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Characterising substructures of finite projective spaces

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On a dit souvent que la géométrie
est l'art de bien raisonner
sur des figures mal faites.
– Henri Poincaré

Dankwoord

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Preface

Finite geometry can be described as the study of any particular incidence structure containing only a finite number of points. While there are many structures that could be called finite geometries, attention is mostly paid to substructures of finite projective and affine spaces. Such spaces may be constructed via linear algebra, i.e. starting from a vector space $V(n+1,q)$ of rank $n+1$ over a finite field $\mathbb{F}_q$, one can construct an n-dimensional projective space $\mathrm{PG}(n,q)$. The affine and projective spaces so constructed are called *Desarguesian.* At the same time, finite projective spaces can also be defined in a purely axiomatical way. However, while for dimension two there do exist non-Desarguesian planes, for dimension three or greater, any finite projective space arises from a vector space over a finite field.

In this branch of finite geometry, different objects of study include vector spaces, incidence structures, affine and projective spaces and various substructures contained in them. This area of research started with the seminal work of Beniamino Segre considering algebraic characterisations of combinatorially defined objects. From an algebraic point of view, a *conic* in a projective plane is an absolutely irreducible algebraic curve of degree two. Segre's celebrated characterisation theorem from 1954 states that in the Desarguesian projective plane $\mathrm{PG}(2,q)$, with q odd, every set of $q+1$ points, no three of which are collinear (also called an *oval*), is a conic. However, for q even, not all ovals in $\mathrm{PG}(2,q)$ are conics. Even more so, there is still no known classification of ovals in the Desarguesian projective plane of even order q, for $q>4$. This problem has inspired the study of other classical sets in projective spaces, such as for example *ovoids*, i.e. sets of q^2+1 points in $\mathrm{PG}(3,q)$, $q>2$, no three of which are collinear. An *elliptic quadric* $Q^-(3,q)$ is a particular example of an ovoid. For q odd, all ovoids in $\mathrm{PG}(3,q)$ correspond to elliptic quadrics. However, for $q>32$ even, the classification of ovoids remains an open problem.

Generalised quadrangles (GQ) are point-line incidence structures that have connections with several other geometrical objects. In particular, ovals and ovoids lead to translation generalised quadrangles. Starting from an oval O in $\mathrm{PG}(2,q)$ and an ovoid $\mathcal{O}$ in $\mathrm{PG}(3,q)$, Tits (1959) constructed two GQ's denoted by $T_2(O)$ and $T_3(\mathcal{O})$. Payne derivation of $T_2(O)$, when q is even, yields a third non-isomorphic GQ written as $T_2^*(O')$, where O' is the unique hyperoval extending O. A generalisation of this construction is the *linear representation* $T_n^*(\mathcal{K})$ of a point set $\mathcal{K}$. Several interesting incidence structures, such as semipartial geometries and (α,β)-geometries, but also interesting graphs, for instance *semisymmetric graphs*, can be obtained from the linear representation of a well-chosen point set.

Thas (1974) generalised $T_2(O)$ and $T_3(\mathcal{O})$ by considering the higher dimensional equivalent of (hyper)ovals and ovoids, called *pseudo-(hyper)ovals* and *pseudo-ovoids*, both are examples of *eggs*. One can construct a translation generalised quadrangle $T(\mathcal{E})$ from every egg $\mathcal{E}$; moreover, every translation generalised quadrangle arises as $T(\mathcal{E})$ for some egg $\mathcal{E}$. The concept of considering a projective space over a smaller subfield is called *field reduction*, this means that points of $\mathrm{PG}(n-1, q^t)$ correspond to $(t-1)$-dimensional subspaces of $\mathrm{PG}(tn-1, q)$. The set consisting of all these induced subspaces forms a *Desarguesian spread*. Applying field reduction to ovals and ovoids leads to so called *elementary* examples of pseudo-ovals and pseudo-ovoids. Of course, eggs obtained in this way do not lead to new generalised quadrangles. Not all eggs are elementary, but a complete classification is not yet attained.

A *generalised linear representation* $T^*_{n,t}(\mathcal{K})$ is an incidence structure similar to a linear representation, but instead of a set of points, a set $\mathcal{K}$ of disjoint $(t-1)$-spaces in $\mathrm{PG}(n, q)$ is considered. If $\mathcal{K}$ corresponds to a pseudo-hyperoval, the corresponding structure is again a generalised quadrangle. If $\mathcal{K}$ is a $(t-1)$-spread in $\mathrm{PG}(2t-1, q)$, this construction corresponds to the *André/Bruck-Bose representation* of an affine translation plane. When the spread is Desarguesian, the affine plane is Desarguesian. The André/Bruck-Bose representation thus provides a nice representation of the Desarguesian plane $\mathrm{PG}(2, q^t)$ as subspaces of $\mathrm{PG}(2t, q)$. This representation has proven to be useful for the construction of substructures of $\mathrm{PG}(2, q^t)$, for instance for the generation of *(translation) arcs* and *unitals*, and can contribute to the classification or characterisation of them.

This thesis contributes to some of the aforementioned combinatorial questions by focussing on *the characterisation of substructures* such as pseudo-caps, eggs, spreads, linear representations, subgeometries and unitals.

Chapter 1 recalls the basic definitions and fundamental results concerning incidence structures and (substructures of) projective spaces over finite fields.

Part I considers characterisations of elementary pseudo-caps and Desarguesian spreads. In Chapter 2 we investigate pseudo-caps and (weak) eggs. We provide conditions on element induced spreads which ensure that these structures are contained in a Desarguesian spread. Next, in Chapter 3, focussing on the Desarguesian spread itself, we obtain a geometric characterisation in terms of the normal elements of the spread.

In Part II, we consider the linear representation $T^*_n(\mathcal{K})$ defined by a point set $\mathcal{K}$ at infinity. We investigate the isomorphism problem for linear representations in Chapter 4. If the set $\mathcal{K}$ contains a frame, then the full automorphism group of this structure is obtained. Using the corresponding incidence graph, we construct new infinite families of semisymmetric graphs in Chapter 5.

Part III focusses on the André/Bruck-Bose representation of $\mathrm{PG}(2, q^n)$ in $\mathrm{PG}(2n, q)$. We investigate the representation of $\mathbb{F}_{q^t}$-sublines and $\mathbb{F}_{q^t}$-subplanes of $\mathrm{PG}(2, q^n)$ in Chapter 6. In Chapter 7 we obtain a characterisation of the ovoidal Buekenhout-Metz unitals of $\mathrm{PG}(2, q^2)$ in terms of its Baer secants.

Lastly, an extended summary in English and a brief overview in Dutch is given, summarising the main results of this thesis.

Contents

1 Preliminaries

This chapter introduces the structures and objects investigated in this thesis. It is not meant as a layman's introduction, but simply recalls basic definitions and fixes notation to avoid ambiguity. We will assume the reader has basic knowledge of combinatorics, finite field theory, linear algebra, graph theory and group theory.

1.1 Finite fields

The *finite field of order* q is unique up to isomorphism and is denoted by $\mathbb{F}_q$. Finite fields exist for all $q = p^h$, p a prime number and $h \geq 1$. When $\mathbb{F}_{q_0}$ is a subfield of $\mathbb{F}_q$, then $q = p^h$ and $q_0 = p^{h_0}$, p prime, with $h_0|h$.

There are various ways of representing finite fields. In this thesis, we will encounter two of them in particular, namely the representation as vector spaces, that is,

$$\mathbb{F}_{q^k} = \{a_0x_0 + a_1x_1 + \cdots + a_{k-1}x_{k-1} \mid a_i \in \mathbb{F}_q\},$$

where $\{x_0, x_1, \ldots, x_{k-1}\}$ is a *basis* of $\mathbb{F}_{q^k}$ over $\mathbb{F}_q$, and the representation as polynomial quotient rings, that is

$$\mathbb{F}_{q^k} = \mathbb{F}_q[x]/(f(x)) = \{a_0 + a_1x + \cdots + a_{k-1}x^{k-1} \mid a_i \in \mathbb{F}_q\},$$

where $f(x)$ is an irreducible monic polynomial of degree k over $\mathbb{F}_q$.

The finite field $\mathbb{F}_q$, $q = p^h$, p prime, has exactly h automorphisms, namely,

$$\sigma_i : \mathbb{F}_q \to \mathbb{F}_q : a \mapsto a^{p^i}, i = 1, 2, \ldots, h.$$

The automorphism group $\mathrm{Aut}(\mathbb{F}_q)$ of $\mathbb{F}_q$ is thus a cyclic group of order h with σ_1 as a generator. The distinct automorphisms of $\mathbb{F}_{q^k}$, trivial over $\mathbb{F}_q$, are given by

$$\sigma'_i : \mathbb{F}_{q^k} \to \mathbb{F}_{q^k} : a \mapsto a^{q^i}, i = 1, 2, \ldots, k.$$

The cyclic group of order k generated by σ_1' is denoted by $\mathrm{Aut}(\mathbb{F}_{q^k}/\mathbb{F}_q)$. It is clear that $\mathrm{Aut}(\mathbb{F}_{q^k})\big/\mathrm{Aut}(\mathbb{F}_{q^k}/\mathbb{F}_q) \cong \mathrm{Aut}(\mathbb{F}_q)$. Depending on the context, the map σ_1 or σ_1' is sometimes called the *Frobenius automorphism*.

