is admissible and that the corresponding GQ is isomorphic to $Q(5, q)$.

2.5.5 The GQ $(S_{xy}^-)^D$

Let $\mathcal{H}$ be a hyperoval of $\mathrm{PG}(2, 2^h)$, $h \geq 1$, and let x, y be two points of $\mathcal{H}$. We can coordinatize $\mathrm{PG}(2, 2^h)$ in such a way that $x \leftrightarrow (1, 0, 0)$, $y \leftrightarrow (0, 1, 0)$ and such that the point with coordinates $(0, 0, 1)$ belongs as well to $\mathcal{H}$. We have then that $\mathcal{H} = \{(1, 0, 0), (0, 1, 0)\} \cup \{(f(\lambda), \lambda, 1) | \lambda \in \mathrm{GF}(q)\}$, where f is a suitable function from $\mathrm{GF}(q)$ to $\mathrm{GF}(q)$. We have the following admissible triple. Let $\mathbf{D}$ be the affine plane $\mathrm{AG}(2, 2^h)$ with point set $\mathrm{GF}(2^h) \times \mathrm{GF}(2^h)$ and let K be the additive group of $\mathrm{GF}(2^h)$. For two points $p_1 = (\alpha_1, \beta_1)$

and $p_2 = (\alpha_2, \beta_2)$ of $\mathrm{AG}(2, 2^h)$, we put $\Delta(p_1, p_2)$ equal to $\frac{[f(\alpha_1)-f(\alpha_2)][\beta_1-\beta_2]}{\alpha_1-\alpha_2}$ if $\alpha_1 \neq \alpha_2$ and 0 otherwise. We will prove in Section 2.7.1 that $(\mathbf{D}, K, \Delta)$ is admissible and that the corresponding GQ is isomorphic to $(S_{xy}^-)^D$, the dual generalized quadrangle of S_{xy}^-.

Problem. Are there other (unknown) GQ's derivable from an admissible triple?

2.6 Ovoids and spreads of GQ's

Let $\mathbf{Q} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a GQ(s, t). An *ovoid* O of $\mathbf{Q}$ is a set of points such that every line of $\mathbf{Q}$ has exactly one point in common with O. The dual notion is that of spread. A *spread* S is a set of lines partitioning the point set. Every ovoid or spread of $\mathbf{Q}$ has $1 + st$ elements. Every set of mutually noncollinear points of $\mathbf{Q}$ has size at most $1 + st$ with equality if and only if this set is an ovoid. A *rosette of ovoids* is a set of s ovoids through a fixed point p of $\mathbf{Q}$ partitioning the set of points at distance 2 from p. A *fan of ovoids* is a set of $s + 1$ ovoids partitioning the point set of $\mathbf{Q}$.

2.6.1 Ovoids

Some classes of GQ's cannot have ovoids.

Theorem 2.6.1

(1) *$W(q)$, q odd, has no ovoids* ([87]).

(2) *Every* GQ *of order (s, s^2), $s \neq 1$, has no ovoids* ([87], [80]).

For several GQ's we will define those ovoids which we will meet later in this thesis.

The $[(s+1) \times (s+1)]$-grid

There are exactly $(s + 1)!$ ovoids.

$Q(4, q)$

Let Q be a nonsingular quadric in $\mathrm{PG}(4, q)$. An elliptic quadric $Q^-(3, q)$ on Q is an ovoid of $Q(4, q)$. Suppose now that q is even, then $Q(4, q) \simeq W(q)$. Note that the points of $W(q)$ are exactly the points of $\mathrm{PG}(3, q)$. If O is an ovoid of $W(q)$, then it is also an ovoid of $\mathrm{PG}(3, q)$, see [84]. Conversely, if O is an ovoid of $\mathrm{PG}(3,q)$, then one can define a symplectic polarity ζ in $\mathrm{PG}(3,q)$ such that O is also an ovoid of the generalized quadrangle ($\simeq W(q)$) related to ζ. A proof of this can be found in [48]. Consider now the case $q=2$ and look at Sylvesters model of $W(2)$ given in Section 2.3. For each $i \in \{1,\dots,6\}$, the set $\{ij | j \neq i\}$ is an ovoid and all ovoids arise this way. Hence, there are exactly six ovoids and every two distinct ovoids meet in a point.

$H(3,q^2)$

Let H be a nonsingular Hermitian variety in $\mathrm{PG}(3,q^2)$. Every nontangent plane intersects H in a Hermitian curve which is an ovoid of $H(3,q^2)$.

$P(\mathbf{Q},x)$

Let $\mathbf{Q}$ be a GQ of order s having a regular point x. Then O is an ovoid of $P(\mathbf{Q},x)$ if and only if $O \cup \{x\}$ is an ovoid of $\mathbf{Q}$, see [71].

S_{xy}^-

Let $\mathcal{H}$ be a hyperoval of $\mathrm{PG}(2,2^h)$, $h \geq 1$, being embedded as a hyperplane α in $\mathrm{PG}(3,2^h)$ and let $x,y \in \mathcal{H}$.

(1) The set of planes through x not containing y forms an ovoid O_{xy} of S_{xy}^-. Similarly, the set of planes through y not containing x forms an ovoid O_{yx} of S_{xy}^-.

(2) Let $\pi \neq \alpha$ be a plane through xy, then the set of affine points in π forms an ovoid O_π of S_{xy}^-.

2.6.2 Spreads

The above given examples of ovoids correspond to spreads in the dual GQ's. We give now some additional examples of spreads.

(A) $T_2^*(O)$
Let O be a hyperoval of the plane $\mathrm{PG}(2,2^h)$, $h \geq 1$, which is embedded as a hyperplane α in $\mathrm{PG}(3,2^h)$. Let $o \in O$ be fixed. The lines of $\mathrm{PG}(3,2^h)$ through o not contained in α defines then a spread S_o of $T_2^*(O)$.

(B) GQ's from the Payne-construction
Let $\mathbf{Q}$ be a GQ of order s having a regular point x, then the hyperbolic lines through x define a spread of $P(\mathbf{Q},x)$.

(C) GQ's from the AT-construction

Let $\mathbf{Q}$ be the GQ arising from the AT $(\mathbf{D}, K, \Delta)$. Let $\mathcal{P}$ denote the point set of $\mathbf{D}$, then the set $S = \{L_x | x \in \mathcal{P}\}$ with $L_x = \{(k, x) | k \in K\}$ is a spread of $\mathbf{Q}$ and called the *associated spread* of the admissible triple.

Definition. A *partial spread* of a Steiner system is a set of mutually disjoint blocks.

Theorem 2.6.2
With each partial spread S_1 of $\mathbf{D}$, there corresponds a spread S_1' of $\mathbf{Q}$.

Proof. Let $S_1 = \{L_1, \ldots, L_r\}$ be a partial spread of $\mathbf{D}$ $(0 \le r \le \frac{1+st}{1+s})$. For each $i \in \{1, \ldots, r\}$, let x_i be a fixed point of L_i. If $L_{(k,x_i)}$ denotes the line $\{(k\delta_{x_i y_i}, y_i) | y_i \mathrm{~I~} L_i\}$ for each $k \in K$ and each $i \in \{1, \ldots, r\}$, then $S_1' = \{L_{(k,x_i)} | k \in K; 1 \le i \le r\} \cup \{L_x | x$ is not incident with $L_i, \forall i \in \{1, \ldots, r\}\}$ is a spread of $\mathbf{Q}$. $\square$

A spread S of a GQ $\mathbf{Q}$ is called *normal* when each pair $\{L, M\} \subseteq S$ is regular and when all the members of the hyperbolic line through $\{L, M\}$ belong as well to S. Dually, one can define *normal ovoids.* Consider now the following problem. Given a nontrivial GQ $\mathbf{Q}$, determine all normal spreads. In the next theorem, we will show a way how these spreads can be determined, but first we define the following linear space $\mathcal{L}(\mathbf{Q})$. The points of $\mathcal{L}(\mathbf{Q})$ are the lines of $\mathbf{Q}$; the lines of $\mathcal{L}(\mathbf{Q})$ are the pencils (the sets of lines through a point of $\mathbf{Q}$) together with the hyperbolic lines (on the set of lines of $\mathbf{Q}$); incidence is the natural one.

Theorem 2.6.3
If S is a normal spread of a nontrivial GQ $\mathbf{Q}$*, then $S = \langle K, L, M\rangle$ for any three non-$\mathcal{L}(\mathbf{Q})$-collinear points $K, L, M \in S$.*

Proof. Let $K, L, M \in S$ be three non-$\mathcal{L}(\mathbf{Q})$-collinear points; then $\langle K, L, M\rangle \subseteq S$ contains at least $s(s+1)+1$ elements. If $N \in S$ lies not in $\langle K, L, M\rangle$, then $\langle K, L, M, N\rangle \subseteq S$ contains at least $s(s^2+s+1)+1$ elements. Hence, $1 + st \ge s^3 + s^2 + s + 1$ or $t \ge s^2 + s + 1$, a contradiction, since $t \le s^2$ holds for an arbitrary nontrivial GQ(s, t). $\square$

A *spread of symmetry* is a spread S satisfying the following property.

For every $K, L \in S$ and every two lines M and N meeting K and L, there exists an automorphism θ fixing each line of S such that $M^\theta = N$.

If S is a spread of symmetry, then S is also a normal spread. Dually, one can define *ovoids of symmetry.*

We will give now all the normal spreads of the following GQ's. It will turn out that these spreads are also spreads of symmetry.

Grids

The two spreads of a grid are spreads of symmetry.

$T_2^*(O)$

Let O be a hyperoval of $\mathrm{PG}(2, 2^h)$, $h \geq 1$. If $h = 1$, then $T_2^*(O)$ is a dual grid and all 24 spreads are spreads of symmetry. If $h > 1$, then we make use of Theorem 3.3.4 of [71]: the pair $\{L, M\}$ of nonconcurrent lines of $T_2^*(O)$ is regular if and only if L and M define the same point of O. In this case $\{L, M\}^{\perp\perp}$ consists of the 2^h lines of $T_2^*(O)$ which go through the point $L \cap M$ and are contained in the plane $\langle L, M \rangle$. Hence, the only normal spreads of $T_2^*(O)$, $q \neq 2$, are the spreads S_o defined by a point $o \in O$. Take such a spread S_o. Consider two lines $K, L \in S_o$ and two lines $M, N \in \{K, L\}^\perp$. There exists a translation of $\mathrm{AG}(3, 2^h)$ mapping M to N and having o as point at infinity (all lines of S_o are then fixed). Hence all normal spreads of $T_2^*(O)$ are also spreads of symmetry.

$AS(q)$

Consider the GQ $AS(q)$, q odd. If $q = 3$, then $AS(3) \simeq Q(5, 2)$ and we will treat this case later. Suppose therefore that $q > 3$. We make now use of Theorem 3.3.5 of [71]. The pair $\{L, M\}$ of nonconcurrent lines of $P(W(q), x)$ is regular if and only if one of the following holds: (i) L and M are hyperbolic lines of $W(q)$ which contain x, or (ii) L and M are concurrent lines in $W(q)$. Te hyperbolic lines of $W(q)$ through x correspond to the lines of type (i) in the model of $AS(q)$ given in Section 2.2.3. The span of two lines of type (i) consists of q lines of type (i). Hence, $AS(q)$ with $q > 3$ has only one normal spread which consists of all lines of type (i). This spread is also a spread of symmetry. The required automorphisms are translations of the affine 3-space (see model given in Section 2.2.3).

$Q(5, q)$

Let S be a normal spread of $Q(5, q)$. This dualizes to a normal ovoid of $H(3, q^2)$. We will prove that O is the intersection of $H(3, q^2)$ with a nontangent plane. Each pair $\{x,y\}$ of noncollinear points of $H(3,q^2)$ is regular and the hyperbolic line through them consists of the $q+1$ points in the intersection of the line xy with the Hermitian variety. Let $x,y,z \in O$, where z is not a member of the hyperbolic line through $\{x,y\}$. From Theorem 2.6.3, it follows that O is contained in the intersection of $H(3,q^2)$ with the plane α through x,y,z. Hence α is nontangent and $O = \alpha \cap H(3,q^2)$. Take coordinates such that $H(3,q^2)$ has equation $X_0^{q+1} + X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0$ and such that α is the plane $X_3 = 0$. The ovoid O is now an ovoid of symmetry: the required automorphisms are of the form $(X_0', X_1', X_2', X_3') = (X_0', X_1', X_2', \lambda X_3')$, where λ is one of the $q+1$ elements of $\mathrm{GF}(q^2)$ for which $\lambda^{q+1} = 1$.

$(S_{xy}^-)^D$

Let $\mathcal{H}$ be a hyperoval of $\mathrm{PG}(2,2^h)$, $h \geq 1$, being embedded as a hyperplane α in $\mathrm{PG}(3,2^h)$ and let $x,y \in \mathcal{H}$. We may suppose that $(S_{xy}^-)^D$ is not isomorphic to $T_2^*(\mathcal{H})$, see [67] when this situation precisely occurs. In this case, O_{xy} and O_{yx} are the only normal spreads, see [70]. We will prove that these ovoids are also ovoids of symmetry. In order that O_{xy} is an ovoid of symmetry, it suffices that there are at least (and hence exactly) 2^h automorphisms of S_{xy}^- fixing each point of O_{xy} (see Theorem 2.7.1). These 2^h automorphisms are induced by the 2^h automorphisms of $\mathrm{PG}(3,2^h)$ which fix each point of α and each line through x.

Let p be a point of a nontrivial $\mathrm{GQ}(s,t)$. A *symmetry about* p is an automorphism of $\mathrm{GQ}(s,t)$ fixing p and every point collinear with p. One can prove that there are at most t such symmetries; if there are exactly t symmetries, then p is called a *center of symmetry.* A center of symmetry is always a regular point. We mention now the following result proved in [36].

Theorem 2.6.4 ([36])
A GQ $\mathbf{Q}$ of order $(s, s+2)$ has a spread of symmetry if and only if $\mathbf{Q} \simeq P(\mathbf{Q}', (\infty))$ with $\mathbf{Q}'$ a $\mathrm{GQ}(s+1)$ and (∞) a center of symmetry of $\mathbf{Q}'$. The spread of symmetry corresponds to the hyperbolic lines through (∞).

Corollary. With each AT $(\mathbf{D}, K, \Delta)$, where $\mathbf{D}$ is an affine plane of order $s+1$, there corresponds a GQ of order $s+1$. This result was already in [18].

2.7 Characterizations of GQ's with an AT-model

2.7.1 Characterization in terms of a group of automorphisms

Let $S = \{L_1, \ldots, L_{1+st}\}$ be a spread of a generalized quadrangle $\mathbf{Q}$ of order (s,t) and let G_S be the group of automorphisms of $\mathbf{Q}$ fixing each line of S. If $\mathbf{Q}$ is an $(s+1) \times (s+1)$-grid, then $|G_S| = (s+1)!$ for both spreads of $\mathbf{Q}$.

Theorem 2.7.1
If $\mathbf{Q}$ is not a grid, then each nontrivial element of G_S has no fixpoints; hence $|G_S| = \frac{s+1}{n}$ with n some nonzero integer.

Proof. Suppose that x is a fixpoint of $\theta \in G_S$.

(a) Suppose $y \sim x$ with $xy \notin S$. If $L \in S$ is incident with y, then $x \mathrm{~I~} xy \mathrm{~I~} y \mathrm{~I~} L$ and $x \mathrm{~I~} xy^\theta \mathrm{~I~} y^\theta \mathrm{~I~} L$ implies that $y^\theta = y$.

(b) Suppose $y \not\sim x$, then x and y have a common neighbour z such that $xz, yz \notin S$. From (a) it follows that $z^\theta = z$ and $y^\theta = y$.

(c) Suppose that $y \sim x$ with $xy \in S$. Take a point z such that $x \not\sim z \sim y$. Then (b) implies that $z^\theta = z$ and (a) implies that $y^\theta = y$.

Hence, the automorphism θ is trivial. The group G_S acts as a permutation group on the set of points of any line of S. If there are n orbits on such a line, then $|G_S| = \frac{s+1}{n}$. $\square$

Corollary 2.7.2
If $\mathbf{Q}$ is not a grid, then S is a spread of symmetry if and only if $|G_S| = s+1$.

Theorem 2.7.3
S is a spread of symmetry of $\mathbf{Q}$ if and only if there exists an AT-model for $\mathbf{Q}$ with S as associated spread.

Proof. We may suppose that $\mathbf{Q}$ is not a grid. If $(\mathbf{D}, K, \Delta)$ is an AT with $\mathbf{Q}$ as corresponding GQ and S as associated spread, then we may suppose that $\{(k,x) | k \in K;\ x \text{ is a point of } \mathbf{D}\}$ is the set of points of $\mathbf{Q}$ and that $S = \{L_x | x \text{ is a point of } \mathbf{D}\}$, where $L_x = \{(k,x) | k \in K\}$. For each $k' \in K$, the map $(k,x) \mapsto (k'k, x)$ is an automorphism of $\mathbf{Q}$ fixing each line of S. Hence $|G_S| \geq |K| = s+1$ and S is a spread of symmetry by Theorem 2.7.1 and Corollary 2.7.2. Conversely, suppose that S is a spread of symmetry (or $|G_S| = s+1$).

(1) Construction of the Steiner system S(2,s+1,1+st).
The points of the Steiner system are the lines of S. The blocks are obtained as follows. Take a line $T \notin S$ of $\mathbf{Q}$. Let $R_1, \ldots, R_{s+1}$ be the lines of S intersecting T; then $\{R_1, \ldots, R_{s+1}\}$ is a line of the Steiner system. It remains to show that every two lines R_1 and R_2 of S are contained in a unique block. Every line intersecting R_1 and R_2 will yield such a block. If T is a line intersecting R_1 and R_2, then $A = \{T^h | h \in G_S\}$ is a set of $s+1$ lines intersecting R_1 and R_2, hence these are all the lines intersecting R_1 and R_2. If $L \in S$ intersects T in a point t, then L intersects T^h in t^h. Hence, every element of A will yield the same block through R_1 and R_2.

(2) Construction of the group K and the map Δ.
We take $K = G_S$. Let $L_1 \in S$ and let p be any point of L_1. Take two points $u = L_m$ and $v = L_n$ of the Steiner system. Let x_1 be the projection of p on the line L_m, let x_2 be the projection of x_1 on the line L_n and finally let x_3 be the projection of x_2 on the line L_1. Now, there exists an element $\theta \in G_S$ such that $x_3 = p^\theta$ and we define $\Delta(u,v) = \delta_{uv} := \theta^{-1}$. It remains to show that the points $x = L_i, y = L_j, z = L_k$ are collinear in the Steiner system if and only if $\delta_{xy}\delta_{yz} = \delta_{xz}$. Put $\delta_{xy} = \alpha^{-1}, \delta_{yz} = \beta^{-1}$ and $\delta_{xz} = \gamma^{-1}$. Denote by $p_l(e)$ $(l \in \{1, i, j, k\})$ the projection of the point e on the line L_l in the generalized quadrangle $\mathbf{Q}$. Put $a = p_i(p), b = p_j(a), c = p_k(b), d = p_k(a)$, then $p^\gamma = p_1(d)$. From $p \sim p_j(p) \sim p_k p_j(p) \sim p^\beta$ and $p^\alpha \sim b \sim c \sim p_1(c)$, it follows that $p^{\beta\alpha} = p_1(c)$. Now,

$$\begin{aligned}
\delta_{xy}\delta_{yz} = \delta_{xz} \quad &\Leftrightarrow \quad \gamma = \beta\alpha \\
&\Leftrightarrow \quad c = d \\
&\Leftrightarrow \quad a, b, c \text{ are on a line } T \\
&\Leftrightarrow \quad x, y, z \text{ are collinear.}
\end{aligned}$$

(3) Isomorphism between $\mathbf{Q}$ and the GQ from $(\mathbf{D}, K, \Delta)$.
Let z be an arbitrary point of $\mathbf{Q}$. Let L_z denote the unique line of S incident with z and let z' denote the projection of z on the fixed line L_1. There exists a unique $g_z \in G_S$ such that $z' = p^{g_z}$. We prove now that the map $z \mapsto (g_z^{-1}, L_z)$ defines an isomorphism between the GQ's. It is clearly a bijection and since both geometries have the same parameters, it suffices to show that adjacency is preserved. So, let z_1 and z_2 be two adjacent points of $\mathbf{Q}$ and put $x = L_{z_1}$ and $y = L_{z_2}$. From $p^{g_{z_1}} \sim z_1 \sim z_2 \sim p^{g_{z_2}}$, it follows that $g_{z_2} = \delta_{xy}^{-1} g_{z_1}$ or $g_{z_2}^{-1} = g_{z_1}^{-1}\delta_{xy}$. Hence $(g_{z_2}^{-1}, y) \sim (g_{z_1}^{-1}, x)$. $\square$

Application. Using the previous theorem, we calculate an AT corresponding to $Q(5,q)$ and to $(S^-_{xy})^D$.

(I) Q(5,q)

We use the dual representation. The points of $H(3,q^2)$ are 1-dimensional subspaces of $V(4,q^2)$. Consider a nonsingular Hermitian form $(\cdot,\cdot)$ in $V(4,q^2)$, i.e. $(\sum_i \lambda_i v_i, \sum_j \mu_j w_j) = \sum_i \sum_j \lambda_i \mu_j^q (v_i, w_j)$, and let ζ be the corresponding polarity of $\mathrm{PG}(3,q^2)$. Take a nontangent plane π and let $\pi^\zeta = \langle \bar{u} \rangle$. Take three arbitrary (but different) points $\langle \bar{a} \rangle, \langle \bar{b} \rangle, \langle \bar{c} \rangle$ of $O = \pi \cap H(3,q^2)$. The tangent plane at $\langle \bar{a} \rangle$ intersects π in a line $\langle \bar{a}, \bar{v} \rangle$. Let $L = \langle \bar{a}, \bar{u} + \lambda \bar{v} \rangle$ be an arbitrary line of $H(3,q^2)$ through $\langle \bar{a} \rangle$. Since $(\bar{u}+\lambda\bar{v}, \bar{u}+\lambda\bar{v}) = 0$, one finds that $\lambda^{q+1} = -\frac{(\bar{u},\bar{u})}{(\bar{v},\bar{v})}$. We determine the line L' of $H(3,q^2)$ through $\langle \bar{b} \rangle$ intersecting L. This line looks like $\langle \bar{b}, \bar{u}+\lambda\bar{v}+\beta\bar{a} \rangle$. An easy calculation yields $\beta = -\lambda\frac{(\bar{v},\bar{b})}{(\bar{a},\bar{b})}$. Hence $L' = \langle \bar{b}, \bar{u}+\lambda\bar{v}' \rangle$ with $\langle \bar{v}' \rangle \in \pi \cap \langle \bar{b} \rangle^\zeta$ independent of λ. Similarly, if we project L' to a line L'' through $\langle \bar{c} \rangle$ and finally L'' to a line L''' through $\langle \bar{a} \rangle$, we will find that $L''' = \langle \bar{a}, \bar{u} + \lambda \bar{v}''' \rangle$ with $\langle \bar{v}''' \rangle \in \pi \cap \langle \bar{a} \rangle^\zeta$ independent of λ. Now $\bar{v}''' = \gamma_1 \bar{a} + \gamma_2 \bar{v}$, hence $L''' = \langle \bar{a}, \bar{u} + \lambda\gamma_2\bar{v} \rangle$ where γ_2 is independent of λ. As before, one has that $(\lambda\gamma_2)^{q+1} = -\frac{(\bar{u},\bar{u})}{(\bar{v},\bar{v})}$ or $\gamma_2^{q+1} = 1$. We calculate now the corresponding admissible triple making use of the previous theorem. The ovoid O is just a unital in π and the Steiner system is an $S(2, q+1, q^3+1)$ (the lines are the intersections of O with nontangent lines). If the above points $\langle \bar{a} \rangle, \langle \bar{b} \rangle, \langle \bar{c} \rangle$ are on a common line, then $[\langle \bar{a} \rangle, \langle \bar{b} \rangle][\langle \bar{b} \rangle, \langle \bar{c} \rangle][\langle \bar{c} \rangle, \langle \bar{a} \rangle]$ is the identical permutation and hence $\gamma_2 = 1$. If the above points $\langle \bar{a} \rangle, \langle \bar{b} \rangle, \langle \bar{c} \rangle$ are not on a common line, then $\bar{v} = \lambda_1\bar{a} + \lambda_2\bar{b} + \lambda_3\bar{c}$ with $\lambda_2(\bar{b},\bar{a}) + \lambda_3(\bar{c},\bar{a}) = 0$. A calculation yields

$$\bar{v}''' = -\lambda_3\frac{(\bar{c},\bar{b})}{(\bar{a},\bar{b})}\bar{a} + \lambda_3\frac{(\bar{c},\bar{b})(\bar{a},\bar{c})}{(\bar{a},\bar{b})(\bar{b},\bar{c})}\bar{b} - \lambda_3\frac{(\bar{c},\bar{b})(\bar{a},\bar{c})(\bar{b},\bar{a})}{(\bar{a},\bar{b})(\bar{b},\bar{c})(\bar{c},\bar{a})}\bar{c},$$

and hence $\gamma_2 = -(\bar{a},\bar{b})^{q-1}(\bar{b},\bar{c})^{q-1}(\bar{c},\bar{a})^{q-1}$. When $\langle \bar{a} \rangle, \langle \bar{b} \rangle, \langle \bar{c} \rangle$ are on a common line, one calculates that $-(\bar{a},\bar{b})^{q-1}(\bar{b},\bar{c})^{q-1}(\bar{c},\bar{a})^{q-1} = 1 = \gamma_2$. By Theorem 2.4.3 ($x \mapsto x^{-1}$ is an automorphism of G_O), it follows that the triple defined in Section 2.5.4 is indeed admissible and that the corresponding GQ is isomorphic to $Q(5,q)$.

(II) $(S^-_{xy})^D$

Let $\mathrm{PG}(2,q)$, $q = 2^h$ with $h \geq 1$, be embedded as a hyperplane π_∞ in $\mathrm{PG}(3,q)$. Let O be a hyperoval in π_∞, let x, y be two points of O, and let $\mathbf{Q}$ be the generalized quadrangle S^-_{xy}. Consider now the following incidence structure $\mathbf{P_x}$:

(1) the points of $\mathbf{P_x}$ are the planes through x,

(2) the lines of $\mathbf{P_x}$ are the lines through x,

(3) incidence is the natural one.

The incidence structure $\mathbf{P_x}$ is then a projective plane of order q. Consider now the line xy as line at infinity $[\infty]$ to construct the associated affine plane $\mathbf{A_{xy}}$. The plane π_∞ corresponds with a point (∞) on $[\infty]$. The points of $\mathbf{A_{xy}}$ correspond with the planes through x not containing y, i.e. with the elements of O_{xy}. Let (X_0, X_1, X_2, X_3) denote coordinates in $\mathrm{PG}(3,q)$ and we may assume that $x = (1,0,0,0)$ and $y = (0,1,0,0)$. Every plane through x not containing y has equation $X_1 + a_1X_2 + a_2X_3$. Let this plane correspond with $(a_1, a_2) \in \mathrm{GF}(q) \times \mathrm{GF}(q)$. This correspondence yields an isomorphism between the affine planes $\mathbf{A_{xy}}$ and $\mathbf{A}$, the latter plane being the one constructed in a well-known way on the set of points $\mathrm{GF}(q) \times \mathrm{GF}(q)$. We may assume that $z = (0,0,1,0)$ is a third point of O. Let o_1 be a fixed point of O_{xy} containing z. We may assume that o_1 has equation $X_1 = 0$. The point o_1 of S^-_{xy} is incident with q lines of S^-_{xy}, namely the lines L_μ, $\mu \in \mathrm{GF}(q)$, where $L_\mu = \{(\mu, 0, \lambda, 1) | \lambda \in \mathrm{GF}(q)\}$, we have also that $O = \{(1,0,0,0), (0,1,0,0)\} \cup \{(f(\lambda), \lambda, 1, 0) | \lambda \in \mathrm{GF}(q)\}$, where $f : \mathrm{GF}(q) \to \mathrm{GF}(q)$ is a bijection such that $f(0) = 0$. Now, let $o_2 : X_1 + \alpha_1 X_2 + \beta_1 X_3 = 0$ and $o_3 : X_1 + \alpha_2 X_2 + \beta_2 X_3 = 0$ be two arbitrary points of O_{xy}. The plane o_2 intersects the hyperoval in two points; one of them is $(1,0,0,0)$, the other is $(f(\alpha_1), \alpha_1, 1, 0)$. The plane o_3 intersects the hyperoval in two points; one of them is (1,0,0,0), the other is $(f(\alpha_2), \alpha_2, 1, 0)$. We suppose that $o_1 \neq o_2 \neq o_3 \neq o_1$.

Case I: $0 \neq \alpha_1 \neq \alpha_2 \neq 0$

Consider the line L_u through o_1. Its intersection with the plane o_2 is equal to the point $(u, 0, \frac{\beta_1}{\alpha_1}, 1)$. The line L'_u through o_2 intersecting L_u is then equal to

$$L'_u = \{(u + \lambda f(\alpha_1), \lambda\alpha_1, \frac{\beta_1}{\alpha_1} + \lambda, 1) | \lambda \in \mathrm{GF}(q)\}.$$

The line L'_u intersects the plane o_3 in the point

$$(u + \frac{\alpha_1\beta_2 - \beta_1\alpha_2}{\alpha_1(\alpha_1 - \alpha_2)} f(\alpha_1), \frac{\alpha_1\beta_2 - \beta_1\alpha_2}{\alpha_1 - \alpha_2}, \frac{\beta_2 - \beta_1}{\alpha_1 - \alpha_2}, 1).$$

The line L''_u through o_3 intersecting L'_u is then equal to

$$L''_u = \{(u + \frac{\alpha_1\beta_2 - \beta_1\alpha_2}{\alpha_1(\alpha_1 - \alpha_2)} f(\alpha_1) + \lambda f(\alpha_2), \frac{\alpha_1\beta_2 - \beta_1\alpha_2}{\alpha_1 - \alpha_2} + \lambda\alpha_2, \frac{\beta_2 - \beta_1}{\alpha_1 - \alpha_2} + \lambda, 1) | \lambda \in \mathrm{GF}(q)\}.$$

The intersection of L''_u with $X_1 = 0$ yields $\lambda = \frac{\beta_1\alpha_2 - \alpha_1\beta_2}{\alpha_2(\alpha_1 - \alpha_2)}$. Hence L_u is mapped to $L_{u'}$, where

$$\begin{aligned} u' &= u + \frac{\alpha_1\beta_2 - \beta_1\alpha_2}{\alpha_1(\alpha_1 - \alpha_2)} f(\alpha_1) + \frac{\beta_1\alpha_2 - \alpha_1\beta_2}{\alpha_2(\alpha_1 - \alpha_2)} f(\alpha_2) \\ &= u + \frac{(\alpha_1\beta_2 - \alpha_2\beta_1)(\alpha_1 f(\alpha_2) - \alpha_2 f(\alpha_1))}{\alpha_1\alpha_2(\alpha_1 - \alpha_2)}. \end{aligned}$$

<u>Conclusion:</u> $\Delta[(\alpha_1, \beta_1), (\alpha_2, \beta_2)] = \frac{(\alpha_1\beta_2 - \alpha_2\beta_1)(\alpha_1 f(\alpha_2) - \alpha_2 f(\alpha_1))}{\alpha_1\alpha_2(\alpha_1 - \alpha_2)}$

Case II: $\alpha_1 \neq \alpha_2 = 0$

Let L''_u be as in the first case, i.e.

$$L''_u = \{(u + \frac{\beta_2 f(\alpha_1)}{\alpha_1}, \beta_2, \frac{\beta_2 - \beta_1}{\alpha_1} + \lambda, 1) | \lambda \in \mathrm{GF}(q)\}.$$

The plane through L''_u and y intersects $X_1 = 0$ in the line $L_{u'}$, where

$$u' = u + \frac{\beta_2 f(\alpha_1)}{\alpha_1}.$$

<u>Conclusion:</u> $\Delta[(\alpha_1, \beta_1), (\alpha_2, \beta_2)] = \frac{\beta_2 f(\alpha_1)}{\alpha_1}$

Case III: $0 = \alpha_1 \neq \alpha_2$

Since $\Delta(p_1, p_2) = \Delta(p_2, p_1)$, the solution follows from the second case.
<u>Conclusion:</u> $\Delta[(\alpha_1, \beta_1), (\alpha_2, \beta_2)] = \frac{\beta_1 f(\alpha_2)}{\alpha_2}$

Case IV: $\alpha_1 = \alpha_2 \neq 0$

Consider the line L'_u given in Case (I). The plane through L'_u and y has equation

$$\{(u + \lambda f(\alpha_1), \lambda\alpha_1 + \lambda', \frac{\beta_1}{\alpha_1} + \lambda, 1) | \lambda, \lambda' \in \mathrm{GF}(q)\}.$$

The intersection of this plane with o_3 yields the line

$$\{(u + \lambda f(\alpha_1), \lambda\alpha_1 + \beta_1 - \beta_2, \frac{\beta_1}{\alpha_1} + \lambda, 1) | \lambda \in \mathrm{GF}(q)\}.$$

Determining the intersection of this line with $X_1 = 0$, we find $\lambda = \frac{\beta_1 - \beta_2}{\alpha_1}$. Hence the line L_u is mapped to the line $L_{u'}$, where $u' = u + \frac{\beta_1-\beta_2}{\alpha_1} f(\alpha_1)$.
Conclusion: $\Delta[(\alpha_1, \beta_1), (\alpha_2, \beta_2)] = \frac{\beta_1-\beta_2}{\alpha_1} f(\alpha_1)$

Case V: $\alpha_1 = \alpha_2 = 0$

We have that

$$\Delta[(\alpha_1, \beta_1), (\alpha_2, \beta_2)] = 0.$$

We apply now Theorem 2.4.2. For every point (α, β) of the affine plane, we define $\delta_{(\alpha,\beta)} := \frac{\beta}{\alpha} f(\alpha)$ if $\alpha \neq 0$ and $\delta_{(\alpha,\beta)} := 0$ otherwise. The new admissible triple is the one which was described in Section 2.5.5.

Remark. Let S be a spread of a $\mathrm{GQ}(s, t)$. If $|G_S| > 1$, then the condition $(s + t) \mid st(s + 1)(t + 1)$ can be strengthened.

Lemma 2.7.4 ([5])
If f is the number of fixpoints of an automorphism θ and if g is the number of points x for which $x^\theta \neq x \sim x^\theta$, then $(t + 1)f + g \equiv 1 + st \pmod{s + t}$.

Corollary 2.7.5
If $|G_S| > 1$, then $(s + t) \mid s(s + 1)(t + 1)$.

Proof. Let θ be any nontrivial element of G_S and apply the previous lemma. $\square$

2.7.2 Characterizations in terms of a group of projectivities

Let $S = \{L_1, \ldots, L_{1+st}\}$ be a spread of a generalized quadrangle $\mathbf{Q}$ of order (s, t). Every two lines L_i and L_j of S define a projectivity $\Delta_{ij} = [L_j, L_1][L_i, L_j][L_1, L_i]$ $(= [L_j, L_1] \circ [L_i, L_j] \circ [L_1, L_i])$ of L_1. The set $A = \{\Delta_{ij} | 1 \leq i, j \leq 1 + st\}$ generates $\Pi_S(L_1)$, the group of projectivities of L_1 with respect to S.

Lemma 2.7.6
If $\mathbf{Q} \simeq W(q)$, then each Δ_{ij} is trivial or has order 2.

Proof. De generalized quadrangle $\mathbf{Q}$ is derived from a symplectic polarity ζ in $\mathrm{PG}(3,q)$. Let $L_1 = \langle \bar{a}, \bar{b} \rangle, L_i = \langle \bar{c}, \bar{d} \rangle$ and $L_j = \langle \bar{e}, \bar{f} \rangle$ where $1 < i, j \leq 1 + q^2, i \neq j$. Since L_1 and L_i are skew, $\bar{e} = \alpha\bar{a} + \beta\bar{b} + \gamma\bar{c} + \delta\bar{d}$ and $\bar{f} = \alpha'\bar{a} + \beta'\bar{b} + \gamma'\bar{c} + \delta'\bar{d}$ for certain $\alpha, \ldots, \delta' \in \mathrm{GF}(q)$. We may also suppose that $\langle \bar{c} \rangle \in \langle \bar{a} \rangle^\zeta$. The plane $\langle \bar{c} \rangle^\zeta = \langle \bar{a}, \bar{c}, \bar{d} \rangle$ intersects L_j in the point $\langle \bar{g} \rangle = \langle (\beta'\alpha - \beta\alpha')\bar{a} + (\beta'\gamma - \beta\gamma')\bar{c} + (\beta'\delta - \beta\delta')\bar{d} \rangle$ and the plane $\langle \bar{g} \rangle^\zeta = \langle \bar{c}, \bar{e}, \bar{f} \rangle$ intersects L_1 in $\langle (\delta'\alpha - \delta\alpha')\bar{a} + (\delta'\beta - \delta\beta')\bar{b} \rangle$, hence $\Delta_{ij}(\bar{a}) = \langle (\delta'\alpha - \delta\alpha')\bar{a} + (\delta'\beta - \delta\beta')\bar{b} \rangle$. The plane $\langle \bar{a} \rangle^\zeta = \langle \bar{a}, \bar{b}, \bar{c} \rangle$ intersects L_j in the point $\langle \bar{h} \rangle = \langle (\delta'\alpha - \delta\alpha')\bar{a} + (\delta'\beta - \delta\beta')\bar{b} + (\delta'\gamma - \delta\gamma')\bar{c} \rangle$, the plane $\langle h \rangle^\zeta = \langle \bar{a}, \bar{e}, \bar{f} \rangle$ intersects L_i in $\bar{k} = \langle (\beta'\gamma - \beta\gamma')\bar{c} + (\beta'\delta - \beta\delta')\bar{d} \rangle$ and the plane $\langle \bar{k} \rangle^\zeta = \langle \bar{c}, \bar{d}, \bar{h} \rangle^\zeta$ intersects L_1 in $\langle (\delta'\alpha - \delta\alpha')\bar{a} + (\delta'\beta - \delta\beta')\bar{b} \rangle$. Hence $\Delta_{ij}(\bar{a}) = \Delta_{ji}(\bar{a})$ and $\Delta_{ij} = \Delta_{ji} = \Delta_{ij}^{-1}$. □

If $\mathbf{Q}$ is a grid, then $\Pi_S(L_1)$ is the trivial group; if $\mathbf{Q}$ is not a grid, then the following theorem says something about the number of elements of $\Pi_S(L_1)$.

Theorem 2.7.7
Suppose that $\mathbf{Q}$ is not a grid. If x, y are points of L_1, then there exists an element $a \in A$, such that $x^a = y$. Hence $|A|, |\Pi_S(L_1)| \geq s + 1$.

Proof. Let $L \neq L_1$ be a fixed line of $\mathbf{Q}$ through y, let L_i be any line of S disjoint with L, let x' be the projection of x on L_i, let x'' be the projection of x' on L and let L_j be the line of S through x''. Now $\Delta_{ij} \in A$ maps x to y. □

Theorem 2.7.8
Let $(\mathbf{D}, K, \Delta)$ be an AT with $\mathbf{Q}$ as corresponding GQ *and S as associated spread. Let L_1 be any line of S. If $\mathbf{Q}$ is not a grid, then $\Pi_S(L_1)$ is isomorphic to K and $|\Pi_S(L_1)| = s + 1$.*

Proof. We have $S = \{L_x | x \in \mathcal{P}\}$ where $L_x = \{(k, x) | k \in K\}$. Let $p \in \mathcal{P}$ be fixed and put $L_1 := L_p$. We can identify the point set of L_1 with the elements of K (consider the bijection $(k, p) \mapsto k$). With this identification the elements Δ_{ij} are right multiplications in the group K. For, let $L_i = L_x$ and $L_j = L_y$. Then $(k, p) \sim (k\Delta(p, x), x) \sim (k\Delta(p, x)\Delta(x, y), y) \sim (k\Delta(p, x)\Delta(x, y)\Delta(y, p), p)$, hence Δ_{ij} corresponds to a (right) multiplication with $\Delta(p, x)\Delta(x, y)\Delta(y, p) \in K$. Since $|A| \geq s + 1$, we have $\Pi_S(L_1) \simeq K$. □

We will prove the converse theorem: if $|\Pi_S(L_1)| = s + 1$, then $\mathbf{Q}$ has an AT-model with S as associated spread.

Lemma 2.7.9
Let X be a set of order $n \geq 1$. If G is a regular group of permutations on X, then there are precisely n permutations of X which commute with every

element of G. These n permutations form a regular group $\tilde{G}$ of permutations on X. Moreover $\tilde{G} \simeq G$.

Proof. Let $x \in X$ be fixed. If h commutes with every element of G, then $hg(x) = gh(x)$ for all $g \in G$, hence the image of x under h determines h completely. Now, for every element $y \in X$, one can define the permutation h_y as follows:

$$h_y g(x) = g(y), \forall g \in G.$$

It is straightforward to check that h_y commutes with every element of G and that h_y is the trivial permutation if it has at least one fixpoint. Put $\tilde{G} = \{h_y | y \in X\}$. For every element g of G, we associate an element g^ϕ of $\tilde{G}$ such that $g^{-1}(x) = g^\phi(x)$. Then

$$(gh)^\phi(x) = (gh)^{-1}(x) = h^{-1}g^{-1}(x) = h^{-1}g^\phi(x) = g^\phi h^{-1}(x) = g^\phi h^\phi(x).$$

Now, let $k \in G$ be arbitrary, then

$$(gh)^\phi k(x) = k(gh)^\phi(x) = kg^\phi h^\phi(x) = g^\phi h^\phi k(x);$$

hence $(gh)^\phi = g^\phi h^\phi$ for all $g, h \in G$ and ϕ defines an isomorphism between G and $\tilde{G}$. □

Theorem 2.7.10
If $\theta \in G_S$ then θ induces a permutation $\bar{\theta}$ on the point set of L_1 that commutes with each element of $\Pi_S(L_1)$. Conversely, if a permutation ϕ on the point set of L_1 commutes with each element of $\Pi_S(L_1)$, then $\phi = \bar{\theta}$ for some $\theta \in G_S$.

Proof. First, we notice that $\phi = \bar{\theta}$ determines θ completely. For, let $x \in L_i$ be an arbitrary point of $\mathbf{Q}$. From $\mathrm{d}(x, [L_i, L_1](x)) \leq 1$, it follows that $\mathrm{d}(\theta(x), \phi[L_i, L_1](x)) \leq 1$ and hence

$$\theta(x) = [L_1, L_i]\phi[L_i, L_1](x). \tag{2.2}$$

Next, let ϕ be a permutation on the point set of L_1 and let us determine under which conditions the map θ defined by (2.2) is an automorphism. If $x \in L_i$ and $y \in L_j$ (with $i \neq j$) are two collinear points, then $y = [L_i, L_j](x)$ and

$$\begin{aligned} \theta(x) \sim \theta(y) \quad &\Leftrightarrow \quad [L_1, L_i]\phi[L_i, L_1](x) \sim [L_1, L_j]\phi[L_j, L_1](y) \\ &\Leftrightarrow \quad [L_i, L_j][L_1, L_i]\phi[L_i, L_1](x) = [L_1, L_j]\phi[L_j, L_1][L_i, L_j](x). \end{aligned}$$

This holds for every $x \in L_i$ if and only if

$$\begin{aligned} [L_j, L_1][L_i, L_j][L_1, L_i]\phi &= \phi[L_j, L_1][L_i, L_j][L_1, L_i] \\ \Delta_{ij}\phi &= \phi\Delta_{ij}. \end{aligned}$$

This holds for all i, j if and only if ϕ commutes with each element of $\Pi_S(L_1)$. $\square$

Theorem 2.7.11
If $|\Pi_S(L_1)| = s+1$ then also $|G_S| = s+1$. Conversely, if $|G_S| = s+1$ and if $\mathbf{Q}$ is not a grid, then $|\Pi_S(L_1)| = s+1$.

Proof. If $|\Pi_S(L_1)| = s+1$, then $\mathbf{Q}$ is not a grid and Theorem 2.7.7 implies that $\Pi_S(L_1)$ is a regular group of permutations of L_1. Lemma 2.7.9 implies then that there are exactly $s+1$ permutations of L_1 which commute with each element of $\Pi_S(L_1)$ and by Theorem 2.7.10 each of these permutations can be extended to an element of G_S. Hence $|G_S| = s+1$. Conversely, suppose that $\mathbf{Q}$ is not a grid and that $|G_S| = s+1$; then Theorem 2.7.1 implies that G_S induces a regular group of permutations on the point set of L_1. Lemma 2.7.9 and Theorem 2.7.10 imply then that $|\Pi_S(L_1)| \leq s+1$ and the result follows by Theorem 2.7.7. $\square$

We prove now the following theorem.

Theorem 2.7.12
If $\Pi_S(L_1)$ is commutative then S is a spread of symmetry and $\mathbf{Q}$ is derivable from an admissible triple.

Proof. We may suppose that $\mathbf{Q}$ is not a grid. Theorems 2.7.7 and 2.7.10 imply that $|G_S| \geq |\Pi_S(L_1)| \geq s+1$, hence $|G_S| = s+1$ by Theorem 2.7.1. So S is a spread of symmetry. $\square$

2.8 Automorphisms of a GQ fixing a spread

With each normal spread S of a GQ(s,t), there is associated a linear space $\mathcal{L}(S)$: the points of $\mathcal{L}(S)$ are the $st+1$ lines of S; the lines of $\mathcal{L}(S)$ are the sets $\{L, M\}^{\perp\perp}$ with L and M two different lines of S; incidence is the natural one. If θ is an automorphism of the GQ fixing S, then θ induces an automorphism of $\mathcal{L}(S)$. If S is the associated spread of an AT $(\mathbf{D}, K, \Delta)$, then $\mathcal{L}(S) \simeq \mathbf{D}$. Hence every automorphism of the GQ fixing S induces an automorphism of $\mathbf{D}$.

Let $\mathbf{Q} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a GQ(s,t) and let $S = \{L_1, \ldots, L_{1+st}\}$ be a spread of $\mathbf{Q}$ (Notice that we number the lines of S). We call L_1 the *base line* of the spread. The set of all automorphisms of $\mathbf{Q}$ fixing S is denoted by $\mathcal{A}_S$ and it is a subgroup of $\mathcal{A} = \mathrm{Aut}(\mathbf{Q})$. The map $\Delta_{ij} := [L_j, L_1][L_i, L_j][L_1, L_i]$ $(i, j \in \{1, \ldots, 1+st\})$ defines a permutation on the point set of L_1; if β is a permutation of $\{1, \ldots, 1+st\}$, then we define $\Delta^{\beta}_{ij} := \Delta_{\beta(i)\beta(j)}$. If $\theta \in \mathcal{A}_S$, then the following permutations can be defined.

(a) A permutation β of $\{1, \ldots, 1+st\}$, such that

$$\theta(L_i) = L_{\beta(i)}, \tag{2.3}$$

for all $i \in \{1, \ldots, 1+st\}$.

(b) Permutations α_i of L_1 such that

$$\theta[L_1, L_i](x) = [L_1, L_{\beta(i)}]\alpha_i(x), \tag{2.4}$$

for all $i \in \{1, \ldots, 1+st\}$ and all $x \in L_1$.

Let $[L_1, L_i](x) \sim [L_1, L_j](x')$ with $x, x' \in L_1$ and $i \neq j$. Hence $x' = \Delta_{ij}(x)$ and from $[L_1, L_{\beta(i)}]\alpha_i(x) \sim [L_1, L_{\beta(j)}]\alpha_j(x') = [L_1, L_{\beta(j)}]\alpha_j \Delta_{ij}(x)$, it follows that $\alpha_i = \Delta^{\beta}_{ji}\alpha_j\Delta_{ij}$. Define

$$\alpha := \alpha_1, \tag{2.5}$$

we get then that

$$\alpha_i = \Delta^{\beta}_{1i}\alpha, \tag{2.6}$$

and the condition $\alpha_i = \Delta^{\beta}_{ji}\alpha_j\Delta_{ij}$ becomes

$$(\Delta^{\beta}_{j1}\Delta^{\beta}_{ij}\Delta^{\beta}_{1i})\alpha = \alpha\Delta_{ij}, \tag{2.7}$$

for all $i, j \in \{1, \ldots, 1+st\}$. So, we have proved the following theorem.

Theorem 2.8.1
If $\theta \in \mathcal{A}_S$, then we can define a permutation α of L_1 and a permutation β of $\{1, \ldots, 1+st\}$ by (2.3), (2.4), (2.5) and these permutations satisfy equation (2.7). Conversely, if (2.7) is satisfied by a permutation α of L_1 and a permutation β of $\{1, \ldots, 1+st\}$, then (2.6) and (2.4) define a permutation $\theta \in \mathcal{A}_S$.

Let $\widehat{G}$ be the set of all permutations α of L_1 for which there exists a permutation β of $\{1, \ldots, 1+st\}$ such that (2.7) is satisfied for all $i, j \in \{1, \ldots, 1+st\}$.

Theorem 2.8.2
If G is commutative, then $\widehat{G}$ is a group.

Proof. Since $\widehat{G}$ is finite, it suffices to prove that $\mu_1\mu_2 \in \widehat{G}$ for every two elements $\mu_1, \mu_2 \in \widehat{G}$. Let β_1 and β_2 be the permutations such that

$$\Delta_{j1}^{\beta_1}\Delta_{ij}^{\beta_1}\Delta_{1i}^{\beta_1} = \mu_1\Delta_{ij}\mu_1^{-1},$$

$$\Delta_{j1}^{\beta_2}\Delta_{ij}^{\beta_2}\Delta_{1i}^{\beta_2} = \mu_2\Delta_{ij}\mu_2^{-1},$$

for all $i, j \in \{1, \ldots, 1+st\}$. One calculates then that

$$\mu_1\mu_2\Delta_{ij}\mu_2^{-1}\mu_1^{-1} = \Delta_{\beta_2(1)1}^{\beta_1}\Delta_{j1}^{\beta_1\beta_2}\Delta_{ij}^{\beta_1\beta_2}\Delta_{1i}^{\beta_1\beta_2}\Delta_{1\beta_2(1)}^{\beta_1},$$

and this is equal to $\Delta_{j1}^{\beta_1\beta_2}\Delta_{ij}^{\beta_1\beta_2}\Delta_{1i}^{\beta_1\beta_2}$ if G is commutative. $\square$

Example 1: the GQ $P(W(q), u)$

We use the AT-model for $P(W(q), u)$, $q = p^h$, given in Section 2.5.2. The associated spread consists of the lines $L_{(x,y)} = \{(z, x, y)|z \in \mathrm{GF}(q)\}$, $x, y \in \mathrm{GF}(q)$. Consider $L_1 = L_{(0,0)}$ as base line. We can identify the point set of L_1 with the elements of $\mathrm{GF}(q)$ and hence with the points of $\mathrm{AG}(1, q)$. The permutations Δ_{ij} of L_1 are then translations of $\mathrm{AG}(1, q)$; indeed if $L_i = L_{(x_1,y_1)}$ and $L_j = L_{(x_2,y_2)}$, then one easily checks that Δ_{ij} is a translation $T(x_1y_2 - x_2y_1)$ over $x_1y_2 - x_2y_1$. Let $\theta \in \mathcal{A}_S$, then θ induces an automorphism of $AG(2, q)$ $(= \mathbf{D})$. Such an automorphism looks like

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x^\sigma \\ y^\sigma \end{bmatrix} + \begin{bmatrix} b \\ c \end{bmatrix}, \tag{2.8}$$

where the map $x \mapsto x^\sigma$ is an automorphism of $\mathrm{GF}(q)$ and $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a nonsingular (2×2)-matrix over $\mathrm{GF}(q)$. One easily checks that $\Delta_{j1}^{\beta}\Delta_{ij}^{\beta}\Delta_{1i}^{\beta} = T(\det(A)k^\sigma)$ if $\Delta_{ij} = T(k)$. From equation (2.7), it follows then that $\alpha(k) = \det(A)k^\sigma + d$, where $d = \alpha(0)$. Hence

$$\alpha_{(x,y)}(k) = (ba_{21} - ca_{11})x^\sigma + (ba_{22} - ca_{12})y^\sigma + (a_{11}a_{22} - a_{12}a_{21})k^\sigma + d,$$

where $\alpha_{(x,y)} := \alpha_i$ with i such that $L_i = L_{(x,y)}$. The automorphisms of $\mathcal{A}_S$ are hence the following maps:

$$\begin{bmatrix} z \\ x \\ y \end{bmatrix}^\theta = \begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & ba_{21} - ca_{11} & ba_{22} - ca_{12} \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} z^\sigma \\ x^\sigma \\ y^\sigma \end{bmatrix} + \begin{bmatrix} d \\ b \\ c \end{bmatrix},$$

where σ is an arbitrary automorphism of $\mathrm{GF}(q)$ and $a_{11}, a_{12}, a_{21}, a_{22}, b, c, d$ are arbitrary elements of $\mathrm{GF}(q)$ satisfying $a_{11}a_{22} - a_{12}a_{21} \neq 0$. The knowledge of the orbit of S under $\mathcal{A}$ gives now a good idea of how close $\mathcal{A}$ is to $\mathcal{A}_S$. The spread S is a normal spread and need to be mapped on a normal spread. If $q \neq 3$ is odd, then S is the only normal spread; hence $\mathcal{A} = \mathcal{A}_S$. If $q = 3$, then $AS(3) \simeq Q(5,2)$ and $\mathcal{A}$ acts transitively on the set of all normal spreads. Suppose now that q is even and consider the model $T_2^*(O)$. If $q \neq 2$, then the elements of $\mathcal{A}$ are induced by automorphisms of $\mathrm{PG}(3,q)$ which fix the hyperoval. If $q = 2$ or $q = 4$, then $\mathcal{A}$ acts transitively on the set of all normal spreads. If $q \geq 8$, then S corresponds with the unique point a of O such that $O \setminus \{a\}$ is a conic; hence $\mathcal{A} = \mathcal{A}_S$.

Example 2: the GQ $Q(5,q)$

We use the AT-model for $Q(5,q)$, $q = p^h$, given in Section 2.5.4 and suppose that $(\bar{a},\bar{b}) = \bar{a}^T\bar{b}^q$ is the nonsingular Hermitian form in $V(3,q^2)$. The associated spread consists of the lines $L_r = \{(k,r) | k \in K\}$, $r \in U$. Consider $L_1 = L_{\langle\bar{a}\rangle}$ as base line. We can identify the points of L_1 with the elements of K. The permutations Δ_{ij} are then multiplications: if $L_i = L_r$ and $L_j = L_s$, then one easily verifies that Δ_{ij} is a multiplication $M(\Delta(r,s))$ with $\Delta(r,s) \in K$. Let $\theta \in \mathcal{A}_S$, then θ induces an automorphism of the design related to U. This automorphism is an automorphism of $PG(2,q^2)$ fixing U, and hence is an element of $P\Gamma U(3,q^2)$ (see [58]). Such an automorphism looks like $\bar{a} \mapsto A\bar{a}^\sigma$, where σ is an automorphism of $\mathrm{GF}(q^2)$ and where A satisfies $A^TA^q = \alpha I$ with $\alpha^{q-1} = 1$. One calculates that $\Delta_{j1}^\beta\Delta_{ij}^\beta\Delta_{1i}^\beta = M(k^\sigma)$ if $\Delta_{ij} = M(k)$. From equation (2.7), it follows then that $\alpha(k) = lk^\sigma$ where $l = \alpha(1)$. Hence $\mathcal{A}_S$ contains $|K| \times |P\Gamma U(3,q^2)| = 2hq^3(q+1)(q^2-1)(q^3+1)$ elements. Since $\mathcal{A}$ acts transitively on the set of $q^3(q-1)(q^2+1)$ normal spreads, $\mathcal{A}$ contains $2hq^6(q^2-1)(q^3+1)(q^4-1)$ elements. The group $P\Gamma U(4,q^2)$ has indeed this order.

Example 3: the GQ $T_2^*(O)$

Let O be a hyperoval of $\mathrm{PG}(2,q)$, $q = 2^h$, and let S be the spread of $T_2^*(O)$ determined by $a \in O$. Let $\theta \in \mathcal{A}_S$, then θ induces an automorphism of the affine plane $\mathbf{D} = \mathrm{AG}(2,q)$ in the related AT-model (see Section 2.5.3). As before, let L_1 denote the base line of S; the points of L_1 can be identified with the elements of $\mathrm{GF}(q)$. For $1 \leq i,j \leq q^2$, one easily checks that

$$\Delta_{ij} = T(\Delta(L_1,L_i) + \Delta(L_i,L_j) + \Delta(L_j,L_1)), \tag{2.9}$$

$$\Delta_{j1}^\beta\Delta_{ij}^\beta\Delta_{1i}^\beta = T(\Delta(L_{\beta(1)},L_{\beta(i)}) + \Delta(L_{\beta(i)},L_{\beta(j)}) + \Delta(L_{\beta(j)},L_{\beta(1)})). \tag{2.10}$$

By equation (2.7), $\Delta_{j1}^{\beta}\Delta_{ij}^{\beta}\Delta_{1i}^{\beta}$ is only dependent of Δ_{ij}. Now, let L_{i_1} and L_{j_1} be such that L_1, L_{i_1}, L_{j_1} are three noncollinear points of $\mathrm{AG}(2,q)$. Let (i_k, j_k), $1 \leq k \leq q-1$, be the $q-1$ pairs such that

(A) $L_{i_k} \neq L_1 \neq L_{j_k}$,

(B) $L_1L_{i_k} = L_1L_{i_1}$ and $L_1L_{j_k} = L_1L_{j_1}$,

(C) $L_{i_k}L_{j_k} \| L_{i_1}L_{j_1}$.

If we put the $(q-1)$ pairs (i_k, j_k) in equations (2.9) and (2.10), then one finds that $\Delta_{j1}^{\beta}\Delta_{ij}^{\beta}\Delta_{1i}^{\beta} = T(lk^{\sigma})$ if $\Delta_{ij} = T(k)$. Here l is a fixed nonzero element of $\mathrm{GF}(q)$ and σ is the automorphism of $\mathrm{GF}(q)$ related (see equation (2.8)) to the above mentioned automorphism of $\mathrm{AG}(2,q)$. From equation (2.7), it follows then that $\alpha(k) = lk^{\sigma} + d$ with $d = \alpha(0)$. Hence $|\widehat{G}| \leq h(q-1)q$. Now, if β corresponds with a homothetic transformation of $\mathrm{AG}(2,q)$ with factor $l \neq 0$ and center L_1, then equation (2.7) has q solutions for α, namely the maps $\alpha(k) = lk + d$ with $d \in GF(q)$. Hence $|\widehat{G}| \geq (q-1)q$. Since $\widehat{G}$ is a group, $|\widehat{G}| = h'(q-1)q$ with $h' \mid h$. The exact value for $|\widehat{G}|$ depends on the kind of hyperoval we are working with.

2.9 The nonexistence of spreads of symmetry in certain GQ's of order (s, s^2)

2.9.1 Generalized quadrangles as group coset geometries.

Let G be a group of order s^2t, $s > 1$, $t > 1$, and let $\mathcal{F}$ be a family of $t+1$ subgroups of G, each of order s. For each $A \in \mathcal{F}$, let A^* be a subgroup of G having st elements and containing A, and put $\mathcal{F}^* := \{A^* | A \in \mathcal{F}\}$. The following incidence structure **S** can then be defined. Points are of three types:

(i) elements of G;

(ii) right cosets of elements of $\mathcal{F}^*$;

(iii) a symbol (∞).

Lines are of two types:

(a) right cosets of elements of $\mathcal{F}$;

(b) symbols $[A]$, where $A \in \mathcal{F}$.

A point g of type (i) is incident with each line Ag of type (a). A point A^*g of type (ii) is incident with $[A]$ and with each line Ahg contained in it. The point (∞) is incident with all lines of type (b). There are no further incidences. The just defined geometry, denoted by $\mathbf{S}(G,\mathcal{F},\mathcal{F}^*)$, is introduced by Kantor ([49]), and called a *group coset geometry*. The following theorem was proved in [49].