Consider the subfield $\mathbb{F}_q$ of $\mathbb{F}_{q^k}$. The *trace map* T_k and *norm map* N_k are defined as follows:

$$T_k : \mathbb{F}_{q^k} \to \mathbb{F}_q : a \mapsto a + a^q + a^{q^2} + \cdots + a^{q^{k-1}},$$
$$N_k : \mathbb{F}_{q^k} \to \mathbb{F}_q : a \mapsto a \cdot a^q \cdot a^{q^2} \cdot \cdots \cdot a^{q^{k-1}}.$$

1.2 Incidence structures

In this thesis, we will encounter projective spaces, linear representations, Laguerre planes and generalised quadrangles. These are all examples of *incidence structures*.

Definition 1.2.1. An *incidence structure* or *incidence geometry* S is a triple $S = (\mathcal{P}, \mathcal{B}, I)$, with $\mathcal{P}$ and $\mathcal{B}$ non-empty disjoint sets and with I the symmetric *incidence relation*, that is $I \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$. The elements of $\mathcal{P}$ are called the *points* of S and the elements of $\mathcal{B}$ are called the *blocks* of S.

We will only encounter incidence structures such that every block of S is uniquely determined by the points incident with it, i.e. the blocks can be identified with the subsets of $\mathcal{P}$ determining them. The incidence relation I becomes symmetrised containment, hence, we will use set theoretical notation and often write $(\mathcal{P}, \mathcal{B})$ instead of $(\mathcal{P}, \mathcal{B}, I)$.

If $(P, B) \in I$, we say that P is *incident* with B, P is *contained* in B or B goes *through* P, and we write this as PIB. We say that two points are *collinear* if they are contained in a block. Dually, we say that two blocks are *concurrent* if and only if they have non-empty intersection.

Actually, after this chapter, we will not use the term blocks. We will consider two types of incidence structures, namely *point-line incidence structures* $(\mathcal{P}, \mathcal{L})$, where the blocks in $\mathcal{L}$ are called *lines*, and *point-line-circle incidence structures* $(\mathcal{P}, \mathcal{L} \cup \mathcal{C})$, with two classes of blocks, namely *lines* in $\mathcal{L}$ and *circles* in $\mathcal{C}$.

Definition 1.2.2. An *isomorphism* between two incidence structures $S = (\mathcal{P}, \mathcal{B}, I)$ and $S' = (\mathcal{P}', \mathcal{B}', I')$ is a pair $\theta = (\theta_1, \theta_2)$, with bijections $\theta_1 : \mathcal{P} \to \mathcal{P}'$ and $\theta_2 : \mathcal{B} \to \mathcal{B}'$, preserving incidence and non-incidence, i.e. $\forall (P, B) \in \mathcal{P} \times \mathcal{B} : PIB \Leftrightarrow \theta_1(P) I' \theta_2(B)$. However, in the rest of this work, we will not use such a strict notation, and just say that θ is an isomorphism without referring to θ_1 and θ_2.

If there exists such an isomorphism, we say that S and S' are *isomorphic* and write $S \cong S'$. An isomorphism from S to itself is called an *automorphism.* The set of all automorphisms of S forms a group, the *automorphism group* of S, and will be denoted by $\mathrm{Aut}(S)$.

A *substructure* $S' = (\mathcal{P}', \mathcal{B}', I')$ of an incidence structure $S = (\mathcal{P}, \mathcal{B}, I)$ is an incidence structure with $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{B}' \subseteq \mathcal{B}$, where I' is the restriction of I on $(\mathcal{P}' \times \mathcal{B}') \cup (\mathcal{B}' \times \mathcal{P}')$.

The *collinearity graph* of an incidence structure $S = (\mathcal{P}, \mathcal{B}, I)$ is the graph with as vertex set the point set $\mathcal{P}$ of S and were two vertices are adjacent if and only if the corresponding points are contained in a block of $\mathcal{B}$.

Definition 1.2.3. An incidence structure is called *connected* if its corresponding collinearity graph is connected.

In this thesis, we will only consider connected incidence structures, moreover, only a specific class of connected incidence structures, namely partial linear spaces.

Definition 1.2.4. A finite connected incidence structure S is called a *partial linear space* of *order* (s, t) if and only if

- every block of S contains exactly $s + 1 > 1$ points,
- every point of S is contained in exactly $t + 1 > 1$ blocks,
- two distinct points are contained in at most one common block.

If every two distinct points are contained in exactly one common block, then S is called a *linear space.*

A *partial geometry* S with parameters s, t, α is a partial linear space of order (s, t) such that for every non-incident point-block pair (P, B) of S, there are exactly $\alpha > 0$ blocks of S incident with P and concurrent with B.

A partial geometry with parameter $\alpha = 1$ is called a *(finite) generalised quadrangle* or *GQ.* A partial geometry with parameter $\alpha = t$ is called a *net.*

A *semipartial geometry* S with parameters s, t, α, μ, is a partial linear space of order (s, t) such that

- for every non-incident point-block pair (P, B) of S, there are either 0 or $\alpha > 0$ blocks of S incident with P and concurrent with B,

- for every pair (P, Q) of non-collinear points of S, there are exactly $\mu > 0$ points of S collinear with both P and Q.

1.3 Projective spaces

1.3.1 Axiomatic projective spaces

Definition 1.3.1. A *projective space* is an incidence structure $S = (\mathcal{P}, \mathcal{L}, I)$ where I satisfies the following axioms.

AX1 Through every two points of $\mathcal{P}$, there is exactly one line of $\mathcal{L}$.

AX2 If P, Q, R, S are distinct points of $\mathcal{P}$ and the lines PQ and RS intersect, then so do the lines PR and QS.

AX3 There are at least 3 points on a line.

A *subspace* of S is a substructure $(\mathcal{P}', \mathcal{L}')$ of S, with $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$, forming a projective space. A subset $X \subseteq \mathcal{P}$ forms the point set of a subspace if every line containing two points of X is a subset of X. By abuse of notation and phrasing, we identify a subspace with its point set. The *dimension* of S is said to be n if n is the largest number for which there exists a strictly ascending chain of subspaces such that their point sets satisfy $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = \mathcal{P}$. In this chain, the subspace corresponding to X_m is said to have dimension m and is called an *m-space* or *m-subspace* of S. Subspaces of dimension $0, 1, 2$ and $n-1$ are also called *points*, *lines*, *planes* and *hyperplanes*, respectively.

For two subspaces U and W of S, the *intersection* $U \cap W$ is the subspace of S consisting of all points that U and W have in common. This can be generalised to the intersection of k subspaces $U_1, \ldots, U_k$ of S, denoted by $U_1 \cap \ldots \cap U_k$.

For two subspaces U and W of S, the *span* $\langle U, W \rangle$ is the smallest subspace containing the points of both U and W. This definition can as well be generalised to the span of k subspaces $U_1, \ldots, U_k$ of S, denoted by $\langle U_1, \ldots, U_k \rangle$. In the case of two distinct points P_1, P_2 of S, the line $\langle P_1, P_2 \rangle$ is also written as P_1P_2.

The *Grassmann identity* for subspaces of a projective space states that for all subspaces U and W of S, we have:

$$\dim(U) + \dim(W) = \dim(\langle U, W \rangle) + \dim(U \cap W).$$

A *triangle* of a projective space S is a set $\{P_1, P_2, P_3\}$ of three non-collinear points P_i. Two triangles $\{P_1, P_2, P_3\}$ and $\{Q_1, Q_2, Q_3\}$ contained in a plane are *in perspective* when the lines P_1Q_1, P_2Q_2 and P_3Q_3 are concurrent in one point.

Definition 1.3.2. A projective space is called *Desarguesian* if for any two triangles $\{P_1, P_2, P_3\}$ and $\{Q_1, Q_2, Q_3\}$ that are in perspective, we have that the points $P_1P_2 \cap Q_1Q_2$, $P_1P_3 \cap Q_1Q_3$ and $P_2P_3 \cap Q_2Q_3$ are collinear.

Every vector space gives rise to a projective space (see Subsection 1.3.2 for the coordinatisation). Hilbert [62] showed that the Desarguesian projective spaces are precisely the spaces arising from a vector space over a division ring. Moreover, in the finite case, Wedderburn [81] proved that a finite division ring is a field, hence, every finite Desarguesian projective space has a corresponding finite field $\mathbb{F}_q$. The Desarguesian n-dimensional projective spaces over the finite field $\mathbb{F}_q$ are denoted by $\mathrm{PG}(n, q)$ and will be introduced more formally in the next section.

It is shown by Veblen and Young [118] that a finite projective space of dimension $n \geq 3$ is always a Desarguesian projective space $\mathrm{PG}(n, q)$ over the finite field $\mathbb{F}_q$. This is not true for the case $n = 2$, i.e. there do exist non-Desarguesian projective planes. In this thesis, we introduce the notion of translation planes; these can provide examples of non-Desarguesian planes. Other examples of non-Desarguesian projective planes, besides translation planes, can be found in [69].

Besides projective spaces, we also consider affine spaces.

Definition 1.3.3. Consider a hyperplane H_∞ of an n-dimensional projective space $S = (\mathcal{P}, \mathcal{L}, I)$. Define $\mathcal{P}'$, respectively $\mathcal{L}'$, as the set of points of $\mathcal{P}$, respectively lines of $\mathcal{L}$, that are not contained in H_∞. Let I' be the restriction of I to $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$. We call $A = (\mathcal{P}', \mathcal{L}', I')$ an *n-dimensional affine space.* We call H_∞ the *hyperplane at infinity* of A.

Affine spaces can also be introduced by using axioms. We will not give these axioms here, but note that the above definition is equivalent. This means the reverse construction is possible, that is, every affine space can be extended to a projective space by adding a hyperplane at infinity. We call this the *projective completion* $\overline{A}$ of the affine space A. For any affine subspace π of an affine space A, the *projective completion* of π, denoted by $\overline{\pi}$, forms a subspace of $\overline{A}$.

The dual of a projective space can be considered in the following way. Given a finite projective space S, its *dual projective space*, denoted by S^D, is the incidence structure whose points and hyperplanes are respectively the hyperplanes and points of S. If S is n-dimensional, then the i-spaces of S^D correspond to the

$(n-1-i)$-spaces of S. The dual of a Desarguesian projective space is again a Desarguesian projective space.

1.3.2 Projective spaces over finite fields

Standard references for finite geometries are *Finite Geometries* by Dembowski [47], *Finite Projective Spaces of Three Dimensions* by Hirschfeld [63], *Projective Geometries over Finite Fields* by Hirschfeld [64] and *General Galois Geometries* by Hirschfeld and Thas [67]. We refer to these for a more general overview than given here.

Let $V(n+1,q)$ denote the vector space of *rank* $n+1$ over the finite field $\mathbb{F}_q$ (note that we use the word 'rank' instead of 'dimension' to avoid confusion with the dimension of a projective space). The *projective space* corresponding to $V(n+1,q)$ is *Desarguesian* and denoted by $\mathrm{PG}(n,q)$. It corresponds to the space $(V(n+1,q)\setminus\{0\})/\sim$, where $\sim$ is the equivalence relation such that $\forall v \in V(n+1,q)\setminus\{0\}, \forall\lambda \in \mathbb{F}_q^* : \lambda v \sim v$. Note that, for sake of convenience, we use the notation $\mathbb{F}_q^*$ for the set $\mathbb{F}_q \setminus \{0\}$. We say $\Pi = \mathrm{PG}(n,q)$ has *dimension* n, often written as $\dim(\Pi) = n$.

For every subspace of $V(n+1,q)$ of rank $k+1$, we can consider the corresponding *subspace* of $\mathrm{PG}(n,q)$ of dimension k. Every k-dimensional subspace of $\mathrm{PG}(n,q)$ is isomorphic to $\mathrm{PG}(k,q)$. If H_∞ is a hyperplane of $\mathrm{PG}(n,q)$, then the corresponding n-dimensional *affine space* $\mathrm{PG}(n,q)\setminus H_\infty$ is denoted by $\mathrm{AG}(n,q)$.

Due to the relation with vector spaces, we can count the number of subspaces of a certain dimension in $\mathrm{PG}(n,q)$ by using the *Gaussian coefficient*, defined as follows:

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \prod_{i=1}^{b} \frac{q^{a-b+i}-1}{q^i-1} = \frac{(q^a-1)\cdots(q^{a-b+1}-1)}{(q^b-1)\cdots(q-1)}.$$

One can easily check that the number of k-dimensional subspaces in $\mathrm{PG}(n,q)$ is equal to $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$.

Let π be a k-dimensional subspace of $\mathrm{PG}(n,q)$, $k \leq n-2$. We consider the following incidence structure $S = (\mathcal{P}, \mathcal{L}, I)$, where I is symmetric containment:

$\mathcal{P}:$ the set of $(k+1)$-subspaces of $\mathrm{PG}(n,q)$ containing π,
$\mathcal{L}:$ the set of $(k+2)$-subspaces of $\mathrm{PG}(n,q)$ containing π.

We call S the *quotient space* of π, and we denote it by $\mathrm{PG}(n,q)/\pi$. Consider an $(n-k-1)$-dimensional subspace π' of $\mathrm{PG}(n,q)$ disjoint from π, and look at the *projection* of $\mathcal{P}$ and $\mathcal{L}$ from π onto π', that is, identify every element μ in $\mathcal{P}$ or $\mathcal{L}$ with $\mu \cap \pi'$. Then it clearly follows that $\mathrm{PG}(n,q)/\pi \cong \pi' \cong \mathrm{PG}(n-k-1,q)$.

We introduce the following notation for the points of projective spaces. Consider the vector space $V \simeq \mathbb{F}_{q^{n_0}} \times \cdots \times \mathbb{F}_{q^{n_s}}$ of rank $\sum_{i=0}^{s} n_i$ over $\mathbb{F}_q$, for some positive integers n_i. A point P of the corresponding projective space defined by the vector $v = (a_0, \ldots, a_s)$, where $a_i \in \mathbb{F}_{q^{n_i}}$, will be written as $(v)_{\mathbb{F}_q}$ or $(a_0, \ldots, a_s)_{\mathbb{F}_q}$, emphasizing the fact that every $\mathbb{F}_q$-multiple of $v = (a_0, \ldots, a_s)$ gives rise to the point P, i.e.

$$(v)_{\mathbb{F}_q} = \{\lambda v \mid \lambda \in \mathbb{F}_q^*\} \text{ and } (a_0, \ldots, a_s)_{\mathbb{F}_q} = \{(\lambda a_0, \ldots, \lambda a_s) \mid \lambda \in \mathbb{F}_q^*\}.$$

1.3.3 Collineations of PG(n,q)

First, recall the following basic definitions from group theory.

Definition 1.3.4. If a group G has a normal subgroup N and the quotient G/N is isomorphic to some group H, we say that G is an *extension* of N by H. This is written as $G = N.H$.

An extension $G = N.H$ which is a *semidirect product* is also called a *split extension* and is denoted by $G = N \rtimes H$. This means that one can find a subgroup $\widetilde{H} \cong H$ in G such that $G = N.\widetilde{H}$ and $N \cap \widetilde{H} = \{e_G\}$.

An extension $G = N.H$ is a *direct product*, written as $G = N \times H$, when G has normal subgroups $\widetilde{H}$ and $\widetilde{N}$ such that $\widetilde{H} \cong H$, $\widetilde{N} \cong N$, $G = \widetilde{N}.\widetilde{H}$ and $\widetilde{N} \cap \widetilde{H} = \{e_G\}$.

A *linear map* on the vector space $V = V(n+1,q)$ is a mapping $V \to V : x \mapsto xA$, with $x \in V$ a row vector and A a non-singular $(n+1) \times (n+1)$-matrix over $\mathbb{F}_q$. By abuse of notation and phrasing, we identify a linear map with the matrix defining it. The group consisting of all linear maps of $V(n+1,q)$, that is, the group consisting of all non-singular $(n+1) \times (n+1)$-matrix over $\mathbb{F}_q$, is called the *general linear group* and is denoted by $\mathrm{GL}(n+1,q)$.

A *semi-linear map* on the vector space $V = V(n+1,q)$ is a mapping $V \to V : x \mapsto x^\sigma A$, with again $x \in V$ a row vector, A a non-singular $(n+1) \times (n+1)$-matrix over $\mathbb{F}_q$ and with $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$. Again, by abuse of notation and phrasing, we identify a semi-linear map with the pair (A, σ) defining it. Moreover, the semi-linear map corresponding to $(A, \mathbb{1})$ is naturally identified with A. The group consisting of all semi-linear maps of $V(n+1,q)$ is denoted by $\Gamma\mathrm{L}(n+1,q)$. It is clear that $\Gamma\mathrm{L}(n+1,q) \cong \mathrm{GL}(n+1,q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$.