Theorem 2.9.1
A group coset geometry $\mathbf{S}(G,\mathcal{F},\mathcal{F}^*)$ is a generalized quadrangle of order (s,t) if and only if the following two conditions are satisfied:

$(K1)$ $AB \cap C = 1$ *whenever A, B, C are distinct elements of $\mathcal{F}$;*

$(K2)$ $A^* \cap B = 1$ *whenever A and B are distinct members of $\mathcal{F}$.*

If the above two conditions are satisfied, then for each $A \in \mathcal{F}$, we have that

$$A^* = \bigcup\{Ag | Ag = A \text{ or } Ag \cap B = \emptyset \text{ for all } B \in \mathcal{F}\},$$

and we can write $\mathbf{S}(G,\mathcal{F}) := \mathbf{S}(G,\mathcal{F},\mathcal{F}^*)$.

2.9.2 Generalized quadrangles arising from q-clans

The following definitions and results were taken from [50], [64] and [66]. Let q be a fixed prime power.

Definitions.

(1) A (2×2)-matrix A over $\mathrm{GF}(q)$ is called *anisotropic matrix* provided $[\alpha_1\,\alpha_2]A[\alpha_1\,\alpha_2]^T = 0$ if and only if $\alpha_1 = \alpha_2 = 0$.

(2) For each $t \in \mathrm{GF}(q)$, let $A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix}$ be an upper triangular (2×2)-matrix. The set $\mathcal{C} = \{A_t | t \in \mathrm{GF}(q)\}$ is called a *q-clan* provided all pairwise differences $A_s - A_t$ $(s,t \in \mathrm{GF}(q), s \neq t)$ are anisotropic.

Suppose now that $\mathcal{C} = \{A_t | t \in \mathrm{GF}(q)\}$ is a q-clan, where $A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix}$. Put $K_t = A_t + A_t^T = \begin{pmatrix} 2x_t & y_t \\ y_t & 2z_t \end{pmatrix}$. Let G denote the group consisting of the set $G = \{(\alpha_1,\alpha_2,c,\beta_1,\beta_2) | \alpha_1,\alpha_2,c,\beta_1,\beta_2 \in \mathrm{GF}(q)\}$, together with the binary operation

$$(\alpha_1,\alpha_2,c,\beta_1,\beta_2)\cdot(\alpha_1',\alpha_2',c',\beta_1',\beta_2') =$$

$$(\alpha_1+\alpha_1', \alpha_2+\alpha_2', c_1+c_1'+\beta_1\alpha_1'+\beta_2\alpha_2', \beta_1+\beta_1', \beta_2+\beta_2').$$

We define now $q+1$ subgroups of G having order q^2:

$$\begin{array}{rcl} A(\infty) & = & \{(0,0,0,\beta_1,\beta_2)|\beta_1,\beta_2 \in \mathrm{GF}(q)\}; \\ A(t) & = & \{(\alpha_1,\alpha_2,\alpha_1^2x_t+\alpha_1\alpha_2y_t+\alpha_2^2z_t, 2\alpha_1x_t+\alpha_2y_t, \alpha_1y_t+2\alpha_2z_t)| \\ & & \qquad\qquad\qquad\qquad\qquad\qquad \alpha_1,\alpha_2 \in \mathrm{GF}(q)\}, \end{array}$$

for all $t \in \mathrm{GF}(q)$, and we put $\mathcal{F} = \{A(i)|i \in \mathrm{GF}(q) \cup \{\infty\}\}$. We define now $q+1$ subgroups of G having order q^3:

$$\begin{array}{rcl} A^*(\infty) & = & \{(0,0,c,\beta_1,\beta_2)|c,\beta_1,\beta_2 \in \mathrm{GF}(q)\}, \\ A^*(t) & = & \{(\alpha_1,\alpha_2,c,2\alpha_1x_t+\alpha_2y_t,\alpha_1y_t+2\alpha_2z_t)|c,\alpha_1,\alpha_2 \in \mathrm{GF}(q)\}, \end{array}$$

for all $t \in \mathrm{GF}(q)$, and we put $\mathcal{F}^* = \{A^*(i)|i \in \mathrm{GF}(q) \cup \{\infty\}\}$. With the above definitions, conditions (K1) and (K2) of Theorem 2.9.1 are satisfied. $\mathbf{S}(G,\mathcal{F})$ is hence a GQ of order (q^2,q). This GQ has the following properties:

(1) the point (∞) is a regular point;

(2) if $\mathbf{S}(G,\mathcal{F})$ is not isomorphic to $H(3,q^2)$, then

  (i) the point (∞) is fixed by each automorphism of $\mathbf{S}(G,\mathcal{F})$, see [72],

  (ii) every ovoid of symmetry O contains the point (∞) (For, let θ be a nontrivial automorphism of $\mathbf{S}(G,\mathcal{F})$ which fixes each point of O, then the only fixpoints of θ are the elements of O.),

  (iii) if $\mathbf{S}(G,\mathcal{F})$ has an ovoid of symmetry then the automorphism group of $\mathbf{S}(G,\mathcal{F})$ is transitive on the set of lines through (∞).

We refer to [89] for the known examples of q-clans. We only give four examples here. For each of these examples, $\mathbf{Q}$ denotes the corresponding GQ.

(1) $q \equiv \pm 2 \pmod 5$, $A_t = \begin{pmatrix} t & 5t^3 \\ 0 & 5t^5 \end{pmatrix}$, $t \in \mathrm{GF}(q)$. It was proved in [69] that $\mathrm{Aut}(\mathbf{Q})$ has two orbits on the lines through (∞). Hence $\mathbf{Q}$ has no ovoid of symmetry.

(2) $q = 5^e > 5$, k is a nonsquare in $\mathrm{GF}(q)$ and $A_t = \begin{pmatrix} t & t^2 \\ 0 & k^{-1}t(1+kt^2)^2 \end{pmatrix}$. It was proved in [69] that $\mathrm{Aut}(\mathbf{Q})$ fixes the line $[A(\infty)]$. Hence $\mathbf{Q}$ has no ovoid of symmetry.

(3) $q = 3^e, e \geq 3$, m is a nonsquare in GF(q) and $A_t = \begin{pmatrix} t & t^3 \\ 0 & -\frac{t^9}{m} - mt \end{pmatrix}$.
It was proved in [69] that Aut$(\mathbf{Q})$ fixes the line $[A(\infty)]$. Hence $\mathbf{Q}$ has no ovoid of symmetry.

(4) Let q be odd, $\sigma \in$ Aut[GF(q)], m be a nonsquare in GF(q) and $A_t = \begin{pmatrix} t & 0 \\ 0 & -mt^\sigma \end{pmatrix}$. The quadrangles so-obtained are called *Kantor-Knuth quadrangles*. The generalized quadrangle $\mathbf{Q}$ is classical (i.e. isomorphic to $H(3, q^2)$) if and only if σ is the identity. We will prove in Section 2.9.4 that every nonclassical Kantor-Knuth quadrangle cannot have an ovoid of symmetry.

2.9.3 Ovoids of symmetry in GQ's derived from q-clans

In this section, we suppose that $\mathbf{Q}$ is a GQ(q^2, q) derived from a q-clan $\mathcal{C} = \{A_t | t \in \text{GF}(q)\}$, where $A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix}$. Since we already determined all ovoids of symmetry of $H(3, q^2)$, we suppose that $\mathbf{Q}$ is not isomorphic to $H(3, q^2)$. Let O be an ovoid of symmetry of $\mathbf{Q}$, then we know that $(\infty) \in O$.

Definitions.

(1) A *net* of *order* k and *degree* r is a partial linear space satisfying

  (i) each point is incident with r lines;

  (ii) each line is incident with k points;

  (iii) if x is a point and L is a line not incident with x, then there is a unique line M incident with x and not concurrent with L.

(2) Let x_1 and x_2 be two points of $\mathbf{Q}$ at distance two from (∞). If $|\{(\infty), x_1\}^\perp \cap \{(\infty), x_2\}^\perp| \geq 2$, then $x_2 \in \{x_1, (\infty)\}^{\perp\perp}$ and $\{(\infty), x_1\}^\perp = \{(\infty), x_2\}^\perp$ (since (∞) is regular). Hence we can define the following dual net $N^*_{(\infty)}$, see [71]:

  - points are the elements of $(\infty)^\perp \setminus \{(\infty)\}$;
  - lines are the sets $\{x, (\infty)\}^\perp$ with $\mathrm{d}(x, (\infty)) = 2$;
  - incidence is the natural one, i.e. containment.

  $N^*_{(\infty)}$ has $q^3 + q^2$ points, each line has $q + 1$ points and each point is incident with q^2 lines. A spread of $N^*_{(\infty)}$ is a set of lines partitioning the point set of $N^*_{(\infty)}$.

Theorem 2.9.2
Let $S = \{L_1, \dots, L_{q^2}\}$ be a spread of $N^*_{(\infty)}$, then $O_S = \{(\infty)\} \cup \{x | \, d(x,(\infty)) = 2$ and $\{x,(\infty)\}^\perp \in S\}$ is an ovoid of $\mathbf{Q}$.

Proof. Let x_1 and x_2 be two collinear points of O_S, then $\{x_1,(\infty)\}^\perp$ and $\{x_2,(\infty)\}^\perp$ have at least one point in common. Hence $\{x_1,(\infty)\}^\perp = \{x_2,(\infty)\}^\perp$ and $x_2 \in \{x_1,(\infty)\}^{\perp\perp}$. This implies that x_1 and x_2 are not collinear, a contradiction. O_S is a set of q^3+1 points, no three of which are collinear. Hence O_S is an ovoid of $\mathbf{Q}$. □

Definition. A spread S of $N^*_{(\infty)}$ is called *regular* if for every line M_1 of $N^*_{(\infty)}$ not belonging to S (hence there exist $q+1$ lines $L_1, \dots, L_{q+1}$ of S intersecting M_1), there are q other lines $M_2, \dots, M_{q+1}$ such that:

- M_i is disjoint with M_j for all $i, j \in \{1, \dots, q+1\}$ with $i \neq j$,
- M_i meets L_j in a point for all $i, j \in \{1, \dots, q+1\}$.

Theorem 2.9.3
There exists a regular spread S of $N^*_{(\infty)}$ such that $O = O_S$.

Proof. We prove that $\{o_1,(\infty)\}^\perp$ and $\{o_2,(\infty)\}^\perp$ are equal or disjoint for every two points o_1, o_2 of $O \setminus \{(\infty)\}$. Suppose $x \in \{o_1,(\infty)\}^\perp \cap \{o_2,(\infty)\}^\perp$ and let θ be a nontrivial automorphism of $\mathbf{Q}$ which fixes each point of O, then $x, x^\theta \in \{o_1,(\infty)\}^\perp \cap \{o_2,(\infty)\}^\perp$ and hence $\{o_1,(\infty)\}^\perp = \{o_2,(\infty)\}^\perp = \{x, x^\theta\}^{\perp\perp}$. Hence $S = \{\{o,(\infty)\}^\perp | o \in O \setminus \{(\infty)\}\}$ is a spread of $N^*_{(\infty)}$. We prove now that S is regular. Let M_1 be an arbitrary line of $N^*_{(\infty)}$ not belonging to S and let $M_1 = \{(\infty), x_1\}^\perp$. Let $\{x_1, \dots, x_{q+1}\}$ denote the orbit of x_1 determined by the group of automorphisms fixing each point of O and put $M_i = \{(\infty), x_i\}^\perp$ for all $i \in \{1, \dots, q+1\}$. The above conditions are then satisfied. □

Every automorphism θ of $\mathbf{Q}$ induces an automorphism $\bar{\theta}$ of $N^*_{(\infty)}$. If θ fixes every point of O, then $\bar{\theta}$ fixes every line of S. If L is a line of $\mathrm{PG}(3,q)$, then there is a geometry H^3_q with points those points of $\mathrm{PG}(3,q)$ not on L, see Section 1.4.4. If $N^*_{(\infty)} \simeq H^3_q$, then the spread S can be extended (by adding the line L) to a spread $\bar{S}$ of $\mathrm{PG}(3,q)$.

Theorem 2.9.4
If $N^*_{(\infty)} \simeq H^3_q$, then $\bar{S}$ is a regular spread of $\mathrm{PG}(3,q)$.

Proof. Let θ_i, $i \in \{1, \dots, q+1\}$, denote the $q+1$ automorphisms of $\mathbf{Q}$ fixing each point of O. The automorphism θ_i corresponds to an automorphism $\bar{\theta}_i$ of $N^*_{(\infty)}$ and an automorphism $\bar{\bar{\theta}}_i$ of $\mathrm{PG}(3,q)$, see Theorem

1.4.3. The automorphism $\bar{\bar{\theta}}_i$ fixes each line of $\bar{S}$. Now, let M be any line of $\mathrm{PG}(3,q)$ not belonging to $\bar{S}$, then M meets the lines $L_1, \ldots, L_{s+1}$ of $\bar{S}$. Put $\{M_1, \ldots, M_{q+1}\} = \{\bar{\bar{\theta}}_i(M) | i \in \{1, \ldots, q+1\}\}$. Clearly $L_1, \ldots, L_{q+1}$ is a regulus of $\mathrm{PG}(3,q)$; hence $\bar{S}$ is regular. $\square$

Theorem 2.9.5
If $N^*_{(\infty)} \simeq H^3_q$, then the group of automorphisms of $\mathbf{Q}$ fixing each point of O is isomorphic to the cyclic group C_{q+1}.

Proof. Let θ_i, $i \in \{1, \ldots, q+1\}$, denote the $q+1$ automorphisms of $\mathbf{Q}$ fixing each point of O. The automorphism θ_i corresponds to an automorphism $\bar{\theta}_i$ of $N^*_{(\infty)}$ and an automorphism $\bar{\bar{\theta}}_i$ of $\mathrm{PG}(3,q)$. The group $\{\bar{\bar{\theta}}_i | i \in \{1, \ldots, q+1\}\}$ is a subgroup of the group of automorphisms of $\mathrm{PG}(3,q)$ which fix each element of $\bar{S}$. The result follows now by Section 1.4.3. $\square$

Remark. By [90] we know that $N^*_{(\infty)} \simeq H^3_q$ if and only if q is even or if $\mathbf{Q}$ is a Kantor-Knuth quadrangle. In the following section we will prove that only classical Kantor-Knuth quadrangles can have ovoids of symmetry. We give now a method for determining all ovoids of symmetry. As we already mentioned, every automorphism θ of $\mathbf{Q}$ induces an automorphism $\bar{\theta}$ of $N^*_{(\infty)}$.

Theorem 2.9.6
Let ϕ be an automorphism of $N^*_{(\infty)}$ fixing each line of a spread S of $N^*_{(\infty)}$, then there exists at most one automorphism θ of $\mathbf{Q}$ such that the following two conditions are satisfied:

(i) *θ fixes each point of O_S,*

(ii) *$\bar{\theta} = \phi$.*

Proof. If θ satisfies (i) and (ii), then it must be equal to the map which we will construct now. Put $(\infty)^\theta = (\infty)$. If $x \in (\infty)^\perp \setminus \{(\infty)\}$, then it corresponds with a point x' of $N^*_{(\infty)}$. Let the point x'^ϕ correspond with the point y of $\mathbf{Q}$, then $x^\theta = y$. Finally, take $z \notin (\infty)^\perp$ and let L_1 and L_2 be two lines through z. The line L_i, $i \in \{1,2\}$, intersects $(\infty)^\perp$ in a point z_i and the ovoid O_S in a point o_i. Now, z^θ is the intersection of the lines $z_1^\theta o_1$ and $z_2^\theta o_2$. $\square$

If we want to determine all ovoids of symmetry of $\mathbf{Q}$, one can proceed as follows:

(1) Determine all regular spreads of $N^*_{(\infty)}$.

(2) For each regular spread S of $N^*_{(\infty)}$, determine the automorphisms of $N^*_{(\infty)}$ fixing each line of S. There have to be at least $q+1$ such automorphisms (including the identity), otherwise S cannot correspond to an ovoid of symmetry.

(3) Calculate the ovoid O_S of $\mathbf{Q}$ corresponding to S.

(4) Check for each of the automorphisms determined in (2) whether it corresponds with an automorphism of $\mathbf{Q}$ which fixes each point of O_S. If we find $q+1$ such automorphisms, then O_S is an ovoid of symmetry.

2.9.4 The nonexistence of ovoids of symmetry in GQ's in nonclassical Kantor-Knuth quadrangles.

Let q be an odd prime power, $\sigma \in \mathrm{Aut}[\mathrm{GF}(q)]$, m be a nonsquare in $\mathrm{GF}(q)$. Let $\mathbf{Q}$ be the Kantor-Knuth quadrangle associated with these parameters. We suppose that $\mathbf{Q}$ is not isomorphic to $H(3,q^2)$, or equivalently, that σ is not the identity. The choice of the nonsquare m in $\mathrm{GF}(q)$ is immaterial, see [65]. Also the GQ $\mathbf{Q}$ obtained by using a particular automorphism σ is isomorphic to the one obtained by using σ^{-1}. The Kantor-Knuth quadrangles have a lot of subquadrangles isomorphic to $W(q)$. For each $(\gamma_1,\gamma_2) \in \mathrm{GF}(q)\times\mathrm{GF}(q)\setminus\{(0,0)\}$, we have the following subquadrangle $S(\gamma_1,\gamma_2)$ isomorphic to $W(q)$ ([65]). Points are of four kinds:

(a) (∞);

(b) $A^*(\infty)\cdot(k\gamma_1, k^\sigma\gamma_2, 0,0,0)$, $k \in \mathrm{GF}(q)$;

(c) $A^*(t)\cdot(0,0,0,2\gamma_1 u, -2\gamma_2 m u^\sigma)$, $t,u \in \mathrm{GF}(q)$;

(d) $(k\gamma_1, k^\sigma\gamma_2, \gamma_1^2 c - \gamma_2^2 m c^\sigma, 2\gamma_1 u, -2\gamma_2 m u^\sigma)$, $k,c,u \in \mathrm{GF}(q)$.

Lines are of three kinds:

(a) $[A(t)]$, $t \in \mathrm{GF}(q)\cup\{(\infty)\}$;

(b) $A(\infty)\cdot(k\gamma_1, k^\sigma\gamma_2, \gamma_1^2 c - \gamma_2^2 m c^\sigma, 0, 0)$, $k,c \in \mathrm{GF}(q)$;

(c) $A(r)\cdot(0,0,\gamma_1^2 c - \gamma_2^2 m c^\sigma, 2\gamma_1 s, -2\gamma_2 m s^\sigma)$, $r,c,s \in \mathrm{GF}(q)$.

Incidence is the one derived from $\mathbf{Q}$.

Theorem 2.9.7 ([65])
Let $\sigma \neq 1$ and let θ be any automorphism of $\mathbf{Q}$. Then $[S(1,0)]^\theta = S(1,0)$ or $[S(1,0)]^\theta = S(0,1)$.

Theorem 2.9.8
If $\sigma \neq 1$, then $\mathbf{Q}$ *has no ovoid of symmetry.*

Proof. Clearly $q \geq 5$. Let $x_1, \ldots, x_{q-1}$ be the $q-1$ points of $S(1,0)$ on the line $[A(\infty)]$ and different from (∞) and $A^*(\infty)(0,0,0,0,0)$. Let θ_1 and θ_2 be two nontrivial automorphisms fixing each point of O. Let $i \in \{1, \ldots, q-1\}$ be fixed. If $\theta_1(x_i)$ and $\theta_2(x_i)$ are no points of $S(1,0)$, then $x_i \in \{\theta_1(x_i), \theta_2(x_i)\}^{\perp\perp} \subseteq S(0,1)$, a contradiction. Hence $\{\theta_1(x_i), x_i\}^{\perp\perp} \subseteq S(1,0)$ for all $i \in \{1, \ldots, q-1\}$. Let $o_1, o_2 \in O \setminus \{(\infty)\}$ such that o_1 is collinear with x_1 and o_2 is collinear with x_2. Then $o_1 \in \{x_1, \theta_1(x_1)\}^{\perp} \subseteq S(1,0)$. Similarly $o_2 \in S(1,0)$. The points $(\infty), o_1$ and o_2 belong to a subquadrangle isomorphic to $W(q)$. These points have then at least one common neighbour, a contradiction. $\square$

2.10 Generalized quadrangles of order $(s, s+2)$ with a lot of grids

In this section we prove that there is a common construction for all GQ's of order $(s, s+2)$ which have the property that every two intersecting lines are contained in a unique subquadrangle of order $(s, 1)$.

2.10.1 Amalgamations of projective planes

All GQ's of order n which have a regular point incident with some regular line can be constructed in the following way, see [61]. Let π_1 and π_2 be two projective planes of order n. Choose in π_i, $i \in \{1,2\}$, an incident point-line pair $((\infty)_i, [\infty]_i)$. Let α be a bijection between the point sets of $[\infty]_1$ and $[\infty]_2$ such that $(\infty)_1^\alpha = (\infty)_2$, and let β be a bijection between the set of lines through $(\infty)_1$ and the set of lines through $(\infty)_2$ such that $[\infty]_1^\beta = [\infty]_2$. Suppose also that the following configuration fails to exist:

(1) A_i, B_i, C_i $(i \in \{1,2\})$ are points of π_i not on $[\infty]_i$;

(2) no three of the points $(\infty)_i, A_i, B_i, C_i$ $(i \in \{1,2\})$ are collinear;

(3) $A_2 \in ((\infty)_1 A_1)^\beta, B_2 \in ((\infty)_1 B_1)^\beta, C_2 \in ((\infty)_1 C_1)^\beta$,

(4) $(A_1B_1 \cap [\infty]_1)^\alpha = A_2B_2 \cap [\infty]_2$, $(A_1C_1 \cap [\infty]_1)^\alpha = A_2C_2 \cap [\infty]_2$ and $(B_1C_1 \cap [\infty]_1)^\alpha = B_2C_2 \cap [\infty]_2$.

The following generalized quadrangle $\mathbf{Q}$ can then be constructed. The points are of two types:

(i) the points of π_1,

(ii) pairs of lines (L_1, L_2), where L_i is a line of π_i different from $[\infty]_i$, and $(L_1 \cap [\infty]_1)^\alpha = L_2 \cap [\infty]_2 \neq (\infty)_2$.

The lines are of two types.

(a) lines of π_2,

(b) pairs of points (x_1, x_2) with $x_1 \notin [\infty]_1$ a point of π_1, $x_2 \notin [\infty]_2$ a point of π_2 and $((\infty)_1 x_1)^\beta = (\infty)_2 x_2$.

Incidence is as follows.

(i,a) A point Q of type (i) is incident with a line L of type (a) if and only if ($Q \in [\infty]_1$ and $Q^\alpha \in L$) or ($(\infty)_2 \in L$ and $Q \in L^{\beta^{-1}}$).

(i,b) A point Q of type (i) is incident with a line (R, S) of type (b) if and only if $Q = R$.

(ii,a) A point (L_1, L_2) of type (ii) is incident with a line L of type (a) if and only if $L = L_2$.

(ii,b) A point (L_1, L_2) of type (ii) is incident with a line (R_1, R_2) of type (b) if and only if $R_1 \in L_1$ and $R_2 \in L_2$.

The generalized quadrangle $\mathbf{Q}$ has order n and is called an *amalgamation* of the projective planes π_1 and π_2. The generalized quadrangle $\mathbf{Q}^D$ which is the dual of $\mathbf{Q}$ is also an amalgamation of two projective planes (namely of the dual planes). The point $[\infty]_2$ is a regular point of $\mathbf{Q}^D$ and we find the following description for $P(\mathbf{Q}^D, [\infty]_2)$. The points of $P(\mathbf{Q}^D, [\infty]_2)$ are the pairs of points (x_1, x_2) with $x_1 \notin [\infty]_1$ a point of π_1, $x_2 \notin [\infty]_2$ a point of π_2 and $((\infty)_1 x_1)^\beta = (\infty)_2 x_2$. Lines are of three types:

(i) points of π_1 not contained in $[\infty]_1$;

(ii) points of π_2 not contained in $[\infty]_2$;

(iii) pairs of lines (L_1, L_2) such that L_1 is a line of π_1 not through $(\infty)_1$, L_2 is a line of π_2 not through $(\infty)_2$ and $(L_1 \cap [\infty]_1)^\alpha = L_2 \cap [\infty]_2$.

Incidence is as follows. The point (x_1, x_2) is incident with the line x of type (i) if and only if $x = x_1$. The point (x_1, x_2) is incident with the line x of type (ii) if and only if $x = x_2$ and the point (x_1, x_2) is incident with the line (L_1, L_2) of type (iii) if and only if $x_1 \in L_1$ and $x_2 \in L_2$.

2.10.2 GQ's with a unique grid through every two intersecting lines

The generalized quadrangle $T_2^*(O)$, with O a hyperoval in $\mathrm{PG}(2, 2^h)$, $h \geq 2$, satisfies the following properties.

(P1) The order is $(s, t) = (s, s + 2)$.

(P2) Every two intersecting lines are contained in a unique subquadrangle of order $(s, 1)$, i.e. a grid.

Property (P2) holds because of the following reason: every subquadrangle of order $(s, 1)$ is the set of affine points in a plane of $\mathrm{PG}(3, 2^h)$ which intersects O in exactly two points, see Theorem 3.3.4 of [71]. In this section, we try to prove a kind of converse theorem, namely the following one.

Theorem 2.10.1
If $\mathbf{Q}$ *is a generalized quadrangle satisfying* (P1) *and* (P2)*, then there exists a generalized quadrangle* $\mathbf{Q}'$ *and a regular point* x *of* $\mathbf{Q}'$ *such that* $\mathbf{Q}$ *is isomorphic to* $P(\mathbf{Q}', x)$*, moreover* $\mathbf{Q}'$ *is an amalgamated generalized quadrangle.*

This theorem has the following corollary.

Corollary 2.10.2
If $\mathbf{Q}$ *is a* $\mathrm{GQ}(s, s + 2)$ *with the property that every two lines are contained in a unique grid, then there exists a projective plane of order* $s + 1$*. Hence, there are no such* GQ*'s with* $s = 5, 9, \ldots$.

In order to prove the theorem, we start with the following lemma.

Lemma 2.10.3
Let $\mathbf{Q}$ *be a* $\mathrm{GQ}(s, t)$ *satisfying* (P2).

(A) There are $\frac{t+1}{2}$ *grids through every two points* x *and* y *at distance* 2*. Hence* t *is odd.*

(B) If $t \neq 1$*, then* $t \geq s + 2$*. If* $t = s + 2$*, then one of the following possibilities occurs for every two grids* $\mathbf{G}_1$ *and* $\mathbf{G}_2$*:*

(1) $\mathbf{G}_1 = \mathbf{G}_2$*;*

(2) $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *are disjoint;*

(3) $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *intersect in a line;*

(4) $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *intersect in an ovoid of* $\mathbf{G}_1$*.*

Proof.

(A) The grids through x and y partition the set of lines through x. For, every line L through x is contained in a unique grid through x and y, namely the grid containing L and the line through y meeting L.

(B) Let $\mathbf{G}_1$ be a fixed grid through a fixed point x. If $\mathbf{G}_2$ is another grid through x such that $\mathbf{G}_1$ and $\mathbf{G}_2$ have a common line L through x, then $\mathbf{G}_1$ and $\mathbf{G}_2$ have no other points in common than those on L. Now, let V be the set of the $\frac{t(t+1)}{2} - 1 - 2(t-1) = \frac{(t-2)(t-1)}{2}$ grids $\mathbf{G}_2 \neq \mathbf{G}_1$ through x, such that $\mathbf{G}_1$ and $\mathbf{G}_2$ have no common line through x. For every $\mathbf{G}_2 \in V$, let $A(\mathbf{G}_2)$ be the set of the $f(\mathbf{G}_2)$ common points of $\mathbf{G}_1$ and $\mathbf{G}_2$ different from x. Suppose that α and β are two collinear points of $A(\mathbf{G}_2)$. Let γ denote that point on $\alpha\beta$ collinear with x, then γx and $\alpha\beta$ are contained in at least two grids, a contradiction. Hence $A(\mathbf{G}_2)$ is a set of mutual noncollinear points and $f(\mathbf{G}_2) \leq s$. Moreover, if $f(\mathbf{G}_2) = s$, then $\mathbf{G}_1$ and $\mathbf{G}_2$ intersect in an ovoid of $\mathbf{G}_1$. Counting the number of pairs $(\mathbf{G}_2, \alpha)$ with $\mathbf{G}_2 \in V$ and $\alpha \in A(\mathbf{G}_2)$ yields

$$\sum_{\mathbf{G}_2 \in V} f(\mathbf{G}_2) = s^2 \frac{t-1}{2}.$$

Since $|V| = \frac{(t-2)(t-1)}{2}$ and $f(\mathbf{G}_2) \leq s$ for all $\mathbf{G}_2 \in V$, we have that $s \leq t-2$. If $s = t-2$, then $f(\mathbf{G}_2) = s$ for all $\mathbf{G}_2 \in V$; hence $\mathbf{G}_1$ and $\mathbf{G}_2$ intersect in an ovoid of $\mathbf{G}_1$ for all $\mathbf{G}_2 \in V$.

$\square$

From now on we suppose that $\mathbf{Q}$ is a $\mathrm{GQ}(s,t)$ satisfying (P1) and (P2).

Lemma 2.10.4
If the point p is not contained in the grid $\mathbf{G}$, then p is contained in exactly two lines disjoint with $\mathbf{G}$ and exactly one grid disjoint with $\mathbf{G}$.

Proof. The points of $\mathbf{G}$ collinear with p form an ovoid of $\mathbf{G}$, hence there are exactly $(s+3)-(s+1) = 2$ lines, say L_1 and L_2, through p disjoint with $\mathbf{G}$. Every grid through p disjoint with $\mathbf{G}$ contains L_1 and L_2. If $\mathbf{G}'$ is the unique grid through L_1 and L_2, then $\mathbf{G}'$ cannot satisfy property (1), (3) or (4) of the previous lemma (with respect to $\mathbf{G}$). Hence $\mathbf{G}$ and $\mathbf{G}'$ are disjoint.
$\square$

Lemma 2.10.5
Let R *be the following relation on the line set:*

$$L \mathrm{~R~} M \iff \begin{cases} L = M, \ \textit{or} \\ (L \cap M = \emptyset) \ \textit{and} \ (L \ \textit{and} \ M \ \textit{are contained in a grid}), \end{cases}$$

then R *is an equivalence relation.*

Proof. It suffices to prove that R is transitive. So, let L, M and N be lines such that $L \mathrm{~R~} M$ and $M \mathrm{~R~} N$. We may suppose that $L \neq M$ and $M \neq N$. Let $\mathbf{G}_1$ be the grid through L and M, and $\mathbf{G}_2$ the grid through M and N. We may suppose that $\mathbf{G}_1 \neq \mathbf{G}_2$. Let x be an arbitrary point of N and let $\mathbf{G}_3$ be the unique grid through x and L. If $\mathbf{G}_2$ and $\mathbf{G}_3$ meet in a line, then this line is necessary N. In this case $N \mathrm{~R~} L$. So, suppose that $\mathbf{G}_2$ and $\mathbf{G}_3$ are not meeting in a line, then $\mathbf{G}_2$ and $\mathbf{G}_3$ intersect in an ovoid of $\mathbf{G}_2$. Hence, there is a point y of M which belongs to $\mathbf{G}_3$, a contradiction. $\square$

For every line L of $\mathbf{Q}$, let E_L be the equivalence class (with respect to R) to which L belongs.

Lemma 2.10.6
E_L is a normal spread of $\mathbf{Q}$ *for every line L.*

Proof. If two lines M_1 and M_2 of E_L would meet, then $M_1 \mathrm{~R~} M_2$ implies that $M_1 = M_2$. If x is a point of $\mathbf{Q}$ not on L, consider then the grid through x and L. The line in this grid through x and not meeting L belongs to E_L. Hence E_L is a spread. Clearly E_L is a normal spread. $\square$

Remark. The GQ has $(s+3)(s+1)^2$ lines and each spread E_L has $(s+1)^2$ lines, hence there are at least $s+3$ normal spreads.

Let L be a line of $\mathbf{Q}$ and consider the normal spread E_L. The incidence structure $\mathbf{A}_L$ with

(i) point set equal to E_L,

(ii) lines of the form $\{K, M\}^{\perp\perp}$ with $K, M \in E_L$ and $K \neq M$,

(iii) natural incidence,

is then an $S(2, s+1, (s+1)^2)$ and hence an affine plane of order $s+1$. Let $L_1, L_2, \ldots, L_{s+3}$ be lines of $\mathbf{Q}$ such that $E_{L_1}, \ldots, E_{L_{s+3}}$ are mutually different equivalence classes. Let x be a point of $\mathbf{Q}$ and let $i, j \in \{1, \ldots, s+3\}$, $i \neq j$. Through x there is a unique line of E_{L_i} and a unique line of E_{L_j}; these two lines determine a grid, which we denote by $\mathbf{G}(x, L_i, L_j)$. The equivalence class E_{L_i}, $i \in \{1, \ldots, s+3\}$, defines an affine plane $\mathbf{A}_i$ and a projective plane $\mathbf{P}_i$.

We can define $\mathbf{P}_1$ as follows. The points of $\mathbf{P}_1$ are of two types.

(I) Lines L with $L \mathrm{R} L_1$.

(II) Lines L_j with $j \neq 1$.

The lines of $\mathbf{P}_1$ are of two types.

(A) Grids $\mathbf{G}(p, L_1, L_j)$ with $j \neq 1$ and p a point of $\mathbf{Q}$.

(B) $[\infty]_1$.

A point of type (I) is incident with a grid $\mathbf{G}(p, L_1, L_j)$ if and only if it belongs to this grid. A point of type (I) is never incident with the line $[\infty]_1$. A point L_j of type (II) is incident with a grid $\mathbf{G}(p, L_1, L_k)$ if and only if $j = k$ and it is always incident with $[\infty]_1$.

Similarly, we can define the points and lines of $\mathbf{P}_2$ as follows. The points of $\mathbf{P}_2$ are of two types.

(I) Lines L with $L \mathrm{R} L_2$.

(II) Lines L_j with $j \neq 2$.

The lines of $\mathbf{P}_2$ are of two types.

(A) Grids $\mathbf{G}(p, L_2, L_j)$ with $j \neq 2$ and p a point of $\mathbf{Q}$.

(B) $[\infty]_2$.

Now, let α be the following bijection from the point set of $[\infty]_1$ to the point set of $[\infty]_2$:

$$\begin{array}{lcl} L_2 & \rightarrow & L_1, \\ L_i & \rightarrow & L_i \text{ if } i \neq 1, 2. \end{array}$$

Let β be the following bijection from the set of lines through L_2 to the set of lines through L_1:

$$\begin{array}{rcl} \mathbf{G}(p, L_1, L_2) & \rightarrow & \mathbf{G}(p, L_1, L_2), \\ [\infty]_1 & \rightarrow & [\infty]_2. \end{array}$$

We prove now that $\mathbf{Q}$ admits a model like the one given at the end of Section 2.10.1. Let p be an arbitrary point of $\mathbf{Q}$. Through p, there is a unique line A of E_{L_1} and a unique line B of E_{L_2}. We make the following identification: $p \leftrightarrow (A, B)$. Clearly $(L_2A)^\beta = L_1B$, where L_2A is the line of $\mathbf{P}_1$ through L_2 and A and where L_1B denotes the line of $\mathbf{P}_2$ through L_1 and B. Let L be an arbitrary line of $\mathbf{Q}$ and let $x \in L$ be fixed. If $L \in E_{L_1}$, then we regard L as point of $\mathbf{P}_1$. If $L \in E_{L_2}$, then we regard L as point of $\mathbf{P}_2$. If $L \in E_{L_i}$ with $i \neq 1, 2$, then we make the following identification: $L \leftrightarrow (\mathbf{G}(x, L_1, L_i), \mathbf{G}(x, L_2, L_i))$. The intersection of the line $\mathbf{G}(x, L_1, L_i)$ with the line $[\infty]_1$ is equal to the point L_i (in $\mathbf{P}_1$). Similarly, the intersection of the line $\mathbf{G}(x, L_2, L_i)$ with the line $[\infty]_2$ is equal to L_i (in $\mathbf{P}_2$). Clearly $(L_i)^\alpha = L_i$.

Now, let p be an arbitrary point of $\mathbf{Q}$ and let L be an arbitrary line of $\mathbf{Q}$. Let $p \leftrightarrow (A, B)$. If $L \in E_{L_1}$, then $p \mathcal{I} L$ if and only if $A = L$. If $L \in E_{L_2}$, then $p \mathcal{I} L$ if and only if $B = L$. Finally, if $L \in E_{L_i}$, $i \neq 1, 2$, then $L \leftrightarrow (\mathbf{G}(x, L_1, L_i), \mathbf{G}(x, L_2, L_i))$ with x an arbitrary point of L. If $p \mathcal{I} L$, then A is a point of $\mathbf{G}(x, L_1, L_i)$ (in $\mathbf{P}_1$) and B is a point of $\mathbf{G}(x, L_2, L_i)$ (in $\mathbf{P}_2$). Conversely suppose that A and B are points of $\mathbf{G}(x, L_1, L_i)$ and $\mathbf{G}(x, L_2, L_i)$ respectively, then L meets A and B; hence $p \mathcal{I} L$. This proves the theorem. $\square$

Remark. With the same notations as above we have that $\{E_{L_1}, \ldots, E_{L_{s+3}}\}$ is a partition of the line set in spreads. Moreover, each spread E_{L_i} is *pivotal*, i.e. E_{L_i} is normal and for every $M, N \in E_{L_i}$, $M \neq N$, $\{M, N\}^\perp \subseteq E_{L_j}$ for some $j \in \{1, \ldots, s+3\}$. Theorem 2.10.1 follows then from Section 3 of [70].

Chapter 3

Basic theory of near polygons

This chapter contains the basic definitions and properties about near polygons which we will use through this thesis. It also contains some theorems and constructions found by the author himself. For proofs and a more complete survey, we refer to the literature about near polygons ([10],[11],[12],[13],[14], [15],[17],[80],[82]).

3.1 Definitions and line-line relation

A *near polygon* is a connected partial linear space satisfying the following property.

(NP) For every point p and every line L, there exists a unique point q on L nearest to p (with respect to the distance in the point graph).

The point q is called *the projection of* p *on* L. If d is the diameter of the point graph, then the near polygon is called a near $2d$-gon. Hence we will talk about near 0-gons, near 2-gons, near quadrangles, near hexagons, near octagons, etc. The near quadrangles are just the generalized quadrangles discussed in the previous chapter. The point graph of a near polygon determines uniquely the near polygon. For, every two adjacent vertices of such graph are contained in a unique maximal clique and these maximal cliques necessary correspond with the lines of the near polygon. Property (NP) describes the point-line relation. Next theorem describes the line-line relation.

Theorem 3.1.1 ([17])
One of the following two cases occurs for any two lines L and M of a near polygon.

(1) *There exists a unique point p on L and a unique point q on M such that $d(l,m) = d(l,p) + d(p,q) + d(q,m)$ for all points l on L and m on M.*

(2) *There exists an $i \in \mathbb{N}$ such that $d(l,M) = d(m,L) = i$ for all points l on L and m on M.*

Two lines are called *parallel* ($\|$) if they satisfy condition (2) of the previous theorem. The relation $\|$ is most of the time not an equivalence relation, see Section 3.3 for characterizations related to this. A near $2d$-gon $\mathbf{S}$ is called *regular* if the following two conditions are satisfied.

(a) $\mathbf{S}$ has order (s,t).

(b) There are constants t_i $(0 \leq i \leq d)$ such that for every two points x and y at distance i, there are exactly $t_i + 1$ points in $\Gamma(x) \cap \Gamma_{i-1}(y)$.

Clearly $t_0 = -1$, $t_1 = 0$ and $t_d = t$. The parameters s, t, t_i $(i \in \{2, \ldots, d-1\})$ are called the *parameters* of the regular near polygon. If x is a point of a near polygon, then easily counting yields that

$$\begin{aligned} |\Gamma_0(x)| &= 1, \\ |\Gamma_k(x)| &= s^k \frac{\prod_{i=0}^{k-1}(t - t_i)}{\prod_{i=1}^{k}(1 + t_i)} \text{ for all } i \in \{1, \ldots, d\}. \end{aligned}$$

The question whether a certain regular near polygon is completely determined by its parameters is always an interesting problem (see e.g. next section). A *thin near polygon* is a near polygon with the property that every line is incident with exactly two points; we can consider these geometries as graphs. In this point of view the thin near polygons are exactly the connected bipartite graphs. A dual grid is a thin near polygon. If $\mathbf{S_1} = (\mathcal{P}_1, \mathcal{L}_1, \mathcal{I}_1)$ and $\mathbf{S_2} = (\mathcal{P}_2, \mathcal{L}_2, \mathcal{I}_2)$ are two partial linear spaces, then the *direct product* of $\mathbf{S_1}$ and $\mathbf{S_2}$ is the partial linear space $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ with $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$ and $\mathcal{L} = (\mathcal{P}_1 \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times \mathcal{P}_2)$. The point (x,y) is incident with the line $(a,L) \in \mathcal{P}_1 \times \mathcal{L}_2$ if and only if $x = a$ and $y\ \mathcal{I}_2\ L$ and it is incident with the line $(M,b) \in \mathcal{L}_1 \times \mathcal{P}_2$ if and only if $y = b$ and $x\ \mathcal{I}_1\ M$. We denote $\mathbf{S}$ also with $\mathbf{S_1} \times \mathbf{S_2}$. Let $\mathrm{d}_1(\cdot,\cdot), \mathrm{d}_2(\cdot,\cdot)$ and $\mathrm{d}(\cdot,\cdot)$ denote the distances in $\mathbf{S_1}, \mathbf{S_2}$ and $\mathbf{S}$ respectively, then $\mathrm{d}((x_1,y_1),(x_2,y_2)) = \mathrm{d}_1(x_1,x_2) + \mathrm{d}_2(y_1,y_2)$. Hence if $\mathbf{S}_i$ $(i \in \{1,2\})$ is a near $2d_i$-gon then the direct product $\mathbf{S} = \mathbf{S_1} \times \mathbf{S_2}$ is a near $2(d_1+d_2)$-gon. Since $\mathbf{S_1} \times \mathbf{S_2} \simeq \mathbf{S_2} \times \mathbf{S_1}$ and $(\mathbf{S_1} \times \mathbf{S_2}) \times \mathbf{S_3} \simeq \mathbf{S_1} \times (\mathbf{S_2} \times \mathbf{S_3})$, also the direct product of $k \geq 1$ partial linear spaces $\mathbf{S_1}, \ldots, \mathbf{S_k}$ is well-defined.

Theorem 3.1.2 ([17])
Let **S** *be a near polygon with the property that every two points at distance* 2 *have at least two common neighbours. If lines of different length occur, then* **S** *is the direct product of a number of near polygons, each of which has a constant length for the lines.*

We also prove the following result about near polygons.

Theorem 3.1.3
Let $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ *be a partial linear space of order* $(s,t) \neq (s,0)$ *satisfying*

(1) *for every point* p *and every line* L *not through* p*, there exists at most one point on* L *collinear with* p*,*

(2) $a = |\Gamma_2(x)|$ *is independent of the point* $x \in \mathcal{P}$*,*

(3) $d(x, L) \leq 2$ *for all* $x \in \mathcal{P}$ *and* $L \in \mathcal{L}$*,*

then $b = |\Gamma_3(x)|$ *is also independent of* $x \in \mathcal{P}$ *and the following inequalities hold:*

- $a \geq s^2 t$,
- $b \leq s(a - s^2 t)$.

Moreover, **S** *is a generalized quadrangle if and only if* $a = s^2 t$ *and* **S** *is a near hexagon if and only if* $a > s^2 t$ *and* $b = s(a - s^2 t)$.

Proof. Clearly $|\Gamma_3(x)| = |\mathcal{P}| - 1 - s(t+1) - |\Gamma_2(x)|$ is independent of $x \in \mathcal{P}$. Take an arbitrary line L and let r be a point of L. There are a points in $\Gamma_2(r)$, $s^2 t$ of these are contained in $\Gamma_1(L)$. Hence $a \geq s^2 t$ and $\Gamma_2(L) \leq (s+1)(a - s^2 t)$. If $a = s^2 t$ then $\Gamma_2(L) = \emptyset$ implies that **S** is a generalized quadrangle. So, suppose that $a \neq s^2 t$, then **S** is a near hexagon if and only if $\Gamma_2(L) = (s+1)(a - s^2 t)$. From $|\Gamma_2(L)| = |\mathcal{P}| - (s+1) - st(s+1) = a + b - s^2 t$, it follows that $b \leq s(a - s^2 t)$ and equality appears if and only if **S** is a near hexagon. □

3.2 Examples of some regular near polygons

3.2.1 Near polygons of Hamming type

A near polygon is of *Hamming type* if it is the direct product of a number of lines. This yields the following description. With a collection of $d \geq 1$ finite sets $X_1, \ldots, X_d$, there is associated the following near $2d$-gon: the points are the elements of the cartesian product $X_1 \times \ldots \times X_d$ and the lines are of the form $\{x_i\} \times \ldots \times \{x_{i-1}\} \times X_i \times \{x_{i+1}\} \times \ldots \times \{x_d\}$, with $i \in \{1, \ldots, d\}$ and $x_j \in X_j$, $j \in \{1, \ldots, d\} \setminus \{i\}$. Incidence is the natural one. If all the sets X_i have the same size n, then the near polygon is regular with parameters $s = n - 1$, $t = d - 1$ and $t_i = i - 1$. The regular near polygons of Hamming type are uniquely determined by their parameters, see Corollary 9.2.3 of [15].

3.2.2 Dual polar spaces

In this section, we mean with a *projective space* an incidence structure such that any three noncollinear points generate a (possible degenerate) projective plane. It is called *irreducible* when there are no lines of length 2; otherwise it is called *reducible*. Now, let $(P_i)_{i \in I}$ be a family of irreducible projective spaces whose point sets are pairwise disjoint. Then the union P of their point sets carries the structure of a projective space whose lines are the lines on each P_i on the one hand and all pairs $\{x_i, y_j\}$ with $x_i \in P_i$, $y_j \in P_j$, $i \neq j$, on the other hand. P is called the *direct sum* of all the P_i's. It is know that a reducible projective space is the direct sum of irreducible projective spaces.

Definition. A *polar space of rank* $n \geq 1$ is a set together with a collection of subsets, called *subspaces*, satisfying the following axioms.

(P1) Any proper subspace, together with the subspaces it contains, is a projective space of dimension at most $n-1$. This projective space may be reducible.

(P2) The intersection of two subspaces is a subspace.

(P3) If L is a subspace of dimension $n - 1$ and p is a point outside L, then there is a unique subspace M through p such that $\dim(L \cap M) = n - 2$; it contains all points q of L with the property that there is a onedimensional subspace through p and q.

(P4) There exist two disjoint subspaces of dimension $n - 1$.

Polar spaces of rank 2 are precisely the nondegenerate generalized quadrangles. With each polar space of rank $n \geq 1$, there is associated a *dual polar space* as follows. The points of the dual polar space are the maximal subspaces of the polar space (i.e. the subspaces of dimension $n - 1$), the lines are the next to maximal subspaces (i.e. the subspaces of dimension $n - 2$) and incidence is reverse containment. The near polygons of Hamming type are examples of dual polar spaces. For each of the following polar spaces, the corresponding dual polar space is a regular near $2n$-gon.

(a) The polar space $W(2n-1,q)$ associated with a nonsingular symplectic polarity in $\mathrm{PG}(2n-1,q)$, $n \geq 2$. Let $W^D(2n-1,q)$ denote the corresponding dual polar space.

(b) The polar space $Q(2n,q)$ associated with a nonsingular quadric in $\mathrm{PG}(2n,q)$, $n \geq 2$. Let $Q^D(2n,q)$ denote the corresponding dual polar space.

(c) The polar space $Q^-(2n+1,q)$ associated with a nonsingular elliptic quadric in $\mathrm{PG}(2n+1,q)$, $n \geq 2$. Let $[Q^-(2n+1,q)]^D$ denote the corresponding dual polar space.

(d) The polar space $Q^+(2n-1,q)$ associated with a nonsingular hyperbolic quadric in $\mathrm{PG}(2n-1,q)$, $n \geq 2$. Let $[Q^+(2n-1,q)]^D$ denote the corresponding dual polar space.

(e) The polar space $H(2n,q)$ associated with a nonsingular Hermitian variety in $\mathrm{PG}(2n,q)$, q square and $n \geq 2$. Let $H^D(2n,q)$ denote the corresponding dual polar space.

(f) The polar space $H(2n-1,q)$ associated with a nonsingular Hermitian variety in $\mathrm{PG}(2n-1,q)$, q square and $n \geq 2$. Let $H^D(2n-1,q)$ denote the corresponding dual polar space.

The parameters are $s = q^e$, $t_i = q^{i-1} + \ldots + q^2 + q$ $(2 \leq i \leq n)$, $t = t_n$, where e is for each of the dual polar spaces as shown in the table.

S	e
$W^D(2n-1,q)$	1
$Q^D(2n,q)$	1
$[Q^-(2n+1,q)]^D$	2
$[Q^+(2n-1,q)]^D$	0
$H^D(2n,q)$	$\frac{3}{2}$
$H^D(2n-1,q)$	$\frac{1}{2}$

If a regular near polygon has these parameters, then it is a dual polar space, see Corollary 3.7.5 and the classification of classical near polygons given in Section 3.6. Note that the dual polar spaces $W^D(2n-1,q)$ and $Q^D(2n,q)$ have the same parameters, but they are isomorphic if and only if q is even.

3.2.3 The near hexagon related to $S(5,8,24)$

Consider the Steiner system $S(5,8,24)$ which we defined in Section 1.4.2. The following near hexagon is associated with this Steiner system: the points are the blocks of the Steiner system, the lines are the sets of three mutually disjoint blocks and incidence is the natural one. The near hexagon is regular with parameters $s = 2$, $t_2 = 2$, $t = 14$ and has 759 vertices. Two points of the near haxagon have distance 0, 1, 2, respectively 3 if the corresponding blocks intersect in 8, 0, 4, respectively 2 points. The near hexagon is completely determined by its parameters, see [12]. This regular near hexagon has the following interesting characterization.

Theorem 3.2.1 ([17])
Let $\mathbf{S}$ *be a regular near hexagon with parameters* s, t_2 *and* t. *Suppose that* $s > 1$ *and* $t_2 > 0$. *Then* $\mathbf{S}$ *is the near hexagon related to* $S(5,8,24)$ *if and only if* $1+t = (t_2+1)(1+st_2)$.

3.2.4 The near hexagon related to the ternary Golay code

Consider the ternary Golay code which we described in Section 1.4.2. This code is a subspace of $\mathbb{F}_3^{12}$. The following near hexagon is associated with this code, see [15]:

(1) the points are the cosets of the code,

(2) the lines are the triples of cosets $\{C_1, C_2, C_3\}$, such that C_i and C_j have representatives which differ in only one position (C_1, C_2 and C_3 differ then in the same position).

(3) incidence is the natural one.

This near hexagon is regular with parameters $s = 2$, $t_2 = 1$ and $t = 11$. The near hexagon is completely determined by its parameters, see [11]. This near hexagon has a linear representation $T_5^*(\mathcal{K})$ in $\mathrm{AG}(6,3)$, where $\mathcal{K}$ is the Coxeter cap in $\mathrm{PG}(5,3)$, see Chapter 6.

3.2.5 The near octagon related to the Hall-Janko group

The following near octagon was first constructed in [23]. Consider the sporadic simple Hall-Janko group of order 604800. Let V_0 be the set of all 315 involutions, whose centralizer contains Sylow 2-subgroups, i.e. groups of order 128. Let Γ be the graph with vertex set V_0, two involutions being adjacent when they commute, or, equivalently, when their product is again an element of V_0. The graph Γ is then the point graph of a regular near octagon (the lines correspond with the maximal cliques). The parameters are $s = 2$, $t_2 = 1$, $t_3 = 3$ and $t = 4$ and the near octagon is uniquely determined by its parameters, see [24].

3.3 Characterizations of the Hamming near polygons in terms of parallellism

Recall that two lines L and M of a near polygon are called parallel if $\mathrm{d}(l, M)$ is independent of the point $l \, \mathcal{I} \, L$.

Lemma 3.3.1
Let $\mathcal{S}_i, 1 \leq i \leq k$, be a number of near polygons and let $\mathcal{S}$ be their direct product. Then parallelism ($\|$) is an equivalence relation in $\mathcal{S}$ if and only if it is an equivalence relation in each $\mathcal{S}_i$.

Proof. We may suppose that $k = 2$, otherwise one can use induction. The result follows now by looking at the following remarks, which are readily obtained. (Here p, q denote points and L, M denote lines.)

- The lines (p, M) and (L, q) are never parallel.
- The lines (p, M) and (q, L) are parallel if and only if M and L are parallel.
- The lines (L, q) and (M, p) are parallel if and only if L and M are parallel.

$\square$

Theorem 3.3.2
If $\mathcal{S}$ is a near polygon, then the following conditions are equivalent.

(1) *$\mathcal{S}$ is a near polygon of Hamming type.*

(2) *Parallelism is an equivalence relation and every two points at mutual distance 2 have at least two common neighbours.*

(3) *For every point x and every line L, there is a unique line through x which is parallel to L.*

Proof.

(1) $\Rightarrow$ (2). Trivial.

(2) $\Rightarrow$ (3). If there were two lines M and M' through x parallel to L, then $M \| M'$, a contradiction. We only need to show that there is at least one line through x parallel to L. Since $\mathcal{S}$ is connected and $\|$ is an equivalence relation, we may suppose that $\mathrm{d}(x, L) = 1$. Let x' be the unique point of L which is collinear with x and let y' be any other point of L. If y is a common neighbour of x and y', different from x, then $xy \| L$.

(3) $\Rightarrow$ (1). We prove by induction that for every two points at mutual distance i, there are exactly i points in $\Gamma_{i-1}(x) \cap \Gamma(y)$. This is true for $i = 0$ and $i = 1$, so suppose $i \geq 2$. Let z be a neighbour of x having distance $i-1$ to y. Let $y_1, \ldots, y_{i-1}$ be the $i-1$ neighbours of y having distance $i-2$ to z. Let N be the line through y parallel with xz and y_i be the point on this line having distance $i-1$ to x, then y_j $(1 \leq j \leq i)$ are i different points collinear with y and having distance $i-1$ to x. If y_{i+1} would be another point having these properties, then $yy_{i+1} \| xz$, so $yy_{i+1} = yy_i$ and $y_i = y_{i+1}$, a contradiction. If all lines of $\mathcal{S}$ have a constant number of points, then the near polygon is regular and has the same parameters as a regular near polygon of Hamming type. Hence, $\mathcal{S}$ is a regular near polygon of Hamming type. If not all lines of $\mathcal{S}$ have a constant number of points, then the theorem follows from Lemma 3.3.1 and Theorem 3.1.2.

$\square$

3.4 The nondegenerate support of a near polygon

A partial linear space is called *degenerate* if there is a point incident with exactly one line. Every such point is called *superthin*. Notice that the definition of degeneracy is not conform to other definitions given in the literature (e.g. degenerate polar spaces); however, this yields no confusion in the sequel. Although a near 0-gon has only one point and no lines, it is by our definition not degenerate. A near 2-gon consists of one line with a number of points on it, hence it is degenerate. The degenerate near quadrangles are just the degenerate generalized quadrangles.

Let $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a partial linear space. We give now three constructions slightly altering the structure of $\mathbf{S}$. Constructions (1) and (2) increase the "amount" of degeneracy, while the inverse construction (3) reduces this amount. Construction (3) can then be used to define the nondegenerate support of $\mathbf{S}$.

(1) Assume $p \in \mathcal{P}$, then we denote by $\mathbf{S}_p = (\mathcal{P}_p, \mathcal{L}_p, \mathcal{I}_p)$ the following partial linear space:

$$\begin{aligned} \mathcal{P}_p &= \mathcal{P} \cup \{(\infty)\}, \text{ where } (\infty) \notin \mathcal{P}; \\ \mathcal{L}_p &= \mathcal{L} \cup \{[\infty]\}, \text{ where } [\infty] \notin \mathcal{L}; \\ \mathcal{I}_p &= \mathcal{I} \cup \{(p, [\infty]), ((\infty), [\infty])\}. \end{aligned}$$

(2) Assume $L \in \mathcal{L}$, then we denote by $\mathbf{S}_L = (\mathcal{P}_L, \mathcal{L}_L, \mathcal{I}_L)$ the following partial linear space:

$$\begin{aligned} \mathcal{P}_L &= \mathcal{P} \cup \{(\infty)\}, \text{ where } (\infty) \notin \mathcal{P}; \\ \mathcal{L}_L &= \mathcal{L}; \\ \mathcal{I}_L &= \mathcal{I} \cup \{((\infty), L)\}. \end{aligned}$$

(3) Assume that $p \in \mathcal{P}$ is a superthin point incident with the line L, then we denote by $\mathbf{S}^p = (\mathcal{P}^p, \mathcal{L}^p, \mathcal{I}^p)$ the following partial linear space:

$$\begin{aligned} \mathcal{P}^p &= \mathcal{P} \setminus \{p\}; \\ \mathcal{L}^p &= \mathcal{L} \setminus \{L\} \text{ if } L \text{ has length 2, while } \mathcal{L}^p = \mathcal{L} \text{ in the other case;} \\ \mathcal{I}^p &= \mathcal{I} \cap (\mathcal{P}^p \times \mathcal{L}^p). \end{aligned}$$

Suppose that p_1 is a superthin point of a (finite) partial linear space $\mathbf{S}$. If p_2 is a superthin point of $\mathbf{S}^{p_1}$, then $(\mathbf{S}^{p_1})^{p_2}$ is again a partial linear space. Repeating this argument, one finally finds a nondegenerate partial linear space $(\ldots(\mathbf{S}^{p_1})\ldots)^{p_l}$, which is called the *nondegenerate support of* $\mathbf{S}$. We prove now that this is a good definition, i.e. that it is independent of the order in which the superthin points are considered.

Proof. We may suppose that $\mathbf{S}$ is connected, otherwise one can reason on each connected component. Suppose that we obtain two nondegenerate partial linear spaces $\mathbf{S}' = (\ldots(\mathbf{S}^{p_1})\ldots)^{p_k}$ and $\mathbf{S}'' = (\ldots(\mathbf{S}^{q_1})\ldots)^{q_l}$. Here p_1 and q_1 are superthin points of $\mathbf{S}$ and p_i $(1 < i \leq k)$ and q_j $(1 < j \leq l)$ are superthin points of $(\ldots(\mathbf{S}^{p_1})\ldots)^{p_{i-1}}$ and $(\ldots(\mathbf{S}^{q_1})\ldots)^{q_{j-1}}$ respectively. Suppose $p_i \notin \{q_1, \ldots, q_l\}$ with i the smallest integer with that property, then p_i is not a superthin point of $\mathbf{S}''$. If p_i is contained in two lines L and M of

$\mathbf{S}''$, then at least one of these lines, say L, is not a line of $(\ldots(\mathbf{S}^{p_1})\ldots)^{p_{i-1}}$. Let $r \neq p_i$ be a point of $\mathbf{S}''$ on L. Then $r \notin \{q_1, \ldots, q_l\}$ and $r = p_{i'}$ with $i' < i$, a contradiction. Hence $\mathbf{S}''$ is the near 0-gon with unique point p_i (since there are no lines through p_i). Hence we may assume that $\mathbf{S}'$ is not isomorphic to a near 0-gon and using the same reasoning as above, we may conclude that $q_j \in \{p_1, \ldots, p_k\}$ for all $j \in \{1, \ldots, l\}$. This implies however that $\mathcal{P} - \{p_i\} = \{q_1, \ldots, q_l\} \subseteq \{p_1, \ldots, p_k\}$, a contradiction. Hence $\{p_1, \ldots, p_k\} = \{q_1, \ldots, q_l\} = A$. Now $\mathbf{S}'$ and $\mathbf{S}''$ have the same points (namely the points not in A) and the same lines (namely the lines of $\mathbf{S}$ having at least two points not in A), which implies that $\mathbf{S}'$ and $\mathbf{S}''$ are isomorphic. $\square$

If p is a superthin point of a near polygon $\mathbf{S}$, then $\mathbf{S}^p$ is again a near polygon. Hence, the nondegenerate support of a near polygon is again a near polygon. If p is an arbitrary point of a near polygon $\mathbf{S}$, then $\mathbf{S}_p$ is also a near polygon. Under certain circumstances, also construction (2) yields a near polygon.