The subgroup of $\mathrm{GL}(n+1,q)$ consisting of all matrices having determinant equal to one, is called the *special linear group* and is denoted by $\mathrm{SL}(n+1,q)$. The sizes of these three groups are the following, for $q = p^h$, p prime, $h \geq 1$:

$$|\Gamma\mathrm{L}(n+1,q)| = hq^{n(n+1)/2}\prod_{i=1}^{n+1}(q^i-1),$$
$$|\mathrm{GL}(n+1,q)| = q^{n(n+1)/2}\prod_{i=1}^{n+1}(q^i-1),$$
$$|\mathrm{SL}(n+1,q)| = q^{n(n+1)/2}\prod_{i=2}^{n+1}(q^i-1).$$

An automorphism of the projective space $\mathrm{PG}(n,q)$, $n \geq 2$, is called a *collineation*. Every semi-linear mapping of $V(n+1,q)$ induces a mapping on the points of the corresponding projective space $\mathrm{PG}(n,q)$, which corresponds to a collineation when $n \geq 2$. Moreover, *the fundamental theorem of projective geometry* states that every collineation of $\mathrm{PG}(n,q)$, $n \geq 2$, arises from a semi-linear map. The group consisting of these maps is called the *collineation group* of $\mathrm{PG}(n,q)$ (even for $n = 1$) and is denoted by $\mathrm{P}\Gamma\mathrm{L}(n+1,q)$. A map arising from a linear map of $V(n+1,q)$ is called a *projectivity* of $\mathrm{PG}(n,q)$. The *projective general linear group* is the subgroup of $\mathrm{P}\Gamma\mathrm{L}(n+1,q)$ containing all projectivities of $\mathrm{PG}(n,q)$ and is denoted by $\mathrm{PGL}(n+1,q)$. We have that $\mathrm{P}\Gamma\mathrm{L}(n+1,q) \cong \mathrm{PGL}(n+1,q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$.

Remark. Note that every semi-linear map (A,σ), A a non-singular $(n+1)\times(n+1)$-matrix over $\mathbb{F}_q$ and $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$, corresponds to an element of $\mathrm{P}\Gamma\mathrm{L}(n+1,q)$ (or $\mathrm{PGL}(n+1,q)$ when $\sigma = \mathbb{1}$). However, every map of $\mathrm{P}\Gamma\mathrm{L}(n+1,q)$ or $\mathrm{PGL}(n+1,q)$ corresponds to several semi-linear maps (A,σ) when $q > 2$.

The subgroup of $\mathrm{PGL}(n+1,q)$ arising from elements of $\mathrm{SL}(n+1,q)$ is denoted by $\mathrm{PSL}(n+1,q)$. Note that all these groups are obtained as quotient groups:

$$\mathrm{P}\Gamma\mathrm{L}(n+1,q) = \Gamma\mathrm{L}(n+1,q)/Z,$$
$$\mathrm{PGL}(n+1,q) = \mathrm{GL}(n+1,q)/Z,$$
$$\mathrm{PSL}(n+1,q) = \mathrm{SL}(n+1,q)/Z',$$

with $Z = \{\lambda I_{n+1} \mid \lambda \in \mathbb{F}_q^*\} \leq \mathrm{GL}(n+1,q)$ and $Z' = \{\lambda I_{n+1} \mid \lambda \in \mathbb{F}_q^*, \lambda^{n+1} = 1\} \leq \mathrm{SL}(n+1,q)$, where I_{n+1} denotes the $(n+1)\times(n+1)$-identity matrix.

Hence, we have the following sizes for these groups:

$$\begin{aligned}
|\mathrm{P\Gamma L}(n+1,q)| &= hq^{n(n+1)/2}\prod_{i=2}^{n+1}(q^i-1),\\
|\mathrm{PGL}(n+1,q)| &= q^{n(n+1)/2}\prod_{i=2}^{n+1}(q^i-1),\\
|\mathrm{PSL}(n+1,q)| &= \frac{1}{d}q^{n(n+1)/2}\prod_{i=2}^{n+1}(q^i-1) \text{ with } d=\gcd(n+1,q-1).
\end{aligned}$$

As $V(1,q^{n+1}) \cong \mathbb{F}_{q^{n+1}} \cong \mathbb{F}_q^{n+1} \cong V(n+1,q)$, the map that multiplies vectors in $V(1,q^{n+1})$ with an element $\alpha \in \mathbb{F}^*_{q^{n+1}}$ induces a linear map $A \in \mathrm{GL}(n+1,q)$ on the vectors of $V(n+1,q)$. It follows that $\mathrm{GL}(n+1,q)$ contains a subgroup isomorphic to $\mathbb{F}^*_{q^{n+1}}$. Singer [105] obtained this in the following way. Let $f(x) = x^{n+1} - m_n x^n - \cdots - m_1 x - m_0$ be an irreducible monic polynomial of degree $n+1$ over $\mathbb{F}_q$ used to construct $\mathbb{F}_{q^{n+1}} \cong \mathbb{F}_q[x]/(f(x))$. Let M be the *companion matrix* of $f(x)$, that is,

$$M = \begin{pmatrix}
0 & 1 & 0 & & \cdots & 0\\
0 & 0 & 1 & & & \vdots\\
 & & & \ddots & & \\
\vdots & \vdots & & & 1 & 0\\
0 & 0 & \cdots & & 0 & 1\\
m_0 & m_1 & \cdots & & m_{n-1} & m_n
\end{pmatrix}.$$

Then it is well known from linear algebra (see for example [68], Theorem 3.3.14) that $f(x)$ is the minimal polynomial of M. Consequently, if we define

$$H = \{a_0 I_{n+1} + a_1 M + \cdots + a_n M^n \mid a_i \in \mathbb{F}_q\},$$

then H has the structure of $\mathbb{F}_{q^{n+1}}$ under usual matrix addition and multiplication. Consider the action of $H \setminus \{0\}$ on $V(n+1,q)$ by right-multiplication, that is, let the matrix act on row vectors from the right. The subgroup in $\mathrm{PGL}(n+1,q)$ inherited by $(H \setminus \{0\}) \leq \mathrm{GL}(n+1,q)$ is denoted by $\mathrm{SG}(n+1,q)$ and is called a *Singer group*. It is a cyclic group of order $\frac{q^{n+1}-1}{q-1}$ acting regularly on the points (and hyperplanes) of $\mathrm{PG}(n,q)$.

A *perspectivity* or *central collineation* of $\mathrm{PG}(n,q)$ is an element of $\mathrm{P\Gamma L}(n+1,q)$ fixing a hyperplane H_∞ of $\mathrm{PG}(n,q)$ pointwise; this hyperplane is called the *axis* of the perspectivity. Every perspectivity also has a *centre*, i.e. a point such that every line through it is stabilised. If the centre belongs to the axis, the perspectivity is called an *elation*. If the centre does not belong to the axis, it is called a *homology*.

Remark. Note that the term *perspectivity* is also used in the setting of non-Desarguesian planes. That is, an incidence preserving map on a projective plane π is called a perspectivity if it fixes a line l_∞ of π (called the axis) pointwise and fixes all lines through a fixed point P (called the center). Also in this context we call the perspectivity an *elation* if the centre belongs to the axis. If the centre does not belong to the axis, it is called a *homology*.

Clearly, the kernel of the action of $\mathrm{P\Gamma L}(n+1, q)$ on a hyperplane H_∞ is the group of all perspectivities with axis H_∞, denoted by $\mathrm{Persp}_q(H_\infty)$. Note that this is in fact a subgroup of $\mathrm{PGL}(n+1, q)$. Similarly, the subgroup of $\mathrm{P\Gamma L}(n+1, q)$, actually of $\mathrm{PGL}(n+1, q)$, consisting of all perspectivities with as centre a point V is denoted by $\mathrm{Persp}_q(V)$. A perspectivity ϕ is uniquely determined by its axis, its centre and one ordered pair $(P, \phi(P))$ for a point P different from the centre and not on the axis. Hence, one can easily count that $|\mathrm{Persp}_q(H_\infty)| = |\mathrm{Persp}_q(V)| = q^n(q-1)$.

The subgroup of $\mathrm{P\Gamma L}(n+1, q)$ consisting of all elations of $\mathrm{PG}(n, q)$ with axis H_∞ is denoted by $\mathrm{Elat}_q(H_\infty)$. One can easily count that $|\mathrm{Elat}_q(H_\infty)| = q^n$. Although the following result is well known, we include a reference for completeness.

Theorem 1.3.5. [94, Lemma 13.]

- *The group* $\mathrm{Elat}_q(H_\infty)$ *is a normal subgroup of* $\mathrm{Persp}_q(H_\infty)$.
- *The group* $\mathrm{Elat}_q(H_\infty)$ *can be identified with the vector space* $V = V(n, q)$.
- *Under this identification, a subgroup of* $\mathrm{Elat}_q(H_\infty)$ *corresponds to a subspace of* V *if and only if it is normalised by* $\mathrm{Persp}_q(H_\infty)$.