Lemma 3.4.1
Let $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a near polygon. If $L \in \mathcal{L}$ is a line which is not parallel to any other line $M \neq L$, then $\mathbf{S}_L$ is a near polygon.

Proof. Choose a point p and a line M of $\mathbf{S}_L$. If $p \neq (\infty)$, then the distance between p and any point of M is not changed by adding (∞) to $\mathcal{P}$. If $p = (\infty)$, we may suppose that $M \neq L$. Let α and β be the unique points of L and M such that $\mathrm{d}(m, l) = \mathrm{d}(l, \alpha) + \mathrm{d}(\alpha, \beta) + \mathrm{d}(\beta, m)$ holds for all points l of L and m of M. The point β is now the unique point of M nearest (in $\mathbf{S}_L$) to (∞). $\square$

Corollary 3.4.2
If $p \in \mathcal{P}$ is a superthin point which is incident with the line L, then $\mathbf{S}_L$ is a near polygon.

Proof. If p is a superthin point of a near polygon which is incident with L, then every line $M \neq L$ is non-parallel to L. Indeed, if $(p, q, \ldots)$ is a shortest path from p to M, then $p, q \in L$ and $\mathrm{d}(q, M) = \mathrm{d}(p, M) - 1$. $\square$

3.5 Quads

3.5.1 Special point sets

In this section $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ denotes an arbitrary partial linear space. A subset X of $\mathcal{P}$ is called a *subspace* if all the points of a line L are in X as soon as two of its points are in X. Every such X induces a partial linear space $\mathbf{S}_X = (X, \mathcal{L}', \mathcal{I}')$. Here $\mathcal{L}'$ denotes the set of lines having all their points in X

and $\mathcal{I}'$=$\mathcal{I}\cap(X\times\mathcal{L}')$. A nonempty subspace X is called *Cameron closed* if $\mathbf{S}_X$ is connected and if all the common neighbours of two different points of X are as well contained in X. A subset X of $\mathcal{P}$ is called *geodetically closed* if X is a subspace and if all the points of a shortest path between two points of X are as well contained in X. The set X is called a *quad* if X is geodetically closed and if X induces a nondegenerate generalized quadrangle. Since no confusion is possible in the sequel, this generalized quadrangle will also be called a quad. Finally, we call a subset X of the point set *classical* if for every point x there is a unique point $\pi_X(x)\in X$ such that $\mathrm{d}(x,y)=\mathrm{d}(x,\pi_X(x))+\mathrm{d}(\pi_X(x),y)$ for all $y\in X$. One easily checks that every classical set X is geodetically closed. We give now some properties of quads in the case that $\mathbf{S}$ is a near polygon.

Theorem 3.5.1 ([80] and [82])
If there is a quad through two points at distance 2, then this quad is unique. As a consequence, every two different quads are either disjoint or meet in a point or line of the near polygon.

Concerning the existence of quads, one can say the following.

Theorem 3.5.2 (Theorem of Yanushka ([80] and [82]))
Let x and y be two points of a near polygon $\mathbf{S}$ at mutual distance 2. If x and y have two common neighbours c and d such that the line xc contains at least three points, then x and y are in a unique quad. This quad contains these points of the near polygon which have distance at most 2 from x,y,c and d.

We already have given the point-line relation (see property (NP)) and the line-line relation (see Theorem 3.1.1), we give now the point-quad relation.

Theorem 3.5.3 ([80] and [82])
Let p be a point of a near polygon $\mathbf{S}$ and let Q be a quad, then exactly one of the following cases occurs.

(1) *There is a unique point $q\in Q$ such that $d(p,r)=d(p,q)+d(q,r)$ for all points $r\in Q$. In this case (p,Q) is called* classical. *The point q is called the projection of p on Q and is denoted by $\pi_Q(p)$, or shortly by $\pi(p)$.*

(2) *The points of Q which are nearest to p form an ovoid of Q. In this case (p,Q) is called* ovoidal.

(3) *The quad Q induces a dual grid. Let A be the set of points of Q at smallest distance k from p. Let B, respectively C, denote those points of Q, that have distance $k+1$, respectively $k+2$ to p. Then*

(a) $|A| \geq 2$ *and* $|C| \geq 1$,

(b) B *and* $A \cup C$ *are the two maximal cocliques of the point graph of* Q.

In this case (p, Q) *is called* thin ovoidal.

3.5.2 Local spaces

The notion of local space was already defined for near polygons which have quads through every two points at distance two. We slightly modify this definition such that local spaces exist for any near polygon. With this new definition, a local space is not necessary a linear space.

Let x be a point of a near polygon $\mathbf{S}$. For a quad $\mathbf{Q}$ through x, let $L_{\mathbf{Q}}$ denote the set of lines through x which are also lines of $\mathbf{Q}$. The following incidence structure $\mathbf{S}_x$, called the *local space at* x, can then be defined:

- the point set of $\mathbf{S}_x$ is the set of lines of $\mathbf{S}$ through x;
- the line set of $\mathbf{S}_x$ is equal to $\{L_{\mathbf{Q}} | \mathbf{Q}$ is a quad through $x\}$;
- incidence is the natural one, i.e containment.

By Theorem 3.5.1, $\mathbf{S}_x$ is a partial linear space. If there are quads through every two points at distance 2, then $\mathbf{S}_x$ is a linear space.

3.6 Dual polar spaces and classical near polygons

Definition. A near $(2d)$-gon, $d \geq 1$, is called *classical* if the following two conditions are satisfied:

(i) every two points at distance 2 are contained in a quad;

(ii) every point-quad relation is classical.

We have the following important theorem.

Theorem 3.6.1 ([22])
The dual polar spaces are precisely the classical near polygons.

Theorem 3.6.2
Let $\mathbf{S}$ *be a near hexagon satisfying the following conditions:*

(1) *every two points at distance* 2 *are contained in a quad,*

(2) *all local spaces are (possible degenerate) projective planes,*

then $\mathbf{S}$ *is a classical near hexagon.*

Proof. Let x be a point of $\mathbf{S}$ and let Q be a quad of $\mathbf{S}$ such that (x, Q) is not classical, then $\mathrm{d}(x, Q) = 2$ and (x, Q) is ovoidal. Let y be a point of Q at distance 2 from x and let Q' be the quad through x and y. The quads Q and Q' determine now two mutually disjoint lines of the local space S_y, a contradiction. $\square$

Given a number of classical near polygons then many others can be constructed. For, if $\mathbf{S}_1$ and $\mathbf{S}_2$ are classical near polygons, then $\mathbf{S}_1 \times \mathbf{S}_2$ is also a classical near polygon. Given a number of polar spaces, then many others can be constructed as shown now. Let $(\Gamma_i)_{i\in I}$ be a family of polar spaces defined on the sets $(P_i)_{i\in I}$ and with $(\mathcal{A}_i)_{i\in I}$ as collection of subspaces. We suppose that all the P_i's are mutually disjoint. A new polar space, called the *direct sum* can then be constructed on the set $P = \bigcup_{i\in I} P_i$. A general form of a subspace is as follows: $\cup_{i\in I} a_i$, where $a_i \in A_i$. The connection between the two constructions is given in the following theorem which is easy to verify.

Theorem 3.6.3
Let Γ_1 *and* Γ_2 *be two polar spaces and let* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *be the dual polar spaces related to it. Let* Γ *be the direct sum of* Γ_1 *and* Γ_2 *and let* $\mathbf{S}$ *be the dual polar space related to it, then* $\mathbf{S} \simeq \mathbf{S}_1 \times \mathbf{S}_2$.

A polar space is called *irreducible* if it is not isomorphic to a direct sum of at least two polar spaces of rank ≥ 1. The classification of classical near polygons is equivalent to the classification of irreducible polar spaces of rank $n \geq 2$. If $n \geq 3$ and if all lines have length at least three, then the irreducible polar space is isomorphic to $W^D(2n-1, q)$, $Q^D(2n, q)$, $[Q^-(2n+1, q)]^D$, $[Q^+(2n-1, q)]$, $H^D(2n, q)$ or $H^D(2n-1, q)$ ([94]). The case where there are some lines of length 2 is settled in the next theorem.

Theorem 3.6.4 ([19])
If Γ *is an irreducible polar space of finite rank having some line of length* 2, *then* Γ *is a dualized projective space.*

Remains to define what a dualized projective space ([19]) is.

Definition. Let $\mathbf{P}$ be an irreducible projective space of dimension $d \geq 1$ and let $\mathbf{P}^*$ be the dual of $\mathbf{P}$ whose points are the $(d-1)$-dimensional subspaces of $\mathbf{P}$ and whose lines are the $(d-2)$-dimensional subspaces of $\mathbf{P}$. If $d = 1$, then we assume that $\mathbf{P}^*$ is a copy of $\mathbf{P}$ disjoint from it. Consider now the following incidence structure $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$. The point set $\mathcal{P}$ is the union of the point sets of $\mathbf{P}$ and $\mathbf{P}^*$. The lines have length 2 and look like $\{x, y\}$, where x is a point of $\mathbf{P}$ and where y is a hyperplane through x. The set $\mathcal{P}$ together with the subspaces of the geometry $\mathbf{S}$ form then a polar space of rank $d+1$. Dual polar spaces obtained in the above described way are called *dualized projective spaces.*

3.7 Near polygons with quads

Let $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a near polygon such that

(1) every line has at least three points,

(2) every two points at distance 2 have at least two common neighbours.

By Yanushka's Theorem (Theorem 3.5.2), every two points at distance 2 are contained in a unique quad. In the sequel, let $\mathbf{Q}$ denote a fixed quad. If $x \in \mathcal{P}$, then the following possibilities can occur, see Theorem 3.5.3.

(a) $(x, \mathbf{Q})$ is classical, in this case there is a unique point $\pi(x)$ in $\mathbf{Q}$ nearest to x;

(b) $(x, \mathbf{Q})$ is ovoidal.

We define now the following sets:

$$\begin{aligned} N_i &:= \{x \in \mathcal{P} |\, \mathrm{d}(x, \mathbf{Q}) = i\}, \\ N_{i,C} &:= \{x \in N_i | x \text{ is classical with respect to } \mathbf{Q}\}, \\ N_{i,O} &:= \{x \in N_i | x \text{ is ovoidal with respect to } \mathbf{Q}\}. \end{aligned}$$

Clearly $N_0 = \mathbf{Q}$, $N_{1,O} = \emptyset$, $N_{d-1,C} = \emptyset$ and $N_d = \emptyset$.

Theorem 3.7.1 ([17], p. 148)

(1) *If L is a line contained in $N_i \cup N_{i+1}$ for some i, then $|L \cap N_i| = 1$.*

(2) *There are no edges between $N_{i,O}$ and $N_{i,C}$ for all i.*

(3) *If L is a line contained in $N_{i,C}$ for some i, then $\pi(L)$ is a line of $\mathbf{Q}$ parallel to L.*

(4) *If L is a line contained in $N_{i,O}$ for some i, then the points of L determine a fan of ovoids in $\mathbf{Q}$.*

(5) *If L is contained in $N_{i,O} \cup N_{i+1}$ for some i, then L is even contained in $N_{i,O} \cup N_{i+1,O}$. In this case all points of L determine the same ovoid of $\mathbf{Q}$.*

(6) *If L is contained in $N_{i,C} \cup N_{i+1,C}$ for some i, then all points of L determine the same point in $\mathbf{Q}$.*

(7) *If L is contained in $N_{i,C} \cup N_{i+1,O}$ for some i, then the points of $L \cap N_{i+1,O}$ determine a rosette of ovoids in $\mathbf{Q}$. The common point of all these ovoids is the point of $\mathbf{Q}$ determined by $L \cap N_{i,C}$.*

Let x be a fixed point of $\mathbf{S}$. For every point $y \in \mathcal{P}$, we define

$S(x,y) = \{L | x \, \mathcal{I} \, L$ and L contains a point at distance $\mathrm{d}(x,y)-1$ from $y\}$.

Theorem 3.7.2 ([17], Lemma 16 (i))
$S(x,y)$ is a subspace of the local space $\mathbf{S}_x$ for all points $y \in \mathcal{P}$.

Theorem 3.7.3 ([17], Theorem 4)
Let x and y be two points at distance i, then there is a unique geodetically closed sub near $2i$-gon $H(x,y)$ containing x and y. Moreover

$$H(x,y) = \{u | S(x,u) \subseteq S(x,y)\}.$$

Suppose now that $\mathbf{S}$ is regular with parameters s, t, t_i $(0 \leq i \leq d)$. The above conditions say that $s \geq 2$ and $t_2 \geq 1$.

Theorem 3.7.4 ([17], p. 158)

$$\begin{aligned} |N_{i,C}| &= (s+1)(1+st_2)s^i \frac{\prod_{j=2}^{i+1}(t-t_j)}{\prod_{j=2}^{i}(1+t_j)} \textit{ for all } i \in \{1,\ldots,d-1\}, \\ |N_{i,O}| &= s^i(s+1)[t_{i+1} - t_2(t_i+1)] \frac{\prod_{j=2}^{i}(t-t_j)}{\prod_{j=2}^{i}(1+t_j)} \textit{ for all } i \in \{2,\ldots,d-1\}. \end{aligned}$$

Corollary 3.7.5

(1) *$t_{i+1} \geq t_2(t_i+1)$ for all $i \in \{0,\ldots,d-1\}$.*

(2) *$\mathbf{S}$ is classical if and only if $t_{i+1} = t_2(t_i+1)$ for all $i \in \{0,\ldots,d-1\}$.*

3.8 Near polygons with good quads

Definition. A quad $\mathbf{Q}$ of a near polygon is called *good* if every point of $\mathbf{Q}$ is incident with the same number of lines; we denote this number by $t_{\mathbf{Q}}+1$. Only quads isomorphic to nonsymmetrical dual grids are not good.

Theorem 3.8.1 ([17], Lemma 19)
Let $\mathbf{S}$ be a near polygon with the property that every two points at distance 2 are contained in a good quad, then each point of $\mathbf{S}$ is incident with the same number of lines.

Proof. Let x and y be two collinear points. The point x (respectively y) is incident with t_x+1 (respectively t_y+1) lines. Now

$$t_x + 1 = 1 + \sum t_{\mathbf{Q}} = t_y + 1,$$

where the summation ranges over all quads $\mathbf{Q}$ through the line xy. Hence x and y are incident with the same number of lines and the result follows by connectedness of $\mathbf{S}$. □

Remark. If $\mathbf{S}$ is a dualized projective space, then every two points at distance two are contained in a quad, but not every point is incident with the same number of lines; hence the condition that quads must be good is necessary.

If $\mathbf{S}=(\mathcal{P},\mathcal{L},\mathcal{I})$ is a near 2-gon or a good GQ, then $|\Gamma_i(p)|$ $(i \in \{0,1,2\})$ is independent of $p \in \mathcal{P}$. We derive a similar property for near hexagons.

Theorem 3.8.2
Let $\mathbf{S}=(\mathcal{P},\mathcal{L},\mathcal{I})$ be a near hexagon such that every two points at distance 2 are contained in a good quad, then $|\Gamma_i(p)|$ $(i \in \{0,1,2,3\})$ is independent of $p \in \mathcal{P}$.

Proof. If not all lines of $\mathbf{S}$ are incident with the same number of points, then Theorem 3.1.2 implies that $\mathbf{S}$ is the direct product of a line with a good GQ. It is straightforward to check that the result is true in this case. Hence we may suppose that all lines are incident with $s+1$ points. The previous theorem implies then that $\mathbf{S}$ has an order (s,t). Now, let $p \in \mathcal{P}$ be a fixed point and put $n_i = |\Gamma_i(p)|$. Then $n_0 = 1, n_1 = s(t+1)$. Let V be the set of quads through p. Counting points in $\Gamma_2(p)$ we find

$$n_2 = s^2 \sum_{x \in V} t_x. \tag{3.1}$$

Counting edges between $\Gamma_2(p)$ and $\Gamma_3(p)$ we find that

$$n_3(t+1) = s^3 \sum_{x \in V} t_x(t - t_x). \tag{3.2}$$

Finally, counting triples $(L_1, L_2, \mathbf{Q})$ where L_1, L_2 are two different lines through p and $\mathbf{Q}$ is the quad through L_1 and L_2, yields

$$t(t+1) = \sum_{x \in V} t_x(t_x + 1). \tag{3.3}$$

Eliminating $\sum t_x$ and $\sum t_x^2$, we find that $n_3 = s(n_2 - s^2 t)$. Together with $v = n_0 + n_1 + n_2 + n_3$ this gives

$$n_2 = \frac{v}{s+1} - 1 + st(s-1), \tag{3.4}$$

$$n_3 = s(\frac{v}{s+1} - st - 1). \tag{3.5}$$

□

Corollary 3.8.3
If $\mathbf{S}$ *is a near hexagon satisfying the property that every two points at distance* 2 *are contained in a quad of order* (s, t_1) *or* (s, t_2), $s \geq 1$ *and* $1 \leq t_1 < t_2$, *then for each* $i \in \{1, 2\}$, *the number of quads of order* (s, t_i) *through a point is independent of that point.*

Proof. This follows from equations (3.1), (3.3) and (3.4). □

Remark. The previous corollary was proved in [14] in the case that $s = 2, t_1 = 1, t_2 = 2$ by using the same double countings as in the proof of Theorem 3.8.2.

3.9 Some characterizations for classical and regular near hexagons

In this section we suppose that $\mathbf{S}$ is a near hexagon of order (s, t) satisfying the property that every two points at distance 2 are contained in a good quad. Recall that every point of a good quad $\mathbf{Q}$ is incident with the same number of lines, this number being denoted by $t_{\mathbf{Q}} + 1$. Let N_x be the total number of quads through a fixed point x, let v denote the total number of points and put $M := \frac{n_2}{s^2} = \frac{1}{s^2}(\frac{v}{s+1} - 1 + st(s-1))$. For a point x of $\mathbf{S}$, we define $T_x + 1 := \max |\Gamma(x) \cap \Gamma(y)|$, where y ranges over all n_2 elements of $\Gamma_2(x)$.

Theorem 3.9.1

(*i*) *If* $\alpha, \beta \in \mathbb{R}$, *then*

$$\sum (t_{\mathbf{Q}}^2 + \alpha t_{\mathbf{Q}} + \beta) = t(t+1) + (\alpha - 1)M + \beta N_x,$$

where the summation ranges over all quads $\mathbf{Q}$ *through* x.

(*ii*) *The inequality* $t(t+1) \geq 2M$ *holds, with equality if and only if every quad is a grid.*

(*iii*) *One has the following lower bound for* N_x:

$$N_x \geq \frac{M^2}{t(t+1) - M},$$

and equality holds if and only if every quad through x *has the same order. Hence,* $\mathbf{S}$ *is a regular near hexagon if and only if the equality holds for every point of the near hexagon.*

(*iv*) *The inequality* $M \geq t+1$ *holds, with equality if and only if* $\mathbf{S}$ *is a Hamming near hexagon.*

(*v*) *The inequality* $[M - (t+1)]T_x \geq t(t+1) - 2M$ *holds with equality if and only if the local space at* x *is a (possible degenerate) projective plane. If* $\mathbf{S}$ *is not a Hamming near hexagon, then the inequality is equivalent to*

$$T_x \geq \frac{t(t+1) - 2M}{M - (t+1)}.$$

Hence $\mathbf{S}$ *is a classical near hexagon if and only if the equality holds for every point* x *of the near hexagon.*

Proof.

(i) The result follows from equations (3.1), (3.3) and (3.4).

(ii) Counting edges between $\Gamma(x)$ and $\Gamma_2(x)$ yields

$$[s(t+1)][st] \geq 2n_2 = 2s^2 M,$$

and equality holds if and only if $|\Gamma(x) \cap \Gamma(y)| = 2$ for every point y at distance 2 from x. The result follows now immediately.

(iii) From (i) it follows that $\sum (t_{\mathbf{Q}} - \gamma)^2 = N_x\gamma^2 - (2M)\gamma + t(t+1) - M$. If not all quads through x have the same order, then the above quadratic equation in γ cannot have a real root, hence the discriminant $D = M^2 + N_x M - t(t+1)N_x$ must be negative. If all quads through x have order (s, t_2), then one calculates that $N_x = \frac{t(t+1)}{t_2(t_2+1)}$, $M = \frac{t(t+1)}{1+t_2}$ and hence $D = 0$.

(iv) For every point $y \in \Gamma_2(x)$, one has that $|\Gamma(x) \cap \Gamma(y)| \leq t$. Counting edges between $\Gamma(x)$ and $\Gamma_2(x)$ yields

$$[s(t+1)][st] \leq [s^2 M]t,$$

from which the above inequality follows. If equality holds, then $\mathbf{S}$ is a regular near polygon with $t_2 = t-1$. The number of quads through a line is then equal to $\frac{t}{t_2} = \frac{t}{t-1}$, hence $t = 2, t_2 = 1$ and $\mathbf{S}$ is a Hamming near hexagon.

(v) Applying the Fundamental Theorem for linear spaces, see [26], to the local space at x yields $N_x \geq t+1$. We have now the following inequalities:

$$\begin{aligned} 0 &\geq \sum (t_{\mathbf{Q}} - 1)(t_{\mathbf{Q}} - T_x) \\ &= \sum (t_{\mathbf{Q}}^2 - (T_x + 1)t_{\mathbf{Q}} + T_x) \\ &= t(t+1) - (T_x + 2)M + T_x N_x \\ &\geq t(t+1) - (T_x + 2)M + T_x(t+1), \end{aligned}$$

from which the above equality readily follows. Suppose now that equality holds, then $t_{\mathbf{Q}}$ is equal to 1 or T_x for every quad through x and the local space at x is a possibly degenerate projective plane. Hence equality holds if and only if the local space at x is a (possible degenerate) projective plane and the theorem follows by Theorem 3.6.2.

□

3.10 A construction method for near polygons

The result of the following theorem is mentioned in [14] in the case where X_1 and X_2 are both classical.

Theorem 3.10.1
Let $\mathbf{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathcal{I}_1)$ and $\mathbf{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathcal{I}_2)$ be two near polygons. Let X_1, respectively X_2, be a classical subset of $\mathcal{P}_1$, respectively a subspace of $\mathbf{S}_2$. Denote by $d_i(\cdot,\cdot), i \in \{1,2\}$ the distance in $\mathbf{S}_i$. Suppose that there is a bijection $\alpha : X_1 \mapsto X_2$ such that $d_2(\alpha(x), \alpha(y)) = d_1(x,y)$ for all points $x, y \in X_1$. Then the partial linear space obtained by identifying the corresponding points and lines in X_1 and X_2 is a near polygon, which we denote by $S[\mathbf{S}_1, \mathbf{S}_2, X_1, X_2, \alpha]$.

Proof. Choose in $S[\mathbf{S}_1, \mathbf{S}_2, X_1, X_2, \alpha]$ a point p and a line L. We want to prove that there is a unique point on L nearest to p. We define $\mathrm{d}(\cdot, \cdot)$ as the distance in $S[\mathbf{S}_1, \mathbf{S}_2, X_1, X_2, \alpha]$. We consider three cases.

(1) Assume that p and L both belong to the same near polygon $\mathbf{S}_i$ $(i = 1, 2)$. As $\mathrm{d}_2(\alpha(x), \alpha(y)) = \mathrm{d}_1(x, y)$ for all points $x, y \in X_1$, there exists for any two points a and b from $\mathcal{P}_i$ $(i = 1, 2)$ a shortest path from a to b only containing points of $\mathbf{S}_i$ and hence $\mathrm{d}(a, b) = \mathrm{d}_i(a, b)$ $(i = 1, 2)$. This implies that L contains a unique point nearest to p.

(2) Assume that $p \in \mathcal{P}_1$ while $L \in \mathcal{L}_2$. Let l_1, l_2 be two points such that $\mathrm{d}(p, l_1) = \mathrm{d}(p, l_2) = \mathrm{d}(p, L)$. Since X_1 is a classical subset, there exists a shortest path from p to l_i, that contains the point $\pi_{X_1}(p)$. Hence $l_1 = l_2$ is the unique point of L nearest to $\pi_{X_1}(p)$ (in $\mathbf{S}_2$).

(3) Finally assume $p \in \mathcal{P}_2$ while $L \in \mathcal{L}_1$. Let l_1 and l_2 be two different points of L such that $\mathrm{d}(p, l_1) = \mathrm{d}(p, l_2) = \mathrm{d}(p, L)$. Define $s_i = \pi_{X_1}(l_i)$. Since X_1 is a classical subset, there exists a shortest path from p to l_i containing the point s_i. If $s_1 = s_2$ then l_1 and l_2 are two different points on L, and both are nearest to s_1, a contradiction. If $s_1 \neq s_2$ then, since X_1 is classical, $\mathrm{d}(s_1, s_2) = 1$ and $\mathrm{d}(l_1, s_1) = \mathrm{d}(l_2, s_2)$. Denote by s_3 the unique point of $s_1 s_2$ nearest to p, then $s_1 \neq s_3 \neq s_2$. Let l_3 denote the unique point of L, for which $\mathrm{d}(l_1, s_1) = \mathrm{d}(l_2, s_2) = \mathrm{d}(l_3, s_3)$. Now, there exists a path of length $\mathrm{d}(p, l_2) - 1$ from p to l_3, a contradiction.

□

Remark. In Section 5.2 we will describe a class of near polygons which can be obtained by successive application of the previous theorem.

Chapter 4

Association schemes associated with near polygons

4.1 Association schemes and distance regular graphs

Definition.
An *association scheme* is a pair $(X, \mathcal{R})$ with X a finite nonempty set and $\mathcal{R} = \{R_0, \ldots, R_n\}$ a partition of $X \times X$ such that the following conditions are satisfied.

(1) $R_0 = \{(x, x) | x \in X\}$.

(2) $R_i^T \in \{R_0, \ldots, R_n\}$ for all $i \in \{0, \ldots, n\}$ $(R_i^T = \{(y, x) | (x, y) \in R_i\})$.

(3) There exist integers p_{jk}^i, called the *parameters of the association scheme*, having the following property: for all $(x, y) \in R_i$, there exist exactly p_{jk}^i elements $z \in X$ such that $(x, z) \in R_j$ and $(z, y) \in R_k$.

If $R_i^T = R_i$, for all $i \in \{0, \ldots, n\}$, then the association scheme is called *symmetric* and we have $p_{jk}^i = p_{kj}^i$ for all $i, j, k \in \{0, \ldots, d\}$. The number $n_i = p_{ii}^0$ is called the *valency* of R_i. For more details on association schemes we refer to [3] or chapter 2 of [15].

Let Γ be a graph of diameter d and let $\mathrm{d}(\cdot, \cdot)$ be the corresponding distance function. The graph Γ is called *distance regular* if and only if there exist parameters a_i, b_i, c_i $(i \in \{0, \ldots, d\})$ such that the following equalities hold for every two vertices x and y of Γ (we put $i = \mathrm{d}(x, y)$ in each of these equalities):

(1) $|\Gamma_i(x) \cap \Gamma(y)| = a_i$,

(2) $|\Gamma_{i+1}(x) \cap \Gamma(y)| = b_i$,

(3) $|\Gamma_{i-1}(x) \cap \Gamma(y)| = c_i$.

We call $k := b_0$ the *valency* of Γ. Clearly $a_0 = 0, c_0 = 0, c_1 = 1, b_d = 0$ and $a_i + b_i + c_i = k$ for all $i \in \{0, \ldots, d\}$. A distance regular graph of diameter 2 is called a *strongly regular graph.*

The relation between distance regular graphs and association schemes is given by the following theorem.

Theorem 4.1.1 (([15]))
Suppose that Γ is a distance regular graph of diameter d and vertex set X. Let $R_i = \{(x, y) \in X \times X |\, d(x, y) = i\}$ for all $i \in \{0, \ldots, d\}$ and let $\mathcal{R} = \{R_0, \ldots, R_d\}$, then $(X, \mathcal{R})$ is a symmetric association scheme.

The following known result gives the relation between distance regular graphs and regular near polygons.

Theorem 4.1.2
A near polygon is regular if and only if its point graph is a distance regular graph.

Proof. Let $\mathbf{S}$ be a regular near $(2d)$-gon with parameters s, t_i $(i \in \{0, \ldots, d\})$ and $t = t_d$. Choose $i \in \{0, \ldots, d\}$ and consider two points x and y at distance i from each other. Put $a_i = |\Gamma_i(x) \cap \Gamma(y)|$, $b_i = |\Gamma_{i+1}(x) \cap \Gamma(y)|$ and $c_i = |\Gamma_{i-1}(x) \cap \Gamma(y)|$. We will prove that a_i, b_i and c_i are only dependent on i. Clearly $c_i = t_i + 1$ and $a_i + b_i + c_i = s(t+1)$. Now, if $z \in \Gamma_i(x) \cap \Gamma(y)$, then yz contains a unique point at distance $i - 1$ from x. There are now $t_i + 1$ lines through y containing a point at distance $i - 1$ from x and on each of these lines, there are $s - 1$ points in $\Gamma_i(x) \cap \Gamma(y)$. Hence $a_i = (s - 1)(t_i + 1)$ and $b_i = s(t + 1) - a_i - c_i = s(t - t_i)$. Conversely, suppose that Γ is the distance regular point graph of a near $2d$-gon $\mathbf{S}$. Every two adjacent vertices of Γ are then contained in a unique clique of size $a_1 + 2$. Hence, every line of $\mathbf{S}$ has length $s + 1 = a_1 + 2$. Clearly $t_i + 1 = c_i$ for all $i \in \{0, \ldots, d\}$. As a consequence $\mathbf{S}$ is regular. □

Corollary 4.1.3
With every regular near polygon, there is associated a symmetric association scheme.

If we have an association scheme (distance regular graph) with certain parameters, then there are conditions on the parameters of this association scheme (distance regular graph), see [3] and [15]. For instance, in the case of a regular near hexagon with parameters (s, t_2, t), $s > 1$, one of the Krein conditions yields the inequality

$$t \leq s^3 + t_2(s^2 - s + 1). \tag{4.1}$$

This inequality is known as the *Mathon bound* (see also [43]).

Theorem 4.1.4 ([80])
Let **S** *be a near hexagon with parameters* $s = 2$, t_2 *and* t. *Then we have the following possibilities for* (t_2, t)*:*

(1) $(t_2, t) = (0, 1)$,	(2) $(t_2, t) = (0, 2)$,
(3) $(t_2, t) = (0, 8)$,	(4) $(t_2, t) = (1, 2)$,
(5) $(t_2, t) = (1, 11)$,	(6) $(t_2, t) = (2, 6)$,
(7) $(t_2, t) = (2, 14)$,	(8) $(t_2, t) = (4, 20)$.

For every of these parameters, there is a unique near hexagon, except for case (2) where there is up to duality only one. The near hexagons with parameters as in (1), (2) or (3) are generalized hexagons. The near hexagons with parameters as in (4), (6) or (8) satisfy $t = t_2(t_2 + 1)$ and hence they are dual polar spaces. One can find a description of these near polygons in Chapter 3. The near hexagon with parameters $(s, t_2, t) = (2, 1, 11)$ is the one related to the extended ternary Golay code. Finally, the near hexagon with parameters $(s, t_2, t) = (2, 2, 14)$ is the one related to the Steiner system $S(5, 8, 24)$.

Theorem 4.1.5 ([80])
Let **S** *be a near hexagon with parameters* $s = 3$, t_2 *and* t. *Then we have one of the following possibilities for* (t_2, t)*:*

(1) $(t_2, t) = (0, 1)$,	(2) $(t_2, t) = (0, 3)$,
(3) $(t_2, t) = (0, 27)$,	(4) $(t_2, t) = (1, 2)$,
(5) $(t_2, t) = (1, 9)$,	(6) $(t_2, t) = (1, 34)$,
(7) $(t_2, t) = (3, 12)$,	(8) $(t_2, t) = (3, 27)$,
(9) $(t_2, t) = (3, 48)$,	(10) $(t_2, t) = (9, 90)$.

Near polygons with parameters as in (1), (2) or (3) are generalized hexagons. Except for case (1) (unique example), it is not known whether they are uniquely determined by their parameters. Near hexagons with parameters as in (4), (7) or (10) are dual polar spaces (since $t = t_2(t_2 + 1)$). We refer to Chapter 3 for a description of these near hexagons. There is no near hexagon with parameters as in (5), see [10]. Whether there exists a near polygon with parameters as in (6) is still an open problem. It is also known that there exists no near hexagon with parameters as in (8) or (9), but this will also follow from the discussion given in Chapter 8, where we try to classify all near hexagons of order $(3, t)$ which have a quad through every two points at distance 2.

4.2 Some relations on the line set of a near hexagon

If $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a near hexagon, then we will define the following relations on $\mathcal{L}$:

$$\begin{aligned}
R_0 &= \{(L, L) | L \in \mathcal{L}\}, \\
R_1 &= \{(L, M) | L \neq M \text{ and } \mathrm{d}(L, M) = 0\}, \\
R_2 &= \{(L, M) | L \| M \text{ and } \mathrm{d}(L, M) = 1\}, \\
R_3 &= \{(L, M) | L \not\| M \text{ and } \mathrm{d}(L, M) = 1\}, \\
R_4 &= \{(L, M) | L \| M \text{ and } \mathrm{d}(L, M) = 2\}.
\end{aligned}$$

For a fixed line L define $n_i(L) = |\{M \in \mathcal{L} | (L, M) \in R_i\}|$ and for a fixed $(L, M) \in R_i$ define $p^i_{jk}(L, M) = |\{N \in \mathcal{L} | (L, N) \in R_j \text{ and } (N, M) \in R_k\}|$. If $n_i(L)$ is independant of L or if no confusion is possible we will write n_i instead of $n_i(L)$. A similar remark holds for the coefficients $p^i_{jk}(L, M)$. As we will see in the next section all the coefficients n_i and allmost all the coefficients p^i_{jk} are constant in the case of a regular near hexagon having quads through every two points at mutual distance 2.

4.3 The numbers n_i and p^i_{jk} in the case of a regular near hexagon with quads

Let $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a regular near hexagon with parameters (s, t_2, t). If $t_2 = 0$, then $\mathbf{S}$ is a generalized hexagon. In this case $R_2 = \emptyset$ and $\mathcal{R} = (R_0, R_1, R_3, R_4)$ defines an association scheme on the set of lines of $\mathbf{S}$. This association scheme is in fact the association scheme associated with the dual of $\mathbf{S}$ (see Corollary 4.1.3). In this section we will suppose that there is a unique quad through every two points at mutual distance 2 (hence $t_2 \geq 1$). This assumption is always satisfied when $s \geq 2, t_2 \geq 1$ (Yanushka's Theorem) or $s = t_2 = 1$.

If x is a fixed point of $\mathbf{S}$, then one calculates that $\Gamma_0(x) = 1, \Gamma_1(x) = s(t+1), \Gamma_2(x) = \frac{s^2t(t+1)}{t_2+1}, \Gamma_3(x) = \frac{s^3t(t-t_2)}{1+t_2}$. Hence

$$v = |\mathcal{P}| = (1+s)(1+st+\frac{s^2t(t-t_2)}{1+t_2})$$

and

$$b = |\mathcal{L}| = (1+t)(1+st+\frac{s^2t(t-t_2)}{1+t_2}).$$

Theorem 4.3.1
The coefficients n_i are constant and equal to the numbers given below:

$$n_0 = 1, n_1 = (s+1)t, n_2 = st_2t,$$

$$n_3 = st(s+1)(t-t_2), n_4 = \frac{s^2t(t-t_2)^2}{1+t_2}.$$

Proof. Let L be a fixed line of $\mathcal{L}$. It is clear that $n_0 = 1$. We have that $n_1 = (s+1)t$, since there are $s+1$ choices for a point on L and t choices for a line $M \neq L$ through that point. Take a fixed point p on L. Counting pairs (q, M) with $q \in \Gamma(p) \cap M$ and $(L, M) \in R_2$ yields $n_2 = st_2t$. Counting triples (r, q, M) with $r \in L, q \in \Gamma(p) \cap M$ and $(L, M) \in R_3$ yields $n_3 = st(s+1)(t-t_2)$. Finally, we have that $n_4 = b - n_0 - n_1 - n_2 - n_3 = \frac{s^2t(t-t_2)^2}{1+t_2}$. □

Corollary 4.3.2
The coefficients p^0_{ij} are constant and equal to $\delta_{ij}n_i$.

Theorem 4.3.3
The coefficients p^1_{ij} are constant and equal to the numbers given below:

$$p^1_{00} = 0, p^1_{01} = p^1_{10} = 1, p^1_{02} = p^1_{20} = 0, p^1_{03} = p^1_{30} = 0, p^1_{04} = p^1_{40} = 0, p^1_{11} = t-1,$$

$$p^1_{12} = p^1_{21} = st_2, p^1_{13} = p^1_{31} = s(t-t_2), p^1_{14} = p^1_{41} = 0, p^1_{22} = st_2(t_2-1),$$

$$p^1_{23} = p^1_{32} = st_2(t-t_2), p^1_{24} = p^1_{42} = 0, p^1_{33} = s(t-t_2)(st_2+t-t_2-1),$$

$$p^1_{34} = p^1_{43} = s^2(t-t_2)^2, p^1_{44} = \frac{s^2(t-t_2)^2(t-t_2-1)}{1+t_2}.$$

Proof. We only need to prove the values for $p^1_{11}, p^1_{12}, p^1_{14}, p^1_{22}, p^1_{24}$ and p^1_{33}. The remaining coefficients are then readily calculated. It is immediate that $p^1_{11} = t-1$ and $p^1_{14} = p^1_{24} = 0$. Let K and L be two lines intersecting in a point p and let $V_{ij} = \{M \in \mathcal{L} | (K, M) \in R_i \text{ and } (L, M) \in R_j\}$. Counting pairs (q,M) where $q \in K \cap M$ and $M \in V_{12}$ gives $p^1_{12} = st_2$. Counting tuples (q_1,q_2,q,M) with $q_1 \in K - \{p\}, q_2 \in L - \{p\}, q \in \Gamma(q_1) \cap \Gamma(q_2) \cap M$ and $M \in V_{22}$, we find that $p^1_{22} = st_2(t_2-1)$. The value of p^1_{23} is then equal to $n_2 - p^1_{20} - p^1_{21} - p^1_{22} - p^1_{24} = st_2(t-t_2)$. If $M \in V_{33}$, let p_1 (and p_2) be those points of M at distance 1 from K (and L). If $p_1 \neq p_2$, then (since $\mathrm{d}(p,p_1) = \mathrm{d}(p,p_2) = 2$) there exists a point p_3 on M such that $\mathrm{d}(p,p_3) = 1$, implying $M \in V_{22}$, a contradiction. Hence two cases can appear.

(1) The point $p_1 (= p_2)$ is collinear with p.
There are $s(t-1)$ choices for a point $q \in \Gamma(p) \backslash (K \cup L)$ and through every such q, there are t lines not intersecting K or L. Hence, the contribution of this case to p^1_{33} equals $A = st(t-1) - 2p^1_{23} - p^1_{22} = s(t-t_2)(t-t_2-1)$.

(2) The point $p_1 (= p_2)$ is not collinear with p.
Counting tuples (q_1,q_2,r,N) with $q_1 \in K \setminus \{p\}, q_2 \in L \setminus \{p\}, r \in N \cap \Gamma(q_1) \cap \Gamma(q_2)$ and $N \in V_{33}$ yields that the contribution of this case to p^1_{33} equals $B = s^2 t_2 (t-t_2)$.

We have now that $p^1_{33} = A + B = s(t-t_2)(st_2 + t - t_2 - 1)$. □

Theorem 4.3.4
The coefficients p^2_{ij} are constant and equal to the numbers given below:

$$p^2_{00} = 0, p^2_{01} = p^2_{10} = 0, p^2_{02} = p^2_{20} = 1, p^2_{03} = p^2_{30} = 0, p^2_{04} = p^2_{40} = 0, p^2_{11} = s+1,$$

$$p^2_{12} = p^2_{21} = (s+1)(t_2-1), p^2_{13} = p^2_{31} = (s+1)(t-t_2), p^2_{14} = p^2_{41} = 0,$$

$$p^2_{22} = st_2^2 - st_2 - t_2 + s, p^2_{23} = p^2_{32} = 0, p^2_{24} = p^2_{42} = st_2(t-t_2),$$

$$p^2_{33} = (s+1)(t-t_2)(2st_2-1), p^2_{34} = p^2_{43} = s(s+1)(t-t_2)(t-2t_2),$$

$$p^2_{44} = s(t-t_2)[\frac{st(t-t_2)}{1+t_2} - t_2 - (s+1)(t-2t_2)].$$

Proof. We only need to prove the values for $p^2_{11}, p^2_{12}, p^2_{14}, p^2_{22}, p^2_{23}$ and p^2_{34}. The remaining coefficients are then readily calculated. It is immediate that $p^2_{11} = s+1$ and $p^2_{14} = 0$. Let K and L be two lines such that $(K,L) \in R_2$ and let Q be the quad through K and L. Finally, let $V_{ij} = \{M \in \mathcal{L} | (K,M) \in R_i \text{ and } (L,M) \in R_j\}$. Counting pairs (p,M) with $p \in K \cap M$ and $M \in V_{12}$ yields $p^2_{12} = (s+1)(t_2-1)$. If $M \in V_{22}$, then every point of M is collinear with two points of Q and hence it is contained in Q. As a consequence p^2_{22} equals the number of lines in Q, not intersecting K or L. Hence $p^2_{22} = st_2^2 - st_2 - t_2 + s$. Suppose $M \in V_{23}$. Let p be the unique point of M at distance 1 from L. As before, p must belong to Q. Now, every line through p parallel with K is contained in Q and hence it is impossible that $(M,L) \in R_3$. So $V_{23} = \emptyset$ and

$p^2_{23} = 0$. Finally, counting tuples (p, q, M) with $p \in K, q \in \Gamma(p) \cap M, M \in V_{34}$ gives $p^2_{34} = s(s+1)(t-t_2)(t-2t_2)$. For, there are $s+1$ choices for p, for fixed p, there are $s(t-t_2)$ choices for $q \notin Q$. Through every such q there are $t+1$ lines, t_2 of them are parallel with K and t_2+1 others have relation R_3 with L. $\square$

Theorem 4.3.5
The coefficients p^3_{ij} are constant and equal to the numbers given below:

$$p^3_{00} = 0, p^3_{01} = p^3_{10} = 0, p^3_{02} = p^3_{20} = 0, p^3_{03} = p^3_{30} = 1, p^3_{04} = p^3_{40} = 0, p^3_{11} = 1,$$

$$p^3_{12} = p^3_{21} = t_2, p^3_{13} = p^3_{31} = st_2 + t - t_2 - 1, p^3_{14} = p^3_{41} = s(t-t_2), p^3_{22} = 0,$$

$$p^3_{23} = p^3_{32} = t_2(2st_2 - 1), p^3_{24} = p^3_{42} = st_2(t - 2t_2),$$

$$p^3_{33} = (s-1)(t-2t_2) + 3st_2(t-2t_2) + s^2(tt_2 + t - t_2 - 2t_2^2),$$

$$p^3_{34} = p^3_{43} = st(s+1)(t-t_2) - p^3_{30} - p^3_{31} - p^3_{32} - p^3_{33},$$

$$p^3_{44} = \frac{s^2t(t-t_2)^2}{1+t_2} - p^3_{41} - p^3_{42} - p^3_{43}.$$

Proof. We only need to prove the values for $p^3_{11}, p^3_{12}, p^3_{14}, p^3_{22}, p^3_{23}$ and p^3_{33}. The remaining coefficients are then readily calculated. Let K and L be two lines such that $(K, L) \in R_3$ and let p and q denote these points of K and L such that $\mathrm{d}(p, q) = 1$. Finally, let $V_{ij} = \{M \in \mathcal{L} | (K, M) \in R_i \text{ and } (L, M) \in R_j\}$. We immediately have that: (i) $p^3_{11} = 1$, (ii) $p^3_{12} = t_2$, (iii) $p^3_{22} = 0$ (since $p^2_{23} = 0$). Counting pairs (r, M) with $r \in K \cap M$ and $M \in V_{14}$ yields $p^3_{14} = s(t-t_2)$. Suppose $M \in V_{23}$. Let $r \in M$ be the point at distance 1 from $u \in L$ and let $v \in K$ be the point collinear with r. There are four cases.

(1) $p = v, u = q$.
The point r is then on the line pq. So, the contribution of this case to p^3_{23} is $A = (s-1)t_2$.

(2) $p = v, u \neq q$.
There are s choices for u. For fixed u, there are t_2 choices for r and through every such r, there are t_2 lines parallel with K. Hence, the contribution of this case to p^3_{23} equals $B = st_2^2$.

(3) $p \neq v, u = q$.
Just like in the second case one calculates that the contribution is equal to $C = st_2(t_2 - 1)$.

(4) $p \neq v, u \neq q$.
There exists a path of length 2 connecting u and v, a contradiction.

Hence $p^3_{23} = A + B + C = t_2(2st_2 - 1)$. Finally, suppose that $M \in V_{33}$. Let u and v be those points of M which are collinear with $u' \in K$ and $v' \in L$. There are eight cases.

(1) $u = v, u' = p, v' = q$.
As before, the contribution is easy to calculate and equal to $A = (s - 1)(t - 2t_2)$.

(2) $u = v, u' = p, v' \neq q$.
The contribution is equal to $B = st_2(t - 2t_2)$.

(3) $u = v, u' \neq p, v' = q$.
The contribution is again equal to $C = st_2(t - 2t_2)$.

(4) $u = v, u' \neq p, v' \neq q$
There exists a path of length 2 connecting u' and v', a contradiction.

(5) $u \neq v, u' = p, v' = q$.
Let Q_1 (Q_2) be the quad through pq and K (pq and L). The line pu is not contained in Q_1 or Q_2. The contribution of this case equals $D = st_2(t - 2t_2)$. For, there are $s(t - 2t_2)$ choices for u and for every such choice there are t_2 choices for v (otherwise v and hence also M is contained in Q_1 or Q_2).

(6) $u \neq v, u' = p, v' \neq q$.
Since $\mathrm{d}(q, u) = \mathrm{d}(q, v) = 2$, there exists a point w on M collinear with q. Hence M is parallel to L, a contradiction.

(7) $u \neq v, u' \neq p, v' = q$.
This is impossible again.

(8) $u \neq v, u' \neq p, v' \neq q$.
There are s choices for u' and s choices for v'. For fixed u' and v', there are $t - t_2$ points a collinear with u', at distance 2 from v' and not collinear with q. For fixed a, there are $1 + t_2$ points b collinear with a and v'. In this way we counted $s^2(t - t_2)(1 + t_2)$ lines ab, $s^2t_2^2$ of them are parallel to K and the others are the lines we are looking for. Hence, the contribution of this case to p^3_{33} is $E = s^2(tt_2 + t - t_2 - 2t_2^2)$.

Summing all the contributions, one finds the value of p^3_{33} mentioned in the theorem. □

Theorem 4.3.6
The coefficients p^4_{ij} with $i \leq 1$ or $j \leq 1$ are constant and equal to the numbers given below.

$$p^4_{00} = 0, p^4_{01} = p^4_{10} = 0, p^4_{02} = p^4_{20} = 0, p^4_{03} = p^4_{30} = 0, p^4_{04} = p^4_{40} = 1, p^4_{11} = 0,$$

$$p^4_{12} = p^4_{21} = 0, p^4_{13} = p^4_{31} = (s+1)(t_2+1), p^4_{14} = p^4_{41} = (s+1)(t-t_2-1).$$

Proof. We only need to prove the values of p^4_{11}, p^4_{12} and p^4_{13}. Clearly $p^4_{11} = p^4_{12} = 0$. Let K and L be two lines such that $(K,L) \in R_4$. Counting pairs (p, M) with $p \in K \cap M$ and $(M, L) \in R_3$ gives $p^4_{13} = (s+1)(t_2+1)$. $\square$

Remarks.

(1) Some of the above values for the intersection numbers are only correct when **S** contains quads through every two points at mutual distance 2. If $t_2 \geq 1$ and **S** would contain two points x and y such that: (i) $\mathrm{d}(x,y) = 2$, (ii) there is no quad through x and y, then s equals 2 (see [15] for examples). Then there exists three common neighbours a, b, c of x and y and a common neighbour z of a and b which is not collinear with c. Then $(xb, yc), (xb, az) \in R_2$ and $(az, yc) \in R_3$, so $p^2_{23}(xb, yc) \neq 0$ (we had $p^2_{23} = 0$ in Theorem 4.3.4).

(2) If the coefficients p^4_{ij} which are not mentioned in the above theorem are also constant, then $\mathcal{R} = (R_0, R_1, R_2, R_3, R_4)$ defines an association scheme. As we will see in the next paragraph it is sufficient for this that p^4_{22} is constant. This last condition is however not always satisfied.

4.4 The case where $\mathcal{R}$ determines an association scheme

Just like in the previous paragraph, let **S** be a regular near hexagon having quads through every two points at mutual distance 2. With the same notations as above, we have the following theorem.

Theorem 4.4.1
The following conditions are equivalent:

(1) *$\mathcal{R}$ determines an association scheme,*

(2) *p^4_{22} is constant.*

In this case, the remaining intersection numbers are equal to

$$p^4_{22} = \frac{(1+t_2)t_2^2}{t-t_2}, p^4_{23} = p^4_{32} = \frac{t_2(1+t_2)(s+1)(t-2t_2)}{t-t_2},$$

$$p^4_{24} = p^4_{42} = \frac{n_2}{n_4}p^2_{44}, p^4_{33} = \frac{n_3}{n_4}p^3_{34}, p^4_{34} = p^4_{43} = \frac{n_3}{n_4}p^3_{44},$$

$$p^4_{44} = n_4 - p^4_{04} - p^4_{14} - p^4_{24} - p^4_{34}.$$

Proof. Clearly (1) implies (2), so suppose that p^4_{22} is constant. Then $p^4_{22} = \frac{n_2 p^2_{24}}{n_4} = \frac{(1+t_2)t_2^2}{t-t_2}$. Let K and L be two lines such that $(K,L) \in R_4$. Counting triples (p,q,M) with $p \in K, q \in \Gamma(p) \cap \Gamma(L)$ and $M \| K$ gives $p^4_{23} + (s+1)p^4_{22} = (1+s)(1+t_2)t_2$. Hence $p^4_{23} = p^4_{32} = \frac{(1+t_2)t_2(s+1)(t-2t_2)}{t-t_2}$. If we are able to prove that p^4_{33} is constant, then the theorem is valid. Let M be a line such that $(K,M),(L,M) \in R_3$. Let u and v be those points that are collinear with $u' \in K$ and $v' \in L$. We have the following cases.

(1) $u \neq v$ and $\mathrm{d}(u',v') = 2$.
Let Q be the quad through u' and v'. Since u' has distance 2 to v and v', there exists a point on vv' collinear with u. Hence $v \in Q$. Also $u \in Q$ for similar reasons and hence $M \subseteq Q$. On the contrary, every line in Q not containing u', v' or any common neighbour of u' and v' has relation R_3 with K and L. Hence, the contribution of this case to p^4_{33} is equal to $A = (s+1)[(1+t_2)(1+st_2) - 2(1+t_2) - (1+t_2)(t_2-1)]$.

(2) $u = v$.
Let B denote the contribution of this case to p^4_{33}. Counting triples (p,q,N) with $p \in K, q \in \Gamma(p) \cap \Gamma(L) \cap M$ and $M \cap (K \cup L) = \emptyset$ yields

$$(s+1)(t_2+1)(t-1) = B + 2p^4_{23} + (s+1)p^4_{22}.$$

Hence B is constant.

(3) $u \neq v$ and $\mathrm{d}(u',v') = 3$.
Let C denote the contribution of this case to p^4_{33}. Counting the tuples (p,q,r,N) with $p \in K, q \in L \cap \Gamma_3(p), r \in \Gamma(p) \cap \Gamma_2(q) \cap \Gamma_1(K) \cap \Gamma_2(L) \cap N$ and $\mathrm{d}(q,N) = 1$ yields

$$(s+1)s(t-t_2-1)(t_2+1) = C + sp^4_{23}.$$

Hence C is constant.

It is now clear that $p^4_{33} = A + B + C$ is constant. □

Theorem 4.4.2
Let $\mathbf{S}$ *be a regular near hexagon.*

(1) *If* $\mathbf{S}$ *is a dual polar space, then* p^4_{22} *is constant.*

(2) *If* $\mathbf{S}$ *is the near hexagon related to* $S(5,8,24)$*, then* p^4_{22} *is constant.*

(3) *If* $\mathbf{S}$ *is a the near hexagon related to the extended ternary Golay code, then* p^4_{22} *is not constant.*

Proof.

(1) If $\mathbf{S}$ has parameters (s, t_2, t), then we prove that $p^4_{22} = t_2 + 1$. Let K and L be two lines such that $\mathrm{d}(K, L) = 2$ and let k and l be two points on K and L such that $\mathrm{d}(k, l) = 2$. If M is a line having relation R_2 with K and M, then there is a point on M collinear with k and l. The quad through that point and the line K necessary contains M. Now, let m be one of the $t_2 + 1$ common neighbours of k and l, and let Q be the quad through m and K. A point $q \neq l$ on L has distance 2 from two points of Q, hence there is a point $q' \in Q$ collinear with q. The line mq' is then the unique line through m having relation R_2 with K and L. This proves that $p^4_{22} = t_2 + 1$.

(2) Let $K = \{K_1, K_2, K_3\}$ and $L = \{L_1, L_2, L_3\}$ be two lines of $\mathbf{S}$ such that $\mathrm{d}(K, L) = 2$. Hence, we may assume that there is some partition $\{X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9\}$ of the point set of $S(5, 8, 24)$ such that

- $|X_1| = |X_5| = |X_9| = 4, |X_i| = 2$ for $i \neq 1, 5, 9$;
- $X_1 \cup X_2 \cup X_3 = K_1, X_4 \cup X_5 \cup X_6 = K_2, X_7 \cup X_8 \cup X_9 = K_3$;
- $X_1 \cup X_4 \cup X_7 = L_1, X_2 \cup X_5 \cup X_8 = L_2, X_3 \cup X_6 \cup X_9 = L_3$.

Suppose $M = \{M_1, M_2, M_3\}$ is a line of $\mathbf{S}$ having relation R_2 with K and L. We may assume that $\mathrm{d}(K_i, M_i) = \mathrm{d}(M_i, L_i) = 1$ for all $i \in \{1, 2, 3\}$. Now, let N be one of the three blocks of the Steiner system collinear with K_1 and L_1. If we consider the possible intersections of N with K_1, K_2, K_3, L_1, L_2 and L_3, we have the following cases.

(a) $|N \cap X_5| = |N \cap X_9| = 4, |N \cap X_6| = |N \cap X_8| = 0$.
In that case $N \triangle L_3 = X_5 \cup X_6 \cup X_3$ is also a block, a contradiction, since there is only one block through $X_5 \cup X_6$.

(b) $|N \cap X_5| = |N \cap X_9| = 3, |N \cap X_6| = |N \cap X_8| = 1$.

(c) $|N \cap X_5| = |N \cap X_6| = |N \cap X_8| = |N \cap X_9| = 2$.

So, the three common neighbours N_1, N_2, N_3 of K_1 and L_1 must satisfy (b) or (c). Since $|N_i \cap N_j|$=4 for $i \neq j$, it is easy to check that one, say N_1, satisfies (c) and the two others satisfy (b). Now consider the possible intersections of M_1 and M_2 with X_9. Since $M_1 \cap M_2 = \emptyset$, we must have that $M_1 = N_1$. A similar reasoning holds for M_2 and M_3. This proves that there is at most one line having relation R_2 with K and L. If p^4_{22} is constant, we found earlier that $p^4_{22} = \frac{(1+t_2)t_2^2}{t-t_2}$. If p^4_{22} is not necessary constant, then $\frac{(1+t_2)t_2^2}{t-t_2}$ is the average value of all the parameters p^4_{22}. Hence, since $\frac{(1+t_2)t_2^2}{t-t_2} = 1$ in this case, one finds that p^4_{22} is constant and equal to 1.

(3) If p^4_{22} would be constant, then $\frac{(1+t_2)t_2^2}{t-t_2} = \frac{1}{5}$ would be integral, a contradiction.

$\square$

The parameters of the association scheme related to $S(5, 8, 24)$

The association scheme is one on 3795 elements. One has that $n_0 = 1, n_1 = 42, n_2 = 56, n_3 = 1008, n_4 = 2688$. The values of the coefficients p^i_{jk} are easily calculated and listed below.

$p^0_{00} = 1$, $p^0_{01} = 0$, $p^0_{02} = 0$, $p^0_{03} = 0$, $p^0_{04} = 0$, $p^0_{10} = 0$, $p^0_{11} = 42$, $p^0_{12} = 0$, $p^0_{13} = 0$, $p^0_{14} = 0$, $p^0_{20} = 0$, $p^0_{21} = 0$, $p^0_{22} = 56$, $p^0_{23} = 0$, $p^0_{24} = 0$, $p^0_{30} = 0$, $p^0_{31} = 0$, $p^0_{32} = 0$, $p^0_{33} = 1008$, $p^0_{34} = 0$, $p^0_{40} = 0$, $p^0_{41} = 0$, $p^0_{42} = 0$, $p^0_{43} = 0$, $p^0_{44} = 2688$, $p^1_{00} = 0$, $p^1_{01} = 1$, $p^1_{02} = 0$, $p^1_{03} = 0$, $p^1_{04} = 0$, $p^1_{10} = 1$, $p^1_{11} = 13$, $p^1_{12} = 4$, $p^1_{13} = 24$, $p^1_{14} = 0$, $p^1_{20} = 0$, $p^1_{21} = 4$, $p^1_{22} = 4$, $p^1_{23} = 48$, $p^1_{24} = 0$, $p^1_{30} = 0$, $p^1_{31} = 24$, $p^1_{32} = 48$, $p^1_{33} = 360$, $p^1_{34} = 576$, $p^1_{40} = 0$, $p^1_{41} = 0$, $p^1_{42} = 0$, $p^1_{43} = 576$, $p^1_{44} = 2112$, $p^2_{00} = 0$, $p^2_{01} = 0$, $p^2_{02} = 1$, $p^2_{03} = 0$, $p^2_{04} = 0$, $p^2_{10} = 0$, $p^2_{11} = 3$, $p^2_{12} = 3$, $p^2_{13} = 36$, $p^2_{14} = 0$, $p^2_{20} = 1$, $p^2_{21} = 3$, $p^2_{22} = 4$, $p^2_{23} = 0$, $p^2_{24} = 48$, $p^2_{30} = 0$, $p^2_{31} = 36$, $p^2_{32} = 0$, $p^2_{33} = 252$, $p^2_{34} = 720$, $p^2_{40} = 0$, $p^2_{41} = 0$, $p^2_{42} = 48$, $p^2_{43} = 720$, $p^2_{44} = 1920$, $p^3_{00} = 0$, $p^3_{01} = 0$, $p^3_{02} = 0$, $p^3_{03} = 1$, $p^3_{04} = 0$, $p^3_{10} = 0$, $p^3_{11} = 1$, $p^3_{12} = 2$, $p^3_{13} = 15$, $p^3_{14} = 24$, $p^3_{20} = 0$, $p^3_{21} = 2$, $p^3_{22} = 0$, $p^3_{23} = 14$, $p^3_{24} = 40$, $p^3_{30} = 1$, $p^3_{31} = 15$, $p^3_{32} = 14$, $p^3_{33} = 258$, $p^3_{34} = 720$, $p^3_{40} = 0$, $p^3_{41} = 24$, $p^3_{42} = 40$, $p^3_{43} = 720$, $p^3_{44} = 1904$, $p^4_{00} = 0$, $p^4_{01} = 0$, $p^4_{02} = 0$, $p^4_{03} = 0$, $p^4_{04} = 1$, $p^4_{10} = 0$, $p^4_{11} = 0$, $p^4_{12} = 0$, $p^4_{13} = 9$, $p^4_{14} = 33$, $p^4_{20} = 0$, $p^4_{21} = 0$, $p^4_{22} = 1$, $p^4_{23} = 15$, $p^4_{24} = 40$, $p^4_{30} = 0$, $p^4_{31} = 9$, $p^4_{32} = 15$, $p^4_{33} = 270$, $p^4_{34} = 714$, $p^4_{40} = 1$, $p^4_{41} = 33$, $p^4_{42} = 40$, $p^4_{43} = 714$, $p^4_{44} = 1900$.

The parameters of some association schemes related to classical near polygons

Consider the classical near hexagons $W^D(5,q)$, $Q^D(6,q)$, $[Q^-(7,q)]^D$, $[Q^+(5,q)]$, $H^D(6,q)$ or $H^D(5,q)$ from Section 3.2.2. The parameters of these near hexagons look like $s = q^e, t_2 = q, t = q(q+1)$, with q a prime power and e equal to 1, 1, 2, 0, $\frac{3}{2}$ or $\frac{1}{2}$. It is now easy to calculate the parameters of the association scheme. The association scheme is one on $(q^2+q+1)(q^{2e+3}+q^{e+2}+q^{e+1}+1)$ elements. One has that $n_0 = 1, n_1 = q(q+1)(q^e+1), n_2 = q^{e+2}(q+1), n_3 = q^{e+3}(q+1)(q^e+1), n_4 = q^{2e+5}$. The values of p^i_{jk} are listed below.

$p^0_{00} = 1$, $p^0_{01} = 0$, $p^0_{02} = 0$, $p^0_{03} = 0$, $p^0_{04} = 0$, $p^0_{10} = 0$, $p^0_{11} = q(q+1)(q^e+1)$, $p^0_{12} = 0$, $p^0_{13} = 0$, $p^0_{14} = 0$, $p^0_{20} = 0$, $p^0_{21} = 0$, $p^0_{22} = q^{e+2}(q+1)$, $p^0_{23} = 0$, $p^0_{24} = 0$, $p^0_{30} = 0$, $p^0_{31} = 0$, $p^0_{32} = 0$, $p^0_{33} = q^{e+3}(q+1)(q^e+1)$, $p^0_{34} = 0$, $p^0_{40} = 0$, $p^0_{41} = 0$, $p^0_{42} = 0$, $p^0_{43} = 0$, $p^0_{44} = q^{2e+5}$, $p^1_{00} = 0$, $p^1_{01} = 1$, $p^1_{02} = 0$, $p^1_{03} = 0$, $p^1_{04} = 0$, $p^1_{10} = 1$, $p^1_{11} = q^2+q-1$, $p^1_{12} = q^{e+1}$, $p^1_{13} = q^{e+2}$, $p^1_{14} = 0$, $p^1_{20} = 0$, $p^1_{21} = q^{e+1}$, $p^1_{22} = q^{e+1}(q-1)$, $p^1_{23} = q^{e+3}$, $p^1_{24} = 0$, $p^1_{30} = 0$, $p^1_{31} = q^{e+2}$, $p^1_{32} = q^{e+3}$, $p^1_{33} = q^{e+2}(q^{e+1}+q^2-1)$, $p^1_{34} = q^{2e+4}$, $p^1_{40} = 0$, $p^1_{41} = 0$, $p^1_{42} = 0$, $p^1_{43} = q^{2e+4}$, $p^1_{44} = q^{2e+4}(q-1)$, $p^2_{00} = 0$, $p^2_{01} = 0$, $p^2_{02} = 1$, $p^2_{03} = 0$, $p^2_{04} = 0$, $p^2_{10} = 0$, $p^2_{11} = q^e+1$, $p^2_{12} = (q-1)(q^e+1)$, $p^2_{13} = q^2(q^e+1)$, $p^2_{14} = 0$, $p^2_{20} = 1$, $p^2_{21} = (q-1)(q^e+1)$, $p^2_{22} = q^{e+2}-q^{e+1}+q^e-q$, $p^2_{23} = 0$, $p^2_{24} = q^{e+3}$, $p^2_{30} = 0$, $p^2_{31} = q^2(q^e+1)$, $p^2_{32} = 0$, $p^2_{33} = q^2(q^e+1)(2q^{e+1}-1)$, $p^2_{34} = q^{e+3}(q-1)(q^e+1)$, $p^2_{40} = 0$, $p^2_{41} = 0$, $p^2_{42} = q^{e+3}$, $p^2_{43} = q^{e+3}(q-1)(q^e+1)$, $p^2_{44} = q^{2e+5}-q^{2e+4}+q^{2e+3}-q^{e+4}$, $p^3_{00} = 0$, $p^3_{01} = 0$, $p^3_{02} = 0$, $p^3_{03} = 1$, $p^3_{04} = 0$, $p^3_{10} = 0$, $p^3_{11} = 1$, $p^3_{12} = q$, $p^3_{13} = q^{e+1}+q^2-1$, $p^3_{14} = q^{e+2}$, $p^3_{20} = 0$, $p^3_{21} = q$, $p^3_{22} = 0$, $p^3_{23} = q(2q^{e+1}-1)$, $p^3_{24} = q^{e+2}(q-1)$, $p^3_{30} = 1$, $p^3_{31} = q^{e+1}+q^2-1$, $p^3_{32} = q(2q^{e+1}-1)$, $p^3_{33} = q(q-1)(q^{2e+1}+3q^{e+1}+q^e-1)+q^{2e+2}$, $p^3_{34} = q^{e+3}(q^{e+1}+q-2)$, $p^3_{40} = 0$, $p^3_{41} = q^{e+2}$, $p^3_{42} = q^{e+2}(q-1)$, $p^3_{43} = q^{e+3}(q^{e+1}+q-2)$, $p^3_{44} = q^{e+3}(q^e+2)$, $p^4_{00} = 0$, $p^4_{01} = 0$, $p^4_{02} = 0$, $p^4_{03} = 0$, $p^4_{04} = 1$, $p^4_{10} = 0$, $p^4_{11} = 0$, $p^4_{12} = 0$, $p^4_{13} = (q+1)(q^e+1)$, $p^4_{14} = (q-1)(q+1)(q^e+1)$, $p^4_{20} = 0$, $p^4_{21} = 0$, $p^4_{22} = q+1$, $p^4_{23} = (q+1)(q-1)(q^e+1)$, $p^4_{24} = (q+1)(q^{e+2}-q^{e+1}+q^e-q)$, $p^4_{30} = 0$, $p^4_{31} = (q+1)(q^e+1)$, $p^4_{32} = (q+1)(q-1)(q^e+1)$, $p^4_{33} = q(q+1)(q^e+1)(q^{e+1}+q-2)$, $p^4_{34} = q(q+1)(q^e+1)(q^e+2)$, $p^4_{40} = 1$, $p^4_{41} = (q-1)(q+1)(q^e+1)$, $p^4_{42} = (q+1)(q^{e+2}-q^{e+1}+q^e-q)$, $p^4_{43} = q(q+1)(q^e+1)(q^e+2)$, $p^4_{44} = q^{2e+5}-q^{2e+2}-q^{2e+1}-q^{e+3}-4q^{e+2}-3q^{e+1}-2q^2-q$.

4.5 A characterization theorem

Theorem 4.5.1
If $\mathbf{S}$ *is a regular near hexagon with parameters* (s, t_2, t) *and satisfying:* (i) $s \geq t_2$, (ii) p^4_{22} *is constant, then* $\mathbf{S}$ *is one of the following near hexagons:*

- *a generalized hexagon of order* (s, t),
- *a classical near hexagon (a regular Hamming near hexagon,* $W^D(5, q)$, $Q^D(6, q)$, $[Q^-(7, q)]^D$ *or* $H^D(6, q)$*),*
- *the near hexagon related to* $S(5, 8, 24)$,
- *the near hexagon whose point graph is the incidence graph of the unique biplane of order* 2.

Proof. Suppose that $\mathbf{S}$ is a near hexagon with parameters (s, t_2, t) and satisfying: (i) $s \geq t_2$, (ii) p^4_{22} is constant. If $t_2 = 0$, then $\mathbf{S}$ is a generalized hexagon. So, in the sequel we suppose that $t_2 \geq 1$. If $s = 1$ then $t_2 = 1$, so there are always quads through every two points at mutual distance 2. If $s = t_2 = 1$, then $\frac{(1+t_2)t_2^2}{t-t_2} \in \mathbb{N}$ implies that $t \in \{2, 3\}$. If $t = 2$, then $\mathbf{S}$ is a cube and hence a Hamming near hexagon. If $t = 3$, then $\mathbf{S}$ is the incidence graph of a square 2-design $\mathbf{D}$ (see [15], Theorem 1.3.1). Out of the parameters of $\mathbf{S}$, we derive that $\mathbf{D}$ has 14 points, every block contains $k = 4$ points and every 2 points are contained in $\lambda = 2$ blocks. Hence $\mathbf{D}$ is a biplane of order $n = k - \lambda = 2$. This biplane is easily proved to be unique (see also [6] for more details about this). A description of the incidence graph Γ of $\mathbf{D}$ is as follows. Take an arbitrary vertix and label it ∞. The vertices in $\Gamma_1(\infty)$ are labeled with the singletons of $X = \{1, 2, 3, 4\}$. The vertices of $\Gamma_2(\infty)$ are labeled with the subsets of X having order 2 in such a way that $\{i\} \sim \{j, k\} \iff i = j$ or $i = k$. Finally one can label a point in $\Gamma_3(\infty)$ with the set of two vertices of $\Gamma_2(\infty)$, where that point is not adjacent with. With this description, it is now easy to prove that p^4_{22} is constant and equal to 1. For, let K and L be two lines such that $\mathrm{d}(K, L) = 2$. We may suppose that $K = \{\infty, \{a\}\}$ and $L = \{\{b, c\}, \{\{a, b\}, \{c, d\}\}\}$ with $\{a, b, c, d\} = \{1, 2, 3, 4\}$. It is immediate that $M = \{\{c\}, \{a, c\}\}$ is the unique line having relation R_2 with K and L. If $s \geq 2$ and if $\mathbf{S}$ is no dual polar space, then $1 + t \geq (1 + s)(1 + st_2)$ (see Theorem 3.7.4). Since $s \geq t_2$ we have that $1 + t \geq (1 + t_2)(1 + t_2^2)$, implying $p^4_{22} \leq 1$. Hence $1 + t = (1 + st_2)(1 + s)$ and $s = t_2$. Now, Theorem 3.2.1 implies that $\mathbf{S}$ is the regular near hexagon associated with $S(5, 8, 24)$. $\square$

4.6 An association scheme derived from a $\mathrm{GQ}(s, s^2)$, $s \geq 1$, having a spread of symmetry

Suppose that $\mathbf{Q}$ is a nontrivial $\mathrm{GQ}(s, s^2)$, $s \geq 1$, having a spread of symmetry S and let L be a fixed line of S. Put $G = \Pi_S(L)$. Let $X = S \setminus \{L\}$ and $\mathcal{R} = \{R_0\} \cup \{R_g | g \in G\}$; $(M, N) \in R_0$ if and only if $M = N$, $(M, N) \in R_g$ if and only if $M \neq N$ and $[N, L] \circ [M, N] \circ [L, M] = g$. In the following theorem, we will prove that $(X, \mathcal{R})$ is an association scheme. We have that $p^{\alpha}_{0\beta} = p^{\alpha}_{\beta 0} = 1$ or 0 depending on $\alpha = \beta$ or $\alpha \neq \beta$. For $g, h \in G$, we have that $p^0_{gh} = n_g$ or 0 depending on $gh = e$ or $gh \neq e$. The other parameters together with the parameters n_g are given in the following theorem.

Theorem 4.6.1
$(X, \mathcal{R})$ is an association scheme and the remaining parameters are (g, h, k denote elements of G, different from e):

$$
\begin{array}{lcllcl}
n_e & = & s-1; & n_g & = & s^2-1; \\
p^e_{ee} & = & s-2; & p^g_{ee} & = & 0; \\
p^e_{ge} & = & 0; & p^e_{eg} & = & 0; \\
p^g_{ge} & = & 0; & p^g_{eg} & = & 0; \\
p^e_{gh} & = & 0 \text{ if } gh = e; & p^e_{gh} & = & s+1 \text{ if } gh \neq e; \\
p^g_{eh} & = & 1 \text{ if } h \neq g; & p^g_{he} & = & 1 \text{ if } h \neq g; \\
p^g_{hk} & = & s+1 \text{ if } g \neq hk; & p^g_{hk} & = & 1 \text{ if } g = hk.
\end{array}
$$

Proof. Clearly $R_0^T = R_0$ and $R_l^T = R_{l^{-1}}$ for all $l \in G$. In this proof, we will several times make use of the fact that every three mutually noncollinear points of a $\mathrm{GQ}(s, s^2)$ have exactly $s+1$ common neighbours, see Theorem 2.1.3. When we talk about a hyperbolic line, we always mean a hyperbolic line consisting of $s+1$ lines of the normal spread S. Let x denote a fixed point of L.

We have that $(M, N) \in R_e$ if and only if $M \neq N$ and L, M, N are contained in a hyperbolic line, hence $n_e = s-1, p^e_{ee} = s-2, p^g_{ee} = p^e_{ge} = p^e_{eg} = 0$.

Let $g \in G \setminus \{e\}$ be fixed and let $M \in S \setminus \{L\}$. We prove that each hyperbolic line, different from the one through L and M, contains a unique line N such that $(M, N) \in R_g$. From this it follows that $n_g = s^2-1, p^g_{eg} = p^g_{ge} = 0$ and $p^g_{eh} = p^g_{he} = 1$ if $g \neq h$. So, choose such a hyperbolic line $\mathcal{H}$ and let N' be the line of $\mathbf{Q}$ through x^g meeting each element of $\mathcal{H}$. Let x' be the unique point on M collinear with x, let x'' denote the unique point of N' collinear with x' and let N be the unique line of $\mathcal{H}$ through x''. Further, N is also the unique line of $\mathcal{H}$ with the property that $(M, N) \in R_g$.

Let $g, h \in G \setminus \{e\}$. Let $(K, M) \in R_e$ and suppose that $(K, N) \in R_g$ and $(N, M) \in R_h$. The points $x^{gh[L,M]}, x^{[L,K]}$ and x^g have then a common neighbour on N. Indeed, let $\theta \in G_S$ such that $x^{h^{-1}\theta} = x^g$, then $x^{\theta h^{-1}} = x^g$ and hence $x^\theta = x^{gh}$; $x^{\theta[L,M]}$ and $x^{h^{-1}\theta}$ have now a common neighbour on N. This is impossible if $gh = e$; if $gh \neq e$, then there are $s+1$ such common neighbours and each of them yields a good choice for N. Hence $p^e_{gh} = s+1$ if $gh = e$.

Finally, let $g, h, k \in G \setminus \{e\}$. Let $(K, M) \in R_g$ and suppose that $(K, N) \in R_h$ and $(N, M) \in R_k$; then N contains a common neighbour of $x^{[L,K]}$, x^h and $x^{hk[L,M]}$. If $hk = g$, then this common neighbour is equal to the projection of x^h on the line through $x^{[L,K]}$ and $x^{g[L,M]}$. Hence $p^g_{hk} = 1$ if $g = hk$. If $hk \neq g$, then there are $s+1$ common neighbours and each of these yields a good element N. Hence $p^g_{hk} = s+1$ if $g \neq hk$. $\square$

Remark. The above defined association scheme is symmetric if and only if $g^{-1} = g$ for all $g \in G$. From $ab = a^{-1}b^{-1} = (ba)^{-1} = ba, \forall a, b \in G$, it follows that G is an elementary abelian group and hence isomorphic to the additive group of some finite field with characteristic 2.

Chapter 5

The i-neighbourhood of a point

5.1 Definitions and properties

Let $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a near $2d$-gon. For $x \in \mathcal{P}$ and $i \in \mathbb{N}$, we define $\mathcal{P}_x^i = \{p \in \mathcal{P} \mid \mathrm{d}(x,p) \leq i\}$. This set is always a subspace. Indeed, take a line L which has two points α and β in common with $\mathcal{P}_x^i$. Since $\mathrm{d}(x,\alpha) \leq i$ and $\mathrm{d}(x,\beta) \leq i$, it follows that there must be a point γ on L such that $\mathrm{d}(x,\gamma) \leq i-1$. Hence every point of L has distance at most i to x. The set $\mathcal{P}_x^i$ induces a partial linear space $\mathbf{S}_x^i$, called *the i-neighbourhood of x*. The line set of $\mathbf{S}_x^i$ consists of those lines of $\mathbf{S}$ which have distance at most $i-1$ to x. Clearly $\mathbf{S}_x^0$, $\mathbf{S}_x^1$ and $\mathbf{S}_x^d$ are near polygons. We prove now that $\mathbf{S}_x^2$ is also a near polygon.

Theorem 5.1.1
If x is a point of a near polygon $\mathbf{S}$, then $\mathbf{S}_x^2$ is a near polygon.

Proof. Let $\mathrm{d}_2(\cdot,\cdot)$ denote the distance in $\mathbf{S}_x^2$. Take any point p and any line L of $\mathbf{S}_x^2$. Let p' denote the unique point of L nearest to p (in $\mathbf{S}$). If $\mathrm{d}_2(p,L) = 0$, there is nothing to prove. If $\mathrm{d}_2(p,L) = 1$, then $\mathrm{d}(p,L) = 1$ and p' is the unique point of L nearest to p (in $\mathbf{S}_x^2$). If $\mathrm{d}_2(p,L) = 2$, then $\mathrm{d}(p,L) = 2$ and p' is the unique point of L nearest to p (in $\mathbf{S}_x^2$). So we may suppose that $\mathrm{d}_2(p,L) = 3$. This implies that $\mathrm{d}(p,x) = 2$ and $\mathrm{d}(x,L) = 1$. Let α denote the point of L which is collinear with x. Any path of length ≤ 2 between p and α is contained in $\mathcal{P}_x^2$, so $\mathrm{d}(p,\alpha) = \mathrm{d}_2(p,\alpha) = 3$. Let s denote another point of L for which $\mathrm{d}_2(p,L) = 3$. Let (p,q,r,s) be a path in $\mathbf{S}_x^2$ connecting p and s. If $\mathrm{d}(x,q) = 1$, then $\mathrm{d}(q,\alpha) = 2$ and $\mathrm{d}(q,s) = 2$. So, there is a point β of L, collinear with q. The path (p,q,β) is contained in $\mathcal{P}_x^2$, a contradiction. If $\mathrm{d}(x,r) = 1$, then since $\mathrm{d}(p,x) = 2 = \mathrm{d}(p,r)$ there is a point γ of xr which is collinear with p. Now $\mathrm{d}(\gamma,\alpha) = \mathrm{d}(\gamma,s) = 2$, so there is a point δ of L collinear with γ. The path (p, γ, δ) is a path in $\mathcal{P}_x^2$, a contradiction. So we may suppose that $\mathrm{d}(x,p) = \mathrm{d}(x,q) = \mathrm{d}(x,r) = 2$. There exists a point a on pq and a point b on qr such that $\mathrm{d}(x,a) = \mathrm{d}(x,b) = 1$. Since $\mathrm{d}(b,s) = 2$ and $\mathrm{d}(b,\alpha) \leq 2$, there exists a point c on L collinear with b. Since $\mathrm{d}(p,b) = \mathrm{d}(p,x) = 2$, there exists a point d of xb collinear with p. If $\alpha = c$ then $xb = x\alpha$, so d is collinear with α, contradicting $\mathrm{d}(p,\alpha) = 3$. Suppose $\alpha \neq c$. Since $\mathrm{d}(d,\alpha) = \mathrm{d}(d,c) = 2$, there exists a point e of L which is collinear with d. The path (p,d,e) is contained in $\mathcal{P}_x^2$, a contradiction. This proves that α is the unique point of L such that $\mathrm{d}_2(p,\alpha) = 3$. $\square$

Theorem 5.1.2

(1) *If* $\mathbf{S}$ *is a thin near polygon, then* $\mathbf{S}_x^i$ *is a thin near polygon, for every integer* i.

(2) *If* $\mathbf{S}$ *is a generalized* $2d$*-gon, then* $\mathbf{S}_x^i$ *is a near polygon for every* $i \in \mathbb{N}$.

(3) *If* $\mathbf{S}$ *is a dual polar space, then* $\mathbf{S}_x^i$ *is a near polygon for every* $i \in \mathbb{N}$.

Proof.

(1) The point graph of $\mathbf{S}_x^i$ remains bipartite and connected.

(2) We may suppose that $0 < i < d$. In that case, $\mathbf{S}_x^i$ is a degenerate near polygon, whose nondegenerate support is a point.

(3) Let $\mathrm{d}_i(\cdot,\cdot)$ be the distance in $\mathbf{S}_x^i$. It suffices to prove that $\mathrm{d}(p,q) = \mathrm{d}_i(p,q)$ for all $p,q \in \mathcal{P}_x^i$. Clearly $\mathrm{d}(p,q) \leq \mathrm{d}_i(p,q)$. Consider a path $p = b_0, b_1, \ldots, b_k = q$ of length $k = \mathrm{d}(p,q)$ such that $\mathrm{d}(x,b_0) + \cdots + \mathrm{d}(x,b_k)$ is minimal. If there exists an index j such that $\mathrm{d}(x,b_j) > i$, then one of the following two cases certainly occurs.

   - There exists an index l such that $\mathrm{d}(x,b_l) = m$, $\mathrm{d}(x,b_{l+1}) = \mathrm{d}(x,b_{l+2}) = m+1$. Define Q as the quad through b_l, b_{l+1} and b_{l+2} and define α as the unique point in Q nearest to x. It is impossible that $\alpha = b_l$, otherwise $\mathrm{d}(x,b_{l+2}) = m+2$. The point α is collinear with b_l and has distance $m-1$ to x. Call β the point on $b_l\alpha$ at distance 1 of b_{l+2}. The path (b_l, b_{l+1}, b_{l+2}) can be replaced by (b_l, β, b_{l+2}), a contradiction.
   - There exists an l such that $\mathrm{d}(x,b_l) = \mathrm{d}(x,b_{l+2}) = m, \mathrm{d}(x,b_{l+1}) = m+1$. Define Q as the quad through b_l, b_{l+1} and b_{l+2} and define α as the unique point in Q nearest to x. The point α is a common neighbour of b_l and b_{l+2}, so the path (b_l, b_{l+1}, b_{l+2}) can be replaced by (b_l, α, b_{l+2}), yielding a contradiction. $\square$

One might wonder whether $\mathbf{S}_x^i$ is a near polygon for all choices of $\mathbf{S}, x$ and i. This is not the case, as shown in the following theorem. We refer to Section 3.2.5 for the construction of the regular near octagon involved.

Theorem 5.1.3
Let x be a point of the unique regular near octagon with parameters $s = 2, t = 4, t_2 = 0, t_3 = 3$, then $\mathbf{S}_x^3$ is not a near polygon.

Proof. Let $\mathrm{d}_3(\cdot,\cdot)$ denote the distance in $\mathbf{S}_x^3$. Let $y \in \Gamma_4(x), \{y_1, y_2\} \subset \Gamma_3(x) \cap \Gamma(y), y_1 \neq y_2, y_3 \in \Gamma(y_2) \cap \Gamma_2(x)$ and let y_4 be the third point of the line y_3y_2. Now $y_3 \neq y_1 \neq y_4$, $\mathrm{d}(y_3, y_1) \neq 1 \neq \mathrm{d}(y_4, y_1)$ since $t_2 = 0$, $\mathrm{d}(y_1, y_3) \neq 2$ since $\mathrm{d}(y_1, y_4) \neq 1$ and $\mathrm{d}(y_1, y_4) \neq 2$ since $\mathrm{d}(y_1, y_3) \neq 1$. So $\mathrm{d}(y_1, y_3) = \mathrm{d}(y_1, y_4) = 3$. There are four shortest paths $(y_1, \alpha, \beta, y_i)$ connecting y_1 and y_i $(i = 3, 4)$. Since $t - t_3 = 1$, only two of these paths can have $\mathrm{d}(x, \alpha) = 4$ or $\mathrm{d}(x, \beta) = 4$. This means that $\mathrm{d}_3(y_1, y_3) = \mathrm{d}_3(y_1, y_4) = 3$, but $\mathrm{d}_3(y_1, y_2) \geq 3$, so $\mathbf{S}_x^3$ is not a near polygon. $\square$

Problem. Are there other classes of near polygons for which the i-neighbourhoods are again near polygons?

5.2 Construction of near polygons from partial linear spaces

5.2.1 Main theorem

In this section, we prove the following theorem.

Theorem 5.2.1
Let $\mathbf{G}$ be an arbitrary partial linear space, then there exists a near polygon $\mathbf{S}$ and a point (∞) of $\mathbf{S}$ such that the local space at (∞) is isomorphic to $\mathbf{G}$.

In order to prove this theorem, we have to construct a near polygon from an arbitrary partial linear space. The near polygons from this construction have the property that there exists a point at distance at most 2 from all the other points and this property almost characterizes these near polygons as we shall see in Section 5.3.

5.2.2 Construction

Let $\mathbf{G}$ be an arbitrary partial linear space with point set $\mathcal{A}$ and line set $\mathcal{B}$. We may and will suppose that the lines of $\mathcal{B}$ are sets of points. Let $\mathbf{G}^*$ be the geometry obtained from $\mathbf{G}$ by deleting all the lines of length 2 and call C_i, $1 \leq i \leq n$, the point sets of the connected components of $\mathbf{G}^*$.

Remark. Suppose that $\mathbf{G}$ is the local space at the point (∞) of a near polygon $\mathbf{S}$. Every $\mathbf{G}$-point p corresponds with an $\mathbf{S}$-line through (∞), which is incident with $s(p)+1$ $\mathbf{S}$-points. Every $\mathbf{G}$-line L corresponds with a quad through (∞). If L contains at least three $\mathbf{G}$-points, then the corresponding quad is not a grid. Hence $s(p)$ is independent of the point p of L; we define $s(L) := s(p)$, where p is an arbitrary point of L. If we fix a certain connected component C_i, then $s(p)$ is independent of the point p of C_i; we define $s_i := s(p)$, where p is an arbitrary point of C_i. If L, $|L| \geq 3$, is a $\mathbf{G}$-line contained in C_i, then the quad defined by L is a dual grid or has order $(s_i, |L|-1)$. In both cases there exists a GQ of order $(s_i, |L|-1)$. This remark explains the following definitions.

For every i with $1 \leq i \leq n$ we can choose a natural number $s_i = s(C_i) \geq 1$ as follows.

- If $|C_i| = 1$, choose $s_i \geq 1$ at random.
- If $|C_i| > 1$, choose $s_i \geq 1$ such that for every $\mathbf{G}$-line L, $|L| \geq 3$, contained in C_i, there exists a GQ$(s_i, |L|-1)$. Clearly $s_i = 1$ is always a feasible choice for s_i.

For every $L \in \mathcal{B}$ with $|L| \geq 3$, we define $s(L) = s(C_i)$, where C_i is the component containing L. For every $p \in \mathcal{A}$, we define $s(p) = s(C_i)$, where C_i is the component containing p.

Now we will define a partial linear space $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{I}$ is the inclusion. The elements of $\mathcal{P}$ are as follows:

(A) (∞);

(B) (p, i) with $p \in \mathcal{A}$ and $1 \leq i \leq s(p)$;

(C) (L, i) with $L \in \mathcal{B}$ and $1 \leq i \leq k(L)$, here $k(L) \geq 1$ is at random if $s(p) = 1$ for all points p on L, $k(L) = s(p_1) \cdot s(p_2)$ if $L = \{p_1, p_2\}$ and $s(p_1) \cdot s(p_2) > 1$, $k(L) = [s(L)]^2(|L|-1)$ if $|L| \geq 3$ and $s(L) \neq 1$.

The lines of $\mathcal{L}$ are obtained as follows.

(1) The set $\{(\infty)\} \cup \{(p, i) | 1 \leq i \leq s(p)\}$ is a line for every $p \in \mathcal{A}$.

(2) Consider a line $L \in \mathcal{B}$ with $|L| = 2, L = \{p_1, p_2\}, s(p_1) \cdot s(p_2) > 1$. The set $\{(\infty)\} \cup \{(p_1, i) | 1 \leq i \leq s(p_1)\} \cup \{(p_2, i) | 1 \leq i \leq s(p_2)\} \cup \{(L, i) | 1 \leq i \leq s(p_1) \cdot s(p_2)\}$ contains $(s(p_1) + 1) \cdot (s(p_2) + 1)$ points, so we can give it the structure of an $(s(p_1) + 1) \times (s(p_2) + 1)$-grid by introducing $s(p_1) + s(p_2) + 2$ lines; two of those lines were already defined in (1).

(3) Consider a line $L \in \mathcal{B}$ with $|L| = t_L + 1 \geq 3$ and $s(L) \neq 1$. The set $\{(\infty)\} \cup \{(p, i) | p \in L, 1 \leq i \leq s(L)\} \cup \{(L, i) | 1 \leq i \leq [s(L)]^2 t_L\}$ contains $(1+s(L)) \cdot (1+s(L)t_L)$ elements, so we can give it the structure of a $\mathrm{GQ}(s(L), t_L)$ by introducing $(1 + t_L)(1 + s(L)t_L)$ lines; $1 + t_L$ of those lines were already defined in (1).

(4) Consider a line $L \in \mathcal{B}$ for which $s(p) = 1$ for every point p on L. The set $\{(\infty)\} \cup \{(p, 1) | p \in L\} \cup \{(L, i) | 1 \leq i \leq k(L)\}$ can be given the structure of a dual grid with $\{(p, 1) | p \in L\}$ and $\{(\infty)\} \cup \{(L, i) | 1 \leq i \leq k(L)\}$ as two maximal cocliques of the point graph.

If the below defined assumptions (AS1) and (AS2) are satisfied, then we can add additional points and lines; however we are not obliged to do this.

The first assumption is as follows.

(AS1) There exist a **G**-point p for which $s(p) = 1$.

The following graph Δ can then be defined. The vertices of Δ are **G**-points p for which $s(p) = 1$. Two G-points are adjacent when they are not collinear in **G**.

The second assumption is as follows.

(AS2) There is a set of (at least two) cliques in Δ, such that for every clique of this set, there exists another clique intersecting it in at least two points.

Let $N_1, \ldots, N_l$ denote such a set of l cliques. For each $i \in \{1, \ldots, l\}$, let $u_i \geq 1$ be arbitrary. We add now the following points:

(D) (N_i, j) with $1 \leq i \leq l$ and $1 \leq j \leq u_i$,

and the following lines:

(5) $\{(p, 1), (N_i, j)\}$ with $1 \leq i \leq l, 1 \leq j \leq u_i$ and $p \in N_i$.

Theorem 5.2.2
S *is a near polygon and the local space* $\mathbf{S}_{(\infty)}$ *is isomorphic to* **G**.

Proof. Let $Q_1, \ldots, Q_r$ be all the sets mentioned in (2) and (3) above, and which we gave the structure of a GQ. Put $Q_{r+1} = \{(\infty)\} \cup \{(p,1)|s(p) = 1\} \cup \{(L,i)|s(p) = 1, \forall p \in L \text{ and } 1 \leq i \leq k(L)\} \cup \{(N_i, j)|1 \leq i \leq l \text{ and } 1 \leq j \leq u_i\}$. The set Q_{r+1} is a subspace inducing a point or a connected bipartite graph. We prove by induction that $Q_1 \cup \ldots \cup Q_i$, $1 \leq i \leq r+1$, is a subspace inducing a near polygon. Since $\mathcal{P} = Q_1 \cup \ldots \cup Q_{r+1}$, the theorem is then proved. Suppose $Q_1 \cup \ldots \cup Q_i$, $1 \leq i \leq r$, is a subspace inducing a near polygon $\mathbf{S}_1$ and let $\mathbf{S}_2$ be the near polygon induced by Q_{i+1}. We apply Theorem 3.10.1. Put $X = X_1 = X_2 = (Q_1 \cup \ldots \cup Q_i) \cap Q_{i+1}$ and let α be the identical map. The set X consists of a number of lines through (∞); hence $\mathrm{d}_2(\alpha(x), \alpha(y)) = \mathrm{d}_1(x,y)$ for all points $x, y \in X$. Also, X is a subspace of $\mathbf{S}_2$. It suffices to prove that X is a classical subset of $\mathbf{S}_1$. Take $x \in (Q_1 \cup \ldots \cup Q_i) \setminus X$ arbitrary. Suppose that there are two points $x_1, x_2 \in X$ at smallest distance from x (in $\mathbf{S_1}$). By (NP), x_1 and x_2 are not collinear, hence $x_1 \neq (\infty) \neq x_2$. By (NP), we have $\mathrm{d}(x, (\infty)) = 2$. Let Q_j be the set for which $x \in Q_j$ and let L_i, $1 \leq i \leq 2$, be the line through (∞) and x_i. The lines L_1 and L_2 are contained in Q_j and Q_{i+1}. Let p_i be the $\mathbf{G}$-point corresponding with the $\mathbf{S}$-line L_i. If $i < r$, then p_1 and p_2 are contained in two $\mathbf{G}$-lines, a contradiction. If $i = r$, then $s(p_1) = s(p_2) = 1$. Hence the $\mathbf{G}$-line through p_1 and p_2 satisfies $s(p) = 1$ for all points of this line, but this is impossible since Q_j is not a dual grid.

By construction we have that $\mathbf{S}_{(\infty)} \simeq \mathbf{G}$. (Remark that there are no quads through (∞) and a point of type (5) because of condition (AS2).) $\square$

The above given proof is different from the one given in [31]. This latter proof, which will be given now, makes no use of Theorem 3.10.1.

Proof. Let x denote the point (∞). For every point y at distance 2 from x, we define the set $S(x,y) = \{L \in \mathcal{L} | x \in L \text{ and } L \text{ contains a point collinear with } y\}$. Notice that if m is a point collinear with y but at distance 2 from x, then $S(x,y) = S(x,m)$ is equal to the set of lines through x in the quad defined by x and y.

Let p be a point and let L be a line of $\mathbf{S}$, then we have to prove that there exists a unique point on L nearest to p. This is certainly true if $p \in L$ or $p = x$. So, suppose that $p \neq x$ and $p \notin L$. The following statements are trivial or are readily obtained.

- If $\mathrm{d}(p,x) = 1$ and $x \in L$, then x is the unique point of L nearest to p.
- If $\mathrm{d}(p,x) = 1$ and $x \notin L$, let a be the point of L at distance 1 from x and let b be any other point of L. If $xp \in S(x,b)$, then there exists

a unique point of L at distance 1 from p. If $xp \notin S(x, b)$, then a has distance 2 from p and every other point of L has distance 3 to p.

- If $\mathrm{d}(p, x) = 2, x \in L, L \in S(x, p)$ then there exists a unique point of L at distance 1 from p.

- If $\mathrm{d}(p, x) = 2, x \in L, L \notin S(x, p)$, then x has distance 2 to p and every other point of L has distance 3 to p.

- If $\mathrm{d}(p, x) = 2, x \notin L$, let a be the point of L collinear with x and b any other point of L. If $\mathrm{d}(p, L) = 1$, then there exists a unique point of L at distance 1 from p. Suppose therefore that $\mathrm{d}(p, L) \geq 2$. If $S(x, b) \cap S(x, p) \neq \emptyset$, then there exists a unique point of L at distance 2 from p, all the other points have distance 3 to p. If $S(x, b) \cap S(x, p) = \emptyset$, then a has distance 3 to p and every other point of L has distance 4 to p.

□

Remarks.

(a) If **G** is a partial linear space without isolated points, then near polygons arising from the construction are nondegenerate; hence, the above construction can be used to construct many nondegenerate near polygons.

(b) The main property of the constructed near polygons is the existence of a point at distance at most 2 from all the other points. This property characterizes, up to degeneracy, all these near polygons as we shall show in the following section.

5.3 The 2-neighbourhood of a point

Let $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a near polygon and let x be a fixed point of $\mathcal{P}$. Define the following sets:

$$\begin{aligned} V_2 &= \{y \in \mathcal{P} \mid \mathrm{d}(x, y) = 2 \text{ and } |\Gamma(x) \cap \Gamma(y)| \geq 2\}, \\ V_1 &= \{z \in \mathcal{P} \mid \exists y \in V_2 \text{ such that } z \in \Gamma(x) \cap \Gamma(y)\}, \\ \mathcal{P}' &= \{x\} \cup V_1 \cup V_2. \end{aligned}$$

Theorem 5.3.1
The point set $\mathcal{P}'$ is a subspace.

Proof. Consider a line L having two of its points, say α and β, in $\mathcal{P}'$. We may suppose that L contains at least three points. If x is incident with L, then we may suppose that $\alpha \in V_1$. So, there exists a point $y \in V_2$ such that α is a common neighbour of x and y. By Yanushka's theorem, the points x and y determine a quad Q. Hence, for every point $\gamma \neq x$ of L there exists a point y' of $V_2 \cap Q$ such that γ is a common neighbour of x and y'. If x is not incident with L, we may suppose that $\mathrm{d}(x, \alpha) = 2$. The points x and α determine a quad and once again it is clear that every point of L belongs to V_1 or V_2. □

Let $\mathbf{S}'$ be the partial linear space induced by $\mathcal{P}'$.

Lemma 5.3.2
The nondegenerate support of $\mathbf{S}_x^2$ is $\mathbf{S}'$, hence $\mathbf{S}'$ is a near polygon.

Proof. If $y \in \mathcal{P}$ with $\mathrm{d}(x, y) = 2$ and $y \notin V_2$, then there is a unique line through y which belongs to $\mathcal{L}_x^2$. Hence y does not belong to the nondegenerate support of $\mathbf{S}_x^2$. If $z \in \mathcal{P}$ with $\mathrm{d}(x, z) = 1$ and $z \notin V_1$, then z does not belong to the nondegenerate support of $\mathbf{S}_x^2$ as otherwise this would imply the existence of a point $u \in V_2$, collinear with z. □

Theorem 5.3.3
If $\mathcal{P}' \neq \{x\}$, then $\mathbf{S}'$ is a near polygon which can be obtained by the construction given in the previous section.

Proof. Let $G = (\mathcal{A}, \mathcal{B}, \mathcal{I})$ be the local space of $\mathbf{S}'$ at the point x. Every point p of $\mathcal{A}$ corresponds to a line through x which is incident with $s(p) + 1$ points. Every line L of $\mathcal{B}$ corresponds to a quad through x; let $k(L)$ be the number of points in the quad at distance 2 from x. Let y be a point of $\mathbf{S}'$ at distance 2 from x with the property that there is no quad through x and y. Any common neighbour p of x and y satisfies $s(p) = 1$, hence we can define the graph Δ as above. The set of lines through x containing a point collinear with y defines then a clique N_i in Δ and we say that the clique N_i can be constructed u_i times in the just described way.

The point x of $\mathbf{S}'$ is now the unique point of type (A), the points of $\mathbf{S}'$ at distance 1 from x are the points of type (B), the points of $\mathbf{S}'$ at distance 2 from x which are contained in a quad through x are the points of type (C), and the points of $\mathbf{S}'$ at distance 2 from x which are not contained in a quad through x are the points of type (D). □

Corollary 5.3.4
If $\mathbf{S}$ is a near $2d$-gon which satisfies $\mathrm{d}(x, y) \leq 2$ for all $y \in \mathcal{P}$, then its nondegenerate support, if not a single point, can be obtained by the construction of Section 5.2.2.

Chapter 6

Linear representations of near polygons

Throughout this chapter, Π_∞ is a $\mathrm{PG}(n,q)$, $n \geq 0$, which is embedded as a hyperplane in a projective space $\Pi = \mathrm{PG}(n+1,q)$ and $\mathcal{K}$ is a nonempty set of points of Π_∞.

6.1 Generalities

With every point $p \in \Pi_\infty$, we associate an element $i_{\mathcal{K}}(p) \in \mathbb{N} \cup \{+\infty\}$, called the *(generating) index of p with respect to $\mathcal{K}$*:

- if $p \notin \langle \mathcal{K} \rangle$, then $i_{\mathcal{K}}(p) = +\infty$,
- if $p \in \langle \mathcal{K} \rangle$, then $i_{\mathcal{K}}(p) = m$, where m is the smallest integer with the property that there are m points of $\mathcal{K}$ generating a subspace containing p.

Clearly $i_{\mathcal{K}}(p) = 1$ if and only if $p \in \mathcal{K}$.

Lemma 6.1.1
If x and y are 2 different points of $T_n^*(\mathcal{K})$ and if z is the intersection of xy with Π_∞ then $d(x,y) = i_{\mathcal{K}}(z)$, where $d(\cdot,\cdot)$ denotes the distance in the point graph of $T_n^*(\mathcal{K})$.

Proof.

- $i_{\mathcal{K}}(z) \geq \mathrm{d}(x,y)$.
  This is true for $i_{\mathcal{K}}(z) = 1$ or $i_{\mathcal{K}}(z) = +\infty$. For $2 \leq i_{\mathcal{K}}(z) < +\infty$ we use induction. Assume $i_{\mathcal{K}}(z) = m$ and let $a_1, \ldots, a_m$ be m points of $\mathcal{K}$ such that $z \in \langle a_1, \ldots, a_m \rangle$. The space $\langle y, a_1, \ldots, a_m \rangle$ is then m-dimensional

while $\langle y, a_1, \ldots, a_{m-1}\rangle$ is $(m-1)$-dimensional. Hence, the line xa_m intersects $\langle y, a_1, \ldots, a_{m-1}\rangle$ in a point u. Now $u \neq y$, so uy intersects Π_∞ in a point of $\langle a_1, \ldots, a_{m-1}\rangle$ and by induction $\mathrm{d}(u,y) \leq m-1$. Together with $\mathrm{d}(x,u) = 1$ this implies that $\mathrm{d}(x,y) \leq m$.

- $\mathrm{d}(x,y) \geq i_{\mathcal{K}}(z)$.
  We may suppose that $\mathrm{d}(x,y) < +\infty$. Let $x = a_0, \ldots, a_l = y$ be a path of length $l = \mathrm{d}(x,y)$ between x and y. The points $a_0, \ldots, a_l$ generate a subspace of Π which intersects Π_∞ in a subspace β. Let b_i, $i \in \{1, \ldots, l\}$, be the intersection of the line $a_{i-1}a_i$ with Π_∞, then $b_i \in \mathcal{K}$ and $\langle a_0, b_1, b_2, \ldots, b_l\rangle = \langle a_0, a_1, \ldots, a_l\rangle$. Hence $\beta = \langle b_1, b_2, \ldots, b_l\rangle$ and since $z \in \beta$, we have $i_{\mathcal{K}}(z) \leq l$.

$\square$

Corollary 6.1.2
$T_n^*(\mathcal{K})$ is connected if and only if $\mathcal{K}$ generates Π_∞.

Lemma 6.1.3
Consider in Π_∞ two disjoint subspaces π_1 and π_2 with respective dimensions n_1 and n_2 such that $\langle \pi_1, \pi_2\rangle = \Pi_\infty$ $(n = n_1 + n_2 + 1)$. Let $\mathcal{K}_i$ $(i = 1,2)$ be a nonempty set of points of π_i. Then $T_n^*(\mathcal{K}_1 \cup \mathcal{K}_2)$ is isomorphic to the direct product $T_{n_1}^*(\mathcal{K}_1) \times T_{n_2}^*(\mathcal{K}_2)$.

Proof. Let Π_i $(i = 1,2)$ be an (n_i+1)-dimensional subspace of Π intersecting Π_∞ in π_i. We can consider the points of $T_{n_i}^*(\mathcal{K}_i)$ as the affine points of Π_i. Let a be an arbitrary point of $T_n^*(\mathcal{K}_1 \cup \mathcal{K}_2)$. The space $\langle a, \pi_i\rangle$ intersects Π_{3-i} in a point a_{3-i}. So, with every point a there corresponds a point (a_1, a_2) of the direct product $T_{n_1}^*(\mathcal{K}_1) \times T_{n_2}^*(\mathcal{K}_2)$. This correspondence is bijective. With every point (a_1, a_2) of the direct product there corresponds a unique point a: the point a is the point in the intersection of $\langle a_1, \pi_2\rangle$ with $\langle a_2, \pi_1\rangle$. One easily verifies that this bijection also determines a one to one mapping from the set of lines of $T_n^*(\mathcal{K}_1 \cup \mathcal{K}_2)$ to the set of lines of the direct product $T_{n_1}^*(\mathcal{K}_1) \times T_{n_2}^*(\mathcal{K}_2)$. $\square$

Theorem 6.1.4
Two points of $T_n^*(\mathcal{K})$ at distance 2 of each other have at least two common neighbours.

Proof. Let x and y be two points of $T_n^*(\mathcal{K})$ at mutual distance 2 and let z be a common neighbour. Let u, respectively v, be the intersection point of zx, respectively zy, with Π_∞. The intersection point of the lines uy and vx is then a second common neighbour of x and y. $\square$

6.2 Linear representations of near polygons

$\mathcal{K}$ satisfies *the near polygon property* if for every point $p \in \mathcal{K}$ and for every line L (of Π_∞) through p, there is a unique point $q \in L \setminus \{p\}$ with smallest index.

Theorem 6.2.1
$T_n^*(\mathcal{K})$ is a near polygon if and only if $\mathcal{K}$ satisfies the near polygon property.

Proof. This is trivial for $n = 0$. So, we may suppose that $n > 0$. Suppose that $\mathcal{K}$ satisfies the near polygon property. Let a be a point and let L be a line of $T_n^*(\mathcal{K})$. We may suppose that $a \notin L$, so a and L determine a plane α which intersects Π_∞ in a line L'. Let p be the common point of L and L' and let $f : L \to L'$ be the projection such that $f(x) = y$ if and only if x, y and a are collinear. Since $\mathrm{d}(a, x) = i_{\mathcal{K}}(f(x))$ for all $x \in L \setminus \{p\}$, there is a unique affine point b on L nearest to a. Since there are at least two affine points on L it follows that $\mathrm{d}(a, b) < +\infty$. Conversely, suppose that $T_n^*(\mathcal{K})$ is a near polygon. Take a point p of Π_∞ and a line L of Π_∞ through p. Let M be a line of $T_n^*(\mathcal{K})$ through p and let a be a point in the plane $\langle M, L\rangle$ but not on M or L. Using the projection from M to L, it is immediate that there is a unique point $q \in L \setminus \{p\}$ with smallest index. □

Corollary 6.2.2

(1) *$T_n^*(\mathcal{K})$ is a near $2d$-gon if and only if*

  (a) *for every point $p \in \mathcal{K}$ and every line L (of Π_∞) through p, there is a unique point $r \neq p$ on L with smallest index,*

  (b) *the maximal index of a point of Π_∞ is equal to d.*

(2) *$T_n^*(\mathcal{K})$ is a near $(2d+1)$-gon if and only if*

  (a) *the maximal index of a point of Π_∞ is equal to d,*

  (b) *there exists a point $p \in \mathcal{K}$ and a line L (of Π_∞) through p such that every point on L different from p has index d,*

  (c) *for every point $p \in \mathcal{K}$ and every line L (of Π_∞) through p containing a point ($\neq p$) with index different from d, there is a unique point $r \neq p$ on L with smallest index.*

Proof. Property (1) is trivial. The properties (a), (b) and (c) given in (2) are respectively equivalent to the properties (1), (3) and (2) from Section 1.2.2. □

Concerning the existence of quads, one can say the following.

Theorem 6.2.3
Suppose $q \geq 3$. If $T_n^*(\mathcal{K})$ is a near polygon, then every two points at distance 2 are contained in a unique quad.

Proof. This follows immediate from Theorems 3.5.2 and 6.1.4. $\square$

6.3 Linear representations of generalized quadrangles

Result 6.3.1
$T_n^*(\mathcal{K})$ is a generalized quadrangle if and only if every line of Π_∞ intersects $\mathcal{K}$ in 0 or 2 points.

This yields the following examples.

- If $n = 1$, then Π is a $\mathrm{PG}(2, q)$, Π_∞ is a line of Π and $\mathcal{K}$ is a set of 2 points of Π_∞. In this case $T_1^*(\mathcal{K})$ is a $(q \times q)$-grid.
- If $n = 2$, then Π_∞ is a $\mathrm{PG}(2, q)$ with q even and $\mathcal{K}$ is a hyperoval in Π_∞. In this case $T_2^*(\mathcal{K})$ is a generalized quadrangle of order $(q-1, q+1)$.
- If Π_∞ is a $\mathrm{PG}(n, 2)$ with $n \geq 3$, then the complement of $\mathcal{K}$ is a subspace and hence a hyperplane, since $|\mathcal{K}| = 2^n$.
- There are no examples if Π_∞ is a $\mathrm{PG}(n, q)$ with $n, q \geq 3$. Otherwise, there would exist a 3-dimensional subspace β of Π_∞ such that $\beta \cap \mathcal{K}$ is a cap of β with $q^2 + q + 2$ elements, a contradiction (see Section 1.4.1).

Remark. It is possible to determine all affine embeddings of generalized quadrangles, see [86].

6.4 Some constructions of linear representations of near polygons

The following theorem illustrates how some linear representations of near polygons can be constructed.

Theorem 6.4.1
Consider in Π_∞ two disjoint subspaces Π_1 and Π_2 of dimensions $n_1 \geq 0$ and $n_2 \geq 0$, such that $\Pi_\infty = \langle \pi_1, \pi_2 \rangle$. Let $\mathcal{K}_i$, $i \in \{1, 2\}$, be a set of points in Π_i and put $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$. If $T_{n_i}^*(\mathcal{K}_i)$, $i \in \{1, 2\}$, is a near $2d_i$-gon, then $T_n^*(\mathcal{K})$ is a near $2(d_1 + d_2)$-gon.

Proof. This is immediate since $T_n^*(\mathcal{K}) \simeq T_{n_1}^*(\mathcal{K}_1) \times T_{n_2}^*(\mathcal{K}_2)$. $\square$

Example. Consider in Π_∞ a set $\mathcal{K}$ of $n+1$ points generating the space, then $T_n^*(\mathcal{K})$ is a regular near $2(n+1)$-gon of Hamming type.

Theorem 6.4.2
Assume $n = 2k$, $k \geq 1$ and $q = 2^h$ an even prime power. Take a fixed point p of Π_∞ and consider k planes $\Pi_1, \dots, \Pi_k$ through it which generate Π_∞. Let $\mathcal{H}_i$, $i \in \{1, \dots, k\}$, be a hyperoval of Π_i containing p. Put $\mathcal{K} = \mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k$. Then $T_{2k}^*(\mathcal{K})$ is a near $(2k+2)$-gon having the property that every two points at distance 2 are contained in a unique quad.

Proof. We use induction on k. The theorem is clearly true when $k = 1$, so suppose that $k \geq 1$.

Let L be a line of Π_∞ through p. We may suppose that L is not contained in $\langle \Pi_1, \dots, \Pi_{k-1} \rangle \cup \Pi_k$. The space $\langle L, \Pi_k \rangle$ intersects $\langle \Pi_1, \dots, \Pi_{k-1} \rangle$ in a line L_1 through p and the space $\langle L, \Pi_1, \dots, \Pi_{k-1} \rangle$ intersects Π_k in a line L_2 through p. Clearly $\langle L, L_1 \rangle \cap \Pi_k = L_2$. Let p_i, $i \in \{1, 2\}$, be the unique point of $L_i \setminus \{p\}$ with smallest index (hence $p_2 \in \mathcal{K}$). We now will prove that the point in the intersection of p_1p_2 with L is the unique point of $L \setminus \{p\}$ with smallest index. Let f be the smallest positive integer with the property that there exist f points $a_1, \dots, a_f \in \mathcal{K}$ such that $\langle a_1, \dots, a_f \rangle \cap (L \setminus \{p\}) \neq \emptyset$. Clearly $p \notin \{a_1, \dots, a_f\}$. Suppose that $a_1, \dots, a_l \in \mathcal{H}_1 \cup \ldots \cup \mathcal{H}_{k-1}$ and $a_{l+1}, \dots, a_f \in \mathcal{H}_k$ for some $l \in \{1, \dots, f-1\}$. Every point r of $\langle a_1, \dots, a_f \rangle \cap (L \setminus \{p\})$ belongs to a line r_1r_2 with $r_1 \in \langle a_1, \dots, a_l \rangle$ and $r_2 \in \langle a_{l+1}, \dots, a_f \rangle$. Clearly $r_1 \in L_1$, $r_2 \in L_2$ and $i_{\mathcal{K}}(r) = f = l + (f - l) = i_{\mathcal{K}}(r_1) + i_{\mathcal{K}}(r_2)$. It follows immediately that $L \setminus \{p\}$ contains a unique point with smallest index, namely the point $L \cap p_1p_2$.

Let $r \in \mathcal{H}_k \setminus \{p\}$ and let L be a line of Π_∞ through r. We may suppose that L is not contained in Π_k. The space $\langle L, \Pi_k \rangle$ intersects $\langle \Pi_1, \dots, \Pi_{k-1} \rangle$ in a line L' through p. Let f be the smallest integer with the property that there exist f points $a_1, \dots, a_f \in \mathcal{K}$ such that $\langle a_1, \dots, a_f \rangle \cap (L \setminus \{r\}) \neq \emptyset$. Clearly $r \notin \{a_1, \dots, a_f\}$. Suppose that $a_1, \dots, a_l \in (\mathcal{H}_1 \cup \ldots \cup \mathcal{H}_{k-1})$ and $a_{l+1}, \dots, a_f \in \mathcal{H}_k \setminus \{p\}$ for some $l \in \{1, \dots, f\}$. If $l = f$, then L meets $\langle \Pi_1, \dots, \Pi_{k-1} \rangle$ in a point of L'. This point is a point of $L \setminus \{r\}$ with smallest index. Suppose $l \neq f$, then every point s of $\langle a_1, \dots, a_f \rangle \cap (L \setminus \{r\})$ belongs to a line s_1s_2 with $s_1 \in \langle a_1, \dots, a_l \rangle$ and $s_2 \in \langle a_{l+1}, \dots, a_f \rangle$. Clearly $s_1 \in L'$ and $s_1 \notin L$. The plane $\langle s_1, L \rangle$ meets Π_k in a line L'' through r. Let s_1' be the second point of $\mathcal{H}_k$ on L''. The line s_1s_1' meets $L \setminus \{r\}$. Hence $f = l + 1 = i_{\mathcal{K}}(s_1) + 1$. The following conclusions are now straightforward. If L meets $\langle \Pi_1, \dots, \Pi_{k-1} \rangle$ in a point, then this point is the unique point of $L \setminus \{r\}$ with smallest index. If L does not meet $\langle \Pi_1, \dots, \Pi_{k-1} \rangle$, consider then the intersection L' of $\langle L, \Pi_k \rangle$ with $\langle \Pi_1, \ldots, \Pi_{k-1} \rangle$. Let s_1 be the unique point of $L' \setminus \{p\}$ with smallest index. The plane $\langle s_1, L \rangle$ intersects Π_k in a line L'' through r. Let s_1' be the second point of $\mathcal{K}$ on L''. The intersection of the line $s_1 s_1'$ with L is then the unique point of $L \setminus \{r\}$ with smallest index. So far, we have proved that $T_n^*(\mathcal{K})$ is a near polygon.

We prove now (again by induction on k) that $k+1$ is the maximum index of a point of Π_∞. This statement is true for $k = 1$; so suppose that $k > 1$. Take a point $r \neq p$ and consider the line pr. Above, we showed how we can construct the point of pr with smallest index. By induction we have that $i_{\mathcal{K}}(r) \leq k+1$. Take now a point t_1 of $\langle \Pi_1, \ldots, \Pi_{k-1} \rangle$ with index k and let $t_2 \in \mathcal{H}_k \setminus \{p\}$. If t_3 is a point of $t_1 t_2 \setminus \{t_1, t_2\}$, then $i_{\mathcal{K}}(t_3) = k+1$. This proves that $T_n^*(\mathcal{K})$ is a near $(2k+2)$-gon.

If $q \geq 3$, then we already know that $T_n^*(\mathcal{K})$ satisfies the property that every two points at distance 2 are contained in a unique quad. We construct now explicitely the quads and this will show that this property also holds if $q = 2$. Let x and y be two points of $T_n^*(\mathcal{K})$ at distance 2. The line xy intersects Π_∞ in a point z. If $z \in \Pi_i$, $i \in \{1, \ldots, k\}$, then the affine points in $\langle x, \Pi_i \rangle$ define a geodetically closed subspace of $T_n^*(\mathcal{K})$ inducing a GQ isomorphic to $T_2^*(\mathcal{H}_i)$. If the above defined set would not be geodetically closed, then Π_i is contained in a 3-space β such that $|\beta \cap \mathcal{K}| \geq q+4$, a contradiction. If $z \in z_1 z_2$ with $z_1 \in \mathcal{H}_i \setminus \{p\}$ and $z_2 \in \mathcal{H}_j \setminus \{p\}$, $i \neq j$, then the affine points in the plane $\langle z_1, z_2, x \rangle$ define a geodetically closed subspace of $T_{2k}^*(\mathcal{K})$ inducing a grid. If the above defined set would not be geodetically closed, then $z_1 z_2$ is contained in a plane α such that $|\alpha \cap \mathcal{K}| \geq 4$, a contradiction. $\square$

We construct some examples in the case $q = 2$.

Theorem 6.4.3
Let Π_∞ be a $\mathrm{PG}(n,2)$, $n \geq 0$, embedded as a hyperplane in $\Pi = \mathrm{PG}(n+1,2)$, let π be a hyperplane in Π_∞ and let $\mathcal{K}$ be a set of points of $\Pi_\infty \setminus \pi$ generating Π_∞. Then $T_n^*(\mathcal{K})$ is a near polygon.

Proof. The geometry $T_n^*(\mathcal{K})$ is connected since $\mathcal{K}$ generates Π_∞. In the space Π there are two hyperplanes through π and different from Π_∞. Every line of $T_n^*(\mathcal{K})$ has one point in common with each of the two hyperplanes and as a consequence $T_n^*(\mathcal{K})$ is a thin near polygon since it is a connected bipartite graph. $\square$

Now, let Π_∞ be a $\mathrm{PG}(n,2)$ with $n \in \mathbb{N} \setminus \{0\}$. Take a point p of Π_∞ and consider $k \geq 1$ subspaces $\pi_1, \ldots, \pi_k$ of dimension at least 1 through it which generate Π_∞. Let n_i ($i \geq 1$) be the number of i-dimensional subspaces, hence $\sum_{i \geq 1} n_i = k$. We make the assumption that $n = \sum_{i \geq 1} i n_i$. Now, choose in each π_i a hyperplane not through p and let V_i be the set of points of π_i not in that hyperplane. Finally, put $\mathcal{K} = V_1 \cup \ldots \cup V_k$. Then $T_n^*(\mathcal{K})$ is a near $(2k+2)$-gon. Indeed, in each of the k subspaces π_k, we had chosen a hyperplane. The subspace π of Π_∞ generated by those hyperplanes is itself a hyperplane of Π_∞ and the result follows from the previous theorem.

6.5 Linear representations of near hexagons

In this section, we suppose that $n \geq 2$ and $q \geq 3$. Our aim is to determine all sets $\mathcal{K}$ such that $T_n^*(\mathcal{K})$ is a near hexagon.

6.5.1 Necessary and sufficient conditions

From the results of Section 6.2, we may conclude the following result.

Result 6.5.1
$T_n^*(\mathcal{K})$ is a near hexagon if and only if the following conditions are satisfied.

(1) *Every line of Π_∞ intersects $\mathcal{K}$ in 0, 1 or 2 points, hence $\mathcal{K}$ is a cap of Π_∞.*

(2) *There is a tangent line and on every tangent line there is a unique point different from the intersection point, through which there is a secant line.*

Proof. We prove that conditions (1) and (2) are equivalent to conditions (1a) and (1b) from Corollary 6.2.2 (the case $d = 3$). Suppose that (1) and (2) hold. Let $p \in \mathcal{K}$ and let L be any line through p. If L is a secant line, then by (1) there exists a unique point of $\mathcal{K}$ on L. This point is the unique point of $L \setminus \{p\}$ with smallest index (i.e. index 1). If L is a tangent line, then there exists a unique point of $L \setminus \{p\}$ through which there is a secant line. This point is the unique point of $L \setminus \{p\}$ with smallest index (i.e. index 2). Hence (a) is satisfied. Take now an arbitrary point r of Π_∞. We will prove that $i_\mathcal{K}(r) \leq 3$. We may suppose that $r \notin \mathcal{K}$. Let $p \in \mathcal{K}$ be arbitrary. If rp is a secant line, then $i_\mathcal{K}(r) = 2$. If rp is a tangent line, then there exists a unique secant line p_1p_2 (with $p_1, p_2 \in \mathcal{K}$) meeting rp in a point different from p. Since $r \in \langle p, p_1, p_2 \rangle$, we have that $i_\mathcal{K}(r) \leq 3$. It suffices now to prove that there exists a point r with $i_\mathcal{K}(r) \geq 3$. Such a point always exists on every tangent line. Suppose now that properties (1a) and (1b) from Corollary 6.2.2 hold. If L is a line intersecting $\mathcal{K}$ in more than two points, then (a) would not be satisfied. Hence (1) is satisfied. Take now a point r with $i_\mathcal{K}(r) = 3$, then rp is a tangent line for every $p \in \mathcal{K}$. Let L be an arbitrary tangent line with tangent point p. There exists a unique point r in $L \setminus \{p\}$ with smallest index. This index is necessary equal to 2 and r is the unique point through $L \setminus \{p\}$ through which there is a secant line. Hence also (2) is satisfied. $\square$

6.5.2 The known examples

If we consider only linear representations of near hexagons in $\mathrm{AG}(n+1, q)$ with $q > 2$, then the following examples are known.