Consider a hyperplane H_∞ of $\mathrm{PG}(n, q)$ and the corresponding affine space $\mathrm{AG}(n, q)$. Every map of $\mathrm{P\Gamma L}(n+1, q)$ fixing the hyperplane H_∞ setwise induces an automorphism of $\mathrm{AG}(n, q)$. The respective subgroups of $\mathrm{P\Gamma L}(n+1, q)$ and $\mathrm{PGL}(n+1, q)$ containing these automorphisms of $\mathrm{AG}(n, q)$ are denoted by $\mathrm{A\Gamma L}(n, q)$ and $\mathrm{AGL}(n, q)$; the latter is called the *affine general linear group. The fundamental theorem of affine geometry* states that every automorphism of $\mathrm{AG}(n, q)$, $n \geq 2$, has a corresponding map in the affine group $\mathrm{A\Gamma L}(n, q)$. It is clear that $\mathrm{Persp}_q(H_\infty) \leq \mathrm{AGL}(n, q)$.

Definition 1.3.6. Two point sets $\mathcal{K}$ and $\mathcal{K}'$ of $\mathrm{PG}(n, q)$ are called *projectively equivalent* or PGL-*equivalent* if and only if there is an element $\phi \in \mathrm{PGL}(n+1, q)$

such that $\phi(\mathcal{K}) = \mathcal{K}'$. The sets are called *isomorphic* or PΓL-*equivalent* if and only if there is an element $\phi \in \mathrm{P\Gamma L}(n+1, q)$ such that $\phi(\mathcal{K}) = \mathcal{K}'$. In this case, by abuse of notation, we write $\mathcal{K} \cong \mathcal{K}'$.

1.3.4 Special subsets of projective spaces

In this subsection, we define several subsets of finite projective spaces, which we will encounter throughout this thesis.

Consider a point set $\mathcal{K}$ in $\mathrm{PG}(n, q)$. A line of $\mathrm{PG}(n, q)$ intersecting $\mathcal{K}$ in 0, 1 or ≥ 2 points, is respectively called an *external* line, a *tangent* line or a *secant* line to $\mathcal{K}$. We note that in some contexts, for example for quadrics and Hermitian varieties, lines that are completely contained in $\mathcal{K}$ are also called tangent lines; we will however not use this phrasing.

We call a point set of $\mathrm{PG}(n, q)$, a *point set in general position*, if every subset of $n+1$ points spans the full space. A *basis* of $\mathrm{PG}(n, q)$ is a set of $n+1$ points in general position. A *frame* of $\mathrm{PG}(n, q)$ is a set of $n+2$ points in general position.

Definition 1.3.7. An *n-dimensional subgeometry* of $\mathrm{PG}(n, q)$ of order q_0 is a set of $(q_0^{n+1} - 1)/(q_0 - 1)$ points whose homogeneous coordinates, with respect to a well-chosen frame of $\mathrm{PG}(n, q)$, are in a subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$. In this case we call it an *$\mathbb{F}_{q_0}$-subgeometry.* A k-dimensional subgeometry of a k-space contained in $\mathrm{PG}(n, q)$ is just called a *k-dimensional subgeometry* of $\mathrm{PG}(n, q)$.

When $\mathbb{F}_q = \mathbb{F}_{q_0^2}$, an $\mathbb{F}_{q_0}$-subgeometry is also called a *Baer subgeometry.*

Note that, for $n > 1$, the inherited incidence structure of an n-dimensional $\mathbb{F}_{q_0}$-subgeometry of $\mathrm{PG}(n, q)$ is isomorphic to the projective space $\mathrm{PG}(n, q_0)$. The group $\mathrm{PGL}(n+1, q)$ acts sharply transitively on the (ordered point sets of) frames of $\mathrm{PG}(n, q)$, hence, $n+2$ points in general position define a unique n-dimensional $\mathbb{F}_{q_0}$-subgeometry of $\mathrm{PG}(n, q)$.

Definition 1.3.8. If a point set S contains a frame of $\mathrm{PG}(n, q)$, then its *closure* $\widehat{S}$ consists of the points of the smallest n-dimensional subgeometry of $\mathrm{PG}(n, q)$ containing all the points of S.

The closure $\widehat{S}$ of a point set S can be constructed recursively as follows:

(i) determine the set $\mathcal{A}$ of all subspaces of $\mathrm{PG}(n, q)$ spanned by an arbitrary number of points of S;

(*ii*) determine the set $\widehat{S}$ of points that occur as the exact intersection of two subspaces in $\mathcal{A}$, if $\widehat{S} \neq S$ replace S by $\widehat{S}$ and go to (*i*), otherwise stop.

For $n = 2$, this recursive construction coincides with the definition of the closure of a set of points in a plane containing a quadrangle, given in [69, Chapter XI].

Polar spaces and in particular quadrics are well-studied objects in finite geometry. The only quadrics of importance for this thesis are the non-singular quadrics in $\mathrm{PG}(3, q)$, which are defined as follows.

An *elliptic quadric* $Q^-(3, q)$ in $\mathrm{PG}(3, q)$ is a set of points whose coordinates $(x_0, x_1, x_2, x_3)_{\mathbb{F}_q}$, with respect to a well-chosen frame, satisfy the standard equation $f(x_0, x_1) + x_2x_3 = 0$, where f is an irreducible homogeneous quadratic form over $\mathbb{F}_q$.

A *hyperbolic quadric* $Q^+(3, q)$ in $\mathrm{PG}(3, q)$ is a set of points whose coordinates $(x_0, x_1, x_2, x_3)_{\mathbb{F}_q}$, with respect to a well-chosen frame, satisfy the standard equation $x_0x_1 + x_2x_3 = 0$. Clearly, the points of this hyperbolic quadric are covered by two sets of $q + 1$ disjoint lines, namely the set of lines arising as the intersection of $ax_0 - bx_2 = 0$ and $bx_1 - ax_3 = 0$, $(a, b) \neq (0, 0)$, and the set of lines arising as the intersection of $cx_0 - dx_3 = 0$ and $dx_1 - cx_2 = 0$, $(c, d) \neq (0, 0)$. Each of these sets is called a 1-*regulus* or a *regulus*.

Definition 1.3.9. A *line-blocking set* B in $\mathrm{PG}(n, q)$ is a set of points such that every line of $\mathrm{PG}(n, q)$ contains at least one point of B.

The point set of a hyperplane of $\mathrm{PG}(n, q)$ forms the classical example of a line-blocking set.

Definition 1.3.10. A *cap* of $\mathrm{PG}(n, q)$ is a set of points such that every three points span a plane. A cap of size k is called a k-*cap*. A k-cap is *complete* if it is not contained in a $(k + 1)$-cap.

Caps are mostly studied in $\mathrm{PG}(2, q)$ and in $\mathrm{PG}(3, q)$. A k-cap of $\mathrm{PG}(2, q)$ satisfies $k \leq q + 1$ for q odd and $k \leq q + 2$ for q even. We call a $(q + 1)$-cap in $\mathrm{PG}(2, q)$ an *oval* and a $(q + 2)$-cap a *hyperoval*. A *conic* is the set of points whose coordinates $(x_0, x_1, x_2)_{\mathbb{F}_q}$, with respect to a well-chosen frame, satisfy the standard equation $x_0^2 + x_1x_2 = 0$. It is clear that a conic is an oval, moreover, Segre [101] proved the converse in odd characteristic.

Theorem 1.3.11. [101] *Every oval in* $\mathrm{PG}(2, q)$*,* q *odd, is a conic.*

When q is even, the $q + 1$ tangent lines to an oval in $\mathrm{PG}(2, q)$ are concurrent in one point (see [21]). This intersection point is called the *nucleus* of the oval. An oval in $\mathrm{PG}(2,q)$, q even, extends uniquely to a hyperoval by adding its nucleus. A conic together with its nucleus is called a *hyperconic*. There are multiple examples of hyperovals that are not hyperconics, and all hyperovals in $\mathrm{PG}(2,q)$, $q \leq 32$ even, have been classified, see [58, 85, 92]. However, unlike the q odd case, the classification of (hyper)ovals in $\mathrm{PG}(2,q)$, $q > 32$ even, remains an open problem. For more information on hyperovals, we refer to the survey paper [90] (2003).

Every k-cap of $\mathrm{PG}(3,q)$, $q > 2$, satisfies $k \leq q^2+1$, and a (q^2+1)-cap of $\mathrm{PG}(3,q)$ is often called an *ovoid*. An elliptic quadric $Q^-(3,q)$ is an ovoid, and Barlotti [8] and Panella [88] independently proved the converse in odd characteristic.