(1) Let Π_∞ be a desarguesian projective plane $\mathrm{PG}(2, q)$ and let $\mathcal{K}$ be a set of three noncollinear points, then $T_2^*(\mathcal{K})$ is a near hexagon of Hamming type.

(2) Let Π_∞ be a $\mathrm{PG}(3, q)$ with q even and let $\mathcal{K}$ consists of a hyperoval in a plane of Π_∞ and one other point not in that plane, then $T_3^*(\mathcal{K})$ is a near hexagon which is a direct product of a line with a generalized quadrangle of order $(q-1, q+1)$, see Lemma 6.1.3.

(3) Consider the near hexagon related to the extended ternary Golay code C, see Section 3.2.4. The points are the cosets from the 6-dimensional affine space $\mathbb{F}_3^{12}/C$. Two cosets are adjacent whenever they contain vectors that differ in only 1 position, the line through them is the line of the affine space connecting them. The embedding of this affine space in a 6-dimensional projective space Π defines a hyperplane Π_∞. It is clear that there is a set $\mathcal{K}$ of 12 points in Π_∞ such that the near hexagon is isomorphic to $T_5^*(\mathcal{K})$.

(4) Let Π_∞ be a $\mathrm{PG}(4, q)$ with q even and let $\mathcal{K}$ be a union of two hyperovals whose carrying planes α and α' meet each other in a point p of $\mathcal{K}$. By Theorem 6.4.2, $T_4^*(\mathcal{K})$ is a near hexagon.

If there exists another linear representation of a near hexagon, then we will prove that the following holds for this linear representation.

(A) $n \geq 7$

(B) $q \geq 16$ and even

(C) There exists a quad isomorphic to a generalized quadrangle $T_2^*(\mathcal{O})$ and every such quad has a rosette of ovoids. Moreover, if there exists a quad isomorphic to $T_2^*(\mathcal{O})$ with $\mathcal{O}$ a regular hyperoval, then $q \geq 64$ and a new type of ovoid in $\mathrm{PG}(3, q)$ exists.

6.5.3 Conditions on the set $\mathcal{K}$

We suppose here that $T_n^*(\mathcal{K})$ is a near hexagon, although some of the results will also hold when we only require that $T_n^*(\mathcal{K})$ is a near polygon.

Lemma 6.5.2
Every plane of Π_∞ *intersects* $\mathcal{K}$ *in* 0, 1, 2, 3 *or* $q+2$ *points. If there exists a plane* α *such that* $|\alpha \cap \mathcal{K}| = q+2$, *then* $\alpha \cap \mathcal{K}$ *is a hyperoval, hence* q *is even.*

Proof. Assume $p \in \alpha \cap K$ then every line of α through p has at most one point ($\neq p$) in common with $\mathcal{K}$, hence $|\alpha \cap \mathcal{K}| \leq q+2$. Suppose $3 < |\alpha \cap \mathcal{K}| < q+2$. Let a, b, c, d be four different points of $\alpha \cap \mathcal{K}$. On a tangent line L in α through a, there are at least two points ($\neq a$) with smallest index, a contradiction. $\square$

Definition. A plane of Π_∞ is called *thick* whenever it intersects $\mathcal{K}$ in $q+2$ points.

Lemma 6.5.3
If β *is a* 3*-space through a thick plane, then* $|\beta \cap \mathcal{K}| \in \{q+2, q+3\}$.

Proof. Let α denote the thick plane. Since $\beta \cap \mathcal{K}$ is a cap of β, we have that $|\beta \cap \mathcal{K}| \leq q^2+1$. Suppose $|\beta \cap \mathcal{K}| \geq q+4$. Take two points of $\beta \cap \mathcal{K}$ not contained in α. The line L through those points intersects α in a point a. Denote by $M_1, \ldots, M_{\frac{q+2}{2}}$ the secant lines in α through a. The plane through L and M_i is thick, hence $|\beta \cap \mathcal{K}| \geq \frac{q+2}{2}q+2 = \frac{q^2+2q+4}{2}$. Take a point $b \in \alpha \cap \mathcal{K}$. Since $q^2+q+1 > q^2 \geq |\beta \cap \mathcal{K}|-1$, there exists a tangent line L' of β through b. Every plane through L' has at most three points in common with $\mathcal{K}$. Hence $|\beta \cap \mathcal{K}| \leq 2(q+1)+1 = 2q+3$, contradicting $|\beta \cap \mathcal{K}| \geq \frac{q^2+2q+4}{2}$. $\square$

Lemma 6.5.4

(1) *Two thick planes cannot meet in a line.*

(2) *Two thick planes cannot meet in a point not belonging to* $\mathcal{K}$.

(3) *If two thick planes* α *and* β *meet each other in a point of* $\mathcal{K}$, *then the points of* $\mathcal{K}$ *in the* 4*-space through* α *and* β *are all contained in* $\alpha \cup \beta$.

(4) *If two thick planes* α *and* β *are disjoint, then the points of* $\mathcal{K}$ *in the* 5*-space through* α *and* β *are all contained in* α *or* β.

Proof.

(1) If two thick planes meet in a line, then the 3-space through these planes has at least $2q+2$ points in common with $\mathcal{K}$, a contradiction.

(2) Suppose that α and β are two thick planes meeting in a point $p \notin \mathcal{K}$. Take a line L through p, contained in α and having two points in common with $\mathcal{K}$. Now, the 3-space through L and β has at least $q+4$ points, a contradiction.

(3) Suppose that α and β are two thick planes meeting in a point $p \in \mathcal{K}$. Suppose that the 4-space through α and β contains a point $r \in \mathcal{K}$ not contained in $\alpha \cup \beta$. Let Ω be the 3-space through α and r. This 3-space meets β in a line L through p. Hence, Ω contains at least $q+4$ points, a contradiction.

(4) Let α and β be two disjoint thick planes. Suppose that the 5-space through α and β contains a point $p \in \mathcal{K}$ not contained in $\alpha \cup \beta$. Let r (respectively s) be the intersection of β (respectively α) with the 3-space through p and α (respectively through p and β). It is impossible that $r, s \in \mathcal{K}$. If one of the points, say r, is in $\mathcal{K}$, then the 3-space through p and α contains at least $q+4$ points of $\mathcal{K}$, a contradiction. Hence $r, s \notin \mathcal{K}$, but this is also impossible, since $rs \setminus \{p\}$ contains then two points with smallest index.

$\square$

Theorem 6.5.5
Two points x and y at mutual distance 2 have 2 or $q+2$ common neighbours, x and y are in a unique quad isomorphic to a grid or a GQ of type $T_2^*(\mathcal{O})$. If there are no thick planes, then all quads are grids and the near hexagon is regular.

Proof. Since x and y are not collinear, the line xy intersects the hyperplane in a point $z \notin \mathcal{K}$. If there is no secant line through z, then $\mathrm{d}(x,y) = 3$, a contradiction. From Lemma 6.5.2, it follows that there are two possibilities, either z is incident with one secant line or with $\frac{q+2}{2}$ secant lines. Each secant line yields a contribution 2 to the number of common neighbours. For, if a secant line through z intersects $\mathcal{K}$ in u and v, then $ux \cap vy$ and $uy \cap vx$ are two common neighbours of x and y. If there is only one secant line L through z, then the affine points of the plane through L and x are the points of a grid $\mathcal{Q}$. The set of the points of $\mathcal{Q}$ is geodetically closed. For, suppose that a, b, c is a path where a, b are noncollinear points of $\mathcal{Q}$ and c is no point of $\mathcal{Q}$. The plane through a, b, c will intersect Π_∞ in a secant line M. The plane through L and M is thick, contradicting the fact that there is only one secant line through z. If there are $\frac{q+2}{2}$ secant lines through z, then these lines are contained in a thick plane α of Π_∞. The affine points of the space through x and α are the points of a generalized quadrangle $\mathcal{Q}$. The set of points of $\mathcal{Q}$ is geodetically closed. For, suppose that a, b, c is a path where a, b are noncollinear points of $\mathcal{Q}$ and c is no point of $\mathcal{Q}$. The plane through a, b, c intersects Π_∞ in a secant line M. Now, the 3-space through α and M contradicts Lemma 6.5.3. □

Lemma 6.5.6
If $k = |\mathcal{K}|$ and if N is the total number of thick planes of Π_∞, then

$$\frac{1}{2}Nq(q^2-1) + \frac{q^n-1}{q-1} - (k-1) = \frac{1}{2}(q-1)(k-1)(k-2).$$

Proof. Let $p \in \mathcal{K}$ be fixed. Let N_1 be the total number of thick planes through p. Counting pairs (L, α) where L is a tangent line through p and α is a plane through L intersecting $\mathcal{K}$ in 3 points, yields

$$(N-N_1)[(q^2+q+1)-(q+2)]\frac{q+2}{2}+\frac{q^n-1}{q-1}-(k-1)-(N-N_1)[(q^2+q+1)-(q+2)]$$

$$= (q-1)[\frac{(k-1)(k-2)}{2} - N_1\frac{q(q+1)}{2}],$$

from which the above equality follows. □

Corollary 6.5.7
The following congruences and inequalities are valid:

(1) $k \equiv n+1 \pmod{q-1}$,

(2) $k \equiv 2, 3 \pmod{q}$,

(3) $k \equiv 1, 3 \pmod{q+1}$ *if n is even and* $q+1 \mid (k-2)^2$ *if n is odd,*

(4) $\frac{q^n-1}{q-1} - (k-1) \leq \frac{1}{2}(q-1)(k-1)(k-2)$,

(5) $k \geq \sqrt{2}q^{\frac{n-2}{2}}$.

Proof. The equation of the previous lemma implies immediately (4). If we consider the equation as a congruence modulo $q-1$ or $q+1$, we find immediately (1) and (3). If we multiply both sides by 2 and then consider it as a congruence modulo q, we will find (2). From (4), it follows now that $q^{n-1} \leq \frac{1}{2}(q-1)(k-1)(k-2) + (k-1)$. Hence $q^{n-1} \leq \frac{1}{2}qk^2$. □

6.5.4 The near hexagons with a $T_2^*(\mathcal{O})$-quad

Assume that the near hexagon $T_n^*(\mathcal{K})$ has a thick plane, hence $n \geq 3$ and $q \geq 4$ is even. From Corollary 6.1.2 it follows that $k = |\mathcal{K}| \geq q + n$. We put $k' = k - q - 2$.

Theorem 6.5.8
If $n = 3$, then $\mathcal{K}$ consists of a hyperoval in some plane and one other point not in that plane.

Proof. This is an immediate corollary of Lemma 6.5.3. $\square$

Theorem 6.5.9
If $n = 4$, then $\mathcal{K}$ is the union of two hyperovals whose carrying planes meet each other in a point belonging to both hyperovals.

Proof. Let α denote a thick plane. Assume $|\Pi_\infty \cap \mathcal{K}| = q + 4$ and let a and b be the two points of $\mathcal{K}$ not in α, then ab is skew to α (see Lemma 6.5.3). There are $q^3 - 1$ lines through a skew to α and different from ab. On every such line there is a unique point ($\neq a$), through which there is a secant line. This secant line is one of the $q + 2$ lines through b and a point of $\alpha \cap \mathcal{K}$. Hence $q^3 - 1 \leq (q + 2)(q - 1)$, a contradiction. So, we may suppose that $|\Pi_\infty \cap \mathcal{K}| \geq q + 5$. Let a, b, c denote three points not contained in α. The plane β through these points meets α in a point d. If d does not belong to $\mathcal{K}$, then da does not satisfy property (2) of Result 6.5.1. Hence $d \in \mathcal{K}$ and β is thick. By Lemma 6.5.4 all points of $\mathcal{K}$ are contained in $\alpha \cup \beta$. $\square$

Theorem 6.5.10
There are no examples for $n = 5$.

Proof. Let $n = 5$ and assume $\mathcal{K}$ is a set of points in $\mathrm{PG}(5, q)$ yielding a near hexagon with a quad of type $T_2^*(\mathcal{O})$. We will find a contradiction. The proof uses several steps.

Step 1. *There are at least two thick planes.*
Proof. Assume that there is only one thick plane. From Lemma 6.5.6, it follows that

$$2q^4 + 2q^3 + 2q^2(1 - k') = k'(k' - 1)(q - 1),$$

which implies $q^2 \mid k'$ or $q^2 \mid k' - 1$. Hence $k' \geq q^2$ and $2q^4 + 2q^3 + 2q^2(1 - q^2) \geq q^2(q^2 - 1)(q - 1)$, a contradiction. $\square$

Step 2. *Two thick planes α and β meet each other in a point of $\mathcal{K}$.*
Proof. Suppose that α and β are disjoint, then by (4) of Lemma 6.5.4 all the points of $\mathcal{K}$ are contained in $\alpha \cup \beta$. Take a point $b \in \beta \setminus \mathcal{K}$ and a point

$a \in \alpha \cap \mathcal{K}$. Let $L \neq ab$ be a tangent line through a in the 3-space through b and α, then L does not satisfy property (2) of Result 6.5.1. $\square$

Step 3. *Through every point p of $\mathcal{K}$, there are at most two thick planes.*
Proof. Suppose that there are three thick planes α, β and γ through p. The 4-space through α and β has no other points in common with $\mathcal{K}$ then those in α or β. Since every line of γ through p intersects $\mathcal{K}$ in a second point, γ must intersect the 4-space in the point p, a contradiction. $\square$

In the sequel let α and β be two thick planes intersecting $\mathcal{K}$ in a point. Let Ω be the 4-space through α and β.

Step 4. *If a and b are two points of $\mathcal{K}$ not in Ω then there exists a thick plane containing them.*
Proof. Let c be the intersection point of the line ab with Ω. The 3-space through α and c intersects β in a line L. Let x be the second point of $\mathcal{K}$ on L. The plane through L and c intersects α in a line M. Let y be the second point of $\mathcal{K}$ on M. If x, c and y are not collinear then the line cx does not satisfy property (2) of Result 6.5.1. Hence c, x, y are collinear and the plane containing x, c, y, a, b is thick. $\square$

Step 5. *There are at least two points of $\mathcal{K}$ not in Ω.*
Proof. It is impossible that $\langle \mathcal{K} \rangle = \Omega$, since every near polygon is connected. Suppose r is the unique point of $\mathcal{K}$ not in Ω. Let s be a point of Ω through which there is no secant line. Then rs contradicts property (2) of Result 6.5.1. $\square$

In the sequel, let r be a fixed point of $\mathcal{K}$ not in Ω. Then the points of $\mathcal{K}$ not in $\Omega \cup \{r\}$ are partitioned by the thick planes through r.

Step 6: *There is a unique thick plane through r.*
Proof. Suppose that γ and δ are thick planes through r. Let s (respectively t) be a point of $\gamma \cap \mathcal{K}$ (respectively $\delta \cap \mathcal{K}$) not in $\Omega \cup \{r\}$. Now, the thick plane through s and t has at most four points (the intersections with $\alpha, \beta, \gamma, \delta$), a contradiction. $\square$

Step 7. *We derive the contradiction.*
From the previous steps it follows that all the points of $\mathcal{K}$ are contained in three thick planes α, β, γ. Through p there are $q^4 - q^2$ lines not contained in Ω and not intersecting γ. On every such line there is a unique point ($\neq p$) through which there is a secant. Hence $q^4 - q^2 \leq 2q^2$, a contradiction. $\square$

Lemma 6.5.11
Let Q be a quad of type $T_2^*(\mathcal{O})$. This quad determines a thick plane α in the hyperplane Π_∞. Then, the following statements are equivalent:

(1) $\Gamma_2(Q) = \emptyset$,

(2) *Every 3-space of* Π_∞ *through* α *contains* $q+3$ *points of* $\mathcal{K}$,

(3) $k = q + 2 + \frac{q^{n-2}-1}{q-1}$.

Proof. Let x be an arbitrary point of $\mathbf{S}$ not in Q. The point x and the quad Q determine a 4-space in Π which intersects Π_∞ in a 3-space through α. Now, there is a point of Q collinear with x if and only if this 3-space contains a point of $\mathcal{K}$ not in α. □

Theorem 6.5.12
Let Q *be a quad of type* $T_2^*(\mathcal{O})$. *If* $n \geq 6$, *then* $\Gamma_2(Q) \neq \emptyset$.

Proof. Suppose $\Gamma_2(Q) = \emptyset$, then by the previous lemma we have $k = q+2+\frac{q^{n-2}-1}{q-1}$. Let $x \in \Gamma(Q)$ be a point collinear with $y \in Q$. Let $L_1, \ldots, L_{q+2}$ be the $q+2$ lines of Q through y. Since $\Gamma_2(Q) = \emptyset$, every line through x is contained in a quad through x and L_i for some $i \in \{1, \ldots, q+2\}$. Hence $q + 2 + \frac{q^{n-2}-1}{q-1} \leq 1 + (q+1)(q+2)$. This implies $n \leq 5$, a contradiction. □

Corollary 6.5.13
If $n \geq 6$ *then every quad* Q *of type* $T_2^*(\mathcal{O})$ *has a rosette of ovoids.*

Proof. Let x denote a point of $\Gamma_2(Q)$, then x is ovoidal with respect to Q. Let $y \in \Gamma(Q)$ be a point collinear with x and let y' denote the point of Q collinear with y. By Theorem 3.7.1 all points ($\neq y$) of the line xy are ovoidal with respect to Q and the $q-1$ ovoids defined by these points form a rosette of ovoids with y' as common point. □

Corollary 6.5.14
If $n \geq 6$ *then* $q \neq 4$.

Proof. If $q = 4$, then the unique generalized quadrangle of order (3,5) would have a rosette of ovoids. In the model $T_2^*(\mathcal{O})$ an ovoid of $\mathrm{GQ}(3,5)$ is the set of affine points in a plane that meets the carrying plane of $\mathcal{O}$ in a line exterior to $\mathcal{O}$, see [68]. Hence two ovoids of $\mathrm{GQ}(3,5)$ are disjoint or meet in 4 points. This proves the corollary. □

Lemma 6.5.15
The following inequality is valid:

$$(\frac{q^{n-2}-1}{q-1} - k')q^2(q+2) \geq (q-1)k'(k' - q^2 - 2q - 1).$$

Proof. Let Q be a quad of type $T_2^*(\mathcal{O})$. The points of Q are the affine points of a 3-space β which intersects Π_∞ in a thick plane α. In Π_∞, there are $(\frac{q^{n-2}-1}{q-1} - k')$ 3-spaces β' through α which contain only $q+2$ points. All the $q^4 - q^3$ affine points of $\langle \beta, \beta' \rangle$ not in Q are points of $\Gamma_2(Q)$ and all points of $\Gamma_2(Q)$ are obtained this way. Hence, the number of points of $\Gamma_2(Q)$ equals $(\frac{q^{n-2}-1}{q-1} - k')(q^4 - q^3)$. Counting pairs (x, y) with $x \in \Gamma(Q)$ and $y \in \Gamma_2(Q)$ yields

$$(\frac{q^{n-2}-1}{q-1} - k')(q^4 - q^3)q^2(q+2) \geq q^3(q-1)^2 k'(k' - q^2 - 2q - 1).$$

□

Corollary 6.5.16

$$k' < \sqrt{q}(1 + \frac{3}{q})q^{\frac{n-2}{2}}$$

Proof. The inequality from the previous theorem is equivalent to

$$\frac{q^{n-2}-1}{q-1}q^2(q+2) \geq (q-1)k'(k' - q^2 - 2q - 1 + \frac{q^2(q+2)}{q-1}).$$

From $\frac{q^2(q+2)}{q-1} = q^2 + 3q + 3 + \frac{3}{q-1}$ it follows that

$$q^n \frac{q+2}{(q-1)^2} > k'(k' + q + 2). \tag{6.1}$$

If $q \neq 4$, then $2q^2 - 11q + 6 > 0$, which is equivalent to $\frac{1}{q} + \frac{6}{q^2} > \frac{q+2}{(q-1)^2}$. Hence (6.1) implies $q^{n-2}(q+6) > k'^2$, from which the stated inequality readily follows. □

Theorem 6.5.17
There are no examples for $n = 6$.

Proof. Assume there is an example, hence $q \geq 8$. We shall derive a contradiction in several steps.

Step 1. *One of the following three cases occurs:*

$$\begin{array}{llll}
(1) & k = 2q^2 + 3q + 2, & k' = 2q^2 + 2q, & N = 2q^2 + 6q - 1, \\
(2) & k = 3q^2 + 2q + 2, & k' = 3q^2 + q, & N = 7q^2 + q, \\
(3) & k = 2q^2 + 2q + 3, & k' = 2q^2 + q + 1, & N = 2q^2 + 2q + 2.
\end{array}$$

Proof. From the congruences in Corollary 6.5.7 it follows that $k = mq(q^2 - 1) + f(q)$ with $m \in \mathbb{Z}$ and $f(q) \in \{2q^2+3q+2, 3q^2+2q+2, q^2+3q+3, 2q^2+2q+3\}$. From $f(q) < 7q^2$ and $7q^2 < q(q^2-1)$ it follows that $m \geq 0$. Suppose now that $m \geq 1$. Since $f(q) \geq q^2+3q+3$, we have that $k \geq q^3+q^2+2q+3$ or $k' \geq q^3+q^2+q+1$, a contradiction since $k' \leq q(q+3)\sqrt{q}$ (see Corollary 6.5.16). We prove now that $k = q^2+3q+3$ cannot occur. As $k \geq \sqrt{2}q^2$ (see Corollary 6.5.7) $q \leq 8$. For $q = 8$, $k = q^2+3q+3 = 91$ contradicts inequality (4) of the same corollary. □

Step 2. *Let Γ denote the graph with vertices the thick planes. Two thick planes are adjacent whenever they are disjoint. Then Γ is the disjoint union of cliques. The size of each clique is at most $q+2$.*
Proof. To prove that Γ is a disjoint union of cliques, it is enough to prove the following property: if $\alpha \sim \beta$ and $\gamma \not\sim \alpha$ then $\gamma \not\sim \beta$. So, let α and β be two disjoint thick planes. The 5-space through α and β has no other point in common with $\mathcal{K}$ then those in $\alpha \cup \beta$. If γ intersects α in a point p, then $p \in \mathcal{K}$. Now γ intersects the 5-space in a line L. Let p' be the second point of $\mathcal{K}$ on this line, then $p' \in \beta$. Hence γ meets β. Let C be a clique of Γ. Since $k < N(q+2)$, there exists a thick plane δ which is no member of C. Now, each member of C meets δ. Hence $|C| \leq q+2$. □

Step 3. *Two thick planes meet each other.*
Proof. Suppose that there are two disjoint thick planes α and β and let C denote the component of α and β. If $\gamma \notin C$ is a thick plane then γ meets α and β. Hence $N \leq |C| + (q+2)^2 \leq q^2+5q+6$, a contradiction. □

Step 4. *There are at most $q+2$ thick planes through a point p.*
Proof. If all thick planes pass through p then $k \geq 1 + N(q+1)$, a contradiction. So, let α be a thick plane not containing p. Now, every thick plane through p intersects α in a point of $\mathcal{K}$. Hence there are at most $q+2$ planes through p. □

Step 5. *We derive the contradiction.*

Let p denote the intersection of two thick planes α and β. There are at most $q+2$ thick planes through p. Every thick plane not through p has a point in common with α and β. Hence $N \leq (q+2) + (q+1)^2$, a contradiction. □

Consider now the case $n \geq 7$. Every quad of type $T_2^*(\mathcal{O})$ has a rosette of ovoids. By Section 2.6.1, there exists a set of $q-1$ ovoids in $T_2(\mathcal{O}')$, which two by two intersect in two points ($\mathcal{O}' = \mathcal{O} \setminus \{x\}$ where x is any point of $\mathcal{O}$). Suppose now that $\mathcal{O}'$ is a conic (hence $\mathcal{O}$ is a regular hyperoval), then $T_2(\mathcal{O}')$ is isomorphic to $W(q)$. An ovoid N in $T_2(\mathcal{O}')$ is equivalent to an ovoid N' in $\mathrm{PG}(3,q)$. If N' is an elliptic quadric or a Tits ovoid, then every other ovoid intersects it in an odd number of points ([2]), a contradiction. Hence N' is an ovoid different from an elliptic quadric or a Tits ovoid. This implies $q \geq 64$. If $q \leq 8$, then every hyperoval $\mathcal{O}$ is regular. Hence, there are no examples of linear representations of near hexagons for $n \geq 7$ and $q = 8$.

6.5.5 The near hexagons without a $T_2^*(\mathcal{O})$-quad

Theorem 6.5.18
If $q \neq 2$ and if there is no thick plane, then one of the following two cases appears:

(1) *$n = 2$ and $\mathcal{K}$ is a set of three noncollinear points,*

(2) *$n = 5, q = 3$ and $\mathcal{K}$ is a Coxeter cap in* $\mathrm{PG}(5,3)$.

Proof. From Lemma 6.5.6, it follows that $k = |\mathcal{K}| = \frac{3q-5+\sqrt{8q^n+q^2-6q+1}}{2(q-1)}$. Hence

$$y^2 = 8q^n + q^2 - 6q + 1 \tag{6.2}$$

for some $y \in \mathbb{Z}$. From Theorem 6.5.5, it is clear that $T_n^*(\mathcal{K})$ is a regular near hexagon with parameters $s = q-1, t_2 = 1, t = k-1$. The Mathon bound $t+1 \leq (s^2-s+1)(s+1+t_2)$, see equation (4.1), is then equivalent to

$$\sqrt{8q^n+q^2-6q+1} \leq 2(q^2-3q+3)(q^2-1)-(3q-5). \tag{6.3}$$

If $n \geq 8$, then one easily checks that (6.3) yields a contradiction; hence $n \leq 7$. If $n = 2$ then $8q^n + q^2 - 6q + 1 = (3q-1)^2$ and $|\mathcal{K}| = 3$. We allready met this example. It yields the regular near hexagon of Hamming type. So we may assume that $3 \leq n \leq 7$. The equation (6.2) is equivalent to

$$8q^n - 8q^2 = (y-3q+1)(y+3q-1), \tag{6.4}$$

so $8q^2 \mid (y-3q+1)(y+3q-1)$. Let $q = p^h$ with p prime. If $p \neq 2$, then p is not a common divisor of $y-3q+1$ and $y+3q-1$. If $p = 2$, then 4 is not a common divisor of $y-3q+1$ and $y+3q-1$. Hence, in any case q^2 is a divisor of one of the factors. Since y is determined up to its sign, we may suppose that $y = xq^2 - 3q + 1$ with $x \in \mathbb{Z}$. The equation (6.4) becomes

$$-8q^{n-2} + x^2q^2 - 6xq + 2x + 8 = 0. \tag{6.5}$$

Suppose $n = 3$, then (6.5) becomes $x^2q^2 - (6x+8)q + 2x + 8 = 0$. If $x = 0$, then $q = 1$, a contradiction. Assume $x \neq 0$, let $z = |x|q$ and we have then that $z^2 - 6z - 8z - 2z + 8 \leq 0$. So $z \leq 15$ and $q \leq 13$. One finds no solutions for these values of q.

Suppose $n = 4$, then (6.5) becomes $(x^2 - 8)q^2 - 6xq + 2x + 8 = 0$. If $|x| \leq 3$, then $q \leq 13$ since q is a divisor of $2x + 8$. If $|x| \geq 4$ (so $x^2 - 8 \geq \frac{x^2}{2}$) and if we put $z = |x|q$ again, we find that $\frac{z^2}{2} - 8z + 8 \leq 0$. So $z \leq 14$ and again $q \leq 13$. One finds no solutions for these values of q.

Now, since q is a divisor of $2x + 8$, put $x = \frac{uq}{2} - 4$ with $u \in \mathbb{Z}$. The equation (6.5) becomes

$$-8q^{n-3} + \frac{u^2q^3}{4} - 4uq^2 + (16 - 3u)q + 24 + u = 0. \tag{6.6}$$

Suppose $n = 5$, then (6.6) becomes $\frac{u^2q^3}{4} - (4u + 8)q^2 + (16 - 3u)q + 24 + u = 0$. If $u = 0$, then $q \leq 23$, since q divides $24 + u$. If $u \neq 0$, put $z = |u|q$, we have then that $\frac{z^3}{4} - 4z^2 - 8z^2 - 16z - 3z^2 - 24z - z^2 \leq 0$ or $\frac{z^2}{4} - 16z - 40 \leq 0$. Hence $z \leq 66$ and $q \leq 64$. Only $q = 3$ gives a solution: $q = 3, u = 0, x = -4, y = -44, k = 12$. We will prove that $\mathcal{K}$ is a Coxeter cap in $\mathrm{PG}(5, 3)$, but we will first treat the cases $n = 6$ and $n = 7$.

Suppose $n = 6$, then (6.6) becomes $(\frac{u^2}{4} - 8)q^3 - 4uq^2 + (16 - 3u)q + 24 + u = 0$. If $|u| \leq 7$, then $q \leq 31$ since q is a divisor of $24 + u$. If $|u| \geq 8$ (so $\frac{u^2}{4} - 8 \geq \frac{u^2}{8}$) and if we put $z = |u|q$ again, we find $\frac{z^3}{8} - 4z^2 - 16z - 3z^2 - 24z - z^2 \leq 0$ or $\frac{z^2}{8} - 8z - 40 \leq 0$. Hence $z \leq 68$ and $q \leq 8$. Only $q = 4$ gives a solution: $q = 4, u = 8, x = 12, y = 181$, but in this case k is not an integer.

Suppose $n = 7$. Now, since q is a divisor of $24 + u$, put $u = vq - 24$ with $v \in \mathbb{Z}$. The equation (6.6) becomes

$$\frac{v^2q^4}{4} - (12v + 8)q^3 + (144 - 4v)q^2 + (96 - 3v)q + 88 + v = 0. \tag{6.7}$$

If $v = 0$, then $q \leq 83$, since q divides $88 + v$. If $v \neq 0$, let $z = |v|q$, then $\frac{z^4}{4} - 12z^3 - 8z^3 - 144z^2 - 4z^3 - 96z^2 - 3z^3 - 88z^2 - z^3 \leq 0$ or $\frac{z^2}{4} - 28z - 328 \leq 0$. Hence $z \leq 122$ and $q \leq 121$. This case gives no new solutions.

Hence, only the case $n = 5, q = 3$ and $k = 12$ remains to be considered. If the corresponding near hexagon exists, then it must be the regular near hexagon related to the extended ternary Golay code, since both near hexagons have the same parameters (there is only one near hexagon with these parameters, see [11]). The points of $\mathrm{PG}(5, 3)$ are the 1-spaces of $V(6,3)$. We call a point of $\mathrm{PG}(5,3)$ of type (w,d) if w is the weight of the point and d is the difference between the number of coordinates equal to 1 and the number of coordinates equal to -1. So, points can be of the following 15 types: (1,1), (2,2), (2,0), (3,3), (3,1),(4,4), (4,2), (4,0), (5,5), (5,3), (5,1), (6,6), (6,4), (6,2), (6,0). Take now 6 linear independant points of $\mathcal{K}$ and choose coordinatization in such a way that these are the points of type (1,1). There is then a secant line through every point of weight 2. The determination of the set $\mathcal{K}$ will go on in two steps.

Step 1. $\mathcal{K}$ can have only points of weight 1 and 5.
Proof. If there would be a point of weight 2 in $\mathcal{K}$, then there exist 3 points which are collinear, a contradiction.

Let $\mathcal{K}$ contain a point a of weight 3. We may assume that $a = \langle e_1+e_2+e_3 \rangle$, where e_i is the vector with all entries but the i-th one equal to 0, the i-th entry being equal to 1. Then the plane $\langle e_1, e_2, e_3 \rangle$ is thick, as it has at least four points in $\mathcal{K}$, namely a and $\langle e_i \rangle$ for $i = 1, 2, 3$, a contradiction. Thus, $\mathcal{K}$ contains no points of weight 3.

Let $\mathcal{K}$ contain a point a of weight 4. We may assume that $a = \langle v \rangle$ with $v = e_1 + e_2 + e_3 + e_4$. The plane $X = \langle v, e_1, e_2 \rangle$ meets the line $\langle e_4, e_3 \rangle$ in $b = \langle e_3 + e_4 \rangle$. However, X is not thick. Hence the line ab is tangent to $\mathcal{K}$ at a. By (2) of Result 6.5.1, just one point of $ab \setminus \{a\}$ belongs to a secant to $\mathcal{K}$. However, $ab \setminus \{a\}$ contains two such points, namely $\langle e_1 + e_2 \rangle$ and $\langle e_3 + e_4 \rangle$, a contradiction. Therefore, all points of $\mathcal{K}$ have weight either 1, 5 or 6.

Suppose now that there exists a point $\langle u \rangle$ of type 6 in $\mathcal{K}$. We can choose coordinatization such that $u = \sum_{i=1}^{6} e_i$. We will prove that it is impossible that there are other points in $\mathcal{K}$, hence we have a contradiction. Clearly no point of type (6,4) and (5,5) belongs to $\mathcal{K}$, otherwise three points of $\mathcal{K}$ are collinear. Let a be a point of $\mathcal{K}$ of type (5,3). We may assume that $a = \langle v \rangle$ with $v = e_1 + e_2 + e_3 + e_4 - e_5$. Then a belongs to the plane $\langle u, e_5, e_6 \rangle$ and the latter is thick, a contradiction. Thus, points of type (5,3) are ruled out from $\mathcal{K}$.

Assume that $a \in \mathcal{K}$ is of type (5,1), for instance $a = \langle v \rangle$ with $v = e_1 + e_2 + e_3 - e_4 - e_5$. The plane $X = \langle v, u, e_6 \rangle$ meets the line $\langle e_4, e_5 \rangle$ in the point $b = \langle e_4 + e_5 \rangle$. As $\mathcal{K}$ contains no points of weight 2 or 3, the line $L = \langle e_4 + e_5, e_6 \rangle$ is tangent to $\mathcal{K}$ in $\langle e_6 \rangle$. However, the line $\langle e_4, e_5 \rangle$ is a secant for $\mathcal{K}$ and meets L in b. The line $\langle u, v \rangle$ is also secant and it meets L in $\langle e_4 + e_5 - e_6 \rangle$. Thus, at least two points of L different from $\langle e_6 \rangle$ belong to secant lines of $\mathcal{K}$, contrary to (2) of Result 6.5.1. Therefore, points of type (5,1) are also ruled out from $\mathcal{K}$.

Assume that $a \in \mathcal{K}$ is of type (6,2), for instance $a = \langle v \rangle$ with $v = e_1 + e_2 + e_3 + e_4 - e_5 - e_6$. Then the plane $X = \langle u, e_5, e_6 \rangle$ is thick, a contradiction.

Finally, let $a \in \mathcal{K}$ be of type (6,0), for instance $a = \langle v \rangle$ with $v = e_1 + e_2 + e_3 - e_4 - e_5 - e_6$. The plane $X = \langle u, v, e_4 \rangle$ meets the line $\langle e_5, e_6 \rangle$ in $b = \langle e_5 + e_6 \rangle$. The line ab is tangent to $\mathcal{K}$ at a and it meets the line $\langle u, e_4 \rangle$ in $c = \langle e_1 + e_2 + e_3 - e_4 + e_5 + e_6 \rangle$. Thus, b and c are distinct points of $ab \setminus \{a\}$ belonging to secant lines of $\mathcal{K}$, again contrary to (2) of Result 6.5.1.

As a consequence, we remark that if one takes 7 arbitrary points of $\mathcal{K}$ then there are 6 points between them which are in a hyperplane. Otherwise, one can choose the coordinatization in such a way that all points of type (1,1) and (6,6) are elements of $\mathcal{K}$, but this is impossible by step 1.

Step 2. Construction of the set $\mathcal{K}$.
As above, we assume to have chosen coordinates in such a way that six of the 12 points of $\mathcal{K}$ correspond to the vectors $e_1, e_2, ..., e_6$ of the canonical basis of $V(6,3)$. In every of the 6 remaining points there is a unique position where there is a "0".

Assume there are two points $a = \langle u \rangle$ and $b = \langle v \rangle$ where this "0" takes the same position. We may assume that $u = e_1 + e_2 + e_3 + e_4 + e_5$ and $v = e_1 + e_2 + e_3 + \varepsilon e_4 - e_5$, with $\varepsilon = 1$ or -1. If $\varepsilon = 1$, then a, b and $\langle e_5 \rangle$ are collinear points of $\mathcal{K}$, a contradiction. Thus, $\varepsilon = -1$. Then $a, b, \langle e_4 \rangle$ and $\langle e_5 \rangle$ are points of $\mathcal{K}$ in the plane $\langle u, v, e_4 \rangle$, which is forced to be thick, a contradiction. Therefore, no two points as above exist in $\mathcal{K}$.

Hence, we may suppose that the coordinates of the 12 points look like (up to the sign):

$$(1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0),$$

$$(0,0,0,1,0,0), (0,0,0,0,1,0), (0,0,0,0,0,1),$$

$$(0,*,*,*,*,*), (*,0,*,*,*,*), (*,*,0,*,*,*),$$

$$(*,*,*,0,*,*), (*,*,*,*,0,*), (*,*,*,*,*,0).$$

Every "*" in the above coordinates must be replaced by 1 or -1. Take now the 7 points $(1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0), (0,0,0,1,0,0), (0,0,0,0,1,0), (0,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5), (\varepsilon_1',0,\varepsilon_2',\varepsilon_3',\varepsilon_4',\varepsilon_5')$, then there is a set A of 6 points between them which are in a hyperplane and the remaining point can not be contained in that hyperplane, since the 7 points above generate Π_∞. Clearly $(1,0,0,0,0,0), (0,1,0,0,0,0), (0,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5), (\varepsilon_1',0,\varepsilon_2',\varepsilon_3',\varepsilon_4',\varepsilon_5') \in A$. If we suppose that $(0,0,0,0,1,0) \notin A$, then $\varepsilon_2 = -\varepsilon_2', \varepsilon_3 = -\varepsilon_3', \varepsilon_4 = \varepsilon_4', \varepsilon_5 = \varepsilon_5'$ or $\varepsilon_2 = \varepsilon_2', \varepsilon_3 = \varepsilon_3', \varepsilon_4 = -\varepsilon_4', \varepsilon_5 = -\varepsilon_5'$, hence $\varepsilon_2\varepsilon_2' + \varepsilon_3\varepsilon_3' + \varepsilon_4\varepsilon_4' +$

$\varepsilon_5\varepsilon_5' = 0$. A similar reasoning shows that any 2 of the points of weight 5 are orthogonal. We still have some freedom in coordinatization, for instance we may suppose that $(0, -1, -1, -1, -1, -1) \in \mathcal{K}$. Since (0,-1,-1,-1,-1,-1) and (*,0,*,*,*,*) are orthogonal (and using the remaining freedom in coordinatization) we may suppose that $(1, 0, 1, -1, -1, 1) \in \mathcal{K}$. Since (0,-1,-1,-1,-1,-1), (1,0,1,-1,-1,1) and (*,*,0,*,*,*) are orthogonal (and using the remaining freedom in coordinatization) we may suppose that $(1, 1, 0, 1, -1, -1) \in \mathcal{K}$. From the orthogonality of the points of weight 5, the remaining points are readily obtained. These are the points (1,-1,1,0,1,-1), (1,-1,-1,1,0,1) and (1,1,-1,-1,1,0). We did the coordinatization in such a way that the 12 points are just the 12 columns of the matrix that defined the extended ternary Golay code. Since we know that there is a linear representation of the near hexagon related to this code, we know that the set of 12 points given above satisfies the required conditions. From the previous reasoning, it also follows that all sets of points in $\mathrm{PG}(5, 3)$ which satisfy the required conditions are equivalent. $\square$

Remark.

The proof of Theorem 6.5.18 can be shortened by using results of [20], moreover here $q = 2$ is allowed. Let $p \in \mathcal{K}$ and let α be a hyperplane of Π_∞ not containing p, then the projection of $\mathcal{K}$ from p on α will yield a cap $\mathcal{K}'$ with $k - 1$ points. It is easy to prove that $T^*_{n-1}(\mathcal{K}')$ is a partial quadrangle with parameters $(s, t, \mu) = (q - 1, k - 2, 2)$. In [20], an almost complete classification of partial quadrangles with a linear representation is given, see also [34] for a survey. The classification is complete for $\mu = 2$. One only has to check whether these partial quadrangles are indeed coming from a projection of a near hexagon. Besides the projections of the examples mentioned in Theorem 6.5.18, there is one other example in the case $q = 2$ in which case $\mathcal{K}'$ is a set of 5 points in $\mathrm{PG}(3, 2)$ no 4 of which are contained in a hyperplane, i.e. $\mathcal{K}'$ is a frame of $\mathrm{PG}(3, 2)$. The set $\mathcal{K}$, yielding the corresponding near hexagon $T^*_4(\mathcal{K})$, is a frame of $\mathrm{PG}(4, 2)$, i.e. a set of 6 points no 5 of which are contained in a hyperplane. Note that this proof relies on the classification of the linear representations of the partial quadrangles.

6.6 Characterizations

Definition.

(1) For $m \in \mathbb{N}$, let $f_m(x, y)$ be the unique polynomial of degree m in x such that $f_m(i, y) = \frac{y^{i-1}-1}{y-1}$ for all $i \in \{1, \ldots, m + 1\}$. It is straightforward

to check that $f_m(x,y) = \sum_{j=1}^{m} \binom{x-1}{j}(y-1)^{j-1}$ and that $f_m(m+2,y) = \frac{y^{m+1}-1}{y-1} - (y-1)^m$.

(2) For $m \in \mathbb{N} \setminus \{0\}$, let $g_m(x,y) = \frac{1}{y}\binom{x-1}{m}(y-1)^{m-1} + f_{m-1}(x,y)$.

Lemma 6.6.1

$$g_m(x,y) = \tfrac{1}{y}[f_m(x+1,y) - 1].$$

Proof. A calculation shows that

$$f_m(x+1,y) = \sum_{j=1}^{m} \binom{x}{j}(y-1)^{j-1} = \sum_{j=1}^{m} \binom{x-1}{j}(y-1)^{j-1} + \sum_{j=1}^{m} \binom{x-1}{j-1}(y-1)^{j-1}$$

$$= 1 + \sum_{j=1}^{m} \binom{x-1}{j}(y-1)^{j-1} + \sum_{j=1}^{m-1} \binom{x-1}{j}(y-1)^{j} = y g_m(x,y) + 1.$$

□

For $m \geq 1, y > 1$ and $1 \leq x_1 < x_2$, one has that $f_m(x_1,y) < f_m(x_2,y)$ and $g_m(x_1,y) < g_m(x_2,y)$. Hence the inequalities of the following theorem yield upper bounds for the value k.

Theorem 6.6.2

Let V be a set of $k > 0$ points of $\mathrm{PG}(n,q)$ with the property that no l of them are contained in an $(l-2)$-flat $(n \geq l-2 \geq 0)$.

(1) *If $l = 2m+1$, then $f_m(k,q) \leq \frac{q^n-1}{q-1}$ and equality holds if and only if $T_n^*(V)$ is a near $2(m+1)$-gon.*

(2) *If $l = 2m$ and $(n,m) \neq (0,1)$, then $g_m(k,q) \leq \frac{q^n-1}{q-1}$ and equality holds if and only if $T_n^*(V)$ is a near $(2m+1)$-gon.*

Proof. (1) Let $l = 2m+1$, then $m \geq 1$. If $k \leq 2m$, then

$$f_m(k,q) \leq f_m(2m,q) = \sum_{j=1}^{m} \binom{2m-1}{j}(q-1)^{j-1}$$

$$\leq \sum_{j=1}^{2m-1} \binom{2m-1}{j}(q-1)^{j-1} = \frac{q^{2m-1}-1}{q-1} \leq \frac{q^n-1}{q-1}.$$

Suppose $k \geq 2m+1$. In what follows we will make use of the fact that no j points of V (with $1 \leq j \leq 2m+1$) are contained in a $(j-2)$-flat. Now, fix a point p of V. A line L through p is said to be of type $i \geq 1$ if $i = \min i_V(r)$, where r ranges over all points of $L \setminus \{p\}$. We prove by induction that for $1 \leq i \leq m$, the number of lines through p with type at most i equals $f_i(k,q)$. This is true for $i = 1$ since $f_1(k,q) = k-1$. Suppose that $2 \leq i \leq m$, then the number of lines through p with type at most i equals

$$f_{i-1}(k,q) + \binom{k-1}{i}\left[\frac{q^i-1}{q-1} - f_{i-1}(i+1,q)\right]$$

and hence

$$f_i(k,q) = f_{i-1}(k,q) + \binom{k-1}{i}(q-1)^{i-1}.$$

Now, since there are $\frac{q^n-1}{q-1}$ lines through p, we must have $\frac{q^n-1}{q-1} \geq f_m(k,q)$. If $T_n^*(V)$ is a near $2(m+1)$-gon, then all the lines through p have type at most m, hence equality holds. Conversely, suppose that equality holds. If $k \leq 2m$, then one calculates that $m = n = 1$ and $k = 2$. In this case $T_1^*(V)$ is a grid and hence a generalized quadrangle. Assume $k \geq 2m+1$, let p be a point of V and let L be an arbitrary line through p. All the points $r \neq p$ on L with smallest index satisfy $i_V(r) \leq m$. Since no $2m+1$ points are contained in a $(2m-1)$-flat, there exists a unique point on L with smallest index. Hence $T_n^*(V)$ is a near polygon. To prove that it is a near $2(m+1)$-gon, it suffices to prove the existence of a point of index $m+1$. Such a point exists on a line of type m through p.
(2) Let $l = 2m$ (then $m \geq 1$) and $(n,m) \neq (0,1)$. If $k \leq 2m-1$, then

$$g_m(k,q) \leq g_m(2m-1,q) = \frac{1}{q}\binom{2m-2}{m}(q-1)^{m-1} + \sum_{j=1}^{m-1}\binom{2m-2}{j}(q-1)^{j-1}$$

$$\leq \sum_{j=1}^{m}\binom{2m-2}{j}(q-1)^{j-1} \leq \sum_{j=1}^{2m-2}\binom{2m-2}{j}(q-1)^{j-1} \leq \frac{q^{2m-2}-1}{q-1} \leq \frac{q^n-1}{q-1}.$$

Suppose $k \geq 2m$. Now, fix a point p of V. A line L through p is said to be of type $i \geq 1$ if $i = \min i_V(r)$, where r ranges over all points of $L \setminus \{p\}$. As before, $f_{m-1}(k,q)$ equals the number of lines through p with type at most $m-1$ and the number of lines of type m through p equals at most

$$\frac{1}{q}\binom{k-1}{m}[\frac{q^m-1}{q-1} - f_{m-1}(m+1,q)] = \frac{1}{q}\binom{k-1}{m}(q-1)^{m-1}.$$

Hence $\frac{q^n-1}{q-1} \geq g_m(k,q)$. If $T_n^*(V)$ is a near $(2m+1)$-gon, then every line through p is counted exactly once, hence equality holds. Conversely, suppose that equality holds. If $k \leq 2m-1$, then the above inequalities imply that $n=0, m=1, k=1$, a contradiction. Assume $k \geq 2m+1$ and let p be an arbitrary point of V. Since equality holds, we have the following properties for lines L through p ($i_V(L)$ denotes the type of L):

(a) $i_V(L) \leq m$,

(b) if $i_V(L) < m$, then there exists a unique point r of $L \setminus \{p\}$ such that $i_V(r) = i_V(L)$,

(c) if $i_V(L) = m$, then $i_V(r) = m$ for all points r of $L \setminus \{p\}$,

(d) there exists an L with $i_V(L) = m$.

This implies that $T_n^*(V)$ is a near $(2m+1)$-gon. □

Remark. Let V be a set of $k > 0$ points of $\mathrm{PG}(n,q)$ with the property that no $2m+1$ of them are contained in a $(2m-1)$-flat ($n \geq 2m-1, m \geq 1$). Fix $v \in V$ and let $\mathrm{PG}(n-1,q)$ be a hyperplane of $\mathrm{PG}(n,q)$ not through v. The projection (from v) of V onto $\mathrm{PG}(n-1,q)$ will yield a set V' with the property that no $2m$ points are contained in a $(2m-2)$-flat. Hence $g_m(k-1,q) \leq \frac{q^{n-1}-1}{q-1}$. From Lemma 6.6.1, it follows then that $f_m(k,q) \leq \frac{q^n-1}{q-1}$. Hence, one of the inequalities of the previous theorem follows from the other. It follows also that if $T_n^*(V)$ is a near $2(m+1)$-gon, then $T_{n-1}^*(V')$ is a near $(2m+1)$-gon, with an exception when $n=1, m=1, |V|=k=2$.

Some special cases

The case $l=2$

In this case there are no restrictions on the set V. Since $g_1(k,q) = \frac{k-1}{q}$, the inequality $g_1(k,q) \leq \frac{q^n-1}{q-1}$ is equivalent with $k \leq \frac{q^{n+1}-1}{q-1}$.

The case $l=3$

In this case we are looking for caps V in $\mathrm{PG}(n,q), n \geq 1$. For $n=1$, we have $k=|V| \leq 2$ and equality is always possible. For $n=2$, we have $k \leq q+2$. Equality is possible when q is even and in this case V is a hyperoval. If $q=2$ and $n \geq 1$, then $k \leq 2^n$ and equality holds if and only if V is the complement of a hyperplane. For $q \neq 2$ and $n \geq 3$, $k \leq 1 + \frac{q^n-1}{q-1}$. If equality holds then every line meets V in 0 or 2 points. If we take a 3-dimensional space β in $\mathrm{PG}(n,q)$ having at least one point in common with V, then this would imply the existence of a (q^2+q+2)-cap in β, a contradiction.

The case $l = 4$

In this case we are looking for sets of points in $\mathrm{PG}(n, q)$, $n \geq 2$, with the property that no four of them are contained in a plane. If the upper bound is achieved then $T_n^*(V)$ is a near pentagon. If x and y are two points of $T_n^*(V)$ at mutual distance 2, then the intersection point z of the line xy with $\mathrm{PG}(n, q)$ is contained in a unique 2-secant. Hence x and y have exactly two common neighbours. This implies that $T_n^*(V)$ is a partial quadrangle with parameters $(s, t, \mu) = (q - 1, k - 1, 2)$. From the classification given in [20] (see also [34]), it follows that there are only two possibilities.

(1) $\mathrm{PG}(n, q) = \mathrm{PG}(3, 2)$ and V is projectively equivalent with $\{(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(1, 1, 1, 1)\}$.

(2) $\mathrm{PG}(n, q) = \mathrm{PG}(4, 3)$ and V is an 11-cap which is the projection of a Coxeter cap in $\mathrm{PG}(5, 3)$.

With $g_2(k, q) = \frac{(k-1)(k-2)(q-1)}{2q} + (k - 1)$, it is an easy calculation to show that the upper bound is indeed attained in both cases. Hence we have the following theorem.

Theorem 6.6.3
If V is a set of points in $\mathrm{PG}(n, q)$, $n \geq 2$, with the property that no four of them are contained in a plane, then $k = |V| \leq \frac{(q-3)+\sqrt{8q^{n+1}+q^2-6q+1}}{2(q-1)}$ and equality holds if and only if V is one of the two examples given above.

The case $l = 5$

In this case we are looking for sets of points in $\mathrm{PG}(n, q), n \geq 3$ with the property that no five of them are contained in a 3-dimensional space. If the upper bound is attained then $T_n^*(V)$ is a near hexagon. By the remark given after Theorem 6.6.2, it follows that the projection of V on a hyperplane is necessary one of the two examples from the case $l = 4$. We have then the following possibilities.

(1) $\mathrm{PG}(n, q) = \mathrm{PG}(5, 3)$.
From Theorem 6.5.18, it follows that V is the Coxeter cap in $\mathrm{PG}(5, 3)$.

(2) $\mathrm{PG}(n, q) = \mathrm{PG}(4, 2)$.
In this case V is projectively equivalent with $\{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1), (1, 1, 1, 1, 1)\}$. This case gives a thin near hexagon $T_4^*(V)$ with 32 points.

Hence, we have the following theorem.

Theorem 6.6.4
If V is a set of points in $\mathrm{PG}(n,q), n \geq 3$, *with the property that no five of them are contained in a* 3*-dimensional space, then* $k = |V| \leq \frac{(3q-5)+\sqrt{8q^{n+1}+q^2-6q+1}}{2(q-1)}$ *and equality holds if and only if V is one of the two examples given above.*

The case $l \geq 6$

We put $l = 2m$ or $l = 2m+1$ depending on l even or odd. Hence $m \geq 3$. We will prove that equality can never hold when $q \neq 2$, so let V be a counterexample. First of all we prove that every two points x and y of $T_n^*(V)$ at distance $i \leq m$ are always contained in a unique geodetically closed sub near $2i$-gon. The intersection point of xy with $\mathrm{PG}(n,q)$ yields a point z and there is a unique set $\{v_1,\ldots,v_i\}$ of i points of V such that $z \in \langle v_1,\ldots,v_i\rangle$. The linear representation $T_{i-1}^*(\{v_1,\ldots,v_i\})$ (in the space $\langle x, v_1,\ldots,v_i\rangle$) determines then a sub near $2i$-gon of Hamming type through x and y. The condition that no l points of V are contained in an $(l-2)$-flat implies that this sub near $2i$-gon is geodetically closed. In this way, one can define quads and hexes and all the hexes are classical. By a remark in [13], page 195, $T_n^*(V)$ is a dual polar space and hence a classical near $2(m+1)$-gon which is regular with parameters $(s,t,t_2,t_3,\ldots,t_m) = (q-1,k-1,1,2,\ldots,m-1)$. From $t = t_{m+1} = t_2(1+t_m) = m$, it follows that $k = m+1$. Since $f_m(k,q) = \frac{q^n-1}{q-1}$ and $f_m(m+1,q) = \frac{q^m-1}{q-1}$, it follows that $m = n$, a contradiction since $n \geq 2m-1$. □

6.7 Some properties of the regular near hexagon with parameters $s = 2, t_2 = 1, t = 11$

6.7.1 A general theorem on Cameron closed subspaces

Let Π_∞ be a $\mathrm{PG}(n,q)$ which is embedded as a hyperplane in a $\mathrm{PG}(n+1,q)$ Π. Let $\mathcal{K}$ be a nonempty set of points of Π_∞ and let V be a Cameron closed subset of $T_n^*(\mathcal{K})$. Finally, denote by $\mathcal{A}$ the partial linear space induced by V.

Theorem 6.7.1
There exists an m-dimensional subspace α of Π_∞ such that $\mathcal{A} \simeq T_m^*(\alpha \cap \mathcal{K})$.

Proof. We prove by induction on k that all the affine points of $\langle x_0,\ldots,x_k\rangle$ $(k \geq 0)$ are in V if $x_0,\ldots,x_k$ are $k+1$ points of V such that $\mathrm{d}(x_i,x_{i+1}) = 1$, $i \in \{0,\ldots,k-1\}$. This is true for $k \leq 1$, hence suppose that $k \geq 2$.

We define inductively the $k+1$ points $y_0, \dots, y_k$ as follows: $y_k = x_k$; for $i \in \{1, \dots, k-1\}$, $y_i = x_i$ whenever y_{i+1}, x_i and x_{i-1} are collinear, otherwise let y_i be the unique neighbour of y_{i+1} and x_{i-1} such that $y_{i+1}x_i \cap y_i x_{i-1} \subseteq \Pi_\infty$ and $y_{i+1}y_i \cap x_i x_{i-1} \subseteq \Pi_\infty$; finally, put $y_0 = x_0$. By induction, one can prove that $y_i \in V$ for all $i \in \{0, \dots, k\}$. By induction, we also have that all affine points of $\langle x_0, \dots, x_{k-1} \rangle$ and $\langle y_1, \dots, y_k \rangle$ belong to V. Put $\{z_i\} = \Pi_\infty \cap x_i x_{i+1}$, $i \in \{0, \dots, k-1\}$, then $\langle x_0, \dots, x_{k-1} \rangle = \langle z_0, \dots, z_{k-2}, x_{k-1} \rangle$ and $\langle y_1, \dots, y_k \rangle = \langle z_0, \dots, z_{k-2}, x_k \rangle$. We may also suppose that $\langle x_0, \dots, x_{k-1} \rangle \neq \langle x_0, \dots, x_k \rangle$. This implies that $\langle x_0, \dots, x_{k-1} \rangle \cap \langle y_1, \dots, y_k \rangle = \langle z_0, \dots, z_{k-2} \rangle$. Now, let a be an arbitrary affine point of $\langle x_0, \dots, x_k \rangle$. The line az_{k-1} intersects $\langle x_0, \dots, x_{k-1} \rangle$ and $\langle y_1, \dots, y_k \rangle$ in points b and c respectively. Since $b \neq c$ and $b, c \in V$, all affine points of the line bc are in V, in particular $a \in V$.

Let β be the subspace of Π generated by the points of V and put $\alpha = \beta \cap \Pi_\infty$. It is clear now that $\mathcal{A}$ is isomorphic to $T_m^*(\alpha \cap \mathcal{K})$, where m is the dimension of α. $\square$

6.7.2 The Cameron closed subspaces of the regular near hexagon with parameters $s = 2, t_2 = 1, t = 11$

We apply the previous theorem to determine all Cameron closed subspaces of the regular near hexagon which is the linear representation of the Coxeter cap. So, assume that $\mathcal{K}$ is a Coxeter cap in $\mathrm{PG}(5,3)$. If V is a Cameron closed subspace of $\mathbf{S} = T_5^*(\mathcal{K})$, then V is the set of affine points of an $(m+1)$-dimensional subspace of Π which intersects Π_∞ in an m-dimensional subspace α $(m \geq -1)$. Since the partial linear space $\mathbf{A}$ induced by V is connected, one has $|\alpha \cap \mathcal{K}| \geq m+1$ (see Corollary 6.1.2). Since distance in $\mathbf{S}_V$ is equal to the corresponding distance in $\mathbf{S}$, $\mathbf{A}$ is a near $2k$-gon with $k \leq 3$. Moreover, $\mathbf{A}$ has a linear representation and all quads of $\mathbf{A}$ are grids. By Section 6.5, we have the following possibilities for V.

(1) V consists of one point of $\mathbf{S}$ $(m = -1)$.

(2) V is the set of points on a line of $\mathbf{S}$. This corresponds with the case $m = 0$ and $|\alpha \cap \mathcal{K}| = 1$.

(3) V is a quad of $\mathbf{S}$ (i.e. a grid). This corresponds with the case $m = 1$ and $|\alpha \cap \mathcal{K}| = 2$.

(4) V is the set of points of a near hexagon of Hamming type. This corresponds to the case $m = 2$ and $|\alpha \cap \mathcal{K}| = 3$.

(5) V is the set of all points of $\mathbf{S}$.

6.7.3 The ovoids of the regular near hexagon with parameters $s = 2, t_2 = 1, t = 11$

Let $\mathcal{K}$ be a Coxeter cap in $\Pi_\infty = \mathrm{PG}(5,3)$ and let $\mathbf{S} = T_5^*(\mathcal{K})$.

Theorem 6.7.2
If α is a hyperplane of Π such that $\alpha \cap \Pi_\infty$ is disjoint from $\mathcal{K}$, then the set of affine points of α form an ovoid of $\mathbf{S}$. Conversely, every ovoid is obtained in this way.

Proof. Let α be a hyperplane such that $\alpha \cap \Pi_\infty$ is disjoint from $\mathcal{K}$. Since every line of $\mathbf{S}$ intersects α in 1 point, the set of affine points of α form an ovoid of $\mathbf{S}$. Conversely, let O be an ovoid of $\mathbf{S}$ and let o_1 and o_2 be two points of O. We prove that all the affine points of the line o_1o_2 are points of O. Assume $\mathrm{d}(o_1, o_2) = 2$ and let x and y be those points of $\mathcal{K}$ such that xy and o_1o_2 are concurrent. The points x and y are uniquely determined since no plane of Π_∞ intersects $\mathcal{K}$ in more than four points. Consider now the grid $T_1^*(\{x,y\})$ (where the representation is in the plane $\langle x, y, o_1, o_2\rangle$). The ovoid O induces an ovoid in this grid. The grid has six ovoids, just like there are six lines in the plane $\langle x, y, o_1, o_2\rangle$ intersecting xy not in x or y. The affine points on every such line form an ovoid of the grid. This proves that all affine points of the line o_1o_2 are points of O. Assume $\mathrm{d}(o_1, o_2) = 3$, let x, y, z be three points of $\mathcal{K}$ such that $\langle x, y, z\rangle$ and the line o_1o_2 meet each other (x, y, z are not necessary uniquely determined). Consider the near hexagon $T_2^*(\{x, y, z\})$ (the representation is in the 3-dimensional space $\langle x, y, z, o_1, o_2\rangle$). This near hexagon has 12 ovoids, just like there are 12 planes in $\langle x, y, z, o_1, o_2\rangle$, disjoint with $\{x, y, z\}$. As before, this implies that every affine point of o_1o_2 is a point of O. The points of O generate now a space α and every affine point of α is a point of O. Since O is an ovoid ($|O| = 243$), α is a hyperplane of Π such that $\alpha \cap \Pi_\infty$ is disjoint with $\mathcal{K}$. $\square$

There are exactly 12 hyperplanes in $\mathrm{PG}(5,3)$ carrying no point of $\mathcal{K}$, see [97]. In the dual space of $\mathrm{PG}(5,3)$, these hyperplanes define again a Coxeter-cap. Hence, there are 36 ovoids and 12 partitions in ovoids.

Chapter 7

Glued near polygons

7.1 Constructions of glued near polygons

Let k be a nonzero positive integer. For every $i \in \{1, \ldots, k\}$, let $\mathbf{Q}_i = (\mathcal{P}_i, \mathcal{L}_i, \mathcal{I}_i)$ be a generalized quadrangle of order (s, t_i), let $S_i = \{L_1^{(i)}, \ldots, L_{1+st_i}^{(i)}\}$ be a spread of $\mathbf{Q}_i$ and let θ_i be a bijection from $L_1^{(1)}$ to $L_1^{(i)}$. Notice that a special line $L_1^{(i)}$ is taken in each spread S_i, $i \in \{1, \ldots, k\}$; this line is called the *base line* of S_i. For $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, 1 + st_i\}$, let $p_j^{(i)}$ be the projection on the line $L_j^{(i)}$ in the generalized quadrangle $\mathbf{Q}_i$.

Consider now the following graph Γ with vertex set $L_1^{(1)} \times S_1 \times \ldots \times S_k$. Two different vertices $(x, L_{i_1}^{(1)}, \ldots, L_{i_k}^{(k)})$ and $(y, L_{j_1}^{(1)}, \ldots, L_{j_k}^{(k)})$ are adjacent if and only if the following two conditions are satisfied:

(i) there exists an $l \in \{1, \ldots, k\}$ such that $i_m = j_m$ for all $m \in \{1, \ldots, k\} \setminus \{l\}$,

(ii) for every l as in (i), we have that $p_{i_l}^{(l)} \circ \theta_l(x)$ and $p_{j_l}^{(l)} \circ \theta_l(y)$ are collinear points in the generalized quadrangle $\mathbf{Q}_l$.

If $i_m = j_m$ for all $m \in \{1, \ldots, k\}$, then the two vertices are adjacent (condition (ii) is satisfied for every $l \in \{1, \ldots, k\}$).

Remarks.

(a) In the sequel, the composition $f \circ g$ of two functions f and g is shortly denoted by fg.

(b) In the above definition of Γ, we used a certain ordering in the set $\{\mathbf{Q}_1, \ldots, \mathbf{Q}_k\}$ of generalized quadrangles (the one induced by the indices). Γ can be retrieved using the same spreads and base lines, but a different ordering of the GQ's; the maps θ_i must then be changed. Indeed, let σ be an arbitrary permutation of $\{1,\ldots,k\}$ and let $\mathbf{Q}_{\sigma(1)},\ldots,$ $\mathbf{Q}_{\sigma(k)}$ be the new ordering of the GQ's. Let ϕ be an arbitrary bijection between $L_1^{(\sigma(1))}$ and $L_1^{(1)}$. Then $\theta'_{\sigma(i)} = \theta_{\sigma(i)}\phi$ is a map from $L_1^{(\sigma(1))}$ to $L_1^{(\sigma(i))}$. In the same way as we defined Γ, we can define now a graph Γ' which is isomorphic to Γ; the isomorphism from Γ to Γ' is given by $(x, L_1, \ldots, L_k) \mapsto (\phi^{-1}(x), L_{\sigma(1)}, \ldots, L_{\sigma(k)})$.

(c) If we take σ equal to the identity in (b), then we still can take ϕ arbitrarly. Hence, Γ can be retrieved if we choose θ_1 equal to the identical permutation of $L_1^{(1)}$.

(d) Under certain conditions, see Theorem 7.1.3, the graph Γ can be retrieved by starting with k arbitrary base lines (using the same GQ's and spreads).

Lemma 7.1.1
Through every two adjacent vertices of Γ, there is a unique maximal clique. This clique has size $s+1$.

Proof. Let $\alpha = (x, L_{i_1}^{(1)}, \ldots, L_{i_k}^{(k)})$ and $\beta = (y, L_{j_1}^{(1)}, \ldots, L_{j_k}^{(k)})$ be two adjacent vertices of Γ. Let l be an element of $\{1,\ldots,k\}$ such that $i_m = j_m$ for all $m \in \{1,\ldots,k\} \setminus \{l\}$. We determine now how the common neighbours $\gamma = (z, L_{f_1}^{(1)}, \ldots, L_{f_k}^{(k)})$ look like. If $i_m = j_m \neq f_m$ for a certain element $m \in \{1,\ldots,k\} \setminus \{l\}$, then $i_l = f_l = j_l$ and $p_{i_m}^{(m)}\theta_m(x) \sim p_{f_m}^{(m)}\theta_m(z) \sim p_{j_m}^{(m)}\theta_m(y)$, implying that $p_{i_m}^{(m)}\theta_m(x) = p_{i_m}^{(m)}\theta_m(y)$, $x = y$ or $\alpha = \beta$, a contradiction. Hence $i_m = j_m = f_m$ for all elements m of $\{1,\ldots,k\} \setminus \{l\}$. The condition $p_{i_l}^{(l)}\theta_l(x) \sim p_{f_l}^{(l)}\theta_l(z) \sim p_{j_l}^{(l)}\theta_l(y)$ holds if and only if $p_{f_l}^{(l)}\theta_l(z)$ is a point of the line connecting $p_{i_l}^{(l)}\theta_l(x)$ with $p_{j_l}^{(l)}\theta_l(y)$. Hence there are $s-1$ solutions for z (or γ). These $s-1$ solutions for γ form together with α and β the unique clique of size $s+1$. $\square$

Let $\mathbf{S}$ be the partial linear space with points the vertices of Γ, with lines the maximal cliques of Γ and with natural incidence. If $\mathbf{S}$ is a near polygon, then it is called a *glued near polygon*. We will examine now under what conditions this precisely happens. For $i \in \{1,\ldots,k\}$, let G_i be the group of projectivities of $L_1^{(i)}$ with respect to S_i. The group G_i is generated by the permutations $\Delta_{jk}^{(i)} : L_1^{(i)} \to L_1^{(i)} : x \mapsto p_1^{(i)}p_k^{(i)}p_j^{(i)}(x)$, $1 \leq j,k \leq 1+st_i$.

Remarks.

(1) If $k = 1$, then $\mathbf{S} \simeq \mathbf{Q}_1$. Indeed, the map $(x, L_j^{(1)}) \mapsto p_j^{(1)}\theta_1(x)$ defines an isomorphism between $\mathbf{S}$ and $\mathbf{Q}_1$.

(2) Let $\alpha = (x, L_{i_1}^{(1)}, \ldots, L_{i_k}^{(k)})$ and $\beta = (y, L_{j_1}^{(1)}, \ldots, L_{j_k}^{(k)})$ be two points of $\mathbf{S}$ and suppose that l is an element of $\{1, \ldots, k\}$ such that $i_m = j_m$ for all $m \in \{1, \ldots, k\} \setminus \{l\}$, then α and β are collinear if and only if $p_{i_l}^{(l)}\theta_l(x) \sim p_{j_l}^{(l)}\theta_l(y)$ or if and only if

$$y = \theta_l^{-1}\Delta_{i_l j_l}^{(l)}\theta_l(x).$$

In the next theorem 0 stands for the trivial group and $[\theta_i^{-1}G_i\theta_i, \theta_j^{-1}G_j\theta_j]$ is the group generated by all commutators $[\theta_i^{-1}g_i\theta_i, \theta_j^{-1}g_j\theta_j]$ with $g_1 \in G_1$ and $g_2 \in G_2$.

Theorem 7.1.2
The partial linear space $\mathbf{S}$ *is a glued near hexagon if and only if* $[\theta_i^{-1}G_i\theta_i, \theta_j^{-1}G_j\theta_j] = 0$ *for all* $i, j \in \{1, \ldots, k\}$ *with* $i \neq j$.

Proof. We may suppose that $k \geq 2$. First, suppose that $\mathbf{S}$ is a near polygon. Consider the point $\alpha = (x, L_{i_1}^{(1)}, \ldots, L_{i_k}^{(k)})$ and the line $L = \{(y, L_{j_1}^{(1)}, \ldots, L_{j_k}^{(k)}) | y \ \mathcal{I}_1 \ L_1^{(1)}\}$. Take two distinct indices $m, n \in \{1, \ldots, k\}$. Suppose that $i_m \neq j_m$, $i_n \neq j_n$ and $i_l = j_l$ for all $l \in \{1, \ldots, k\} \setminus \{m, n\}$. Clearly L contains no points collinear with α. Since L contains the points $((\theta_m^{-1}\Delta_{i_m j_m}^{(m)}\theta_m)(\theta_n^{-1}\Delta_{i_n j_n}^{(n)}\theta_n)(x), L_{j_1}^{(1)}, \ldots, L_{j_k}^{(k)})$ and $((\theta_n^{-1}\Delta_{i_n j_n}^{(n)}\theta_n)(\theta_m^{-1}\Delta_{i_m j_m}^{(m)}\theta_m)(x), L_{j_1}^{(1)}, \ldots, L_{j_k}^{(k)})$ which have both distance 2 from α, it follows that

$$(\theta_m^{-1}\Delta_{i_m j_m}^{(m)}\theta_m)(\theta_n^{-1}\Delta_{i_n j_n}^{(n)}\theta_n)(x) = (\theta_n^{-1}\Delta_{i_n j_n}^{(n)}\theta_n)(\theta_m^{-1}\Delta_{i_m j_m}^{(m)}\theta_m)(x),$$

for all $n, m \in \{1, \ldots, k\}$ with $m \neq n$ and for all $x \ \mathcal{I}_1 \ L_1^{(1)}$. It follows now immediately that $[\theta_i^{-1}G_i\theta_i, \theta_j^{-1}G_j\theta_j] = 0$ for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$.

Conversely, suppose that $[\theta_i^{-1}G_i\theta_i, \theta_j^{-1}G_j\theta_j] = 0$ for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$.

Consider the point $\alpha = (x, L_{i_1}^{(1)}, \ldots, L_{i_k}^{(k)})$ and the line $L = \{(y, L_{j_1}^{(1)}, \ldots, L_{j_k}^{(k)}) | y \ \mathcal{I}_1 \ L_1^{(1)}\}$. If N is the number of positions in which $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_k)$ differ, then α has distance N from L. If the conditions of the theorem are satisfied, then L contains a unique point at distance N from α, namely the point $(\prod_{a \in A}(\theta_a^{-1}\Delta_{i_a j_a}^{(a)}\theta_a)(x), L_{j_1}^{(1)}, \ldots, L_{j_k}^{(k)})$. Here A denotes the set of all indices $a \in \{1, \ldots, k\}$ for which $i_a \neq j_a$ and $\prod_{a \in A}(\theta_a^{-1}\Delta_{i_a j_a}^{(a)}\theta_a)$ denotes the composition of $N = |A|$ permutations of $L_1^{(1)}$.

Now, suppose that M is a fixed line of $\mathbf{Q}_1$ not belonging to the spread S_1. Consider the point $\alpha = (x, L_{i_1}^{(1)}, \ldots, L_{i_k}^{(k)})$ and the line $L = \{(y, L_{j_1}^{(1)}, \ldots, L_{j_k}^{(k)})|$ $y \ \mathcal{I}_1 \ L_1^{(1)}$ and $p_{j_1}^{(1)}\theta_1(y) \ \mathcal{I}_1 \ M\}$ ($j_2, \ldots, j_k$ are fixed). Let A denote the set of indices $a \in \{2, \ldots, k\}$ for which $i_a \neq j_a$, then the set of points of L nearest to α is equal to the set of points of L nearest to $(\prod_{a\in A}(\theta_a^{-1}\Delta_{i_a j_a}^{(a)}\theta_a)(x), L_{i_1}^{(1)}, L_{j_2}^{(1)}, \ldots, L_{j_k}^{(k)})$. Let $y' \in L_1^{(1)}$ and $j_1' \in \{1, \ldots, 1+st_1\}$ be such that $p_{j_1'}^{(1)}\theta_1(y')$ is the unique point of M nearest to $(p_{i_1}^{(1)}\theta_1)(\prod_{a\in A}(\theta_a^{-1}\Delta_{i_a j_a}^{(a)}\theta_a)(x))$, then $(y', L_{j_1'}^{(1)}, L_{j_2}^{(1)}, \ldots, L_{j_k}^{(k)})$ is the unique point of L nearest to α. □

Remark. The maximal distance between two points α and β of $\mathbf{S}$ is equal to $k+1$. This distance is obtained if $\alpha = (x, L_{i_1}^{(1)}, \ldots, L_{i_k}^{(k)})$ and $\beta = (y, L_{j_1}^{(1)}, \ldots, L_{j_k}^{(k)})$, where $i_l \neq j_l$ for all $l \in \{1, \ldots, k\}$ and where $y \neq \prod_{a\in\{1,\ldots,k\}} (\theta_a^{-1}\Delta_{i_a j_a}^{(a)}\theta_a)(x)$.

Conclusion. If $\mathbf{S}$ is a near polygon, then $\mathbf{S}$ is a near $(2k+2)$-gon.

Theorem 7.1.3
If $\mathbf{S}$ is a near polygon, then $\mathbf{S}$ can be retrieved starting from any choice of the k base lines, with good chosen maps θ_i, $i \in \{1, \ldots, k\}$.

Proof. Let $L_{\alpha_1}^{(1)}, \ldots, L_{\alpha_k}^{(k)}$ be the new base lines of the k spreads. With each point $\mu = (x, L_{i_1}^{(1)}, \ldots, L_{i_k}^{(k)})$ of $\mathbf{S}$, we associate the $(k+1)$-tuple $\mu' = (x', L_{i_1}^{(1)}, \ldots, L_{i_k}^{(k)})$, where $x' = p_{\alpha_1}^{(1)} \prod_{a\in\{1,\ldots,k\}}(\theta_a^{-1}\Delta_{i_a\alpha_a}^{(a)}\theta_a)(x) \in L_{\alpha_1}^{(1)}$. If $\nu = (y, L_{j_1}^{(1)}, \ldots, L_{j_k}^{(k)})$ is a second point of $\mathbf{S}$, then μ and ν are adjacent if and only if $y = \theta_l^{-1}\Delta_{i_l j_l}^{(l)}\theta_l(x)$. Here l denotes an element of $\{1, \ldots, k\}$ with the property that $i_m = j_m$ for all $m \in \{1, \ldots, k\} \setminus \{l\}$. We rewrite this condition as follows:

$$\begin{aligned}
y' &= p_{\alpha_1}^{(1)} \left(\prod_{a\in\{1,\ldots,k\}} (\theta_a^{-1}\Delta_{j_a\alpha_a}^{(a)}\theta_a) \right) (\theta_l^{-1}\Delta_{i_l j_l}^{(l)}\theta_l)x \\
&= p_{\alpha_1}^{(1)} \left(\prod_{a<l} (\theta_a^{-1}\Delta_{i_a\alpha_a}^{(a)}\theta_a) \right) (\theta_l^{-1}\Delta_{j_l\alpha_l}^{(l)}\Delta_{i_l j_l}^{(l)}\theta_l) \left(\prod_{a>l} (\theta_a^{-1}\Delta_{i_a\alpha_a}^{(a)}\theta_a) \right) (x) \\
&= p_{\alpha_1}^{(1)}\theta_l^{-1}\Delta_{j_l\alpha_l}^{(l)}\Delta_{i_l j_l}^{(l)}\Delta_{\alpha_l i_l}^{(l)}\theta_l p_1^{(1)}(x') \\
&= (p_{\alpha_1}^{(1)}\theta_l^{-1}p_1^{(l)})(p_{\alpha_l}^{(l)}p_{j_l}^{(l)}p_{i_l}^{(l)})(p_{\alpha_l}^{(l)}\theta_l p_1^{(1)})(x') \\
&= (\theta_l'^{-1}(p_{\alpha_l}^{(l)}p_{j_l}^{(l)}p_{i_l}^{(l)})\theta_l')(x').
\end{aligned}$$

Hence, we can take the maps $\theta_l' : L_{\alpha_1}^{(1)} \to L_{\alpha_l}^{(l)}$, $l \in \{1, \ldots, k\}$, as new maps between the new base lines. □

Remark. Suppose that $\mathbf{S}$ is a near $(2k+2)$-gon with $k \geq 2$. If we restrict ourselves to the generalized quagrangles $\mathbf{Q}_1, \ldots, \mathbf{Q}_{k-1}$ together with their corresponding spreads, base lines and maps, then by Theorem 7.1.2, we find a near polygon $\mathbf{S}'$. If $\mathbf{Q}_k$ is a grid, then $\mathbf{S}$ is the direct product of $\mathbf{S}'$ with a line of length $s+1$.

In Chapter 6, we defined the following class of near polygons. Let Π_∞ be a $\mathrm{PG}(2k, q)$, $k \geq 2$, which is embedded as a hyperplane in Π. Consider in Π_∞ k planes $\alpha_1, \ldots, \alpha_k$ intersecting in a point p and generating Π_∞. Let $\mathcal{H}_i$, $i \in \{1, \ldots, k\}$, be a hyperoval in α_i containing p. Then $T^*_{2k}(\mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k)$ is a near polygon. We will prove now that this near polygon is glued.

Theorem 7.1.4
$T^*_{2k}(\mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k)$ is a glued near polygon.

Proof. Let a be a fixed point of $\Pi \setminus \Pi_\infty$ and put $A_i = \langle a, \alpha_i \rangle$. The set of affine points of A_i is a geodetically closed subspace and induces a generalized quadrangle $\mathbf{Q}_i$ isomorphic to $T^*_2(\mathcal{H}_i)$. Let S_i be the spread of $\mathbf{Q}_i$ determined by the point $p \in \Pi_\infty$ (S_i is the set of lines of A_i through p), let $L_1^{(i)}$ be the line pa and let θ_i be the identical map. The q translations of the affine space on A_i, which have p as corresponding point at infinity, induce q automorphisms of $\mathbf{Q}_i$ which fix each line of S_i; the group G_i is then equal to the group of the q translations of the affine line $pa \setminus \{p\}$. Hence, with the above choices for $\mathbf{Q}_i$, S_i, $L_1^{(i)}$ and θ_i ($i \in \{1, \ldots, k\}$), we obtain a glued near polygon $\mathbf{S}$.

We define now an isomorphism θ between $T^*_{2k}(\mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k)$ and $\mathbf{S}$. Let x be an arbitrary point of $\Pi \setminus \Pi_\infty$.

(1) Let $\phi(x)$ denote the unique point of pa nearest to x (in the near polygon $T^*_{2k}(\mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k)$).

(2) For $i \in \{1, \ldots, k\}$, the space $\langle x, \alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_k \rangle$ intersects $\langle a, \alpha_i \rangle$ in a line $\phi_i(x)$ through p. Clearly $\phi_i(x) \in S_i$.

Put $\theta(x) = (\phi(x), \phi_1(x), \ldots, \phi_k(x))$. We prove now that θ is a bijection. Let $(x, L_1, \ldots, L_k)$ be an element of $pa \times S_1 \times \ldots \times S_k$. The intersection of the k spaces $\langle \alpha_1, \ldots, \alpha_{i-1}, L_i, \alpha_{i+1}, \ldots, \alpha_k \rangle$, $i \in \{1, \ldots, k\}$, is a line L through p. We prove now that pa and L are parallel lines of the near polygon $T^*_{2k}(\mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k)$. We may suppose that $pa \neq L$. The plane $\langle L, pa \rangle$ intersects Π_∞ in a line L' through p. Let z be the point of $L' \setminus \{p\}$ with smallest index. For every point u of pa, there exists a point u' on L for which $\mathrm{d}(u, u') = \mathrm{d}(pa, L)$ (z, u and u' are collinear), hence L and pa are parallel. Let y be the unique point of L nearest to x (in $T^*_{2k}(\mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k)$), then y is the unique point of $\Pi \setminus \Pi_\infty$ for which $\theta(y) = (x, L_1, \ldots, L_k)$. Since $\mathbf{S}$ and

$T^*_{2k}(\mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k)$ have the same order, it suffices to show that θ preserves adjacency. Let x and y be two collinear points of $T^*_{2k}(\mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k)$, the line xy meets then one of the hyperovals, say $\mathcal{H}_l$. Clearly $\phi_m(x) = \phi_m(y)$ for all $m \in \{1, \ldots, k\} \setminus \{l\}$. We prove that x is classical with respect to $\mathbf{Q}_l$ and that the unique point x' of $\mathbf{Q}_l$ nearest to x is contained in the line $\phi_l(x)$ of S_l. This is clear if $x \in A_l$; suppose therefore that $x \notin A_l$. The 4-dimensional space $\langle x, A_l \rangle$ intersects Π_∞ in a threedimensional space β through α_l, which itself intersects $\langle \alpha_1, \ldots, \alpha_{l-1}, \alpha_{l+1}, \ldots, \alpha_k \rangle$ in a line M through p. Let u be the unique point of $M \setminus \{p\}$ with smallest index (say μ). The point u is then also the unique point of $\beta \setminus \alpha_l$ with smallest index. Hence, there is a unique point x' in A_l nearest to x, namely the intersection of xu with A_l, and $\mathrm{d}(x, x') = \mu$. The point x' is clearly contained in $\phi_l(x)$. By the point-quad relation, see Theorem 3.5.3, we know that this point is equal to the projection of $\phi(x)$ on the line $\phi_l(x)$. Similarly, there is a unique point y' in $\mathbf{Q}_l$ nearest to y. It remains to prove , see condition (ii) on page 132, that x' and y' are collinear points of $\mathbf{Q}_l$. Since $\mathrm{d}(x, x') = \mathrm{d}(y, y') = \mu$ and $\mathrm{d}(x, x') + \mathrm{d}(x', y') = \mathrm{d}(x, y') \leq \mathrm{d}(x, y) + \mathrm{d}(y, y') = \mu + 1$, it follows that $\mathrm{d}(x', y') \leq 1$. If $x' = y'$, then there is a point z on the line xy at distance $\mu - 1$ from x', a contradiction. $\square$

7.2 On the classification of glued near polygons

Lemma 7.2.1
Let X be a set of order $n \geq 1$ and let G_1 and G_2 be regular groups of permutations on X, then there exists a permutation $\theta \in Sym(X)$ such that $G_2 = \theta^{-1} G_1 \theta$ if and only if G_1 is isomorphic to G_2. Moreover, if ϕ is an isomorphism from G_1 to G_2, then there exist n permutations θ such that $\phi(g_1) = \theta^{-1} g_1 \theta$ for all $g_1 \in G_1$.

Proof. If $G_2 = \theta^{-1} G_1 \theta$, then the map ϕ defined by $\phi(g_1) = \theta^{-1} g_1 \theta, \forall g_1 \in G_1$, defines an isomorphism from G_1 to G_2. Conversely, suppose that $\phi : G_1 \to G_2$ is an isomorphism. Let $x \in X$ be fixed. The condition $\theta[\phi(g_1)](x) = g_1 \theta(x), \forall g_1 \in G_1$ implies that θ is completely determined as soon as $\theta(x)$ is known. For all $y, z \in X$, we define

$$\theta^{\phi}_y(z) = [\phi^{-1}(g_2)](y),$$

where g_2 is the unique element of G_2 such that $g_2(x) = z$. It is straightforward to check that θ^{ϕ}_y satisfies the required conditions. $\square$

We use the same notations as in the previous section and we assume that

(1) **S** is a near polygon,

(2) $k \geq 2$ (see Remark (1) on 134),

(3) none of the GQ's $\mathbf{Q}_1, \ldots, \mathbf{Q}_k$ is a grid (see Remark on page 136).

Theorem 7.2.2

(1) *$G_i \simeq G_j$ for all $i, j \in \{1, \ldots, k\}$.*

(2) *S_i is a spread of symmetry of $\mathbf{Q}_i$.*

(3) *G_i is commutative if $k \geq 3$.*

Proof. Let i, j be two distinct elements of $\{1, \ldots, k\}$. Since $[\theta_i^{-1} G_i \theta_i, \theta_j^{-1} G_j \theta_j] = 0$, see Theorem 7.1.2, every element $\theta_i \theta_j^{-1} g \theta_j \theta_i^{-1}$ with $g \in G_j$ will define an automorphism of $\mathbf{Q}_i$ fixing each line of S_i (see Theorem 2.7.10). By Theorems 2.7.1 and 2.7.7, $|G_j| = s + 1$ and S_j is a spread of symmetry of $\mathbf{Q}_j$. This proves (2). Since $G_i \simeq \theta_i^{-1} G_i \theta_i$, $G_j \simeq \theta_j^{-1} G_j \theta_j$, (1) follows from Lemma 2.7.9. Now, suppose that $k \geq 3$ and put $G = \theta_1^{-1} G_1 \theta_1$. Since $[G, \theta_i^{-1} G_i \theta_i] = 0$ for all $i \in \{2, \ldots, k\}$, $\theta_i^{-1} G_i \theta_i = \tilde{G}$. Since $k \geq 3$, $[\tilde{G}, \tilde{G}] = 0$ implying that $\tilde{G} = G$ is commutative. $\square$

Remark. Since S_i is a spread of symmetry of $\mathbf{Q}_i$, there exists an AT-model $(\mathbf{D}_i, K_i, \Delta_i)$ for $\mathbf{Q}_i$ with associated spread S_i. The group G_i, which is the group of projectivities of $L_1^{(i)}$ with respect to S_i, is isomorphic to K_i, see Theorem 2.7.8.

We can construct glued near $(2k+2)$-gons, $k \geq 2$, in the following way.

(1) Take a group K of order $s + 1 \geq 2$ for which there exists an admissible triple $(\mathbf{D}, K, \Delta)$, where $\mathbf{D}$ is a linear space different from a line. If $k \geq 3$, then K has to be commutative.

(2) Let $(\mathbf{D}_i, K, \Delta_i)$, $i \in \{1, \ldots, k\}$, be a sequence of k, not necessarily distinct, admissible triples. Let $\mathbf{Q}_i$ denote the corresponding generalized quadrangle and let S_i be the associated spread. Let $L_1^{(i)}$ be an arbitrary line of S_i and let θ_1 be an arbitrary permutation of $L_1^{(1)}$. As before, let G_i denote the group of projectivities of $L_1^{(i)}$ with respect to S_i. Define $G := \theta_1^{-1} G_1 \theta_1$. By Theorem 7.1.2, one has to find bijections $\theta_i : L_1^{(1)} \to L_1^{(i)}$, $i \geq 2$, satisfying $\theta_i^{-1} G_i \theta_i = \tilde{G}$. We now prove that the number of possible θ_i's is equal to $(s+1) \times |\operatorname{Aut}(K)|$. Let ϕ_i denote an arbitrary bijection from $L_1^{(1)}$ to $L_1^{(i)}$, then $\theta_i^{-1} G_i \theta_i = \tilde{G}$ is equivalent to $(\theta_i^{-1} \phi_i)(\phi_i^{-1} G_i \phi_i)(\phi_i^{-1} \theta_i) = \tilde{G}$. Since $\phi_i^{-1} G_i \phi_i \simeq K$ and $\tilde{G} \simeq G \simeq G_1 \simeq K$, the equation has $(s+1) \times |\operatorname{Aut}(K)|$ solutions

for $\phi_i^{-1}\theta_i$ (see Lemma 7.2.1). As a consequence, the set of equations $\theta_2^{-1}G_2\theta_2 = \tilde{G}, \ldots, \theta_k^{-1}G_k\theta_k = \tilde{G}$ has $(s+1)^{k-1}|\operatorname{Aut}(K)|^{k-1}$ solutions for $(\theta_2, \ldots, \theta_k)$.

Collecting the previous results, we have the following theorem.

Theorem 7.2.3
Let $\mathbf{Q}_1, \ldots, \mathbf{Q}_k$ be k generalized quadrangles having all the same line size $s+1$. Suppose that none of these GQ's is isomorphic to a grid, and let S_i, $i \in \{1, \ldots, k\}$, be a spread of $\mathbf{Q}_i$. Then there is a glued near polygon derived from the $\mathbf{Q}_i$'s and the S_i's if and only if there is a group K of order $s+1$ such that

(1) *The group of automorphisms of $\mathbf{Q}_i$, $i \in \{1, \ldots, k\}$, fixing each line of S_i is isomorphic to K.*

(2) *K is commutative if $k \geq 3$.*

Moreover, the number of nonisomorphic glued near polygons which can be obtained this way is at most $(s+1)^{k-1}|\operatorname{Aut}(K)|^{k-1}$.

Problem. What is the number of nonisomorphic glued near polygons, given the GQ's and their spreads?

In order to tackle this problem, one must find conditions under which two glued near polygons are isomorphic. To determine these automorphisms, one can use the fact that substructures must be mapped to substrucures of the same kind. Good candidates for these substructures are the sub near polygons, which we will study in the next section. We will then emphasize on near hexagons in Sections 7.4 and 7.5. In fact, one can get better estimates than the one obtained in Theorem 7.2.3, see Theorem 7.5.1.

7.3 Sub near polygons

We use the same notations as in Section 7.1.

Theorem 7.3.1
Let $\mathbf{S}$ be a glued near polygon, then every two points at distance 2 have at least two common neighbours.

Proof. Let $\alpha = (x, L_{i_1}^{(1)}, \ldots, L_{i_k}^{(k)})$ and $\beta = (y, L_{j_1}^{(1)}, \ldots, L_{j_k}^{(k)})$ be two points at distance 2. We may suppose that one of the following two cases occurs.

(1) If $i_1 \neq j_1$, $i_2 \neq j_2$, $i_m = j_m$ for all $m \in \{3, \ldots, k\}$, $y = (\prod_{a\in\{1,2\}}(\theta_a^{-1}\Delta^{(a)}_{i_aj_a}\theta_a))(x)$, then $((\theta_1^{-1}\Delta^{(1)}_{i_1j_1}\theta_1)(x), L^{(1)}_{j_1}, L^{(2)}_{i_2}, \ldots)$ and $((\theta_2^{-1}\Delta^{(1)}_{i_2j_2}\theta_2)(x), L^{(1)}_{i_1}, L^{(2)}_{j_2}, \ldots)$ are two common neighbours of α and β.

(2) Let $i_1 \neq j_1$, $i_m = j_m$ for all $m \in \{2, \ldots, k\}$, $y \neq (\theta_1^{-1}\Delta^{(1)}_{i_1j_1}\theta_1)(x)$. Since $p^{(1)}_{j_1}\theta_1(y)$ and $p^{(1)}_{i_1}\theta_1(x)$ are two noncollinear points of $\mathbf{Q}_1$, they have $t_1 + 1$ common neighbours. If $p^{(1)}_{\delta_1}\theta_1(z_1)$ and $p^{(1)}_{\delta_2}\theta_1(z_2)$ are two of these neighbours, then $(z_1, L^{(1)}_{\delta_1}, L^{(2)}_{i_2}, \ldots)$ and $(z_2, L^{(1)}_{\delta_2}, L^{(2)}_{i_2}, \ldots)$ are two common neighbours of α and β.

$\square$

If $s \geq 2$, then Theorem 3.7.3 implies the existence of geodetically closed sub near polygons. We will give now an explicit description of these sub near polygons, and this will prove that they also exist if $s = 1$. So, let $\alpha = (x, L^{(1)}_{i_1}, \ldots, L^{(k)}_{i_k})$ and $\beta = (y, L^{(1)}_{j_1}, \ldots, L^{(k)}_{j_k})$ be two points at distance δ. Let $H(\alpha, \beta)$ be the unique smallest geodetically closed subspace containing α and β (i.e. the intersection of all geodetically closed subspaces containing α and β). We may suppose that one of the following cases occurs.

(1) Let $i_m \neq j_m$ for all $m < \delta$, $i_m = j_m$ for all $m \geq \delta$, $y \neq \left(\prod_{a\in\{1,\ldots,\delta-1\}}(\theta_a^{-1}\Delta^{(a)}_{i_aj_a}\theta_a)\right)(x)$. The point $\gamma = (\prod_{a\in\{2,\ldots,\delta-1\}}(\theta_a^{-1}\Delta^{(a)}_{i_aj_a}\theta_a)(x), L^{(1)}_{i_1}, L^{(2)}_{j_2}, \ldots, L^{(k)}_{j_k})$ lies on a shortest path from α to β. Hence, $H(\alpha, \beta)$ contains $H(\beta, \gamma)$. By remark (1) on page 134, we know that $H(\beta, \gamma)$ is the quad consisting of all the points $(z, L^{(1)}_{f_1}, L^{(2)}_{j_2}, \ldots, L^{(k)}_{j_k})$, where $z\ \mathcal{I}\ L^{(1)}_1$ and $f_1 \in \{1, \ldots, 1 + st_1\}$. Repeating the above argument, one finds $\{(z, L^{(1)}_{f_1}, \ldots, L^{(\delta-1)}_{f_{\delta-1}}, L^{(\delta)}_{j_\delta}, \ldots, L^{(k)}_{j_k}) | z\ \mathcal{I}\ L^{(1)}_1$ and $f_i \in \{1, \ldots, 1 + st_i\}$ for all $i \in \{1, \ldots, \delta-1\}\} \subseteq H(\alpha, \beta)$, and we have equality since the first set is classical and hence geodetically closed. Moreover, $H(\alpha, \beta)$ induces a sub near (2δ)-gon which is also a glued one.

(2) Let $i_m \neq j_m$ for all $m \in \{1, \ldots, \delta\}$, $i_m = j_m$ for all $m \in \{\delta + 1, \ldots, k\}$, $y = \left(\prod_{a\in\{1,\ldots,\delta\}}(\theta_a^{-1}\Delta^{(a)}_{i_aj_a}\theta_a)\right)(x)$. The point $\gamma = ((\theta_1^{-1}\Delta^{(1)}_{i_1j_1}\theta_1)(x), L^{(1)}_{j_1}, L^{(2)}_{i_2}, \ldots, L^{(k)}_{i_k})$ lies on a shortest path from α to β. Since $\mathrm{d}(\alpha, \gamma) = 1$, every point of the line $\alpha\gamma$ is contained in $H(\alpha, \beta)$. Now, S_1 is a normal spread since it is also a spread of symmetry. If $L^{(1)}_{l_1} \in sp(L^{(1)}_{i_1}, L^{(1)}_{j_1})$, i.e. if $L^{(1)}_{l_1}$ belongs to the hyperbolic line through $L^{(1)}_{i_1}$ and $L^{(1)}_{j_1}$, then $((\theta_1^{-1}\Delta^{(1)}_{i_1l_1}\theta_1)(x), L^{(1)}_{l_1}, L^{(2)}_{i_2}, \ldots, L^{(k)}_{i_k}) \in H(\alpha, \beta)$. Repeating this argument, one finds that $H'(\alpha, \beta) := \{\ (\prod_{a\in\{1,\ldots,\delta\}}(\theta_a^{-1}\Delta^{(a)}_{i_al_a}\theta_a)(x), L^{(1)}_{l_1}, \ldots,$

$L^{(\delta)}_{l_\delta}, L^{(\delta+1)}_{i_{\delta+1}}, \ldots, L^{(k)}_{i_k}) \mid L^{(\mu)}_{l_\mu} \in sp(L^{(\mu)}_{i_\mu}, L^{(\mu)}_{j_\mu})$ for all $\mu \in \{1, \ldots, \delta\}$ $\} \subseteq H(\alpha, \beta)$. We prove now that $H'(\alpha,\beta)$ is geodetically closed; hence $H'(\alpha,\beta) = H(\alpha,\beta)$. Let $\gamma_3 = (x_3, L^{(1)}_{h_1}, L^{(2)}_{h_2}, \ldots, L^{(k)}_{h_k})$ be a point on a shortest path from $\gamma_1 = (x_1, L^{(1)}_{f_1}, \ldots, L^{(k)}_{f_k}) \in H'(\alpha,\beta)$ to $\gamma_2 = (x_2, L^{(1)}_{g_1}, \ldots, L^{(k)}_{g_k}) \in H'(\alpha,\beta)$. Since $x_1 = \left(\prod_{a\in\{1,\ldots,\delta\}}(\theta_a^{-1}\Delta^{(a)}_{i_a f_a}\theta_a)\right)(x)$ and $x_2 = \left(\prod_{a\in\{1,\ldots,\delta\}}(\theta_a^{-1}\Delta^{(a)}_{i_a g_a}\theta_a)\right)(x)$, it follows that $x_2 = \left(\prod_{a\in\{1,\ldots,\delta\}}(\theta_a^{-1}\Delta^{(a)}_{i_a g_a}\Delta^{(a)}_{f_a i_a}\theta_a)\right)(x_1)$. Since S_1 is a normal spread, $x_2 = \left(\prod_{a\in\{1,\ldots,\delta\}}(\theta_a^{-1}\Delta^{(a)}_{f_a g_a}\theta_a)\right)(x_1)$. For every point γ of $\mathbf{S}$, let $\bar{\gamma}$ denote the k-tuple obtained by deleting the element (of $L^{(1)}_1$) in the first position of γ. Let $A_{ij} \subseteq \{1, \ldots, k\}$ be the set of positions in which $\bar{\gamma}_i$ and $\bar{\gamma}_j$ differ ($1 \leq i < j \leq 3$). Then $|A_{12}| = \mathrm{d}(\gamma_1,\gamma_2) = \mathrm{d}(\gamma_1,\gamma_3) + \mathrm{d}(\gamma_3,\gamma_2) \geq |A_{13}| + |A_{23}|$. At the other hand, $|A_{12}| \leq |A_{13}| + |A_{23}|$, implying $A_{12} = A_{13} \cup A_{23}$, $A_{13} \cap A_{23} = \emptyset$, $\mathrm{d}(\gamma_1,\gamma_3) = |A_{13}|$ and $\mathrm{d}(\gamma_2,\gamma_3) = |A_{23}|$. Moreover $h_i \in \{f_i, g_i\}$ for all $i \in \{1, \ldots, k\}$, implying $L^{(\mu)}_{h_\mu} \in sp(L^{(\mu)}_{i_\mu}, L^{(\mu)}_{j_\mu})$ for all $\mu \in \{1, \ldots, \delta\}$, and $x_3 = \left(\prod_{1\leq a\leq\delta}(\theta_a^{-1}\Delta^{(a)}_{f_a h_a}\theta_a)\right)(x_1) = \left(\prod_{1\leq a\leq\delta}(\theta_a^{-1}\Delta^{(a)}_{f_a h_a}\Delta^{(a)}_{i_a f_a}\theta_a)\right)(x) = \left(\prod_{1\leq a\leq\delta}(\theta_a^{-1}\Delta^{(a)}_{i_a h_a}\theta_a)\right)(x)$. Hence $\gamma_3 \in H'(\alpha,\beta)$. It is also immediate that $H(\alpha,\beta)$ induces a sub near (2δ)-gon of Hamming type (by considering the map $\gamma \mapsto \bar{\gamma}$).

For every point γ_1 not inside $H(\alpha,\beta)$, there exists a point γ_2 in $H(\alpha,\beta)$ such that $\mathrm{d}(\gamma_1,\gamma_2) > \delta$. Collecting all the previous calculations, we have the following Theorem.

Theorem 7.3.2
Every two points of a glued near polygon at distance δ of each other are contained in a unique geodetically closed sub near (2δ)-gon. This sub near polygon is also a glued one.

Remark. Suppose that $\mathbf{S}$ is a glued near hexagon and suppose that none of the GQ's $\mathbf{Q}_1, \ldots, \mathbf{Q}_k$, $k \geq 2$, is a grid. By Theorem 7.2.3 we know then that S_i is a spread of symmetry of $\mathbf{Q}_i$, $i \in \{1, \ldots, k\}$. This means that there are exactly $s+1$ automorphisms of $\mathbf{Q}_i$ fixing each line of S_i. These automorphisms can be recovered as follows in the near polygon. We may assume that $k = 2$ (otherwise take a suitable sub near hexagon) and that $i = 1$. For fixed $j \in \{1, \ldots, 1+st_2\}$, the set $A_j = \{(x, L^{(1)}_l, L^{(2)}_j) | x \; \mathcal{I} \; L^{(1)}_1, 1 \leq l \leq 1 + st_1\}$ is the point set of a quad $\mathbf{A}_j$ isomorphic to $\mathbf{Q}_1$ and since it is also a classical set, one can define projections $[A_{j_1}, A_{j_2}] : A_{j_1} \to A_{j_2}$ for all $j_1, j_2 \in$

$\{1,\ldots,1+st_2\}$. These projections preserve collinearity. It is straightforward to check that the map $[A_{j_2},A_1][A_{j_1},A_{j_2}][A_1,A_{j_1}]$, $1\le j_1,j_2\le 1+st_2$, defines an automorphism of $\mathbf{A}_1$ fixing each line of the spread $\{M_1,\ldots,M_{1+st_1}\}$; here M_l denotes the line with point set $\{(x,L_l^{(1)},L_1^{(2)})|x\ \mathcal{I}\ L_1^{(1)}\}$.

7.4 Isomorphisms between glued near hexagons

Introduction. Let $\mathbf{S}$ be the glued near hexagon derived from $\mathbf{Q}_1$, $\mathbf{Q}_2$, $S_1=\{L_1^{(1)},\ldots,L_{1+st_1}^{(1)}\}$, $S_2=\{L_1^{(2)},\ldots,L_{1+st_2}^{(2)}\}$, θ_1 and θ_2. Similarly, let $\mathbf{S}'$ be the glued near hexagon derived from $\mathbf{Q}'_1$, $\mathbf{Q}'_2$, $S'_1=\{L_1'^{(1)},\ldots,L_{1+st_1}'^{(1)}\}$, $S'_2=\{L_1'^{(2)},\ldots,L_{1+st_2}'^{(2)}\}$, θ'_1 and θ'_2. Suppose that $\mathbf{S}\simeq\mathbf{S}'$. By considering quads in $\mathbf{S}$ and $\mathbf{S}'$, either ($\mathbf{Q}_1\simeq\mathbf{Q}'_1$ and $\mathbf{Q}_2\simeq\mathbf{Q}'_2$) or ($\mathbf{Q}_1\simeq\mathbf{Q}'_2$ and $\mathbf{Q}_2\simeq\mathbf{Q}'_1$). Since $\mathbf{S}$ can be obtained starting from any ordering of the GQ's, we may suppose that $\mathbf{Q}_1=\mathbf{Q}'_1$ and $\mathbf{Q}_2=\mathbf{Q}'_2$. Points of $\mathbf{S}$ are triples. If we take a quad $\mathbf{A}$ consisting of points with fixed third coordinate ($\mathbf{A}\simeq\mathbf{Q}_1$), and we intersect $\mathbf{A}$ with all quads consisting of points with fixed second coordinate, then one finds a spread of $\mathbf{A}$ equivalent to S_1. We may hence suppose that $S_1=S'_1$ and $S_2=S'_2$. By Theorem 7.2.2, we may suppose that $L_1^{(1)}=L_1'^{(1)}$ and $L_1^{(2)}=L_1'^{(2)}$. We may also suppose that θ_1 and θ'_1 are identical maps.

For $i\in\{1,2\}$, let $\mathbf{Q}_i$ be a GQ different from a grid, let S_i be a spread of $\mathbf{Q}_i$ and let $L_1^{(i)}$ be a base line of S_i. Finally, let θ_1 be the identical map. For every map $\theta_2: L_1^{(1)}\to L_1^{(2)}$, we denote by $\mathbf{S}(\theta_2)$ the partial linear space obtained by applying the construction given in Section 7.1. We consider now the following interesting problem.

Problem. Let θ and θ' be two maps from $L_1^{(1)}$ to $L_1^{(2)}$. Under which conditions are $\mathbf{S}(\theta)$ and $\mathbf{S}(\theta')$ isomorphic near hexagons?

As before, let G_i, $i\in\{1,2\}$, denote the group of projectivities of $L_1^{(i)}$ with respect to S_i. Assume that $\mathbf{S}(\theta)$ and $\mathbf{S}(\theta')$ are indeed near hexagons. Hence

$$[\theta^{-1}G_2\theta,G_1]=0, \tag{7.1}$$

and

$$[\theta'^{-1}G_2\theta',G_1]=0. \tag{7.2}$$

We apply the following change of notation. The maps $\Delta_{ij}^{(1)}$, $i,j\in\{1,\ldots,1+st_1\}$, will shortly be denoted by δ_{ij}, while the maps $\Delta_{ij}^{(2)}$, $i,j\in\{1,\ldots,1+$

$st_2\}$, will be denoted by Δ_{ij}. The notations δ^β_{ij} and Δ^β_{ij} (with β a suitable permutation) are then used to denote $\delta_{\beta(i)\beta(j)}$ and $\Delta_{\beta(i)\beta(j)}$ as we already did in Section 2.8. The points of $\mathbf{S}(\theta)$ and $\mathbf{S}(\theta')$ are triples having three coordinates. A set of points with fixed third coordinate is a quad; quads arising this way are said to be of *type I.* Similarly, *quads of type II* are set of points with fixed second coordinate. The remaining quads (which are grids) are said to be of *type III.* If Φ is an isomorphism between $\mathbf{S}(\theta)$ and $\mathbf{S}(\theta')$, then either Φ preserves the quad-type, or it interchanges quads of type (I) and (II).

(A) Let Φ be quad-type preserving, then there exist:

(a) a permutation β of $\{1, \dots, 1 + st_1\}$,

(b) a permutation γ of $\{1, \dots, 1 + st_2\}$,

(c) permutations α_{ij} of $L^{(1)}_1$,

such that
$$\Phi[(x, L^{(1)}_i, L^{(2)}_j)] = (\alpha_{ij}(x), L^{(1)}_{\beta(i)}, L^{(2)}_{\gamma(j)}),$$
for all $x \in L^{(1)}_1$, for all $i \in \{1, \dots, 1 + st_1\}$ and for all $j \in \{1, \dots, 1 + st_2\}$. We define
$$\alpha := \alpha_{11}.$$
Consider in $\mathbf{S}(\theta)$ the adjacent points $(x, L^{(1)}_i, L^{(2)}_j)$ and $(\delta_{ik}(x), L^{(1)}_k, L^{(2)}_j)$ with $i \neq k$. Then $(\alpha_{ij}(x), L^{(1)}_{\beta(i)}, L^{(2)}_{\gamma(j)}) \sim (\alpha_{kj}\delta_{ik}(x), L^{(1)}_{\beta(k)}, L^{(2)}_{\gamma(j)})$ in $\mathbf{S}(\theta')$ and hence
$$\alpha_{ij} = \delta^\beta_{ki}\alpha_{kj}\delta_{ik}. \tag{7.3}$$
Consider in $\mathbf{S}(\theta)$ the adjacent points $(x, L^{(1)}_i, L^{(2)}_j)$ and $(\theta^{-1}\Delta_{jk}\theta(x), L^{(1)}_i, L^{(2)}_k)$ with $j \neq k$, then $(\alpha_{ij}(x), L^{(1)}_{\beta(i)}, L^{(2)}_{\gamma(j)}) \sim (\alpha_{ik}\theta^{-1}\Delta_{jk}\theta(x), L^{(1)}_{\beta(i)}, L^{(2)}_{\gamma(k)})$ in $\mathbf{S}(\theta')$ and hence
$$\alpha_{ij} = \theta'^{-1}\Delta^\gamma_{kj}\theta'\alpha_{ik}\theta^{-1}\Delta_{jk}\theta. \tag{7.4}$$
From (7.3) and (7.4), it follows that
$$\alpha_{ij} = \delta^\beta_{1i}\alpha_{1j} = \delta^\beta_{1i}\theta'^{-1}\Delta^\gamma_{1j}\theta'\alpha, \tag{7.5}$$
for all $i \in \{1, \dots, 1 + st_1\}$ and for all $j \in \{1, \dots, 1 + st_2\}$. From (7.2), (7.3) and (7.5) it follows that
$$(\delta^\beta_{k1}\delta^\beta_{ik}\delta^\beta_{1i})\alpha = \alpha\delta_{ik}, \tag{7.6}$$
for all $i, k \in \{1, \dots, 1 + st_1\}$. From (7.2), (7.4) and (7.5) it follows that
$$(\Delta^\gamma_{k1}\Delta^\gamma_{jk}\Delta^\gamma_{1j})(\theta'\alpha\theta^{-1}) = (\theta'\alpha\theta^{-1})\Delta_{jk}, \tag{7.7}$$

for all $j, k \in \{1, \ldots, 1 + st_2\}$.

(B) If Φ interchanges the quads of type (I) and (II), then we may suppose that $\mathbf{Q}_2 = \mathbf{Q}_1$, $S_2 = S_1$ and $L_1^{(2)} = L_1^{(1)}$. Using another ordering of the GQ's, we can consider Φ as a quad-type preserving isomorphism from $\mathbf{S}(\theta^{-1})$ to $\mathbf{S}(\theta')$. Equation (7.7) becomes then

$$(\Delta_{k1}^{\gamma}\Delta_{jk}^{\gamma}\Delta_{1j}^{\gamma})(\theta'\alpha\theta) = (\theta'\alpha\theta)\Delta_{jk}, \tag{7.8}$$

for all $j, k \in \{1, \ldots, 1 + st_2\}$.

We have now proved the following theorem.

Theorem 7.4.1

(1) *If the pairs $(\mathbf{Q}_1, S_1)$ and $(\mathbf{Q}_2, S_2)$ are not equivalent (i.e. if there exists no isomorphism from $\mathbf{Q}_1$ to $\mathbf{Q}_2$ mapping S_1 to S_2), then $\mathbf{S}(\theta)$ and $\mathbf{S}(\theta')$ are isomorphic glued near hexagons if and only if θ and θ' satisfy* (7.1), (7.2) *and* (7.7) *for some permutation α of $L_1^{(1)}$ satisfying* (7.6), *some permutation β of $\{1, \ldots, 1 + st_1\}$ and some permutation γ of $\{1, \ldots, 1 + st_2\}$.*

(2) *If the pairs $(\mathbf{Q}_1, S_1)$ and $(\mathbf{Q}_2, S_2)$ are equivalent, then we may suppose that $\mathbf{Q}_2 = \mathbf{Q}_1$, $S_2 = S_1$ and $L_1^{(2)} = L_1^{(1)}$. In this case $\mathbf{S}(\theta)$ and $\mathbf{S}(\theta')$ are isomorphic glued near hexagons if and only if θ and θ' satisfy* (7.1), (7.2), (7.7) *or* (7.1), (7.2), (7.8) *for some permutation α of $L_1^{(1)}$ satisfying* (7.6), *some permutation β of $\{1, \ldots, 1+st_1\}$ and some permutation γ of $\{1, \ldots, 1 + st_2\}$.*

Theorem 7.4.2

Equations (7.1), (7.6) *and* (7.7) *imply equation* (7.2). *Similarly, if $\mathbf{Q}_2 = \mathbf{Q}_1$, $S_2 = S_1$ and $L_1^{(2)} = L_1^{(1)}$, then equations* (7.1), (7.6) *and* (7.8) *imply equation* (7.2).

Proof. Suppose that (7.1), (7.6) and (7.7) hold, then

$$\alpha^{-1}[\theta'^{-1}G_2\theta', G_1]\alpha \overset{(7.6)}{=} \alpha^{-1}[\theta'^{-1}G_2\theta', \alpha G_1\alpha^{-1}]\alpha = [\alpha^{-1}\theta'^{-1}G_2\theta'\alpha, G_1]$$

$$\overset{(7.7)}{=} [\alpha^{-1}\theta'^{-1}(\theta'\alpha\theta^{-1})G_2(\theta'\alpha\theta^{-1})^{-1}\theta'\alpha, G_1] = [\theta^{-1}G_2\theta, G_1] \overset{(7.1)}{=} 0.$$

The other statement is proved similarly. □

Remark. As in Section 2.8, one can define the sets $\widehat{G}_1$ and $\widehat{G}_2$; equations (7.6), (7.7) and (7.8) become then $\alpha \in \widehat{G}_1$, $\theta'\alpha\theta^{-1} \in \widehat{G}_2$ and $\theta'\alpha\theta \in \widehat{G}_2$.

7.5 The number of nonisomorphic glued near hexagons

Let $(\mathbf{D}_1, K, \Delta_1)$ and $(\mathbf{D}_2, K, \Delta_2)$ be two admissible triples. Here $\mathbf{D}_1$ and $\mathbf{D}_2$ are two linear spaces with constant line size $s+1$ and respective point sets $\mathcal{P}_1$ and $\mathcal{P}_2$; $\mathbf{D}_1$ and $\mathbf{D}_2$ are supposed to be different from lines such that the corresponding GQ's $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are not grids. Recall that K is a group of order $s+1$ and that $\Delta_1 : \mathcal{P}_1 \times \mathcal{P}_1 \to K$ and $\Delta_2 : \mathcal{P}_2 \times \mathcal{P}_2 \to K$ are two maps such that the admissibility condition is satisfied. As we have seen in Section 7.2, near hexagons can be derived from these two AT's. Let S_1 and S_2 be the associated spreads of the AT's. Let $x_1 \in \mathcal{P}_1$ and $x_2 \in \mathcal{P}_2$ be fixed and let $L_1^{(1)} = \{(k, x_1) | k \in K\}$ and $L_1^{(2)} = \{(k, x_2) | k \in K\}$. With every bijection f from $L_1^{(i)}$ to $L_1^{(j)}$, $i, j \in \{1, 2\}$, there corresponds a permutation f' of K such that $f((k, x_i)) = (f'(k), x_j)$ for all $k \in K$. This can be applied to the maps θ, θ', $\Delta_{ij}^{(1)}$, $\Delta_{ij}^{(2)}$, α, $g_1 \in G_1$, $g_2 \in G_2$, $\widehat{g}_1 \in \widehat{G}_1$, $\widehat{g}_2 \in \widehat{G}_2$, etc. In the sequel we will regard all these maps as permutations of K. Conditions (7.1), (7.2), (7.6), (7.7) and (7.8) remain then the same. To every permutation $\theta : K \to K$ corresponds a map $\theta_2 : L_1^{(1)} \to L_1^{(2)}$ and hence a partial linear space $\mathbf{S}(\theta)$ by Section 7.1 (suppose again that θ_1 is the identical map). Let $\mathcal{A}$ be the set of all $(s+1)|\operatorname{Aut}(K)|$ permutations θ of K for which $\mathbf{S}(\theta)$ is a near hexagon. Let G be the group of permutations of K consisting of all right multiplications, then $K \simeq G$ and $G_1 = G_2 = G$ by Theorem 2.7.8. In Chapter 2, we saw an overview of all known GQ's which have an AT-model. We have the following possibilities for K.

(1) $K \simeq C_{q+1}$ with q a prime power.

(2) $K \simeq C_p \times \ldots \times C_p$ being the additive group of $\mathrm{GF}(p^h)$.

Sometimes classes (1) and (2) overlap; in this case $h = 1$ and $q + 1 = p$. This only happens if p is a prime of the form $2^{2^l} + 1$ (a so-called Mersenne prime); the only known such primes are $p = 3$, $p = 5$, $p = 17$, $p = 257$ and $p = 65537$. In each of the above cases K and hence G is commutative. Equation (7.1) becomes then $G\theta = \theta G$ and has as solution all the maps $\theta : K \to K : x \mapsto k \cdot \chi(x)$, with $k \in K$ and χ an automorphism of K. The number of solutions for θ in both cases is then as follows.

(1) If $K \simeq C_{q+1}$, then there are $(q+1)\,\phi(q+1)$ solutions with ϕ the Euler indicator.

(2) If K is isomorphic to the additive group of $\mathrm{GF}(p^h)$, then there are $p^h |GL(h, p)|$ solutions.

Let N denote the number of glued near hexagons arising from $\mathbf{Q}_1$, $\mathbf{Q}_2$, S_1 and S_2. We will make estimates for N.

Theorem 7.5.1
If $(\mathbf{Q}_1, S_1)$ *is equivalent to* $(\mathbf{Q}_2, S_2)$*, then*

$$\frac{(s+1)|\operatorname{Aut}(K)|}{2|\widehat{G}_1||\widehat{G}_2|} \leq N \leq \frac{(s+1)|\operatorname{Aut}(K)|}{\max(|\widehat{G}_1|, |\widehat{G}_2|)};$$

if not so, then we even have that

$$\frac{(s+1)|\operatorname{Aut}(K)|}{|\widehat{G}_1||\widehat{G}_2|} \leq N \leq \frac{(s+1)|\operatorname{Aut}(K)|}{\max(|\widehat{G}_1|, |\widehat{G}_2|)}.$$

Proof. Let θ be one of the $(s+1)|\operatorname{Aut}(K)|$ maps of $\mathcal{A}$. There are at most $|\widehat{G}_1||\widehat{G}_2|$ maps θ' satisfying equations (7.6) and (7.7), and a same remark holds for equations (7.6) and (7.8). This proves the lower bounds for N. Now, consider again equations (7.6) and (7.7). Let α and β be identical permutations, and let θ' be one of the $|\widehat{G}_2|$ maps for which $\theta'\theta^{-1} \in \widehat{G}_2$, then $\theta' \in \mathcal{A}$ by Theorem 7.4.2. Hence $N \leq \frac{(s+1)|\operatorname{Aut}(K)|}{|\widehat{G}_2|}$ and the upper bound follows by symmetry. □

Corollary 7.5.2

(1) *If* $\mathbf{Q}_1$ *and* $\mathbf{Q}_2$ *are dual grids, then only one glued near hexagon can be constructed.*

(2) *Let* p *be a prime of the form* $2^{2^l} + 1$*, then there is only one glued near hexagon arising from the generalized quadrangles* $AS(p)$ *and* $Q(5, p-1)$*.*

Proof.

(1) Up to an automorphism there is only one spread of symmetry in $\mathbf{Q}_1$ and $\mathbf{Q}_2$. For these spreads we have $s+1 = 2$, $|K| = 2$, $|\operatorname{Aut}(K)| = 1$ and $|\widehat{G}_1| = |\widehat{G}_2| = 2$; hence $N = 1$.

(2) Up to an automorphism there is only one spread of symmetry in each GQ. For these spreads the group K is cyclic of order p, e.g. $K = \mathrm{GF}(p)$ equiped with the addition of the field. By results of Section 2.8, $\widehat{G}_1$ consists of all permutations $K \to K : x \mapsto kx + k'$ with $k \in K \setminus \{0\}$ and $k' \in K$. Again by Section 2.8, $\widehat{G}_2$ consists of all permutations $K \to K : x \mapsto kx^{2^i}$ with $k \in K$ and $i \in \{0, \ldots, 2^l - 1\}$. Hence $|\widehat{G}_1| = p(p-1)$ and $|\widehat{G}_2| = p2^l$. By Theorem 7.5.1, $0 < N \leq \frac{p(p-1)}{p(p-1)} = 1$. □

Remark. If $p = 3$, then $AS(p) \simeq Q(5, p-1)$ and the corresponding glued near hexagon was already known before (see [14]).

Sometimes the lower bound given in Theorem 7.5.1 can be sharpened.