Theorem 1.3.12. [8, 88] *Every ovoid of* $\mathrm{PG}(3,q)$*,* q *odd, is an elliptic quadric* $Q^-(3,q)$*.*

In even characteristic, besides the elliptic quadrics, there exists another class of ovoids, namely the *Tits ovoids*. These ovoids exist in $\mathrm{PG}(3,q)$, $q = 2^{2e+1}$, $e \geq 1$, and are projectively equivalent to the point set

$$\{(1,s,t,st+s^{\sigma+2}+t^{\sigma})_{\mathbb{F}_q} \mid s,t \in \mathbb{F}_q\} \cup \{(0,0,0,1)_{\mathbb{F}_q}\},$$

where $\sigma : \mathbb{F}_q \to \mathbb{F}_q : x \mapsto x^{2^{e+1}}$. The only known examples of ovoids in $\mathrm{PG}(3,q)$, q even, are the elliptic quadrics and the Tits ovoids. Moreover, these are the only existing ovoids for $q \leq 32$, see [8, 52, 84, 86, 87]. However, unlike the q odd case, there is no classification of ovoids in $\mathrm{PG}(3,q)$ for $q > 32$ even.

A set of points in general position is also called an arc.

Definition 1.3.13. An *arc* of $\mathrm{PG}(n,q)$ is a set of points such that every subset of $n+1$ points spans $\mathrm{PG}(n,q)$. An arc of size k is called a k*-arc*. A k-arc is *complete* if it is not contained in a $(k+1)$-arc.

Clearly, in $\mathrm{PG}(2,q)$ every cap is an arc and vice versa.

Bush [28] proved that an arc in $\mathrm{PG}(n,q)$, $n > q-2$, has size at most $n+2$. An arc attaining this bound is equivalent to a frame of $\mathrm{PG}(n,q)$.

It is conjectured that an arc in $\mathrm{PG}(n,q)$, $2 \leq n \leq q-2$, has at most $q+1$ points, unless q is even and $n=2$ or $n=q-2$, in which case it has size at most $q+2$. This is the well-known *MDS-conjecture*, in view of its coding-theoretical description. The conjecture is known to be true for many values of q and n. For a summary of these results we refer to [65], more recent results can be found in [4] and [6].

The classical example of an arc of size $q+1$ is given by the normal rational curve.

Definition 1.3.14. A *normal rational curve* in $\mathrm{PG}(n,q)$, $2 \leq n \leq q-2$, is a $(q+1)$-arc PGL-equivalent to the $(q+1)$-arc

$$\{(0,\ldots,0,1)_{\mathbb{F}_q}\} \cup \{(1,t,t^2,t^3,\ldots,t^n)_{\mathbb{F}_q} \mid t \in \mathbb{F}_q\}.$$

There are few examples of $(q+1)$-arcs that are not normal rational curves, we will consider some of them in Chapter 5. In [65], an overview can be found of results showing that for many values of q and n, every $(q+1)$-arc in $\mathrm{PG}(n,q)$ is a normal rational curve.

Definition 1.3.15. A *unital* U in $\mathrm{PG}(2,q^2)$ is a set of q^3+1 points such that every line of $\mathrm{PG}(2,q^2)$ contains either exactly 1 point or $q+1$ points of U.

An example of a unital in $\mathrm{PG}(2,q^2)$ is given by the set of absolute points of a unitary polarity, called a *classical unital* (or *Hermitian curve*). That is, a classical unital in $\mathrm{PG}(2,q^2)$ is a set of q^3+1 points projectively equivalent to the set of points whose coordinates $(x_0,x_1,x_2)_{\mathbb{F}_{q^2}}$ satisfy equation $x_0^{q+1}+x_1^{q+1}+x_2^{q+1}=0$.

Note that every unital in $\mathrm{PG}(2,4)$ is classical. In $\mathrm{PG}(2,q^2)$, $q>2$, there are examples of non-classical unitals. Even so, every known unital, including the classical unital, arises as an *ovoidal Buekenhout-Metz unital*, first introduced by Buekenhout in [27] and extended by Metz in [83]. The exact construction of this unital will be given in Chapter 7. Every unital in $\mathrm{PG}(2,q^2)$, with $q=2,3,4$, corresponds to an ovoidal Buekenhout-Metz unital, see [7, 93]. For $q>4$, the classification of unitals remains an open problem.

All previously defined sets can also be considered in the dual projective space. Structures obtained in this way are called *dual caps*, *dual ovals*, *dual ovoids*, *dual arcs*, *dual unitals*, and so on. We denote the dual of a subspace M or a set of subspaces $\mathcal{O}$ by M^D and $\mathcal{O}^D$.

Remark. Note that, when q is odd, the set of tangent lines to an oval $\mathcal{O}$ in a finite projective plane π of order q forms an oval in the dual plane π^D. Sometimes, in other contents, this oval is called the dual oval of $\mathcal{O}$, written as $\mathcal{O}^D$. However, we will never use this meaning or notation.

1.4 Field reduction and Desarguesian spreads

A *partial* $(n-1)$-*spread* of $\Pi = \mathrm{PG}(N-1,q)$ is a set of mutually disjoint $(n-1)$-spaces. An $(n-1)$-*spread* of Π is a partial spread partitioning the points of Π, i.e. every point of Π is contained in exactly one spread element.

By an easy counting argument, we see that an $(n-1)$-spread in $\mathrm{PG}(N-1,q)$ can exist only if n divides N. The following construction of a *Desarguesian spread* by Segre [103] shows the well-known fact that this condition is also sufficient.

A Desarguesian $(n-1)$-spread of $\mathrm{PG}(rn-1,q)$ can be obtained by applying *field reduction* to the points of $\mathrm{PG}(r-1,q^n)$. The underlying vector space of the projective space $\mathrm{PG}(r-1,q^n)$ is $V(r,q^n)$. When considering $V(r,q^n)$ as a vector space over $\mathbb{F}_q$, we obtain a vector space isomorphic to $V(rn,q)$, which in its turn corresponds to the projective space $\mathrm{PG}(rn-1,q)$. This is the concept of field reduction, namely every point of $\mathrm{PG}(r-1,q^n)$ corresponds to an $(n-1)$-subspace of $\mathrm{PG}(rn-1,q)$. This set of subspaces $\mathcal{D}$ forms an $(n-1)$-spread of $\mathrm{PG}(rn-1,q)$, which is called a *Desarguesian spread.* A different but equivalent construction of a Desarguesian spread, in terms of its *indicator set*, will be considered in Chapter 6.

The $(n-1)$-space of the spread $\mathcal{D}$, corresponding to the point P of $\mathrm{PG}(r-1,q^n)$, will be denoted by $\mathcal{F}_{r,n,q}(P)$. Analogously, for a subset π of $\mathrm{PG}(r-1,q^n)$, we define $\mathcal{F}_{r,n,q}(\pi) := \{\mathcal{F}_{r,n,q}(P) \mid P \in \pi\}$. We refer to $\mathcal{F}_{r,n,q}$ as the *field reduction map* from $\mathrm{PG}(r-1,q^n)$ to $\mathrm{PG}(rn-1,q)$. When there is no ambiguity, we simplify notation by writing $\mathcal{F}$ instead of $\mathcal{F}_{r,n,q}$.

A field reduction map in terms of coordinates can be considered as follows. A point P of $\mathrm{PG}(r-1,q^n)$ defined by the vector $(x_1,x_2,\ldots,x_r) \in (\mathbb{F}_{q^n})^r$ is denoted by $(x_1,x_2,\ldots,x_r)_{\mathbb{F}_{q^n}}$. Since we can identify the vector space $\mathbb{F}_q^{rn}$ with $(\mathbb{F}_{q^n})^r$, every point of $\mathrm{PG}(rn-1,q)$ also has a corresponding vector $(y_1,y_2,\ldots,y_r) \in (\mathbb{F}_{q^n})^r$, but here the point is denoted by $(y_1,y_2,\ldots,y_r)_{\mathbb{F}_q}$. This means, we can consider a field reduction map such that the point $(x_1,\ldots,x_r)_{\mathbb{F}_{q^n}}$ of $\mathrm{PG}(r-1,q^n)$ corresponds to the $(n-1)$-space $\{(\alpha x_1,\ldots,\alpha x_r)_{\mathbb{F}_q} \mid \alpha \in \mathbb{F}_{q^n}\}$ of $\mathrm{PG}(rn-1,q)$.