Theorem 7.5.3
If $\widehat{G}_1 = \widehat{G}_2$ is a group, then $1 + \mu \frac{(s+1)|\operatorname{Aut}(K)| - |\widehat{G}_1|}{|\widehat{G}_1||\widehat{G}_2|} \leq N$. Here $\mu = \frac{1}{2}$ or $\mu = 1$ depending on whether $(\mathbf{Q}_1, S_1)$ is equivalent to $(\mathbf{Q}_2, S_2)$ or not.

Proof. Let $\theta \in \widehat{G}_1$, then also $\theta \in \mathcal{A}$ (see equations (7.1) and (7.6)). Now, by equations (7.7) and (7.8), we have that $\mathbf{S}(\theta') \simeq \mathbf{S}(\theta)$ if and only if $\theta' \in \widehat{G}_1$. The two parts in the lower bound of N correspond to the two parts in the expression $\mathcal{A} = \widehat{G}_1 \cup (\mathcal{A} \setminus \widehat{G}_1)$ (see also the proof of Theorem 7.5.1). $\square$

Remark. Theorem 2.8.2 gives sufficient conditions for $\widehat{G}_1$ and $\widehat{G}_2$ to be a group.

Corollary 7.5.4
Let $\mathbf{Q}_1 = \mathbf{Q}_2 = P(W(q), x)$, $q = p^h$, and let $S_1 = S_2$ be the spread consisting of all hyperbolic lines through x. Then $N = 1$ if $q = 4$ or if q is a prime; otherwise $N > 1$.

Proof. The group K is cyclic of order q, e.g. $K = \mathrm{GF}(q)$ equiped with the addition of the field. Applying Theorems 7.5.1 and 7.5.3 yields

$$1 + \frac{q|GL(h,p)| - h(q-1)q}{2h^2(q-1)^2q^2} \leq N \leq \frac{q|GL(h,p)|}{h(q-1)q}.$$

Note that $|GL(h,p)| = (p^h - 1)(p^h - p) \dots (p^h - p^{h-1})$. The group $\widehat{G}_1$ consists of all maps $K \to K : x \mapsto kx^{p^i} + k'$ where $k \in \mathrm{GF}(q) \setminus \{0\}$, $k' \in \mathrm{GF}(q)$ and $i \in \{0, \dots, h-1\}$, see Section 2.8. If $q = 4$ or if q is prime, then $N = 1$ by the above inequalities. If $h \geq 2$ and $q \neq 4$, then $|GL(h,p)| \geq (p^h - 1)(p^h - p) > h(q-1)$; hence $N > 1$. $\square$

For fixed h, the lower bound in the previous theorem behaves like $\frac{1}{2h^2} p^{h^2 - 3h}$ for great values of p; hence many glued near hexagons will arise. We treat now the case $h = 2$, p odd.

Glued near hexagons arising from $P(W(p^2), x)$, p odd

Let $\mathbf{Q}_1 = \mathbf{Q}_2 = P(W(p^2), x)$, p an odd prime, and let $S_1 = S_2$ be the unique spread of symmetry. The group K is cyclic of order p^2, e.g. $K = \mathrm{GF}(p^2)$ equiped with the addition of the field. The $p^2(p^2 - 1)(p^2 - p)$ elements of $\mathcal{A}$

are the maps $\theta : z \mapsto az + bz^p + c$ with $a, b, c \in \mathrm{GF}(p^2)$ and $a^{p+1} \neq b^{p+1}$. If $z \mapsto a'z + b'z^p + c'$ is the inverse map, then one easily checks that

$$a' = \frac{a^p}{a^{p+1} - b^{p+1}}, b' = -\frac{b}{a^{p+1} - b^{p+1}}, c' = \frac{bc^p - a^p c}{a^{p+1} - b^{p+1}}.$$

The elements of $\widehat{G}_1 = \widehat{G}_2$ are the maps $K \to K : x \mapsto kx^{p^i} + k'$ where $k \in \mathrm{GF}(p^2) \setminus \{0\}$, $k' \in \mathrm{GF}(p^2)$ and $i \in \{0, 1\}$. Let $\theta : z \mapsto a_1 z + b_1 z^p + c_1$ and $\theta' : z \mapsto a_2 z + b_2 z^p + c_2$ be elements of $\mathcal{A}$. If $a_1 = 0$ or $b_1 = 0$, then $\mathbf{S}(\theta) \simeq \mathbf{S}(\theta')$ if and only if $a_2 = 0$ or $b_2 = 0$. If $a_1 \neq 0 \neq a_2$, then one checks that $\mathbf{S}(\theta) \simeq \mathbf{S}(\theta')$ if and only if

(I) $a_2 \neq 0 \neq b_2$ and $(\frac{a_2}{b_2})^{p+1} = (\frac{a_1}{b_1})^{p+1}$; or

(II) $a_2 \neq 0 \neq b_2$ and $(\frac{a_2}{b_2})^{p+1} = (\frac{b_1}{a_1})^{p+1}$.

If $\theta \in \mathcal{A}$, then (i) $a = 0$ or $b = 0$, (ii) $a \neq 0 \neq b$ and $(\frac{a}{b})^{p+1} = -1$, or (iii) $a \neq 0 \neq b$ and $(\frac{a}{b})^{p+1} \neq \pm 1$. Hence

$$N = 1 + 1 + \frac{p-3}{2} = \frac{p+1}{2}.$$

Glued near hexagons arising from $Q(5, q)$

Let $\mathbf{Q}_1$ and $\mathbf{Q}_2$ be generalized quadrangles isomorphic to $Q(5, q)$, $q = p^h$. As we have seen in Section 2.6.2, there is up to an automorphism only one spread of symmetry in $Q(5, q)$. For such a spread, we calculate the number of glued near hexagons arising. The group K is cyclic of order $q + 1$, e.g. $K = \{x \in \mathrm{GF}(q^2) | x^{q+1} = 1\}$ equiped with the multiplication of the field $\mathrm{GF}(q^2)$. The $(q+1)\,\phi(q+1)$ elements of $\mathcal{A}$ are the maps $x \to lx^n$ with $l \in K$ and $(n, q+1) = 1$. The elements of $\widehat{G}_1 = \widehat{G}_2$ are the maps $K \to K : k \mapsto lk^{p^\sigma}$ with $l \in K$ and $\sigma \in \mathbb{N}$. Let $\theta : k \mapsto l_1 k^{n_1}$ and $\theta' : k \mapsto l_2 k^{n_2}$ be two elements of $\mathcal{A}$ satisfying (7.6),(7.7) or (7.6),(7.7). Let $\alpha(k) = mk^{p^{\sigma_1}}$ and $\mu(k) = nk^{p^{\sigma_2}}$. If $\mu = \theta' \alpha \theta^{-1}$, then $l_2 = n l_1^{p^{\sigma_2}} (m^{n_2})^{-1}$ and $n_2 p^{\sigma_1} \equiv n_1 p^{\sigma_2} \pmod{q+1}$; similarly, if $\mu = \theta' \alpha \theta$, then $n_1 n_2 p^{\sigma_1} \equiv p^{\sigma_2} \pmod{q+1}$. Hence $\mathbf{S}(\theta) \simeq \mathbf{S}(\theta')$ if and only if at least one of the following conditions is satisfied:

(I) $n_1 \equiv n_2 p^\eta \pmod{q+1}$ for some $\eta \in \mathbb{N}$,

(II) $n_1 n_2 \equiv p^{\eta'} \pmod{q+1}$ for some $\eta' \in \mathbb{N}$.

If (I) and (II) are satisfied, then $n_1^2 \equiv p^{\eta''} \pmod{q+1}$ for some $\eta'' \in \mathbb{N}$. Suppose that there are R elements $n \in \{1, \ldots, q\}$, $(n, q+1) = 1$, such that

$n^2 \equiv p^{\eta''} \pmod{q+1}$ for some $\eta'' \in \mathbb{N}$, then the number of nonisomorphic glued near hexagons is equal to

$$N = \frac{(q+1)\,\phi(q+1) - (q+1)R}{4h(q+1)} + \frac{(q+1)R}{2h(q+1)} = \frac{\phi(q+1)+R}{4h}.$$

The tabel below gives the value of N for all prime powers $q \leq 19$.

q	N	q	N	q	N
2	1	7	2	13	2
3	1	8	1	16	2
4	1	9	1	17	2
5	1	11	2	19	3

The other known glued near hexagons

The other known glued near hexagons arise if $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are of one of the following two types (hence four possible combinations).

(1) $T_2^*(O)$ with O a hyperoval in $\mathrm{PG}(2, 2^h)$.

(2) $(S_{xy}^-)^D$ associated with a hyperoval in $\mathrm{PG}(2, 2^h)$.

The group K is isomorphic to the additive group of $\mathrm{GF}(2^h)$. By Theorems 7.5.1 and 7.5.3 estimates for the number of glued near hexagons can be made. We have $s = 2^h - 1$, $|\operatorname{Aut}(K)| = |GL(2,h)|$; the values of $|\widehat{G}_1|$ and $|\widehat{G}_2|$ depend on the kind of hyperoval we are working with ($|\widehat{G}| = h'2^h(2^h-1)$ with $h' \mid h$ in the case of $T_2^*(O)$, see Section 2.8). Again, lots of glued near hexagons can be constructed.

7.6 Characterizations

7.6.1 Local spaces with a thin point

Definition. For $u, v \in \mathbb{N} \setminus \{0\}$, let $\mathbf{S}_{u,v} = (\mathcal{P}_{u,v}, \mathcal{L}_{u,v}, \mathcal{I}_{u,v})$ be the following linear space:

- $\mathcal{P}_{u,v} = \{\alpha, \beta_1, \ldots, \beta_u, \gamma_1, \ldots, \gamma_v\}$,
- $\mathcal{L}_{u,v} = \{\{\alpha, \beta_1, \ldots, \beta_u\}, \{\alpha, \gamma_1, \ldots, \gamma_v\}\} \cup \{\{\beta_i, \gamma_j\} | 1 \leq i \leq u \text{ and } 1 \leq j \leq v\}$,
- $\mathcal{I}_{u,v}$ is the symmetrized containment.

$\mathbf{S}_{u,v}$ is a linear space with a *thin point* (i.e. a point incident with only two lines). Conversely, every linear space with a thin point is obtained in this way. If $\mathbf{S}$ is a glued near hexagon, then $\mathbf{S}_x \simeq \mathbf{S}_{t_1,t_2}$ for all points x of $\mathbf{S}$. Recall that a quad is called good if all points are incident with the same number of lines.

Theorem 7.6.1
Let $\mathbf{S}$ be a near hexagon satisfying the following properties:

- *every two points at distance 2 are contained in a quad,*
- *if all lines of $\mathbf{S}$ are thin, then all quads are good,*
- *there exists a point x of $\mathbf{S}$ such that $\mathbf{S}_x \simeq \mathbf{S}_{1,r}$ for some $r \in \mathbb{N} \setminus \{0\}$,*

then $\mathbf{S}$ is the direct product of a line with a nondegenerate GQ.

Proof. If not all lines of $\mathbf{S}$ have the same number of points, then $\mathbf{S}$ is the direct product of a line with a GQ, see Theorem 3.1.2. Hence, by Theorem 3.8.1, we may assume that $\mathbf{S}$ has order (s,t) with $t = r+1$. Consider through x a quad R_x containing t lines through x and let L_x be the remaining line through x. Every point z of R_x is incident with exactly one line L_z which is not in R_x. Let $y \in L_x \setminus \{x\}$ be fixed. Let M_1 and M_2 be two lines through y different from L_x and let R_y be the quad through M_1 and M_2. The quad through M_i ($i \in \{1,2\}$) and L_x intersects R_x in a line M_i'. Now, let u be one of the $s^2(t-1)$ points of R_x at distance 2 from x. Let u_i ($i \in \{1,2\}$) be the unique point on M_i' collinear with u. The quad through uu_i and L_{u_i} is a grid. Let u_i' be the intersection of L_{u_i} with M_i and let v_i be the unique neighbour of u_i' and u different from u_i. The point v_i is then the unique point of L_u at distance 2 from y. This implies that $v = v_1 = v_2$. Since v is collinear with the points u_1' and u_2' of R_y, v is itself contained in R_y. Hence $|\Gamma_2(y) \cap R_y| \geq s^2(t-1)$. This implies that R_y is a GQ of order $(s,t-1)$ containing all lines through y, except the line L_x and that $R_y \simeq R_x$. The result follows now immediately. $\square$

7.6.2 Characterizations of glued near hexagons

Theorem 7.6.2
Let $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a near hexagon satisfying the following properties:

- *every two points at distance 2 are contained in a quad,*
- *if all lines of $\mathbf{S}$ are thin, then all quads are good,*
- *there exists a point x such that $\mathbf{S}_y$ has a thin point for all $y \in \Gamma(x)$,*

then $\mathbf{S}$ *is the direct product of a line with a nondegenerate* GQ *or* $\mathbf{S}$ *is a glued near hexagon.*

Proof. If not all lines of $\mathbf{S}$ have the same number of points, then $\mathbf{S}$ is the direct product of a line with a nondegenerate GQ. Hence, by Theorem 3.8.1 we may assume that $\mathbf{S}$ has an order (s,t). If $\mathbf{S}_y$ (with $y \in \mathcal{P}$) is a linear space with a thin point, then we may suppose that $\mathbf{S}_y$ contains a unique thin point which we denote by L_y, otherwise the result would follow from Theorem 7.6.1. The line L_y is then contained in exactly two quads. The following properties hold.

(a) If y is a point for which $\mathbf{S}_y$ is a linear space with a thin point, then $\mathbf{S}_{y'} \simeq \mathbf{S}_y$ and $L_{y'} = L_y$ for all points $y' \in L_y$.
Indeed, suppose $\mathbf{S}_y \simeq \mathbf{S}_{t_1,t_2}$ with $t_1, t_2 > 1$ and $t = t_1 + t_2$. The point L_y of $\mathbf{S}_{y'}$ is contained in exactly two lines of $\mathbf{S}_{y'}$, one line has $t_1 + 1$ points, the other $t_2 + 1$ points. Since there are exactly $t_1 + t_2 + 1$ points in $\mathbf{S}_{y'}$, it follows that $\mathbf{S}_{y'} \simeq \mathbf{S}_{t_1,t_2}$.

(b) If y_1, y_2 are points such that $\mathbf{S}_{y_1}, \mathbf{S}_{y_2}$ are linear spaces with a thin point, then L_{y_1} and L_{y_2} are equal or disjoint. (This follows immediately from (a).)

(c) There exists a point $y \in \Gamma(x)$ such that $x \in L_y$.
Indeed, suppose that this is not true. Let $y \in \Gamma(x)$ be fixed. Let $\mathbf{Q}$ be the quad of order (s,t') through xy and L_y. There are $s(t'+1)$ points $z_i \in \mathbf{Q}$ collinear with x. These give rise to $s(t'+1)$ lines L_{z_i} and all these lines are different and hence disjoint by (b). Suppose that L_z is not contained in $\mathbf{Q}$ for a certain $z \in \Gamma(x) \cap \mathbf{Q}$, then $\mathbf{S}_z$ contains at least three thick lines (namely the line defined by $\mathbf{Q}$ and the two lines of $\mathbf{S}_z$ through L_z); a contradiction, since $\mathbf{S}_z$ is a linear space with a unique thin point. Hence, all lines L_z are contained in $\mathbf{Q}$ and there are at least $(s+1)(st'+s)$ points in $\mathbf{Q}$, but this is again impossible.

Let $y \in \Gamma(x)$ such that $x \in L_y$. Hence $\mathbf{S}_x$ is also a linear space with a unique thin point L_x. Let $\mathbf{Q}_1$ and $\mathbf{Q}_2$ be the two quads through L_x with respective orders (s,t_1) and (s,t_2). In $\mathbf{Q}_i$ $(i = 1,2)$, there are st_i points z collinear with x and not on L_x. These give rise to st_i disjoint lines L_z which together with L_x form a spread S_i of $\mathbf{Q}_i$. Put $S_i = \{L_1^{(i)}, \ldots, L_{1+st_i}^{(i)}\}$ with $L_1^{(i)} = L_x$. Finally, let θ_i, $i \in \{1,2\}$, be the identical permutation of L_x. By the construction given in Section 7.1, we can define now a partial linear space $\mathbf{S}'$. We will prove that $\mathbf{S} \simeq \mathbf{S}'$, hence $\mathbf{S}$ is glued.
First we prove that every point u of $\mathbf{S}$ has distance at most 1 to each $\mathbf{Q}_i$ $(i \in \{1,2\})$. Let u' be the unique point of L_x nearest to u; we may suppose

that $\mathrm{d}(u, u') = 2$. Since $\mathbf{S}_{u'} \simeq \mathbf{S}_x$, it follows that the quad through u and u' intersects each $\mathbf{Q}_i$ in a line. This proves that each $\mathbf{Q}_i$ contains a point collinear with u. For $i \in \{1, 2\}$ and $u \in \mathcal{P}$, let $p_i(u)$ denote the unique point of $\mathbf{Q}_i$ nearest to u.
Next we prove that all the local spaces $\mathbf{S}_u$ are isomorphic to $\mathbf{S}_{t_1,t_2}$. Since for all $u \in \mathbf{Q}_i$, L_u is contained in exactly two quads ($\mathbf{Q}_i$ and another quad), we have that $G_u \simeq \mathbf{S}_{t_1,t_2}$. Let u be a point of $\mathbf{S}$ not contained in $\mathbf{Q}_1 \cup \mathbf{Q}_2$. Let $u' = p_1(u)$ and $u'' = p_2(u)$. The local space $\mathbf{S}_u$ contains $t_1 + t_2 + 1$ points, a line with $t_1 + 1$ points (determined by the quad through uu'' and $L_{u''}$) and a line with $t_2 + 1$ points (determined by the quad through uu' and $L_{u'}$). From this it follows that $\mathbf{S}_u \simeq \mathbf{S}_{t_1,t_2}$. Hence L_u is defined for all $u \in \mathcal{P}$ and all these lines determine a spread of $\mathbf{S}$. Each L_u is contained in exactly two quads. One quad intersects $\mathbf{Q}_2$ in a line and is isomorphic to $\mathbf{Q}_1$. The other quad intersects $\mathbf{Q}_1$ and is isomorphic to $\mathbf{Q}_2$. Note that the isomorphisms are defined by the projections p_i, $i \in \{1, 2\}$.
We consider now the following map $\Delta : \mathcal{P} \mapsto L_x \times S_1 \times S_2, \Delta(u) = (\gamma(u), \delta_1(u), \delta_2(u))$, where $\gamma(u)$ is the unique point of L_x nearest to u and $\delta_i(u)$ ($i \in \{1, 2\}$) is the unique line of S_i incident with $p_i(u)$. Let $(a, L_1, L_2) = (\gamma(u), \delta_1(u), \delta_2(u))$ and put a_i ($i \in \{1, 2\}$) equal to the projection of a on the line L_i of $\mathbf{Q}_i$. If $L_1 = L_x$, then $u = a_2$; if $L_2 = L_x$, then $u = a_1$; if $L_1 \neq L_x \neq L_2$, then u is the common neighbour of a_1 and a_2 different from a. This proves that Δ is a bijection. Since $\mathbf{S}$ and $\mathbf{S}'$ have the same order, it suffices to prove that Δ preserves adjacency in the point graph of the geometries. Let x and x' be two adjacent points of $\mathbf{S}$. If x and x' are contained in a quad intersecting $\mathbf{Q}_2$, then $\delta_2(x) = \delta_2(x')$. If $\delta_1(x) = L_i^{(1)}$ and $\delta_1(x') = L_j^{(1)}$, then $p_i^{(1)}\theta_1(\gamma(x)) = p_1(x)$ and $p_j^{(1)}\theta_1(\gamma(x')) = p_1(x')$. Since $p_1(x) \sim p_1(x')$, $p_i^{(1)}\theta_1(\gamma(x)) \sim p_j^{(1)}\theta_1(\gamma(x'))$. A similar reasoning holds if x and x' are contained in a quad intersecting $\mathbf{Q}_1$.
$\square$

Theorem 7.6.3
Let $\mathbf{S}$ be a near hexagon satisfying the following properties:

- *every two points at distance 2 are contained in a quad,*
- *if all lines of $\mathbf{S}$ are thin, then all quads are good,*
- *there exists a point x such that $\mathbf{S}_x$ has a thin point and such that $\mathbf{S}_y$ contains the same number of lines for all $y \in \Gamma(x)$,*

then $\mathbf{S}$ is the direct product of a line with a nondegenerate GQ *or $\mathbf{S}$ is a glued near hexagon.*

Proof. Just like before, we may suppose that $\mathbf{S}$ has an order (s,t). Theorem 3.8.2 implies that the number of points in $\Gamma_2(y)$ is independent of the point y of $\mathbf{S}$. For $y \in \Gamma(x)$, let V_y denote the set of quads through y. Now,

$$\sum_{\mathbf{Q}\in V_y} 1, \sum_{\mathbf{Q}\in V_y} s^2 t_{\mathbf{Q}}, \sum_{\mathbf{Q}\in V_y} t_{\mathbf{Q}}(t_{\mathbf{Q}}+1),$$

are respectively equal to the number of quads through y, the number of points in $\Gamma_2(y)$ and $t(t+1)$, hence these expressions are independent of $y \in \Gamma(x)$. Let L_x be a thin point of $\mathbf{S}_x$ and let $\mathbf{Q}_1$ and $\mathbf{Q}_2$ be the two quads through L_x with respective orders (s,t_1) and (s,t_2); $t = t_1 + t_2$. Let $z \neq x$ be a second point of L_x. If $y \in \mathbf{Q}_1 \cap \Gamma(x)$, then

$$\sum_{\mathbf{Q}\in V_y} (t_{\mathbf{Q}}-1)(t_2-t_{\mathbf{Q}}) = \sum_{\mathbf{Q}\in V_z} (t_{\mathbf{Q}}-1)(t_2-t_{\mathbf{Q}}) = (t_1-1)(t_2-t_1).$$

Let $V_y' = V_y \setminus \{\mathbf{Q}_1\}$, then

$$\sum_{\mathbf{Q}\in V_y'} (t_{\mathbf{Q}}-1)(t_2-t_{\mathbf{Q}}) = 0.$$

Since there are only $t+1$ lines through y and $\mathbf{Q}_1$ has t_1+1 lines through y, it follows that $1 \leq t_{\mathbf{Q}} \leq t_2$ for all $\mathbf{Q} \in V_y'$. This implies that $t_{\mathbf{Q}} = 1$ or $t_{\mathbf{Q}} = t_2$ for all $\mathbf{Q} \in V_y'$. By Theorem 7.6.1, we may suppose that $t_1, t_2 \neq 1$. From

$$\sum_{\mathbf{Q}\in V_y} 1 = \sum_{\mathbf{Q}\in V_z} 1,$$

and

$$\sum_{\mathbf{Q}\in V_y} t_{\mathbf{Q}} = \sum_{\mathbf{Q}\in V_z} t_{\mathbf{Q}},$$

it follows now that the number of quads $\mathbf{Q}$ of V_y' with $t_{\mathbf{Q}} = t_2$ is equal to 1. This implies that $\mathbf{S}_y \simeq \mathbf{S}_{t_1,t_2}$ for all $y \in \Gamma(x) \cap \mathbf{Q}_1$. A similar reasoning shows that this is also true for $y \in \Gamma(x) \cap \mathbf{Q}_2$. The result follows now from the previous theorem. $\square$

Chapter 8

Near polygons with four points on a line

8.1 The examples

In this chapter we classify, up to four open cases, all near hexagons with four points on a line ($s = 3$) and having the property that every two points at distance two have at least two common neighbours. The corresponding classification with $s = 2$ was done in [14]. By Yanushka's Theorem, these near hexagons satisfy the property that every two points at distance two are contained in a (necessary unique) quad. We describe now all known examples and divide them into the following three classes.

(A) THE REGULAR CLASSICAL NEAR HEXAGONS.
The direct product of three lines of length 4 is a classical near hexagon with parameters $s = 3$, $t_2 = 1$ and $t = 2$. The other examples are the dual polar spaces $W^D(5,3)$, $Q^D(6,3)$, $H^D(5,9)$. The parameters of these regular near hexagons are:

Type	**s**	$\mathbf{t_2}$	**t**
$W^D(5,3)$	3	3	12
$Q^D(6,3)$	3	3	12
$H^D(5,9)$	3	9	90

(B) THE IRREGULAR CLASSICAL NEAR HEXAGONS.
These near hexagons are the direct product of a line of length 4 with one of the following GQ's:

(1) $W(3)$,

(2) $Q(4,3)$,

(3) $T_2^*(\mathcal{H})$, with $\mathcal{H}$ the unique (up to projective equivalence) hyperoval in $\mathrm{PG}(2,4)$,

(4) $Q(5,3)$.

Note that the direct product of a line of length 4 with a (4×4)-grid is a regular near hexagon of Hamming type.

(C) THE NONCLASSICAL NEAR HEXAGONS.
There are two examples known and both are glued near hexagons.

(I) The following example has been defined in Chapter 6, page 108. Consider in a $\mathrm{PG}(4,4)$ two planes α_1 and α_2 intersecting in a point p. Let O_i, $i \in \{1,2\}$, be a hyperoval in α_i containing p. The linear representation $T_4^*(O_1 \cup O_2)$ is then a near hexagon. Since there is a unique hyperoval in $\mathrm{PG}(2,4)$ (up to projective equivalence) and since the stabilizer of such a hyperoval in the group of all automorphisms of $\mathrm{PG}(2,4)$ acts transitively on the set of its points, there is up to isomorphism only one such near hexagon.

(II) Put $K = \{x \in \mathrm{GF}(9) | x^4 = 1\}$. Consider in $V(3,9)$ a nonsingular Hermitian form $(\cdot,\cdot)$ and let U be the corresponding unital in $\mathrm{PG}(2,9)$. Let $\alpha = \langle \bar{a} \rangle$ be a fixed point of U. For two points $\beta = \langle \bar{b} \rangle$ and $\gamma = \langle \bar{c} \rangle$ of U, we define

$$\begin{aligned} \Delta(\beta,\gamma) &= -(\bar{a},\bar{b})^2(\bar{b},\bar{c})^2(\bar{c},\bar{a})^2 \text{ if } \alpha \neq \beta \neq \gamma \neq \alpha, \\ &= 1 \text{ otherwise.} \end{aligned}$$

This is a good definition. Indeed, if we replace $\bar{b}$ by $\mu\bar{b}$ and $\bar{c}$ by $\lambda\bar{c}$ with $\mu, \lambda \in \mathrm{GF}(9) \setminus \{0\}$, then the above value for $\Delta(\beta,\gamma)$ is not changed. We define now a graph Γ with vertex set $K \times U \times U$. Two different vertices (k_1,α_1,β_1) and (k_2,α_2,β_2) are adjacent if and only if one of the following conditions is satisfied:

(a) $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$,

(b) $\alpha_1 = \alpha_2$, $\beta_1 \neq \beta_2$ and $k_2 = k_1\Delta(\beta_1,\beta_2)$,

(c) $\alpha_1 \neq \alpha_2$, $\beta_1 = \beta_2$ and $k_2 = k_1\Delta(\alpha_1,\alpha_2)$.

Every two adjacent vertices of Γ are contained in a unique maximal clique. The geometry, with points the vertices of Γ, with lines the maximal cliques of Γ, and with natural incidence, is then a near hexagon.

8.2 The GQ's with $s = 3$ and their ovoids

We recall that there are five GQ's with $s = 3$, namely the following ones.

(1) The (4×4)-grid

(2) $W(3)$

(3) $Q(4, 3)$

(4) $T_2^*(H)$ with H the unique (up to projective equivalence) hyperoval in $\mathrm{PG}(2, 4)$

(5) $Q(5, 3)$

The following lemma settles the existence of ovoids and of rosettes and fans of ovoids in these GQ's.

Lemma 8.2.1

(a) $W(3)$ *and* $Q(5, 3)$ *have no ovoids.*

(b) $T_2^*(H)$ *contains ovoids but no rosettes of ovoids.*

(c) $Q(4, 3)$ *contains ovoids but no fans of ovoids.*

Proof.

(a) This is a special case of Theorem 2.6.1.

(b) Let H be a hyperoval in π which is embedded as hyperplane in $\mathrm{PG}(3, 4)$. Consider a plane α in $\mathrm{PG}(3, 4)$ intersecting π in a line exterior to H. It follows that the points of α not in π form an ovoid of $T_2^*(H)$. Conversely, every ovoid of $T_2^*(H)$ arises this way (see [68]). Hence, two different ovoids are disjoint or intersect in four points. As a consequence no rosette of ovoids occurs.

(c) Let Q be a nonsingular quadric in $\mathrm{PG}(4, 3)$. If π is a hyperplane of $\mathrm{PG}(4, 3)$ intersecting Q in an elliptic quadric, then $\pi \cap Q$ is an ovoid of $Q(4, 3)$. Conversely, every ovoid is obtained this way (see e.g. [17]). Let O_1 and O_2 be two ovoids and let α_1 and α_2 be the hyperplanes such that $O_1 = \alpha_1 \cap Q$ and $O_2 = \alpha_2 \cap Q$. Now, $O_1 \cap O_2 = (\alpha_1 \cap \alpha_2) \cap Q \neq \emptyset$; hence, $Q(4, 3)$ has no fan of ovoids.

$\square$

In the sequel, we will always assume that $\mathbf{S}$ is a near hexagon satisfying the following properties:

(1) all lines of **S** have 4 points;

(2) every two points at distance two have at least two common neighbours.

Definition. A quad of **S** is called *big* if every point of **S** has distance at most 1 from **Q**.

Lemma 8.2.1 has now the following corollaries.

Corollary 8.2.2
If **Q** *is a quad of* **S** *which is not big, then* **Q** *is isomorphic to the* (4×4)*-grid or to* $Q(4,3)$*.*

Proof. Let x be a point at distance 2 from **Q**, then $(x, \mathbf{Q})$ is ovoidal. Since $W(3)$ and $Q(5,3)$ have no ovoids, **Q** is not isomorphic to one of these GQ's. Now, let L be a line through x having a point at distance 1 from **Q**, then L determines a rosette of ovoids, see Theorem 3.7.1. Hence **Q** is not isomorphic to $T_2^*(H)$. □

Corollary 8.2.3
If a quad **Q** *of* **S** *is isomorphic to* $Q(4,3)$*, then no line of* **S** *is contained in* $\Gamma_2(\mathbf{Q})$*.*

Proof. If the line L would be contained in $\Gamma_2(\mathbf{Q})$, then this line determines a fan of ovoids, see Theorem 3.7.1; a contradiction. □

8.3 The classification for some special cases

In this section we give the classification for some special cases. These results will be used during the classification in the following section and it will turn out that all known examples belong to one of the three special cases considered here.

8.3.1 The case of classical near hexagons

Supppose **S** is a classical near hexagon. By the classification of classical near polygons, see Section 3.6, it follows that **S** is one of the classical near hexagons mentioned in Section 8.1.

8.3.2 The case of glued near hexagons

Suppose now that $\mathbf{S}$ is a glued near hexagon arising from the GQ's $\mathbf{Q}_1$ and $\mathbf{Q}_2$, the spreads S_1 and S_2, the base lines $L_1^{(1)}$ and $L_1^{(2)}$, and the map $\theta_2 : L_1^{(1)} \rightarrow L_1^{(2)}$ (as explained in Chapter 7, we may suppose that θ_1 is the identical permutation of $L_1^{(1)}$). From the classification results obtained in Chapter 7 we know that $\mathbf{S}$ is one of the following examples.

(1) $\mathbf{Q}_1$ is a (4×4)-grid and $\mathbf{Q}_2$ is a GQ of order $(3, t)$ not isomorphic to $Q(4, 3)$ (this GQ has no spread). In this case $\mathbf{S}$ is the direct product of $\mathbf{Q}_2$ with a line of size 4.

(2) $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are isomorphic to $T_2^*(H)$. Up to an automorphism of $T_2^*(H)$, there is a unique choice for S_1 and S_2. From Corollary 7.5.4 it follows that there is a unique example (up to isomorphism). This example is the nonclassical near hexagon (I) given on page 155.

(3) $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are isomorphic to $Q(5, 3)$. Up to an automorphism of $Q(5, 3)$ there is a unique choice for S_1 and S_2. From Section 7.5, it follows that there is a unique example. With the same notations as in Section 8.1, we can define a graph Γ' with vertex set $K \times U$. Two different vertices (k_1, α_1) and (k_2, α_2) are adjacent if and only if the following conditions are satisfied:

   (a) $\alpha_1 = \alpha_2$;

   (b) $\alpha_1 \neq \alpha_2$ and $k_2 = k_1 \Delta(\alpha_1, \alpha_2)$.

   Every two adjacent vertices of Γ' are contained in a unique maximal clique. The geometry, with points the vertices of Γ', with lines the maximal cliques of Γ', and with natural incidence, is then a generalized quadrangle $\mathbf{Q}$ isomorphic to $Q(5, 3)$, see Section 2.5.4. If we put $L_\alpha = \{(k, \alpha) | k \in K\}$ for every $\alpha \in U$, then $S = \{L_\alpha | \alpha \in U\}$ is a spread of $\mathbf{Q}$. The unique glued near hexagon is then obtained by putting

   - $\mathbf{Q}_1 = \mathbf{Q}_2 = \mathbf{Q}$;
   - $S_1 = S_2 = S$;
   - $L_1^{(1)} = L_1^{(2)} = L_{\langle \bar{a} \rangle}$;
   - θ_2 the identical map of $L_1^{(1)}$.

   This example is the nonclassical near hexagon (II) given on page 155.

8.3.3 The case of regular near hexagons

In Chapter 4, we already gave the following classification result.

Theorem 8.3.1 ([80])
If $\mathbf{S}$ *is a regular near hexagon with parameters* $s = 3$, $t_2 > 0$ *and* t, *then one of the following possibilities occur:*

(1) $t_2 = 1$ *and* $t = 2$,	(2) $t_2 = 3$ *and* $t = 12$,
(3) $t_2 = 9$ *and* $t = 90$,	(4) $t_2 = 1$ *and* $t = 9$,
(5) $t_2 = 1$ *and* $t = 34$,	(6) $t_2 = 3$ *and* $t = 27$,
(7) $t_2 = 3$ *and* $t = 48$.	

The parameters given in (1), (2) and (3) satisfy $t = t_2^2 + t_2$ and hence correspond to classical near hexagons (see Corollary 3.7.5). It is known that no near hexagon exists with parameters as in (4) (see [10]). It is also known that there are no near hexagons with parameters as in (6) or (7), but this will also follow from the treatment given here. Whether there exists a regular near hexagon with parameters $(s, t, t_2) = (3, 34, 1)$ is still an open problem.

8.4 The classification

By Theorem 3.8.1 every point of $\mathbf{S}$ is incident with the same number of lines, say $t + 1$ lines. We distinguish between two cases.

8.4.1 Case I: all quads of type $Q(4,3)$ are big

In this section we suppose that all quads of type $Q(4,3)$ are big. We first prove the following lemma. We refer to the definition of the linear spaces $\mathbf{S}_{u,v}$ given in Section 7.6.1, page 149.

Lemma 8.4.1
If $\mathbf{G}$ *is a linear space satisfying the following properties:*

(a) $\mathbf{G}$ *has at least two lines,*

(b) *every line has size* 2, 4, 6 *or* 10,

(c) *if two lines are disjoint, then they have both size* 2,

then $\mathbf{G}$ *is one of the following examples:*

(1) K_n *with* $n \geq 3$,

(2) *a projective plane of order* 3, 5 *or* 9,

(3) $\mathbf{S}_{t_1,t_2}$ *with* $t_1, t_2 \in \{1, 3, 5, 9\}$.

Proof.

- If all lines are thin, then we have example (1). If all lines have the same size $\alpha+1 \neq 2$, then (a), (b) and (c) imply that $\mathbf{G}$ is a projective plane of order 3, 5 or 9.

- Suppose that the line sizes $\alpha+1$ and $\beta+1$ occur with $2 < \alpha+1 < \beta+1$. Let L_1 and L_2 be two lines of sizes $\alpha+1$ and $\beta+1$ respectively. If a is a point not on L_1 (respectively L_2), then there are exactly $\alpha+1$ (respectively $\beta+1$) lines through a. Since $\alpha+1 \neq \beta+1$, all points are contained in $L_1 \cup L_2$ and hence $\mathbf{G} \simeq \mathbf{S}_{\alpha,\beta}$.

- Suppose now that 2 and $\alpha+1 \in \{4, 6, 10\}$ are the only line sizes that occur. Consider a line L of size $\alpha+1$ and let N be the set of points not on L. If $|N| = 1$, then $\mathbf{G} \simeq \mathbf{S}_{1,\alpha}$. If $|N| > 1$, then $|N| \geq \alpha$. If $|N| = \alpha$, then $\mathbf{G} \simeq \mathbf{S}_{\alpha,\alpha}$. Suppose therefore that $|N| > \alpha$. Let $L' \neq L$ be any line of size $\alpha+1$ and let $p \notin L \cup L'$. Through p there are α or $\alpha+1$ lines of size $\alpha+1$. This implies that $|N| = \alpha^2 - \alpha + 1$ or $|N| = \alpha^2$. If $|N| = \alpha^2$, then every line of $\mathbf{G}$ has size $\alpha+1$, a contradiction. If $|N| = \alpha^2 - \alpha + 1$, then $\mathbf{G}$ induces a linear space $\mathbf{A}$ on the set N which is an $S(2, \alpha, \alpha^2 - \alpha + 1)$ and hence a projective plane of order $\alpha - 1$. The linear space $\mathbf{A}$ contains $\alpha^2 - \alpha + 1$ lines and every such line determines a point on L. If L_1 and L_2 are two different lines of $\mathbf{A}$, then they determine different points on L, hence $\alpha^2 - \alpha + 1 < \alpha + 1$, a contradiction.

□

Lemma 8.4.2
Every local space of $\mathbf{S}$ *satisfies* (a), (b) *and* (c) *from the previous lemma.*

Proof. Clearly (a) and (b) are satisfied. Let $\mathbf{Q}_1$ and $\mathbf{Q}_2$ be two quads intersecting only in a point x. Every point of $\mathbf{Q}_2$ at distance 2 from x has distance 2 from $\mathbf{Q}_1$. Our assumption and Corollary 8.2.2 imply then that $\mathbf{Q}_1$ is the (4×4)-grid. A similar reasoning proves that $\mathbf{Q}_2$ is the (4×4)-grid. □

Corollary 8.4.3
If $\mathbf{S}$ *contains a local space not isomorphic to a* K_n *with* $n \geq 3$*, then* $t+1 \in \{5, 7, 9, 11, 13, 15, 19, 31, 91\}$.

Lemma 8.4.4
All local spaces of $\mathbf{S}$ *are isomorphic.*

Proof. If a local space is isomorphic to $\mathbf{S}_{1,\alpha}$ with $\alpha \in \{1,3,5,9\}$, then by Theorem 7.6.1 **S** is the direct product of a GQ with a line and the lemma is true in this case. We therefore suppose that no local space is isomorphic to a $\mathbf{S}_{1,\alpha}$ for some $\alpha \in \{1,3,5,9\}$. We may also suppose that there exists a point x for which $\mathbf{S}_x$ is not isomorphic to K_{t+1}. These assumptions imply that $t+1 \neq 5$. If $t+1$ is equal to 7, 9, 11, 15, 19, respectively 31, then $\mathbf{S}_x \simeq \mathbf{S}_{3,3}$, $\mathbf{S}_{3,5}$, $\mathbf{S}_{5,5}$, $\mathbf{S}_{5,9}$, $\mathbf{S}_{9,9}$, respectively $\mathrm{PG}(2,5)$. If y is a point collinear with x, then $\mathbf{S}_y$ is not isomorphic to K_{t+1}. Hence $\mathbf{S}_y \simeq \mathbf{S}_x$ and the result follows by connectedness of **S**. If $t+1 = 91$, then a similar reasoning yields that all local spaces are projective planes of order 9 and **S** is then classical by Theorem 3.6.2; hence all local spaces are isomorphic to $\mathrm{PG}(2,9)$ (see Section 8.3.1). If $t+1 = 13$, then $\mathbf{S}_x \simeq \mathbf{S}_{3,9}$ or $\mathbf{S}_x \simeq \mathrm{PG}(3,2)$. If y is a point collinear with x, then $\mathbf{S}_y \simeq \mathbf{S}_x$ and the result follows once again by connectedness of **S**. $\square$

The following theorem completes the classification (for Case I), up to the open case appearing in the regular case.

Theorem 8.4.5
S *is a regular, a classical or a glued near hexagon.*

Proof. If all local spaces are isomorphic to K_{t+1}, then **S** is regular with $t_2 = 1$. If all local spaces are (possible degenerate) projective planes, then **S** is a classical near hexagon by Theorem 3.6.2. If all local spaces are isomorphic to $\mathbf{S}_{u,v}$ with $u, v \geq 2$, then **S** is glued by Theorem 7.6.2. $\square$

8.4.2 Case II: there is a quad of type $Q(4,3)$ which is not big

We suppose that there is a quad of type $Q(4,3)$ which is not big. If v is the number of vertices of **S**, then $v > 40 + 120(t-3)$, hence every quad of type $W(3)$ or $Q(4,3)$ is not big. Since $W(3)$ has no ovoids, quads of type $W(3)$ do not occur. Some of the following lemmas are just adapted versions of results mentioned in [14].

Lemma 8.4.6
Let x be a point at distance 2 from a quad **Q** *of type $Q(4,3)$, then x is contained in $\alpha = \frac{39-t}{2}$ grids meeting* **Q** *and $\beta = \frac{t-19}{2}$ quads of type $Q(4,3)$ meeting* **Q**. *Hence $19 \leq t \leq 39$ and t is odd.*

Proof. The point x determines an ovoid O of **Q**. Let o and o' be two different points of O. Let $\mathbf{Q}_1$ be the quad through x and o. The point o' is ovoidal with respect to $\mathbf{Q}_1$. Hence $\mathbf{Q}_1$ is either a (4×4)-grid or a GQ isomorphic to

$Q(4,3)$, see Corollary 8.2.2. As a consequence, every quad through x meeting $\mathbf{Q}$ determines 2 or 4 lines through x. It is impossible that a line through x is contained in at least two such quads. If a line through x is not contained in one such quad, then this line determines a fan of ovoids in $\mathbf{Q}$, contradicting Corollary 8.2.3. The lemma follows now from the following equalities:

$$\begin{aligned} \alpha + \beta &= 10, \\ 2\alpha + 4\beta &= t+1. \end{aligned}$$

□

Lemma 8.4.7
There are no quads of type $W(3)$, $T_2^*(H)$ and $Q(5,3)$.

Proof. As we already mentioned, quads of type $W(3)$ cannot occur. Suppose that a quad of type $Q(5,3)$ occurs. Consider a local space $\mathbf{S}_x$ with a line L of size 10 and let p be a point of $\mathbf{S}_x$ not on L. Every line through p meets L and contains an even number of points. Since there are 10 such lines the number t must be even, contradicting Lemma 8.4.6. A totally similar argument yields that there are no quads of type $T_2^*(H)$. □

Lemma 8.4.8
There are constants a and b such that each point is contained in a grids and b quads of type $Q(4,3)$. One has that

$$a = \frac{t(t+1)}{2} - 6b$$

and

$$v = 18t^2 - 6t + 4 - 108b.$$

Proof. Let x be a any point of $\mathbf{S}$ and suppose that x is contained in a grids and b quads of type $Q(4,3)$. There are $9a + 27b$ points at distance 2 from x and $v - 1 - 3(t+1) - 9a - 27b$ points at distance 3 from x. Counting triples $(L_1, L_2, \mathbf{Q})$, where L_1 and L_2 are two different lines through x and where $\mathbf{Q}$ is a quad through L_1 and L_2, yields $t(t+1) = 2a + 12b$ or $a = \frac{t(t+1)}{2} - 6b$. Counting pairs (y,z) with $\mathrm{d}(y,z) = 1$, $\mathrm{d}(x,y) = 2$ and $\mathrm{d}(x,z) = 3$ yields

$$((v-1) - 3(t+1) - 9a - 27b)(t+1) = 27a(t-1) + 81b(t-3).$$

From $a = \frac{t(t+1)}{2} - 6b$, it follows that $v = 18t^2 - 6t + 4 - 108b$. □

Lemma 8.4.9
$t \neq 39$

Proof. Suppose $t = 39$ and let x be any point of $\mathbf{S}$. The local space $\mathbf{S}_x$ has lines of cardinality two or four. We will call them 2-lines, respectively 4-lines. By Lemma 8.4.6, a 2-line and a 4-line of $\mathbf{S}_x$ always intersect. Now, take a 4-line L and a point u not on L. Through u, there is a 4-line M disjoint with L. Let v be a point not on $L \cup M$ and let N be a 4-line through v not meeting $L \cup M$. Any 2-line of $\mathbf{S}_x$ will meet L, M and N; hence there are only 4-lines. $\mathbf{S}$ is then a regular near hexagon with parameters $s = 3$, $t_2 = 3$ and $t = 39$, contradicting Theorem 8.3.1. □

Lemma 8.4.10
If $t = 19$, then $b \leq 13$.

Proof. Let x be any point of $\mathbf{S}$. From Lemma 8.4.6, it follows that every two different 4-lines of the local space $\mathbf{S}_x$ always meet. Consider now two different 4-lines L_1 and L_2 and let p be their common point. If all 4-lines go through p, then $b \leq 6$. If there exists a line L not through p, then there are at most four 4-lines through p (since every such line meets L). Since every 4-line of $\mathbf{S}_x$ meets L_1 and L_2, we have that $b \leq 4 + 3 \times 3 = 13$. □

Let N_2 be the number of points which are ovoidal with respect to a quad of type $Q(4,3)$; one easily checks that

$$N_2 = 18(t^2 - 7t + 18 - 6b).$$

Lemma 8.4.11
Let L be a line of size 4 in the local space $\mathbf{S}_x$. If $t \neq 21$, then there are $\frac{1}{2160}(t-19)N_2$ lines of size 4 and $\frac{1}{720}(39-t)N_2$ lines of size 2 disjoint with L; hence these numbers are integers.

Proof. (a) Let Q be the quad corresponding with L. For a point z of Q, let A_z be the set of quads of type $Q(4,3)$ intersecting Q only in the point z. Let y be a point of Q at distance 2 from x. Counting pairs (R,S) with $R \in A_x, S \in A_y$ and $|R \cap S| = 1$ yields $|A_x|\frac{t-21}{2} = |A_y|\frac{t-21}{2}$. Hence $|A_x| = |A_y|$. By connectedness of the noncollinearity graph of $Q(4,3)$ it follows that $|A_z| = |A_x|$ for all points z of Q. Since there are $\frac{1}{27}(\frac{t-19}{2})N_2$ quads of type $Q(4,3)$ intersecting Q in only one point, it follows that $|A_x| = \frac{1}{2160}(t-19)N_2$.
(b) The numbers $|\Gamma_2(z)|$, $|\Gamma_2(z) \cap Q|$ and $|\Gamma_2(z) \cap \Gamma(Q)|$ are independent of $z \in Q$. Hence also $|\Gamma_2(z) \cap \Gamma_2(Q)|$ is independent of $z \in Q$. Hence $|\Gamma_2(x) \cap \Gamma_2(Q)| = \frac{1}{4}N_2$. Counting $|\Gamma_2(x) \cap \Gamma_2(Q)|$ in another way yields $27\frac{1}{2160}(t-19)N_2 + 9\delta = \frac{1}{4}N_2$, where δ is the number of 2-lines disjoint from L. Hence $\delta = \frac{1}{720}(39-t)N_2$. □

Corollary 8.4.12
$t \neq 23$, $t \neq 29$ and $t \neq 35$

Proof. If $t \neq 21$, then $\frac{1}{2160}(t-19)N_2 = \frac{1}{60}(t-19)[\frac{t(t-7)}{2}+9-3b] \in \mathbb{N}$; hence $3 \mid t(t-19)(t-7)$, from which the corollary follows. $\square$

Lemma 8.4.13

$$\frac{20t(t+1)-(39-t)(t^2-7t+18)}{6(t+1)} \geq b \geq \frac{(t-19)(t^2-7t+18)+120}{6(t+1)}$$

Proof. Let Q be a quad of type $Q(4,3)$. There are $\frac{1}{54}(t-19)N_2$ quads of type $Q(4,3)$ which intersect Q in exactly one point; this number is at most $40(b-1)$, from which the lower bound for b follows. There are $\frac{1}{18}(39-t)N_2$ grids which intersect Q in exactly one point; hence this number is at most $40a$, from which the upper bound for b follows. $\square$

We also have the following lower bound for b.

Lemma 8.4.14

$$b \geq \frac{(t+1)(t^3-26t^2+151t-702)}{6(t^2+2t-319)}$$

Proof. Let x be any point of $\mathbf{S}$. For every point p of the local space $\mathbf{S}_x$, let α_p denote the number of 4-lines through the point p. Elementary counting yields

$$\begin{aligned} \sum 1 &= t+1, \\ \sum \alpha_p &= 4b, \\ \sum \alpha_p(\alpha_p-1) &= b[(b-1)-\frac{1}{2160}(t-19)N_2], \end{aligned}$$

where the summation ranges over all points p of $\mathbf{S}_x$. The inequality $\sum(\alpha_p - \frac{4b}{t+1})^2 \geq 0$ yields the bound for b. $\square$

Remark. If $t \neq 19, 21$, then the lower bound of b given in Lemma 8.4.14 is stronger then the one given in Lemma 8.4.13. Collecting the results of the above lemmas, we find the following bounds for b:

- $t = 19$: $1 \leq b \leq 13$;
- $t = 21$: $6 \leq b \leq 27$;
- $t = 25 : 30 \leq b \leq 41$;

- $t = 27 : 42 \leq b \leq 50$;
- $t = 31 : 67 \leq b \leq 71$;
- $t = 33 : 81 \leq b \leq 84$;
- $t = 37 : b = 113$.

Note that in each case the number of grids is equal to $\frac{va}{16}$ while the number of quads of type $Q(4,3)$ is equal to $\frac{vb}{40}$. Hence these numbers are integers.

Lemma 8.4.15
The following congruences hold.

- *If* $t \neq 19, 21$, *then* $b \equiv t^2 - 2t + 3 \pmod 5$.
- *If* $t = 19$ *or* $t = 21$, *then* $b \equiv 0 \pmod 5$ *or* $b \equiv t^2 - 2t + 3 \pmod 5$.
- *If* $t \equiv 1 \pmod 8$, *then* $b \equiv 0 \pmod 4$.
- *If* $t \equiv 3 \pmod 8$, *then* b *is odd.*
- *If* $t \equiv 5 \pmod 8$, *then* $b \equiv 2 \pmod 4$.

Proof. From $\frac{vb}{40} \in \mathbb{N}$, it follows that $b(3t^2 - t - 1 - 3b) \equiv 0 \pmod 5$. Hence $b \equiv 0 \pmod 5$ or $6t^2 - 2t - 2 - 6b \equiv t^2 - 2t + 3 - b \equiv 0 \pmod 5$. If $t \neq 19, 21$, then $(t - 19)N_2 \equiv 0 \pmod 5$ and, since $t \neq 29, 39$, we even can say $N_2 \equiv 0 \pmod 5$, from which it follows that $b \equiv t^2 - 2t + 3 \pmod 5$. From $\frac{va}{16} \in \mathbb{N}$, it follows that $(\frac{9t^2-3t}{2} + 1 - 27b)(\frac{t(t+1)}{2} - 6b) \equiv 0 \pmod 4$ from which the remaining congruences readily follow. $\square$

Corollary. From the above lemmas there remain the following possibilities:

(1) $(t, b) = (19, 1)$;

(2) $(t, b) = (19, 5)$;

(3) $(t, b) = (19, 11)$;

(4) $(t, b) = (21, 10)$;

(5) $(t, b) = (21, 22)$;

(6) $(t, b) = (27, 43)$;

(7) $(t, b) = (31, 67)$.

The quad-quad relation

For a quad Q of $\mathbf{S}$, let $\Gamma_{\leq 1}(Q) := Q \cup \Gamma_1(Q)$. Now, let Q and Q' be two quads. We will summarize the possibilities of $\Gamma_{\leq 1}(Q) \cap Q'$ regarded as substructure of Q'. If a line of Q' meets two points of $\Gamma_{\leq 1}(Q)$, then it is completely contained in $\Gamma_{\leq 1}(Q)$, see Theorem 3.7.1.

(1) Suppose that Q and Q' are quads of type $Q(4,3)$. Since $Q(4,3)$ has no fan of ovoids, every line of $\mathbf{S}$ meets $\Gamma_{\leq 1}(Q)$. Hence the following possibilities may occur (see also Theorem 2.3.1 of [71]):

(A) $\Gamma_{\leq 1}(Q) \cap Q'$ is an ovoid of Q';

(B) $\Gamma_{\leq 1}(Q) \cap Q'$ consists of the four lines through a fixed point of Q';

(C) $\Gamma_{\leq 1}(Q) \cap Q'$ is a grid of Q';

(D) $\Gamma_{\leq 1}(Q) \cap Q' = Q'$.

(2) Suppose that Q is a quad of type $Q(4,3)$ and that Q' is a grid. With a same reasoning as above, we have the following possibilities for $\Gamma_{\leq 1}(Q) \cap Q'$;

(A) $\Gamma_{\leq 1}(Q) \cap Q'$ is an ovoid of Q';

(B) $\Gamma_{\leq 1}(Q) \cap Q'$ consists of the two lines through a fixed point of Q';

(C) $\Gamma_{\leq 1}(Q) \cap Q' = Q'$.

(3) Suppose that Q and Q' are two grids. We find the following possibilities for $\Gamma_{\leq 1}(Q) \cap Q'$:

(A) $\Gamma_{\leq 1}(Q) \cap Q'$ is a set of i noncollinear points and $i \in \{0, 1, 2, 4\}$;

(B) $\Gamma_{\leq 1}(Q) \cap Q'$ is a line of Q';

(C) $\Gamma_{\leq 1}(Q) \cap Q'$ consists of two intersecting lines;

(D) $\Gamma_{\leq 1}(Q) \cap Q' = Q'$.

The possibility that $\Gamma_{\leq 1}(Q) \cap Q'$ is a set of three noncollinear points is ruled out be a reasoning which one can find in [10].

Theorem 8.4.16
The following cases cannot occur:

(A) $(t,b)=(19,1)$,

(B) $(t,b)=(19,11)$,

(C) $(t,b)=(21,22)$,

(D) $(t,b)=(31,67)$.

Proof.

(A) The case $(t,b)=(19,1)$

Let Q be any quad of type $Q(4,3)$. If G is a grid of $\mathbf{S}$, then $|G\cap\Gamma_2(Q)|\in\{0,9,12\}$. Let M_i, $i\in\{0,9,12\}$, be the number of grids G for which $|G\cap\Gamma_2(Q)|=i$. Through every point of $\Gamma_1(Q)$ there is one line meeting Q, four lines contained in $\Gamma_1(Q)$ and 15 other lines. Hence $M_{12}\le\frac{1}{4}|\Gamma_1(Q)|\frac{15\cdot 14}{2}=50400$. There are $10a-\frac{(39-t)N_2}{72}=640$ grids intersecting Q in a line and $|Q|\frac{(t-3)(t-4)}{2}=4800$ grids intersecting Q in only one point. Counting triples (G,L_1,L_2), where $L_1,L_2\subseteq\Gamma_1(Q)$ are two intersecting lines of the grid G, yields $16(M_0-640-1)+(M_9-4800)\le|\Gamma_1(Q)|\frac{4\cdot 3}{2}=11520$. Hence $M_9\le 16320$ and $M_0\le 1361$. Hence there are at most (50400+16320+1361)=68081 grids, a contradiction, since there are exactly $\frac{va}{16}=72220$ grids.

(B) The case $(t,b)=(19,11)$

Let x be a point of $\mathbf{S}$ and consider the local space $\mathbf{S}_x$. By Lemma 8.4.11 every two 4-lines meet. We will suppose now that there is no point which is incident with exactly three 4-lines and derive a contradiction. There are at most four 4-lines through every point p of $\mathbf{S}_x$. Indeed, there exists a 4-line not containing p and every 4-line through p meets this line. Now, let $M=\{m_1,m_2,m_3,m_4\}$ be a fixed 4-line. Since there are exactly eleven 4-lines, we may suppose that m_i, $i\in\{1,2,3\}$, is incident with four 4-lines and that m_4 is incident with two 4-lines, say M and T. Let L be a 4-line through m_1, different from M and let r be a point of L not on M or T. Every 4-line through r meets M. The line through r and m_i, $i\in\{1,2,3\}$, is a 4-line and the line through r and m_4 is a 2-line, a contradiction. Hence there exists a line K through x, which is contained in exactly three quads of type $Q(4,3)$, say Q_1, Q_2 and Q_3. Let Q_4 be another quad of type $Q(4,3)$ through x and let y be a point of K different from x. There are 10 lines through y contained in $\Gamma_1(Q_4)$; one of these lines, say S, is not contained in $Q_1\cup Q_2\cup Q_3$. Every quad of type $Q(4,3)$ through y, but not through K, meets Q_i, $i\in\{1,2,3\}$, in a line. Hence at least three of its lines through y are contained in $\Gamma_1(Q_4)$.

From the quad-quad relation, it follows then that also the fourth line through y is contained in $\Gamma_1(Q_4)$ and must coincide with S. Hence, there hence are at most 3+3 quads of type $Q(4,3)$ through y, a contradiction.

(C) The case $(t,b) = (21,22)$

For a quad Q of type $Q(4,3)$ and a point $x \in Q$, let $N_{x,Q}$ denote the number of quads of type $Q(4,3)$ intersecting Q only in the point x. We have

$$\sum_Q \sum_x N_{x,Q} = \sum_Q \frac{1}{54}(t-19)N_2 = \frac{vb(t-19)N_2}{2160} = 3vb. \tag{8.1}$$

On the other hand $\sum_Q \sum_x N_{x,Q} = \sum_x \sum_Q N_{x,Q}$. Now, let x be fixed and consider the local space $\mathbf{S}_x$. For a 4-line $L = \{l_1, l_2, l_3, l_4\}$ of $\mathbf{S}_x$, we define $\alpha_L = \alpha_{l_1} + \alpha_{l_2} + \alpha_{l_3} + \alpha_{l_4}$, with α_{l_i} the number of 4-lines through l_i in $\mathbf{S}_x$.

Suppose that there exists a 4-line $L = \{l_1, l_2, l_3, l_4\}$ of $\mathbf{S}_x$ for which $\alpha_L \geq 23$. Let $M \neq L$ be one of the $(\alpha_L - 4)$ 4-lines meeting L, say in the point l_1. There are at least

$$(\alpha_{l_2} - 4) + (\alpha_{l_3} - 4) + (\alpha_{l_4} - 4) = \alpha_L - \alpha_{l_1} - 12 \geq 23 - 7 - 12 = 4$$

4-lines disjoint with M. Hence

$$\sum_Q N_{x,Q} \geq (\alpha_L - 4)4 \geq 76 > 3b. \tag{8.2}$$

Suppose that $\alpha_L \leq 22$ for all 4-lines L of $\mathbf{S}_x$. For every line L of $\mathbf{S}_x$, there are $(25 - \alpha_L)$ 4-lines disjoint with L. Hence

$$\sum_Q N_{x,Q} = \sum_L (25 - \alpha_L) \geq 3b. \tag{8.3}$$

From equations (8.1), (8.2) and (8.3) it follows that $\alpha_L = 22$ for all 4-lines L of $\mathbf{S}_x$. Once again let $L = \{l_1, l_2, l_3, l_4\}$ be a 4-line of $\mathbf{S}_x$ and let $M \neq L$ be one of the 4-lines meeting L, say in l_1. There are at least

$$(\alpha_{l_2} - 4) + (\alpha_{l_3} - 4) + (\alpha_{l_4} - 4) = 10 - \alpha_{l_1}$$

4-lines disjoint with M. Hence $\alpha_{l_1} = 7$. For similar reasons, we may suppose that $\alpha_{l_2} = \alpha_{l_3} = 7$ and $\alpha_{l_4} = 1$. Hence, every 4-line contains a unique point p for which $\alpha_p = 1$. Since there are 22 4-lines and 22 points, all points p of $\mathbf{S}_x$ satisfy $\alpha_p = 1$, a contradiction.