Since $V(r,q^n) \cong V(rn,q)$, the multiplication of vectors in $V(r,q^n)$ with elements $\alpha \in \mathbb{F}_{q^n}^*$ induces a subgroup of $\mathrm{PGL}(rn,q)$, isomorphic to the Singer group $\mathrm{SG}(n,q)$, such that the Desarguesian $(n-1)$-spread in $\mathrm{PG}(rn-1,q)$, under the corresponding field reduction map, is fixed elementwise. In [49], Dye obtained the full projective automorphism group of a Desarguesian $(n-1)$-spread $\mathcal{D}$ in $\mathrm{PG}(rn-1,q)$. This subgroup of $\mathrm{PGL}(rn,q)$ can be obtained as the extension of the inherited collineation group of $\mathrm{PG}(r-1,q^n)$ by the Singer group $\mathrm{SG}(n,q)$, that is

$$\mathrm{PGL}(rn,q)_{\mathcal{D}} \cong (\mathrm{SG}(n,q).\mathrm{PGL}(r,q^n)) \rtimes \mathrm{Aut}(\mathbb{F}_{q^n}/\mathbb{F}_q).$$

Definition 1.4.1. The *Segre map* $\sigma_{l,k} : \mathrm{PG}(l,q) \times \mathrm{PG}(k,q) \to \mathrm{PG}((l+1)(k+1)-$

$1, q)$ is defined by

$$\begin{aligned}\sigma_{l,k}((x_0,\ldots,x_l)_{\mathbb{F}_q},(y_0,\ldots,y_k)_{\mathbb{F}_q})\\ =(x_0y_0,\ldots,x_0y_k,\ldots,x_ly_0,\ldots,x_ly_k)_{\mathbb{F}_q}.\end{aligned}$$

The image of the Segre map is called the *Segre variety* $\mathbf{S}_{l,k}$.
If we give the points of $\mathrm{PG}((l+1)(k+1)-1,q)$ coordinates of the form

$$(x_{00},x_{01},\ldots,x_{0k};x_{10},\ldots,x_{1k};\ldots;x_{l0},\ldots,x_{lk})_{\mathbb{F}_q},$$

$x_{ij}\in\mathbb{F}_q$, then it is clear that the points of the Segre variety $\mathbf{S}_{l,k}$ are exactly the points that have coordinates such that the matrix (x_{ij}), $0\le i\le l$, $0\le j\le k$, has rank 1 (see also [67, Theorem 25.5.7]).

Fix the point $(x_0,\ldots,x_l)_{\mathbb{F}_q}\in\mathrm{PG}(l,q)$. By varying the point $(y_0,\ldots,y_k)_{\mathbb{F}_q}$ of $\mathrm{PG}(k,q)$ and looking at the images under the Segre map, we obtain a k-dimensional space on $\mathbf{S}_{l,k}$. For every point of $\mathrm{PG}(l,q)$ we obtain such a k-space. The set of all these subspaces, which are clearly disjoint, is called a *system.* Similarly, by fixing the point $(y_0,y_1,\ldots,y_k)_{\mathbb{F}_q}\in\mathrm{PG}(k,q)$, we can obtain an l-dimensional space on $\mathbf{S}_{l,k}$ by varying the point $(x_0,x_1,\ldots,x_l)_{\mathbb{F}_q}$ of $\mathrm{PG}(l,q)$; the set of these subspaces is again a system. Spaces of $\mathbf{S}_{l,k}$ from different systems intersect each other in exactly one point.

We have seen that applying the field reduction map $\mathcal{F}_{r,n,q}$ to all points of $\mathrm{PG}(r-1,q^n)$ gives a Desarguesian $(n-1)$-spread of $\mathrm{PG}(rn-1,q)$. The following result shows that applying the field reduction map to a subgeometry of $\mathrm{PG}(r-1,q^n)$ yields one of the two systems of a Segre variety.

Theorem 1.4.2. [75, Theorem 2.4] *Consider a subgeometry $\pi\cong\mathrm{PG}(k-1,q)$ of $\mathrm{PG}(r-1,q^n)$ of order q, then $\mathcal{F}_{r,n,q}(\pi)$ is projectively equivalent to a system of maximal subspaces of a Segre variety $\mathbf{S}_{k-1,n-1}$ contained in the Segre variety $\mathbf{S}_{r-1,n-1}$.*

An $(n-1)$-*regulus* or *regulus* is a set $\mathcal{R}$ of $q+1$ $(n-1)$-spaces contained in $\mathrm{PG}(2n-1,q)$ such that any line meeting 3 elements of $\mathcal{R}$, intersects all elements of $\mathcal{R}$. Such a line is called a *transversal line* to the regulus $\mathcal{R}$. For every point contained in an element of $\mathcal{R}$, there is a unique transversal line. Note that a regulus is actually one of the systems of a Segre variety $\mathbf{S}_{1,n-1}$, that is, a regulus arises from applying the field reduction map $\mathcal{F}_{2,n,q}$ to an $\mathbb{F}_q$-subline of $\mathrm{PG}(1,q^n)$.

It is well known that 3 mutually disjoint $(n-1)$-spaces A,B,C in $\mathrm{PG}(2n-1,q)$ lie on a unique regulus, denoted by $\mathcal{R}(A,B,C)$. In [76], Lavrauw and Zanella showed that this also holds for Segre varieties. Note that a set of $(n-1)$-spaces in $\mathrm{PG}(rn-1,q)$ is said to be *in general position* when any r of them span the full space.

Theorem 1.4.3. [76, Proposition 2, Corollary 1, Proposition 3] *A set of* $r+1$ $(n-1)$*-spaces in* $\mathrm{PG}(rn-1,q)$ *in general position are contained in a unique Segre variety* $\mathbf{S}_{r-1,n-1}$.

An $(n-1)$-spread $\mathcal{S}$ of $\mathrm{PG}(2n-1,q)$ is *regular* if for every three $(n-1)$-spaces A, B, C of $\mathcal{S}$, the elements of $\mathcal{R}(A,B,C)$ are contained in $\mathcal{S}$. Clearly, when $q=2$, every $(n-1)$-spread of $\mathrm{PG}(2n-1,2)$ is regular. When $q>2$, then $\mathcal{S}$ is regular if and only if $\mathcal{S}$ is Desarguesian (see [25]).

An $(n-1)$-spread $\mathcal{S}$ of $\mathrm{PG}(rn-1,q)$ is called *normal* or *geometric* when the subspace spanned by any two spread elements is partitioned by elements of $\mathcal{S}$. Hence, a subspace generated by any number of elements from a normal spread $\mathcal{S}$ is partitioned by elements of $\mathcal{S}$. Clearly, when $r \leq 2$, every $(n-1)$-spread of $\mathrm{PG}(rn-1,q)$ is normal. When $r>2$, then $\mathcal{S}$ is normal if and only if $\mathcal{S}$ is Desarguesian (see [10]).

1.5 The André/Bruck-Bose representation

Definition 1.5.1. A projective plane Π is called a *translation plane* if there exists a line l_∞, the *translation line*, such that the group of elations with axis l_∞ acts transitively on the points of the corresponding affine plane $\Pi \setminus l_\infty$. In this case, $\Pi \setminus l_\infty$ is called an *affine translation plane*.

Remark. If a projective plane has two distinct translation lines, then all its lines are translation lines and the plane must be Desarguesian, see for instance [69, Theorem 6.18].

The *kernel* of a translation plane is a field and its multiplicative group is isomorphic to the group of all homologies with axis l_∞ and centre a fixed point not on l_∞.

André [2] and Bruck and Bose [24] independently found a representation of translation planes of order q^n with kernel containing $\mathbb{F}_q$ in the projective space $\mathrm{PG}(2n,q)$. We refer to this as the *André/Bruck-Bose representation* or the *ABB-representation*. The construction of André was based on group theory, Bruck and Bose gave an equivalent geometric construction, which is the form we use in this thesis and goes as follows.

Let $\mathcal{S}$ be an $(n-1)$-spread in $\mathrm{PG}(2n-1,q)$. Embed $\mathrm{PG}(2n-1,q)$ as a hyperplane H_∞ in $\mathrm{PG}(2n,q)$. Consider the following incidence structure $A(\mathcal{S}) = (\mathcal{P},\mathcal{L},I)$, where the incidence I is natural:

$\mathcal{P}$: the *affine points*, i.e. the points of $\mathrm{PG}(2n,q)\backslash H_\infty$,

$\mathcal{L}$: the n-spaces of $\mathrm{PG}(2n,q)$ intersecting H_∞ in an element of $\mathcal{S}$.

In [24] the authors showed that $A(\mathcal{S})$ is an affine translation plane of order q^n, and conversely, every such translation plane can be constructed in this way. If the spread $\mathcal{S}$ is Desarguesian, the plane $A(\mathcal{S})$ is a Desarguesian affine plane $\mathrm{AG}(2,q^n)$. The projective completion $\overline{A(\mathcal{S})}$ of the affine plane $A(\mathcal{S})$ can be found by adding H_∞ as the line l_∞ at infinity, i.e. the translation line, where the elements of $\mathcal{S}$ correspond to the points of l_∞. Clearly, the projective completion $\overline{A(\mathcal{S})}$ is a Desarguesian projective plane $\mathrm{PG}(2,q^n)$ if and only if the spread $\mathcal{S}$ is Desarguesian.