(D) The case $(t, b) = (31, 67)$

Consider a local space $\mathbf{S}_x$. With the same notations as above we have

$$\begin{aligned} \sum 1 &= 32, \\ \sum \alpha_p &= 268, \\ \sum \alpha_p(\alpha_p - 1) &= 2010, \end{aligned}$$

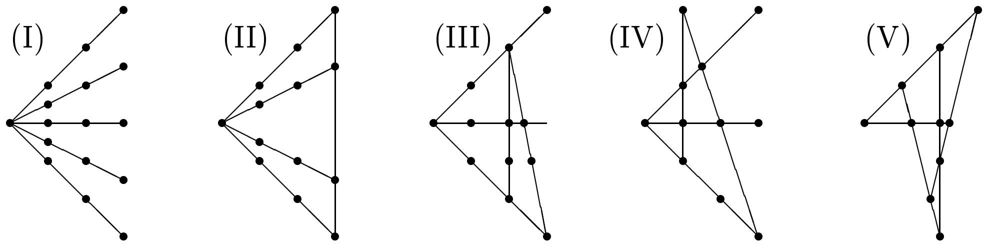
where the summation ranges over all points p of $\mathbf{S}_x$. For every point p of $\mathbf{S}_x$, we have $\alpha_p \leq 10$. Suppose $\alpha_p = 10$ for some point p of $\mathbf{S}_x$; then there exists a unique point q such that p and q are not contained in a 4-line. For every line $\{p, p_1, p_2, p_3\}$ through p, $\alpha_{p_1} + \alpha_{p_2} + \alpha_{p_3} = 24$. As a consequence $\alpha_q = 268 - 10 \times 24 - 10 = 18$, contradicting $\alpha_q \leq 10$. Hence $\alpha_p \leq 9$ for every point p of $\mathbf{S}_x$. Since the average value of the α's is $\frac{67}{8}$, there exists a point p such that $\alpha_p = 9$. Let q_i, $i \in \{1, 2, 3, 4\}$, be those points of $\mathbf{S}_x$ such that p and q_i are not contained in a 4-line. As before, one calculates that $\alpha_{q_1} + \alpha_{q_2} + \alpha_{q_3} + \alpha_{q_4} = 34$; hence $\alpha_{q_i} \in \{7, 8, 9\}$ for all $i \in \{1, 2, 3, 4\}$. For a line $\{p, p_1, p_2, p_3\}$ through p, $\alpha_{p_1} + \alpha_{p_2} + \alpha_{p_3} = 25$; hence $\alpha_{p_i} \in \{7, 8, 9\}$ for all $i \in \{1, 2, 3\}$. With N_i being the number of points r of $\mathbf{S}_x$ for which $\alpha_r = i$, $i \in \{7, 8, 9\}$, we get

$$\begin{aligned} N_7 + N_8 + N_9 &= 32, \\ 7N_7 + 8N_8 + 9N_9 &= 268, \\ 42N_7 + 56N_8 + 72N_9 &= 2010. \end{aligned}$$

Hence, $N_7 = 13$, $N_8 = -6$ and $N_9 = 25$, a contradiction. $\square$

Remark on the case $(t, b) = (19, 5)$

We want to make some remarks about the local spaces for this case. If x is a point of $\mathbf{S}$, then any two 4-lines of the local space $\mathbf{S}_x$ always meet. These 4-lines induce one of the following configurations.



Suppose that Q is a quad of type $Q(4, 3)$. It is impossible that all local spaces $\mathbf{S}_y$, $y \in Q$, are of type (I). Otherwise, the lines of Q which are contained in five quads of type $Q(4,3)$ determine a spread of Q, but $Q(4,3)$ has no spread. We will prove now that no local space is of type (III). Let L be a line of $\mathbf{S}$ which is contained in exactly three quads of type $Q(4,3)$, say Q_1, Q_2 and Q_3. We prove that $\mathbf{S}_x$ is of type (IV) for every point x of L. Let y be a second point of L and let Q_4 be a quad of type $Q(4,3)$ through y, different from Q_1, Q_2 and Q_3. There are 10 lines through x contained in $\Gamma_1(Q_4)$; one of these lines, say K, is not contained in Q_1, Q_2 or Q_3. Let Q_5 be one of the two quads through x different from Q_1, Q_2 and Q_3. At least three of the lines of Q_5 through x are contained in $\Gamma_1(Q_4)$. From the quad-quad relation, it follows that all four lines through x are contained in $\Gamma_1(Q_4)$. Hence the line K is contained in two quads of type $Q(4,3)$, which proves that $\mathbf{S}_x$ is of type (IV).

8.4.3 The main theorem

By collecting the previous results we can now state the main theorem for this chapter.

Theorem 8.4.17
Let $\mathbf{S}$ be a near hexagon satisfying the following properties:

(1) *all lines of $\mathbf{S}$ have 4 points;*

(2) *every two points at distance 2 have at least two common neighbours.*

We distinguish between two cases.

(A) *If $\mathbf{S}$ is classical or glued, then it is isomorphic to one of the ten examples described in Section 8.1.*

(B) *If $\mathbf{S}$ is not classical and not glued, then only quads isomorphic to the (4×4)-grid or to $Q(4,3)$ occur. Moreover, there are numbers a and b such that every point of $\mathbf{S}$ is contained in a grids and b quads isomorphic to $Q(4,3)$. Every point is contained in the same number of lines, say $t+1$ lines. We have then the following possibilities for t, a, b and v (= the number of points):*

- $v = 5848,\ t = 19,\ a = 160,\ b = 5;$
- $v = 6736,\ t = 21,\ a = 171,\ b = 10;$
- $v = 8320,\ t = 27,\ a = 120,\ b = 43;$
- $v = 20608,\ t = 34,\ a = 595,\ b = 0.$

It is still an open problem whether there exist near hexagons with parameters as in (B) of the previous theorem.

Appendix A

The classification of ovoids in the near hexagon associated with $S(5, 8, 24)$

A.1 The problem

In Chapter 1, we defined the Steiner system $S(5, 8, 24)$, which is unique up to isomorphism. It has the following properties.

(1) Any two different blocks intersect in 0, 2 or 4 points.

(2) If two blocks are disjoint, then the complement of their union is again a block.

(3) If two blocks intersect in four points, then their symmetric difference is again a block.

As every t-design is also a $(t-1)$-design, the number of blocks λ_i through i points $(1 \leq i \leq 5)$ can be calculated, and we get the following table.

i	λ_i
1	253
2	77
3	21
4	5
5	1

In Chapter 3, we associated a regular near polygon **S** with this Steiner system. The points are the blocks of this Steiner system while the lines are the triples of disjoint blocks. There are $v = 759$ vertices. Two blocks have distance 1, 2, respectively 3, when they meet in 0, 4, respectively 2 points. An *ovoid* O of **S** is a set of points such that every line of **S** has exactly one point in common with O. **S** has *natural ovoids*, as the following theorem shows. In this theorem and further on, the set X will denote the set of 24 points of the Steiner system $S(5, 8, 24)$. If $i \in \mathbb{N}$, then $\binom{X}{i}$ denotes the set of all subsets of X which have order i.

Theorem A.1.1
If $x \in X$, then the set O_x of all blocks through x defines an ovoid of **S**.

Proof. Let L be any line of **S**. Exactly one of the three blocks of L contains x, which proves the theorem. $\square$

We will prove in this appendix that these natural ovoids are the only ovoids that can occur.

Theorem A.1.2
If O is an ovoid of the near hexagon **S**, *then $O = O_x$ for some point $x \in X$.*

This theorem has already been proved before in [16]. The proof there is short, but relies on results from [39]. W give here a totally different proof. Although the proof is longer, it is more self-contained and many steps are very similar.

A.2 The quads in S

As we already mentioned in Chapter 3, **S** is regular with parameters $(s, t, t_2) = (2, 14, 2)$. The quads are GQ's of order 2 and so isomorphic to $W(2)$. How does these quads appear in the near hexagon? Take therefore two blocks α_1 and α_2 at mutual distance 2 and let **Q** be the quad through these points. The set $\alpha_1 \cap \alpha_2$ (which has order 4) is contained in 5 blocks, say $\alpha_1, \ldots, \alpha_5$. Let $A = \{\alpha_1 \cap \alpha_2, \alpha_1 \setminus \alpha_2, \alpha_2 \setminus \alpha_1, \alpha_3 \setminus \alpha_1, \alpha_4 \setminus \alpha_1, \alpha_5 \setminus \alpha_1\}$. By the above mentioned property (3) of the Steiner system $S(5, 8, 24)$, the union of two elements of A is a block of $S(5, 8, 24)$. Let B be the set of 15 blocks arising this way. Clearly B is a subspace and any two different blocks of B have distance 1 or 2. What is the partial linear space induced by B? By Sylvesters model of $W(2)$, we know that B induces $W(2)$. Since B contains α_1 and α_2, it immediately follows that the 15 blocks in B are exactly the 15 points of the quad **Q**.

A.3 The classification of the ovoids

Let O be any ovoid of $\mathbf{S}$.

Lemma A.3.1
The ovoid O has 253 blocks. If α is a block of O, then $n_2 = |\Gamma_2(\alpha) \cap O| = 140$ and $n_3 = |\Gamma_3(\alpha) \cap O| = 112$.

Proof. The number of lines of $\mathbf{S}$ is equal to

$$\frac{v(t+1)}{s+1} = (t+1)|O|.$$

Hence $|O| = 253$. Counting the number of connections between points of $\Gamma_1(\alpha)$ and $\Gamma_2(\alpha) \cap O$ yields

$$|\Gamma_1(\alpha)|t = (t_2 + 1)|\Gamma_2(\alpha) \cap O|,$$

from which it follows that $n_2 = 140$ and $n_3 = |O| - 1 - n_2 = 112$. □

Lemma A.3.2
Four different elements of X are contained in 1 or 5 elements of O.

Proof. The five blocks through four elements of X form an ovoid O' in a quad $\mathbf{Q}$. Let O'' be the ovoid of $\mathbf{Q}$, induced by O, then O' and O'' have 1 or 5 elements in common (see the classification of the ovoids of $W(2)$ given in Section 2.6). □

Lemma A.3.3
Three different elements of X are contained in at least five elements and at most 21 elements of O.

Proof. Take three elements a, b, c of X. Let d_1 be a fourth element of X and let α be a block of O through $\{a, b, c, \mathrm{d}_1\}$. Let d_2 be an element of X not in α and let β be a block of O through $\{a, b, c, d_2\}$. There are now five blocks of O through $\alpha \cap \beta$. The upper bound is immediate since $\lambda_3 = 21$. □

Lemma A.3.4
Three different elements of X are contained in 5 or 21 elements of O.

Proof. Let $\alpha \in \binom{X}{3}$ be contained in t_α blocks of O. Double counting yields

$$\sum_{\alpha \in \binom{X}{3}} 1 \;=\; \binom{24}{3},$$

$$\begin{aligned}\sum_{\alpha \in \binom{X}{3}} t_\alpha &= |O|\binom{8}{3}, \\ \sum_{\alpha \in \binom{X}{3}} t_\alpha(t_\alpha - 1) &= |O| n_2 \binom{4}{3}.\end{aligned}$$

From the previous lemma and $\sum_\alpha (t_\alpha - 5)(21 - t_\alpha) = 0$, it follows that $t_\alpha = 5$ or $t_\alpha = 21$. □

Remarks.

(1) Let T_1, respectively T_2, be the number of $\alpha \in \binom{X}{3}$ for which $t_\alpha = 5$, respectively $t_\alpha = 21$. From

$$\begin{aligned}T_1 + T_2 &= \binom{24}{3}, \\ 5T_1 + 21T_2 &= |O|\binom{8}{3},\end{aligned}$$

it follows that $T_1 = \binom{23}{3}$ and $T_2 = \binom{23}{2}$.

(2) Let R_1, respectively R_2, denote the number of $\alpha \in \binom{X}{4}$, which are contained in exactly 1, respectively exactly 5, elements of O. From

$$\begin{aligned}R_1 + R_2 &= \binom{24}{4}, \\ R_1 + 5R_2 &= |O|\binom{8}{4},\end{aligned}$$

it follows that $R_1 = \binom{23}{4}$ and $R_2 = \binom{23}{3}$.

Lemma A.3.5
For every $A = \{p_1, p_2, p_3, p_4\} \subseteq X$, there exists an $i \in \{1, 2, 3, 4\}$ such that $A \setminus \{p_i\}$ is contained in exactly 5 blocks of O.

Proof. Suppose the lemma is not true. Let α and β be two different elements of O through A. Take two different points x and y in $\alpha \setminus \beta$ and let z be a point of $\alpha \cap \beta$. Let $\gamma \neq \alpha$ be a block of O through $\{x, y, z\}$, then γ intersects α in a fourth point u. Through $\{x, y, z, u\}$, there are five blocks of O. If $u \in \alpha \setminus \beta$, then $\alpha \setminus \{x, y, z, u\}$ is contained in only 1 block of O, a contradiction. As a consequence $u \in \alpha \cap \beta$. Take a point $v \in \beta \setminus \alpha$. Let D_1 be the block through $[(\alpha \cap \beta) - \{z\}] \cup \{v, y\}$ and D_2 be the block through $[(\alpha \cap \beta) - \{u\}] \cup \{v, x\}$. D_1 and D_2 are two blocks of the ovoid meeting in four points. The block $D_1 \triangle D_2$ is not an element of O, a contradiction, since the set $\{x, y, z, u\}$ is contained in it. □

Lemma A.3.6
If $\{p_1, p_2, p_3, p_4\} \subseteq X$ is contained in exactly 5 elements of O, then there are exactly three subsets of order 3 which are contained in 21 elements of O.

Proof. Counting pairs (A, B) with (i) $A \in \binom{X}{3}$, (ii) $B \in \binom{X}{4}$, (iii) $A \subseteq B$ and (iv) A is contained in 21 elements of O, yields

$$21\, T_2 = \sum k_B,$$

where the sum ranges over all $B \in \binom{X}{4}$, through which there are 5 blocks of O and k_B denotes the number of subsets of order 3, through which there are 21 blocks of O. From this it follows that $\bar{k}_B = 3$, and the result follows from the previous lemma. □

Lemma A.3.7
Every two elements x and y of X are contained in at least 21 elements and in at most 77 elements of O.

Proof. Let $z \in X$ be a third element of X. From the proof of Lemma A.3.3, it follows that there exists a fourth element $u \in X$ such that $\{x, y, z, u\}$ is contained in five elements of O. From the previous lemma, it follows that $\{x, y, z\}$ or $\{x, y, u\}$ is contained in 21 blocks of O. The upper bound is immediate since $\lambda_2 = 77$. □

Lemma A.3.8
Two different elements of X are contained in 21 or 77 blocks of O.

Proof. Let $\alpha \in \binom{X}{2}$ be contained in s_α blocks of O. Double counting yields

$$\begin{aligned}
\sum_{\alpha \in \binom{X}{2}} 1 &= \binom{24}{2}, \\
\sum_{\alpha \in \binom{X}{2}} s_\alpha &= |O| \binom{8}{2}, \\
\sum_{\alpha \in \binom{X}{2}} s_\alpha (s_\alpha - 1) &= |O| n_2 \binom{4}{2} + |O| n_3.
\end{aligned}$$

From the previous lemma and $\sum_\alpha (s_\alpha - 21)(77 - s_\alpha) = 0$, it follows that $s_\alpha = 21$ or $s_\alpha = 77$. □

Remark.
Let S_1, respectively S_2, be the number of $\alpha \in \binom{X}{2}$ for which $s_\alpha = 21$,

respectively $s_\alpha = 77$. From

$$\begin{aligned} S_1 + S_2 &= \binom{24}{2}, \\ 21S_1 + 77S_2 &= |O|\binom{8}{2}, \end{aligned}$$

it follows that $S_1 = \binom{23}{2}$ and $S_2 = \binom{23}{1}$.

Lemma A.3.9
For every $A = \{p_1, p_2, p_3\} \subseteq X$, there exists an $i \in \{1,2,3\}$ such that $A \setminus \{p_i\}$ is contained in exactly 21 blocks of O.

Proof. Suppose that all blocks through $\{p_1, p_2\}$ or through $\{p_2, p_3\}$ belong to O. Choose $q_1, q_2 \notin A$ and q_3 not in the unique block through $\{p_1, p_2, p_3, q_1, q_2\}$. Let B_1, respectively B_2, be the block through $\{p_1, p_2, q_1, q_2, q_3\}$, respectively $\{p_2, p_3, q_1, q_2, q_3\}$. Then $B_1 \triangle B_2 \notin O$, since $B_1, B_2 \in O$. Since $\{p_1, p_3\} \subseteq B_1 \triangle B_2$, the lemma is proved. □

Lemma A.3.10
If $\{p_1, p_2, p_3\} \subseteq X$ is contained in 21 blocks of O, then there are exactly two subsets of order 2 which are contained in 77 blocks of O.

Proof. Counting pairs (A, B) with (i) $A \in \binom{X}{2}$, (ii) $B \in \binom{X}{3}$, (iii) $A \subseteq B$ and (iv) A is contained in 77 elements of O, yields

$$22\, S_2 = \sum l_B,$$

where the sum ranges over all $B \in \binom{X}{3}$, through which there are 21 blocks of O and l_B denotes the number of subsets of order 2, through which there are 77 blocks of O. From this it follows that $\bar{l}_B = 2$, and the result follows from the previous lemma. □

Lemma A.3.11
There are at least 77 blocks of O through every point.

Proof. Let x denote the point. The element x is contained in a set $P \subseteq X$ of order 4, through which there are five elements of O. Let $P' \subseteq P$ be a set of order 3, containing x and contained in 21 blocks of O. Let $P'' \subseteq P'$ be a set of order 2, containing x and contained in 77 blocks of O. This proves the lemma. □

Lemma A.3.12
Every point of X is contained in 77 or 253 blocks of O.

Proof. Let $\alpha \in X$ be contained in μ_α blocks of O. Double counting yields

$$\begin{aligned}
\sum_{\alpha \in X} 1 &= \binom{24}{1}, \\
\sum_{\alpha \in X} \mu_\alpha &= |O| \binom{8}{1}, \\
\sum_{\alpha \in X} \mu_\alpha(\mu_\alpha - 1) &= |O| n_2 \binom{4}{1} + |O| n_3 \binom{2}{1}.
\end{aligned}$$

From the previous lemma and $\sum_\alpha (\mu_\alpha - 77)(253 - \mu_\alpha) = 0$, it follows that $\mu_\alpha = 77$ or $\mu_\alpha = 253$. □

Theorem A.3.13
There exists an $x \in X$, such that $O = O_x$.

Proof. It suffices to prove that there is a point x such that $\mu_x = 253$ (the same notation as in the previous lemma). Let U_1, respectively U_2, be the number of $x \in X$ for which $\mu_x = 77$, respectively $\mu_x = 253$. From

$$\begin{aligned}
U_1 + U_2 &= 24, \\
5U_1 + 21U_2 &= |O| 8,
\end{aligned}$$

it follows that $U_1 = 23$ and $U_2 = 1$. This proves the theorem. □

Samenvatting

1. Elementaire begrippen en eigenschappen

Dit hoofdstuk vangt aan met het begrip incidentiestructuur ([35]). De incidentiestructuren die in deze thesis behandeld worden, zijn steeds eindig verondersteld. Een *schier veelhoek* **S** is een incidentiestructuur die aan de volgende eigenschappen voldoet.

(1) Elke twee punten zijn bevat in ten hoogste een rechte, of equivalent, **S** is een *partiële lineaire ruimte*.

(2) Voor elk punt p en elke rechte L, bestaat er een uniek punt q op L dat het dichtst bij p gelegen is (met betrekking tot de afstand in de puntgraaf van de meetkunde).

Als d de diameter van de puntgraaf is, dan wordt de schier veelhoek een schier $2d$-hoek genoemd. Een schier 0-hoek bestaat dus enkel uit één punt en een schier 2-hoek bestaat uit één rechte met een aantal punten (≥ 2) erop. Er bestaat ook zoiets als een schier $(2d+1)$-hoek, dit is een incidentiestructuur met diameter d die voldoet aan (1), (2') en (3).

(2') Voor elk punt p en elke rechte L waarvoor $\mathrm{d}(p, L) < d$, bestaat er een uniek punt q op L dat het dichtst bij p gelegen is.

(3) Er bestaat een punt p en een rechte L waarvoor $\mathrm{d}(p, L) = d$.

De schier veelhoeken werden ingevoerd in [82] omwille van het verband met bepaalde rechtensystemen in de Euclidische ruimte. Voor een overzicht van wat gekend is over deze meetkunden refereren we naar het derde hoofdstuk van deze thesis of naar de referentie [35] die eveneens een uitgebreide literatuurlijst bevat. In dit inleidende hoofdstuk worden ook enkele definities en eigenschappen gegeven van andere klassen incidentiestructuren zoals de veralgemeende veelhoeken [98] en de projectieve ruimten.

2. Veralgemeende vierhoeken

Schier veelhoeken wiens puntgraaf diameter 2 hebben zijn beter gekend onder de naam van veralgemeende vierhoeken. Op enkele triviale uitzonderingen na hebben alle veralgemeende vierhoeken een *orde* (s,t), dit betekent dat elke rechte incident is met $s+1$ punten en dat elk punt incident is met $t+1$ rechten. Omdat deze incidentiestructuren zo belangrijk zijn in de theorie van de schier veelhoeken wordt een geheel hoofdstuk hieraan gewijd. Toch wordt niet teveel in detail gegaan, voor een uitgebreid overzicht van deze incidentiestructuren verwijzen we naar [71] en [89].

We beschrijven in dit hoofdstuk een nieuwe constructie voor veralgemeende vierhoeken welke gebruik maakt van de zogenaamde *toelaatbare drietallen*. Een drietal $(\mathbf{D}, K, \Delta)$ wordt toelaatbaar genoemd als aan de volgende drie voorwaarden is voldaan.

(1) $\mathbf{D}$ is een lineaire ruimte met orde $(s, t-1)$ (we noteren de puntenverzameling met $\mathcal{P}$).

(2) K is een groep van orde $s+1$ (multiplicatieve notatie).

(3) De afbeelding $\Delta : \mathcal{P} \times \mathcal{P} \to K$ voldoet aan de voorwaarden dat de punten x, y en z collineair zijn als en slechts als $\Delta(x,y)\,\Delta(y,z) = \Delta(x,z)$.

De volgende graaf is dan de puntgraaf van een veralgemeende vierhoek $\mathbf{Q}$ van orde (s,t). De toppenverzameling is $K \times \mathcal{P}$ en twee toppen (k_1, x) en (k_2, y) zijn adjacent als en slechts dan als (i) $x = y$ en $k_1 \neq k_2$, of (ii) $x \neq y$ en $k_2 = k_1\,\Delta(x,y)$. Van volgende veralgemeende vierhoeken is geweten dat zij op dergelijke wijze geconstrueerd kunnen worden.

(1) Het $(s+1)\times(s+1)$-rooster. Hiervoor is $\mathbf{D}$ een rechte met $s+1$ punten en K een willekeurige groep van orde $s+1$. Stel $\Delta(x,y)$ gelijk aan het eenheidselement voor alle punten x en y van $\mathbf{D}$.

(2) Het duale van het $(t+1)\times(t+1)$-rooster. Hiervoor is $\mathbf{D}$ de complete graaf op $t+1$ toppen en K de groep van orde 2. Stel $\Delta(x,y)$ gelijk aan het eenheidselement als en slechts als $x = y$.

(3) De veralgemeende vierhoek $P(W(q), x)$ ([71]). Hiervoor is $\mathbf{D} = \mathrm{AG}(2,q)$ en K de additieve groep van het veld $\mathrm{GF}(q)$. Voor twee punten $r_1 = (x_1, y_1)$ en $r_2 = (x_2, y_2)$ van $\mathbf{D}$ stellen we $\Delta(r_1, r_2) = x_1 y_2 - x_2 y_1$.

(4) De veralgemeende vierhoek $T_2^*(\mathcal{H})$ ([71]). Deze veralgemeende vierhoek is de lineaire representatie van een hyperovaal $\mathcal{H}$ in $\mathrm{PG}(2, 2^h)$ (zie later). We nemen $\mathbf{D} = \mathrm{AG}(2, 2^h)$ en K de additieve groep van het veld $\mathrm{GF}(2^h)$. Weze A een verzameling van $q+2$ punten in $\mathbf{D}$, die opgevat als puntenverzameling van $\mathrm{PG}(2, 2^h)$, een hyperovaal oplevert in $\mathrm{PG}(2, 2^h)$ die equivalent is met $\mathcal{H}$. Noem V de verzameling van de richtingsvectoren bepaald door twee punten van A. Voor elke twee punten x en y van $\mathbf{D}$ bestaat er een unieke $\Delta(x, y) \in \mathrm{GF}(2^h)$ zodat $\overline{xy} = \Delta(x, y)\,\bar{v}$ voor een zekere $\bar{v} \in V$. Dit definieert de afbeelding Δ.

(5) De veralgemeende vierhoek $Q(5, q)$ ([71]). Beschouw een niet-singuliere Hermitische vorm $(\cdot,\cdot)$ in $V(3, q^2)$ en weze U de corresponderende unitaal in $\mathrm{PG}(2, q^2)$. Met deze unitaal is als volgt een lineaire ruimte $\mathbf{D}$ geassocieerd. De punten van $\mathbf{D}$ zijn de punten van U en de rechten van $\mathbf{D}$ zijn de verzamelingen van orde $q+1$ die onstaan als doorsnijding van U met rechten van het projectieve vlak. Stel $K = \{x \in \mathrm{GF}(q^2) | x^4 = 1\}$. Weze $\alpha = \langle \bar{a} \rangle$ een vast punt van U. Voor elke twee punten $\beta = \langle \bar{b} \rangle$ en $\gamma = \langle \bar{c} \rangle$ van U, definiëren we

$$\begin{aligned} \Delta(\beta, \gamma) &= -(\bar{a}, \bar{b})^2 (\bar{b}, \bar{c})^2 (\bar{c}, \bar{a})^2 \text{ als } \alpha \neq \beta \neq \gamma \neq \alpha, \\ &= 1 \text{ in het andere geval.} \end{aligned}$$

Dit is een goede definitie (vervangen we $\bar{b}$ door $\mu\bar{b}$ en $\bar{c}$ door $\lambda\bar{c}$ met $\mu, \lambda \in \mathrm{GF}(q^2) \setminus \{0\}$, dan blijft bovenstaande waarde voor $\Delta(\beta, \gamma)$ onveranderd).

(6) De veralgemeende vierhoek $(S_{xy}^-)^D$ ([66]). Deze is geassocieerd met een hyperovaal $\mathcal{H}$ in $\mathrm{PG}(2, 2^h)$ die de punten x en y bevatten. Neem coördinaten zodanig dat de punten $(1, 0, 0)$, $(0, 1, 0)$ en $(0, 0, 1)$ bevat zijn in $\mathcal{H}$. Er bestaat dan een bijectie $f : \mathrm{GF}(2^h) \to \mathrm{GF}(2^h)$ zodanig dat $\mathcal{H} = \{(1, 0, 0), (0, 1, 0)\} \cup \{(f(\lambda), \lambda, 1) | \lambda \in \mathrm{GF}(2^h)\}$. Weze $\mathbf{D} = \mathrm{AG}(2, 2^h)$ en K de additieve groep van het veld $\mathrm{GF}(2^h)$. Voor twee punten $p_1 = (\alpha_1, \beta_1)$ en $p_2 = (\alpha_2, \beta_2)$ van $\mathrm{AG}(2, 2^h)$ stellen we $\Delta(p_1, p_2)$ gelijk aan $\frac{[f(\alpha_1) - f(\alpha_2)][\beta_1 - \beta_2]}{\alpha_1 - \alpha_2}$ als $\alpha_1 \neq \alpha_2$ en gelijk aan 0 in het andere geval.

Er worden eigenschappen gegeven van veralgemeende vierhoeken die geconstrueerd kunnen worden met behulp van toelaatbare drietallen. In het bijzonder wordt het verband uitgelegd met *spreads* van veralgemeende vierhoeken, dit zijn partities van de puntenverzameling in rechten. Beschikken we over een toelaatbaar drietal $(\mathbf{D}, K, \Delta)$, dan is de verzameling $S = \{L_x | x \in \mathcal{P}\}$, waarbij $L_x = \{(k, x) | k \in K\}$, een spread van $\mathbf{Q}$, die de *geassocieerde spread* van het toelaatbaar drietal genoemd wordt. Een spread S van een veralgemeende vierhoek van orde (s,t) wordt *normaal* genoemd indien elke twee rechten van S bevat zijn in een veralgemeende vierhoek van orde $(s,1)$; S wordt een *symmetriespread* genoemd indien de automorfismegroep van de veralgemeende vierhoek minstens $s+1$ elementen bevat die elke rechte van S fixeren. De spread S geassocieerd aan een toelaatbaar drietal is zowel een normale spread als een symmetriespread. De volgende karakteriseringen kunnen dan bekomen worden.

Stelling 1
Weze $\mathbf{Q}$ *een veralgemeende vierhoek met orde* (s,t).

(1) $\mathbf{Q}$ *heeft een symmetriespread* S *als en slechts dan als* $\mathbf{Q}$ *afgeleid kan worden van een toelaatbaar drietal waarvan* S *de geassocieerde spread is.*

(2) $\mathbf{Q}$ *heeft een symmetriespread* S *als en slechts dan als de projectiviteitengroep van een rechte* $L \in S$ *met betrekking tot* S *orde ten hoogste* $s+1$ *heeft.*

Van alle bovenvermelde veralgemeende vierhoeken (rooster, duaal rooster, $P(W(q),x)$, $T_2^*(\mathcal{H})$, $Q(5,q)$ en $(S_{xy}^{-})^D$) worden alle normale en alle symmetriespreads bepaald. Van bepaalde veralgemeende vierhoeken die afgeleid kunnen worden met behulp van de zogenaamde q-clans (bijvoorbeeld de niet-klassieke Kantor-Knuth veralgemeende vierhoeken) wordt bewezen dat ze geen symmetriespread hebben.

Er wordt een methode aangegeven hoe automorfismen van veralgemeende vierhoeken die een bepaalde spread fixeren kunnen bestudeerd worden. Dit wordt toegepast op sommige van de boven vermelde veralgemeende vierhoeken.

Alle veralgemeende vierhoeken van orde s, die een regulier punt hebben dat incident is met een reguliere rechte, kunnen geconstrueerd worden door *amalgamatie* van twee projectieve vlakken ([61]). Deze veralgemeende vierhoeken worden dan ook *geamalgameerde veralgemeende vierhoeken* genoemd. Is $\mathbf{Q}$ een veralgemeende vierhoek van orde s die een regulier punt x heeft, dan kan een nieuwe veralgemeende vierhoek $P(\mathbf{Q},x)$ gevonden worden die orde $(s-1,s+1)$ heeft ([60]). Deze constructie kan steeds toegepast worden op elke geamalgameerde veralgemeende vierhoek, daar deze steeds een regulier punt heeft. De volgende stelling wordt bewezen.

Stelling 2
Weze $\mathbf{Q}$ *een veralgemeende vierhoek van orde* $(s-1, s+1)$ *met de eigenschap dat elke twee snijdende rechten bevat zijn in een unieke veralgemeende vierhoek van orde* $(s, 1)$*, dan bestaat er een geamalgameerde veralgemeende vierhoek* $\mathbf{Q}'$ *en een regulier punt* x *van* $\mathbf{Q}'$*, zodanig dat* $\mathbf{Q}$ *isomorf is met* $P(\mathbf{Q}', x)$.

3. Basistheorie van de schier veelhoeken

In dit hoofdstuk wordt de basistheorie van de schier veelhoeken besproken. Volgende schier veelhoeken komen veelvuldig in deze thesis voor.

(1) *Klassieke schier veelhoeken of duale polaire ruimten*
Deze zijn geassocieerd met een polaire ruimte Γ. Weze d de maximale dimensie van een deelruimte van Γ. De punten, respectievelijk rechten, van de schier veelhoek $\mathbf{S}$ zijn de d-dimensionale, respectievelijk $(d-1)$-dimensionale deelruimten van Γ (met de natuurlijke incidentie). Voorbeelden van dergelijke klassieke schier veelhoeken worden bekomen door het direct product te nemen van een rechte van lengte $s+1$ met een veralgemeende vierhoek $\mathbf{Q}$. Hiertoe neemt men $s+1$ copieën van $\mathbf{Q}$. Men construeert echter additionele rechten die de corresponderende punten in de $s+1$ copieën verbinden. De juist vermelde constructie werkt eveneens indien $\mathbf{Q}$ een willekeurige partiële lineaire ruimte is. Op inductieve wijze kan men dan het direct product definiëren van $k \geq 1$ rechten. Zulke directe producten zijn eveneens klassieke schier veelhoeken, die *Hamming schier veelhoeken* genoemd worden.

(2) *Schier zeshoek geassocieerd met* $S(5, 8, 24)$
Neem het unieke Steiner systeem $S(5, 8, 24)$. De incidentiestructuur $\mathbf{S}$, met als punten de blokken van $S(5, 8, 24)$ en als rechten de drietallen van onderling disjuncte blokken, is dan een schier zeshoek.

(3) Beschouw de Coxeter-cap $\mathcal{K}$ in $\mathrm{PG}(5, 3)$ ([25],[73]). De volgende incidentiestructuur $T_5^*(\mathcal{K})$ is dan een schier zeshoek. We bedden $\mathrm{PG}(5, 3)$ in als een hypervlak Π_∞ in de ruimte $\Pi = \mathrm{PG}(6, 3)$. De punten van $T_5^*(\mathcal{K})$ zijn de punten van $\Pi \setminus \Pi_\infty$. De rechten van $T_5^*(\mathcal{K})$ zijn al de rechten van Π, niet gelegen in Π_∞, die met $\mathcal{K}$ één punt gemeen hebben. Incidentie is de natuurlijke incidentie.

Een schier veelhoek wordt *regulier* genoemd indien zijn puntgraaf afstandsregulier ([15]) is. Een relatie "parallellisme" kan gedefinieerd worden op de rechtenverzameling van elke schier veelhoek. Twee rechten L en M worden *parallel* genoemd als de afstand van een punt van L tot de rechte M onafhankelijk is van dat punt. We noteren dit ook nog als $L \| M$. De volgende karakterisering kan dan bewezen worden.

Stelling 3
Als **S** *een schier veelhoek is, dan zijn volgende beweringen equivalent.*

(1) **S** *is een Hamming schier veelhoek.*

(2) *Parallellisme is een equivalentierelatie en elke twee punten op afstand twee hebben tenminste twee gemeenschappelijke buren.*

(3) *Voor elk punt x en elke rechte L bestaat er een unieke rechte door x parallel met L.*

Een schier veelhoek wordt *ontaard* genoemd indien deze een *superdun* punt heeft, dit is een punt dat incident is met juist één rechte. Uit een ontaarde schier veelhoek **S** kunnen nieuwe schier veelhoeken geconstrueerd worden. Door opeenvolgende toepassing van deze constructies kan tenslotte een niet-ontaarde schier veelhoek bekomen worden, die uniek bepaald is, en die de *niet-ontaarde ondersteuning van* **S** genoemd wordt.

Eén van de belangrijkste begrippen in de theorie van de schier veelhoeken is het begrip *quad*. Een quad **Q** van een schier veelhoek **S** is een niet-ontaarde veralgemeende vierhoek die deelmeetkunde is van **S** en die bovendien voldoet aan de volgende eigenschap: elk punt van **S** dat collineair is met twee punten van **Q** is een punt van **Q**. De stelling van Yanushka geeft voldoende voorwaarden opdat twee punten bevat zouden zijn in zulk een quad. Deze quads kunnen dan aangewend worden om voor elk punt x van **S** een bepaalde partiële lineaire ruimte $\mathbf{S}_x$ te definiëren, de zogenaamde *locale ruimte in het punt x*.

In dit hoofdstuk worden er eveneens karakteriseringen gegeven van reguliere en klassieke schier zeshoeken in termen van het aantal quads door een punt en het aantal punten op afstand 2 van een vast punt.

4. Associatieschemas geassocieerd met schier veelhoeken

Is $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ een schier zeshoek, dan is $\mathcal{L} \times \mathcal{L} = R_0 \cup \ldots \cup R_4$, waarbij

$$R_0 \;=\; \{(L, L) | L \in \mathcal{L}\},$$

$$\begin{aligned}
R_1 &= \{(L,M)|L \neq M \text{ en } \mathrm{d}(L,M)=0\},\\
R_2 &= \{(L,M)|L\|M \text{ en } \mathrm{d}(L,M)=1\},\\
R_3 &= \{(L,M)|L \not\| M \text{ en } \mathrm{d}(L,M)=1\},\\
R_4 &= \{(L,M)|L\|M \text{ en } \mathrm{d}(L,M)=2\}.
\end{aligned}$$

Stel $\mathcal{R} := \{R_0, R_1, R_2, R_3, R_4\}$. Voor elk koppel $(L,M) \in R_4$ definiëren we $p^4_{22}(L,M)$ als het aantal rechten K waarvoor $(K,L),(K,M) \in R_2$. De volgende stelling geldt dan.

Stelling 4
Veronderstel dat **S** *een schier zeshoek is die aan de volgende eigenschappen voldoet.*

(1) **S** *is regulier met parameters* (s, t_2, t), *d.w.z.* **S** *heeft orde* (s,t) *en elke twee punten op afstand* 2 *hebben juist* $t_2 + 1$ *gemeenschappelijke buren.*

(2) *Elke twee punten op afstand* 2 *zijn bevat in een unieke quad.*

Dan bepaalt $\mathcal{R}$ *een associatieschema* ([3]) *als en dan slechts als* $p^4_{22}(L,M)$ *onafhankelijk is van* $(L,M) \in R_4$.

Voorbeelden van dergelijke associatieschemas worden gegeven, evenals de volgende karakterisering.

Stelling 5
Weze **S** *een reguliere schier zeshoek met parameters* (s, t_2, t) *zodanig dat*

(*i*) $s \geq t_2$,

(*ii*) $p^4_{22}(L,M)$ *onafhankelijk is van* $(L,M) \in R_4$,

dan is **S** *één van de volgende schier zeshoeken:*

- *een veralgemeende zeshoek,*
- *een reguliere klassieke schier zeshoek,*
- *de schier zeshoek geassocieerd met* $S(5, 8, 24)$,
- *de schier zeshoek wiens puntgraaf de incidentiegraaf is van het unieke bivlak van orde* 2 ([6]).

Tenslotte wordt ook bewezen dat er een (niet noodzakelijk symmetrisch) associatieschema verbonden is aan elke veralgemeende vierhoek van orde (s, s^2) die een symmetriespread heeft.

5. De i-buurmeetkunde van een punt

Veronderstel dat x een punt is van een schier veelhoek $\mathbf{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ en stel $i \in \mathbb{N}$. De *i-buurmeetkunde van x* is de deelmeetkunde $\mathbf{S}_x^i$ van $\mathbf{S}$ verkregen door alle punten te nemen met afstand ten hoogste i tot x, evenals alle rechten met afstand ten hoogste $i-1$ tot x. Het is duidelijk dat $\mathbf{S}_x^i$ een schier veelhoek is indien $i \in \{0, 1\}$. We hebben eveneens de volgende eigenschappen bewezen.

Stelling 6
Weze x een punt van een schier veelhoek $\mathbf{S}$.

(1) *$\mathbf{S}_x^2$ is een schier veelhoek.*

(2) *Is $\mathbf{S}$ een dunne schier veelhoek, dan is $\mathbf{S}_x^i$ eveneens een schier veelhoek voor elke $i \in \mathbb{N}$.*

(3) *Is $\mathbf{S}$ een veralgemeende veelhoek, dan is $\mathbf{S}_x^i$ eveneens een schier veelhoek voor elke $i \in \mathbb{N}$.*

(4) *Is $\mathbf{S}$ een klassieke schier veelhoek, dan is $\mathbf{S}_x^i$ eveneens een schier veelhoek voor elke $i \in \mathbb{N}$.*

Een voorbeeld toont echter aan dat niet elke i-buurmeetkunde een schier veelhoek is. In dit hoofdstuk wordt eveneens de volgende stelling bewezen.

Stelling 7
Weze $\mathbf{G}$ een willekeurige partiële lineaire ruimte, dan bestaat er een schier veelhoek $\mathbf{S}$ en een punt (∞) van $\mathbf{S}$ zodanig dat de locale ruimte $\mathbf{S}_{(\infty)}$ isomorf is met $\mathbf{G}$.

Elke lineaire ruimte treedt dus op als locale ruimte. Om voorgaande stelling te bewijzen wordt een oneindige klasse Ω van meetkunden geconstrueerd. Elke meetkunde in Ω heeft de eigenschap dat er een punt bestaat dat afstand ten hoogste 2 heeft tot elk ander punt. De meetkunde $\mathbf{S}$ uit voorgaande stelling kan dan steeds in Ω gekozen worden. Ook de volgende stelling wordt in dit hoofdstuk bewezen.

Stelling 8
Weze (∞) een punt van een schier veelhoek $\mathbf{S}$ welke afstand ten hoogste 2 heeft tot elk ander punt. Is de niet-ontaarde ondersteuning van $\mathbf{S}$ verschillend van een punt, dan is deze bevat in Ω.

6. Lineaire representaties van schier veelhoeken

Veronderstel dat $\mathrm{PG}(n,q)$, $n \geq 0$, ingebed is als een hypervlak Π_∞ in $\Pi = \mathrm{PG}(n+1,q)$ en weze $\mathcal{K}$ een niet ledige verzameling van punten in Π_∞. De *lineaire representatie* $T_n^*(\mathcal{K})$ is dan de volgende partiële lineaire ruimte. De punten van $T_n^*(\mathcal{K})$ zijn de punten van $\Pi \setminus \Pi_\infty$; de rechten van $T_n^*(\mathcal{K})$ zijn deze rechten van Π, die niet bevat zijn in Π_∞, en die Π_∞ snijden in een punt van $\mathcal{K}$; incidentie is deze van Π. Dit hoofdstuk bestudeert het geval wanneer $T_n^*(\mathcal{K})$ een schier veelhoek is, in het bijzonder wordt ingegaan op het speciale geval van de schier zeshoeken. Een voorbeeld van een dergelijke schier zeshoek werd reeds gegeven op pagina 182; hier was $\mathcal{K}$ de Coxeter-cap in $\mathrm{PG}(5,3)$.

Volgende stelling levert een belangrijke klasse van schier veelhoeken op.

Stelling 9
Weze $n = 2k$, $k \geq 1$, en $q = 2^h$ een even priemmacht. Beschouw door een vast punt p van Π_∞ k vlakken $\Pi_1, \ldots, \Pi_k$ die Π_∞ genereren. Weze $\mathcal{H}_i$, $i \in \{1, \ldots, k\}$, een hyperovaal van Π_i die p bevat. Als $\mathcal{K} = \mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k$, dan is $T_{2k}^*(\mathcal{K})$ een schier $(2k+2)$-hoek die de eigenschap heeft dat elke twee punten op afstand 2 bevat zijn in een unieke quad.

Dit hoofdstuk heeft onder andere als doel alle schier zeshoeken te classificeren die een lineaire representatie hebben en dit in het geval $q \geq 3$. Zulke schier zeshoeken hebben steeds quads door elke twee punten op afstand 2. De classificatie is partieel. Ze is wel compleet voor het geval $n \leq 6$.

Stelling 10
Weze $\mathcal{K}$ een verzameling punten in $\mathrm{PG}(n,q)$, $n \leq 6$ en $q \geq 3$, dan is $T_n^*(\mathcal{K})$ een schier zeshoek als en slechts dan als één van de volgende gevallen zich voordoet.

(1) *$\mathcal{K}$ is een verzameling van drie niet collineaire punten in een vlak ($n = 2$).*

(2) *$\mathcal{K}$ bestaat uit een hyperovaal plus een punt niet in het vlak van de hyperovaal ($n = 3$).*

(3) *$\mathcal{K}$ is de unie van twee hyperovalen wiens draagvlakken elkaar ontmoeten in een punt dat behoort tot beide hyperovalen ($n = 4$).*

(4) *$\mathcal{K}$ is de Coxeter-cap in $\mathrm{PG}(5,3)$.*

Geen voorbeelden zijn gekend voor $n \geq 7$ en $q \geq 3$. In dit geval zijn er nochtans wel enkele conclusies te trekken.

Stelling 11
Weze $T_n^*(\mathcal{K})$ een schier zeshoek en veronderstel dat $n \geq 7$ en $q \geq 3$, dan gelden volgende beweringen.

(1) $q \geq 2^h$ *met* $h \geq 4$

(2) *Elke quad is ofwel een rooster ofwel een lineaire representie van een hyperovaal. Quads van het laatste type treden zeker op, evenals een rosette van ovoiden in elk zulke quad. Als er een quad bestaat van type $T_2^*(\mathcal{H})$ met $\mathcal{H}$ een reguliere hyperovaal, dan geldt $q \geq 64$ en een tot heden ongekende ovoide in* $\mathrm{PG}(3,q)$ *moet bestaan.*

Stel $f_m(x,y) = \sum_{j=1}^{m} \binom{x-1}{j}(y-1)^{j-1}$ voor elke $m \in \mathbb{N}$ en $g_m(x,y) = \frac{1}{y}\binom{x-1}{m}(y-1)^{m-1} + f_{m-1}(x,y)$ voor elke $m \in \mathbb{N} \setminus \{0\}$.

Stelling 12
Veronderstel dat $\mathcal{K}$ een verzameling is van $k > 0$ punten in $\mathrm{PG}(n,q)$ die aan de eigenschap voldoen dat geen l ervan bevat zijn in een $(l-2)$-dimensionale ruimte $(n \geq l-2 \geq 0)$.

(1) *Is $l = 2m+1$, dan geldt $f_m(k,q) \leq \frac{q^n-1}{q-1}$ en de gelijkheid treedt op als en slechts dan als $T_n^*(\mathcal{K})$ een schier $2(m+1)$-hoek is.*

(2) *Is $l = 2m$ en $(n,m) \neq (0,1)$, dan geldt $g_m(k,q) \leq \frac{q^n-1}{q-1}$ en de gelijkheid treedt slechts op als en slechts dan als $T_n^*(\mathcal{K})$ een schier $(2m+1)$-hoek is.*

De gelijkheid in voorgaande stelling kan slechts optreden in de volgende gevallen.

$l = 2$: In dit geval bestaat $\mathcal{K}$ uit alle punten van de projectieve ruimte.

$l = 3$: Er zijn drie mogelijkheden.

(1) $\mathcal{K}$ is een verzameling van twee punten in $\mathrm{PG}(1,q)$.

(2) $\mathcal{K}$ is een hyperovaal in $\mathrm{PG}(2,q)$, q even.

(3) $\mathcal{K}$ is het complement van een hypervlak in $\mathrm{PG}(n,2)$, $n \geq 3$.

$l = 4$: Er zijn twee mogelijkheden.

(1) $\mathcal{K}$ is een verzameling van 11 punten in $\mathrm{PG}(4,3)$ die als volgt bekomen wordt. Beschouw een punt x van de Coxeter-cap in $\mathrm{PG}(5,3)$ en een hypervlak α in $\mathrm{PG}(5,3)$ die x niet bevat. De projectie van de Coxeter-cap vanuit het punt x op α levert $\mathcal{K}$ op.

(2) $\mathcal{K}$ is een verzameling van 5 punten in $\mathrm{PG}(3,2)$ waarvan geen 4 bevat zijn in een hypervlak (uniek bepaald op projectieve equivalentie na).

$l = 5$: Er zijn twee mogelijkheden.

(1) $\mathcal{K}$ is de Coxeter-cap in $\mathrm{PG}(5,3)$.

(2) $\mathcal{K}$ is een verzameling van 6 punten in $\mathrm{PG}(4,2)$ waarvan geen 5 bevat zijn in een hypervlak (uniek bepaald op projectieve equivalentie na).

Tenslotte wordt aan de hand van de lineaire representatie een classificatie bekomen van speciale puntenverzamelingen (ovoiden en Cameron gesloten verzamelingen) van de schier zeshoek die geassocieerd is met de Coxeter-cap.

7. Gelijmde schier veelhoeken

Weze k een positief natuurlijk getal. Voor elke $i \in \{1, \ldots, k\}$ beschouwen we de volgende objecten:

- een veralgemeende vierhoek $\mathbf{Q}_i = (\mathcal{P}_i, \mathcal{L}_i, \mathcal{I}_i)$ van orde (s, t_i),
- een spread $S_i = \{L_1^{(i)}, \ldots, L_{1+st_i}^{(i)}\}$ van $\mathbf{Q}_i$,
- een bijectie $\theta_i : L_1^{(1)} \to L_1^{(i)}$.

In elke spread S_i, $i \in \{1, \ldots, k\}$, wordt dus een speciale rechte $L_1^{(i)}$ genomen, die de *basisrechte* van S_i genoemd wordt. De projectie op de rechte $L_j^{(i)}$ in de veralgemeende vierhoek $\mathbf{Q}_i$ wordt genoteerd als $p_j^{(i)}$. De projectiviteitengroep van $L_1^{(i)}$ met betrekking tot S_i wordt genoteerd als G_i en de samenstelling $f \circ g$ van twee functies f en g wordt genoteerd als fg.

Definieer nu de volgende graaf Γ met toppenverzameling $L_1^{(1)} \times S_1 \times \ldots \times S_k$. Twee verschillende toppen $(x, L_{i_1}^{(1)}, \ldots, L_{i_k}^{(k)})$ en $(y, L_{j_1}^{(1)}, \ldots, L_{j_k}^{(k)})$ zijn adjacent als en slechts dan als aan de volgende twee voorwaarden voldaan is.

(i) Er bestaat een $l \in \{1, \ldots, k\}$ zodanig dat $i_m = j_m$ voor elke $m \in \{1, \ldots, k\} \setminus \{l\}$.

(ii) Voor elke l die aan (i) voldoet zijn $p_{i_l}^{(l)}\theta_l(x)$ en $p_{j_l}^{(l)}\theta_l(y)$ collineaire punten in de veralgemeende vierhoek $\mathbf{Q}_l$. (Als $i_m = j_m$ voor elke $m \in \{1, \ldots, k\}$, dan kan l elke waarde van 1 tot en met k aannemen, maar aan voorwaarde (ii) is dan altijd voldaan.)

De volgende stelling kan dan bewezen worden. Hierin wordt de triviale groep met 0 genoteerd en $[\theta_i^{-1}G_i\theta_i, \theta_j^{-1}G_j\theta_j]$ stelt de groep voor die gegenereerd wordt door alle commutatoren $[\theta_i^{-1}g_i\theta_i, \theta_j^{-1}g_j\theta_j]$ waarbij $g_i \in G_i$ en $g_j \in G_j$.

Stelling 13
Γ is de puntgraaf van een schier $(2k+2)$-hoek $\mathbf{S}$ als en slechts dan als $[\theta_i^{-1}G_i\theta_i, \theta_j^{-1}G_j\theta_j] = 0$ voor alle $i, j \in \{1, \ldots, k\}$ met $i \neq j$.

Is $\mathbf{S}$ een schier veelhoek, dan zijn de rechten van $\mathbf{S}$ noodzakelijk de maximale cliques van Γ en $\mathbf{S}$ wordt een *gelijmde schier veelhoek* genoemd. Intuïtief kan men zeggen dat deze schier veelhoek bekomen kan worden door een heleboel veralgemeende vierhoeken aan elkaar te lijmen. Deze veralgemeende vierhoeken verschijnen dan als quads in de schier veelhoek. Is $\mathbf{S}$ een schier veelhoek, dan kan $\mathbf{S}$ bekomen worden

- voor elke keuze van de ordening van de veralgemeende vierhoeken,
- voor elke keuze van de basisrechten in de spreads,
- voor elke keuze van θ_1.

De schier veelhoeken $T_{2k}^*(\mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k)$ die gedefinieerd werden in Stelling 9 zijn voorbeelden van gelijmde schier veelhoeken.

Er wordt ingegaan op de classificatie van gelijmde schier veelhoeken. De volgende stelling wordt bewezen.

Stelling 14
Veronderstel dat aan de volgende voorwaarden voldaan zijn:

(A) $\mathbf{S}$ *is een schier veelhoek;*

(B) $k \geq 2$*;*

(C) *de veralgemeende vierhoeken* $\mathbf{Q}_1, \ldots, \mathbf{Q}_k$ *zijn geen roosters.*

Dan gelden de volgende beweringen:

(1) $G_i \simeq G_j$ *voor alle* $i, j \in \{1, \ldots, k\}$,

(2) S_i *is een symmetriespread van* $\mathbf{Q}_i$,

(3) *als $k \geq 3$ dan is G_i commutatief.*

Laten we nu de veralgemeende vierhoeken, de spreads en de basisrechten fixeren. Zoals eerder opgemerkt mogen we ook θ_1 fixeren. In de thesis wordt aangetoond hoe de vergelijkingen $[\theta_i^{-1}G_i\theta_i, \theta_j^{-1}G_j\theta_j] = 0$, $i, j \in \{1, \ldots, k\}$ met $i \neq j$, op te lossen naar $(\theta_2, \ldots, \theta_k)$. Het kan nu gebeuren dat twee verschillende oplossingen voor $(\theta_2, \ldots, \theta_k)$ toch nog isomorfe schier veelhoeken opleveren. In de thesis worden technieken besproken die het mogelijk maken om het aantal niet isomorfe gelijmde schier zeshoeken te bepalen of af te schatten. Dit wordt toegepast op enkele voorbeelden.

Tenslotte worden de volgende twee karakteriseringen bewezen.

Stelling 15
Veronderstel dat **S** *een schier zeshoek is die aan de volgende voorwaarden voldoet.*

(1) *Elke twee punten op afstand 2 zijn bevat in een quad.*

(2) *Als alle rechten van* **S** *dun zijn, dan hebben alle quads een orde.*

(3) *Er bestaat een punt x zodanig dat alle locale ruimten $\mathbf{S}_y$, y collineair met x, een dun punt bevatten.*

Dan is **S** *het direct product van een rechte met een niet-ontaarde veralgemeende vierhoek of* **S** *is een gelijmde schier zeshoek.*

Stelling 16
Veronderstel dat **S** *een schier zeshoek is die aan de volgende voorwaarden voldoet.*

(1) *Elke twee punten op afstand 2 zijn bevat in een quad.*

(2) *Als alle rechten van* **S** *dun zijn, dan hebben alle quads een orde.*

(3) *Er bestaat een punt x zodanig dat $\mathbf{S}_x$ een dun punt bevat en zodanig dat alle locale ruimten $\mathbf{S}_y$, y collineair met x, eenzelfde aantal rechten bevatten.*

Dan is **S** *het direct product van een rechte met een niet-ontaarde veralgemeende vierhoek of* **S** *is een gelijmde schier zeshoek.*

8. Schier zeshoeken met vier punten op een rechte

In [14] werden alle schier zeshoeken bepaald die aan de volgende voorwaarden voldoen:

(1) alle rechten hebben drie punten;

(2) elke twee punten op afstand 2 zijn bevat in een quad.

Het doel van dit hoofdstuk is de classificatie van alle schier zeshoeken die aan de volgende voorwaarden voldoen:

(1) alle rechten hebben vier punten;

(2) elke twee punten op afstand 2 zijn bevat in een quad.

De classificatie is echter niet compleet. Alle klassieke en gelijmde schier zeshoeken die aan bovenstaande voorwaarden voldoen zijn gekend. De klassieke voorbeelden zijn ofwel een direct product van een rechte met een veralgemeende vierhoek van orde $(3, t)$ (vijf voorbeelden), ofwel zijn het de duale polaire ruimten geassocieerd met de volgende polaire ruimten:

(1) $W(5, 3)$: de polaire ruimte geassocieerd met een symplectische polariteit van $\mathrm{PG}(5, 3)$;

(2) $Q(6, 3)$: de polaire ruimte geassocieerd met een niet-singuliere kwadriek in $\mathrm{PG}(6, 3)$;

(3) $H(5, 9)$: de polaire ruimte geassocieerd met een niet-singuliere Hermitische varieteit in $\mathrm{PG}(5, 9)$.

We beschrijven nu de gelijmde schier zeshoeken die niet klassiek zijn.

- De volgende gelijmde schier zeshoek is het unieke voorbeeld waarbij beide veralgemeende vierhoeken isomorf zijn met $T_2^*(\mathcal{H})$, waarbij $\mathcal{H}$ de unieke hyperovaal is in $\mathrm{PG}(2, 4)$. Beschouw twee vlakken α_1 en α_2 in $\mathrm{PG}(4, 4)$ die elkaar snijden in een punt p. Weze $\mathcal{H}_i$, $i \in \{1, 2\}$, een hyperovaal in α_i die p bevat. De lineaire representatie $T_4^*(\mathcal{H}_1 \cup \mathcal{H}_2)$ is dan een schier zeshoek.

- De volgende gelijmde schier zeshoek is het unieke voorbeeld waarbij beide veralgemeende vierhoeken isomorf zijn met $Q(5, 3)$. Stel $K = \{x \in \mathrm{GF}(9) | x^4 = 1\}$. Beschouw een niet singuliere Hermitische vorm $(\cdot, \cdot)$ in $V(3, 9)$ en weze U de corresponderende unitaal in $\mathrm{PG}(2, 9)$.

Weze $\alpha = \langle \bar{a} \rangle$ een vast punt van U. Voor elke twee punten $\beta = \langle \bar{b} \rangle$ en $\gamma = \langle \bar{c} \rangle$ van U, definiëren we

$$\begin{aligned} \Delta(\beta, \gamma) &= -(\bar{a}, \bar{b})^2 (\bar{b}, \bar{c})^2 (\bar{c}, \bar{a})^2 \text{ als } \alpha \neq \beta \neq \gamma \neq \alpha, \\ &= 1 \text{ in het andere geval.} \end{aligned}$$

We definiëren nu een graaf Γ met toppenverzameling $K \times U \times U$. Twee verschillende toppen (k_1, α_1, β_1) en (k_2, α_2, β_2) zijn adjacent als en slechts dan als aan één van de volgende voorwaarden voldaan is:

(1) $\alpha_1 = \alpha_2$ en $\beta_1 = \beta_2$,

(2) $\alpha_1 = \alpha_2$, $\beta_1 \neq \beta_2$ en $k_2 = k_1 \Delta(\beta_1, \beta_2)$,

(3) $\alpha_1 \neq \alpha_2$, $\beta_1 = \beta_2$ en $k_2 = k_1 \Delta(\alpha_1, \alpha_2)$.

Elke twee adjacente toppen van Γ zijn bevat in een unieke maximale clique. De meetkunde waarvan de punten, respectievelijk de rechten, de toppen en de maximale cliques van Γ zijn, is dan een schier zeshoek.

Het is niet geweten of er voorbeelden zijn indien de schier zeshoek niet klassiek en niet gelijmd is. Toch zegt volgende stelling iets over hun structuur.

Stelling 17
Veronderstel dat **S** *een schier zeshoek is die aan de volgende eigenschappen voldoet:*

(1) *alle rechten van* **S** *hebben vier punten,*

(2) *elke twee punten op onderlinge afstand* 2 *zijn bevat in een quad,*

(3) **S** *is niet klassiek en niet gelijmd,*

dan zijn alle quads ofwel roosters ofwel isomorf met $Q(4,3)$. *Er zijn constanten* a *en* b *zodanig dat elk punt van* **S** *bevat is in* a *roosters en* b *quads die isomorf zijn met* $Q(4,3)$. *Elk punt is incident met een constant (=* $t+1$*) aantal rechten. We hebben dan de volgende mogelijkheden voor* t, a, b *en* v *(= het aantal punten):*

- $v = 5848$, $t = 19$, $a = 160$, $b = 5$;
- $v = 6736$, $t = 21$, $a = 171$, $b = 10$;
- $v = 8320$, $t = 27$, $a = 120$, $b = 43$;
- $v = 20608$, $t = 34$, $a = 595$, $b = 0$.

Appendix: De classificatie van de ovoiden in de schier zeshoek geassocieerd met $S(5,8,24)$

Een *ovoide* van een schier veelhoek is een verzameling punten met de eigenschap dat elke rechte een uniek punt gemeen heeft met deze verzameling. Weze $\mathbf{S}$ nu de schier zeshoek geassocieerd met het Steiner systeem $S(5,8,24)$. De volgende classificatiestelling werd bewezen in [16].

Stelling 18
De verzameling van de blokken door een vast punt van $S(5,8,24)$ *is een ovoide van* $\mathbf{S}$*. Omgekeerd wordt elke ovoide op deze wijze verkregen.*

In de appendix van deze thesis wordt een bewijs hiervan geleverd dat verschillend is van het bewijs in [16].

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