The author of [2] obtained that the full automorphism group of $A(\mathcal{S})$ is a subgroup of $\mathrm{P\Gamma L}(2n+1,q)$ isomorphic to the group extension

$$\mathrm{Persp}_q(H_\infty).\mathrm{P\Gamma L}(2n,q)_{\mathcal{S}}.$$

The *Barlotti-Cofman representation* [10] is a generalisation of the André/Bruck-Bose representation in the following way. Let $\mathcal{S}$ be an $(n-1)$-spread in $\mathrm{PG}(rn-1,q)$. Embed $\mathrm{PG}(rn-1,q)$ as a hyperplane H_∞ in $\mathrm{PG}(rn,q)$. Consider the following incidence structure $BC(\mathcal{S}) = (\mathcal{P},\mathcal{L},I)$, where the incidence I is natural:

$\mathcal{P}$: the *affine points*, i.e. the points of $\mathrm{PG}(rn,q)\backslash H_\infty$,

$\mathcal{L}$: the n-spaces of $\mathrm{PG}(rn,q)$ intersecting H_∞ in an element of $\mathcal{S}$.

The structure $BC(\mathcal{S})$ is sometimes called a *translation Sperner space*. This is a specific type of *Sperner space*, also called a *weak affine space*. When $r > 2$, the structure $BC(\mathcal{S})$ is an affine space if and only if $\mathcal{S}$ is a Desarguesian spread. When $\mathcal{S}$ is Desarguesian, $BC(\mathcal{S})$ is isomorphic to $\mathrm{AG}(r,q^n)$. In this case, by adding H_∞ as the hyperplane at infinity where the elements of $\mathcal{S}$ correspond to its points, one obtains the Desarguesian projective space $\mathrm{PG}(r,q^n)$.

In the same way as for translation planes, one can prove that the automorphism group of $BC(\mathcal{S})$ is a subgroup of $\mathrm{P\Gamma L}(rn+1,q)$ isomorphic to

$$\mathrm{Persp}_q(H_\infty).\mathrm{P\Gamma L}(rn,q)_{\mathcal{S}}.$$

I

Pseudo-ovals, eggs and Desarguesian spreads

Part I consists of two chapters, providing characterisations of elementary pseudo-caps (Chapter 2) and Desarguesian spreads (Chapter 3), both in terms of spread inducing elements.

2

Characterisations of elementary pseudo-caps

In this chapter, we obtain characterisations of *pseudo-ovals* in $\mathrm{PG}(3n-1, q)$, q even, and of *pseudo-caps* and *good (weak) eggs* in $\mathrm{PG}(4n-1, q)$.

These results are joint work with G. Van de Voorde and were published in [98] and [99].

2.1 Preliminaries

We study pseudo-caps, more specifically eggs and pseudo-ovals, in the projective space $\mathrm{PG}(N, q)$. These are the higher dimensional equivalents of caps, ovoids and ovals.

Definition 2.1.1. A *pseudo-cap* is a set $\mathcal{A}$ of $(n-1)$-spaces in $\mathrm{PG}(2n+m-1, q)$ such that any three elements of $\mathcal{A}$ span a $(3n-1)$-space.

When $m = n$, a pseudo-cap is also called a *pseudo-arc*. By [112], a pseudo-arc $\mathcal{A}$ in $\mathrm{PG}(3n-1, q)$ satisfies $|\mathcal{A}| \leq q^n + 1$ for q odd and $|\mathcal{A}| \leq q^n + 2$ for q even. If a pseudo-arc $\mathcal{A}$ has $q^n + 1$ or $q^n + 2$ elements, $\mathcal{A}$ is called a *pseudo-oval* or *pseudo-hyperoval* respectively. When $m = 2n$, a pseudo-cap with $q^{2n} + 1$ elements is called a *pseudo-ovoid*.

Examples of pseudo-caps in $\mathrm{PG}(kn-1, q)$ arise by applying field reduction to caps in $\mathrm{PG}(k-1, q^n)$ and if a pseudo-cap is obtained by field reduction, we call it *elementary*.

An important tool to investigate pseudo-caps is the following observation. Every element E_i of a pseudo-cap $\mathcal{A}$ of $\mathrm{PG}(2n+m-1, q)$ defines a partial $(n-1)$-spread

$$\mathcal{S}_i := \{E_1, \ldots, E_{i-1}, E_{i+1}, \ldots, E_{|\mathcal{A}|}\}/E_i$$

in the quotient space $\mathrm{PG}(n+m-1,q) \cong \mathrm{PG}(2n+m-1,q)/E_i$ and we say that the element E_i *induces* the partial spread $\mathcal{S}_i$. An elementary pseudo-cap is contained in a Desarguesian spread. Hence, every element of an elementary pseudo-cap induces a partial spread which extends to a Desarguesian spread.

Definition 2.1.2. A *weak egg* in $\mathrm{PG}(2n+m-1,q)$ is a pseudo-cap of size q^m+1.

Clearly, pseudo-ovals and pseudo-ovoids are examples of weak eggs. A weak egg $\mathcal{E}$ in $\mathrm{PG}(2n+m-1,q)$ is called an *egg* if each element $E \in \mathcal{E}$ is contained in an $(n+m-1)$-space, T_E, which is skew to every element of $\mathcal{E}$ different from E. The space T_E is called the *tangent space* of $\mathcal{E}$ at E. It is easy to see that when $m=n$, every weak egg is an egg. Eggs are studied mostly because of their one-to-one correspondence with *translation generalised quadrangles* of order (q^n,q^m), see Subsection 2.4.4.

The only known examples of eggs in $\mathrm{PG}(2n+m-1,q)$ have either $m=n$ or $m=2n$, and we have the following theorem restricting the number of possibilities for the parameters m and n.

Theorem 2.1.3. [89, Theorem 8.7.2] *If $\mathcal{E}$ is an egg of $\mathrm{PG}(2n+m-1,q)$, then $m=n$ or $ma=n(a+1)$ with a odd. Moreover, if q is even, then $m=n$ or $m=2n$.*

This explains why the study of eggs is mainly focussed on pseudo-ovals and pseudo-ovoids. In the case of pseudo-ovals, all known examples are elementary. All known examples of pseudo-ovoids in $\mathrm{PG}(4n-1,q)$ are elementary when q is even, but in contrast to the situation for pseudo-ovals, when q is odd, there are non-elementary examples of pseudo-ovoids. For an overview of these examples we refer to [72, Section 3.8]. For both pseudo-ovals and pseudo-ovoids, the classification remains an open problem.

This chapter is organised as follows. First, in Section 2.2 we prove some useful results on Desarguesian spreads. Afterwards, we provide characterisations of elementary pseudo-caps in both $\mathrm{PG}(3n-1,q)$ and $\mathrm{PG}(4n-1,q)$ in terms of the induced (partial) spreads of their elements. That is, in Section 2.3 we consider pseudo-(hyper)ovals in $\mathrm{PG}(3n-1,q)$, q even; and in Section 2.4 we focus on elementary pseudo-caps and (weak) eggs in $\mathrm{PG}(4n-1,q)$.

2.2 Desarguesian spreads

A point of $\mathrm{PG}(r-1, q^n)$ will be denoted by $(x_1, x_2, \ldots, x_r)_{\mathbb{F}_{q^n}}$, where $x_i \in \mathbb{F}_{q^n}$. A point of $\mathrm{PG}(rn-1, q)$ will be written as $(x_1, x_2, \ldots, x_r)_{\mathbb{F}_q}$, where $x_i \in \mathbb{F}_{q^n}$. Under a well-chosen field reduction map, a point $(x_1, x_2, \ldots, x_r)_{\mathbb{F}_{q^n}}$ in $\mathrm{PG}(r-1, q^n)$ corresponds to the $(n-1)$-space $\{(\alpha x_1, \alpha x_2, \ldots, \alpha x_r)_{\mathbb{F}_q} \mid \alpha \in \mathbb{F}_{q^n}\}$ in $\mathrm{PG}(rn-1, q)$, which by abuse of notation will also be denoted by $(x_1, x_2, \ldots, x_r)_{\mathbb{F}_{q^n}}$.

Note that in general, different choices of representations of $\mathrm{PG}(r-1, q^n)$ and $\mathrm{PG}(rn-1, q)$ (or equivalently, different choices for a basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$) give rise to different, but projectively equivalent Desarguesian spreads. To get rid of this ambiguity, in the following definition, we will fix a Desarguesian spread in $\mathrm{PG}(r-1, q^n)$.

Definition 2.2.1. Consider a field reduction map $\mathcal{F} = \mathcal{F}_{2,n,q}$ from $\mathrm{PG}(1, q^n)$ to $\mathrm{PG}(2n-1, q)$ and let $\mathcal{D}$ be the corresponding Desarguesian spread in $\mathrm{PG}(2n-1, q)$. An FR_{q^t}*-subline in* $\mathcal{D}$ is a field reduced $\mathbb{F}_{q^t}$-subline, i.e. it is a set of q^t+1 $(n-1)$-spaces of $\mathrm{PG}(2n-1, q)$, obtained as the image of an $\mathbb{F}_{q^t}$-subline of $\mathrm{PG}(1, q^n)$ under $\mathcal{F}$.

Note that for $t = 1$, an FR_q-subline consists of the $q+1$ spaces of an $(n-1)$-regulus. For $t = n$, an FR_{q^n}-subline is simply the set of all elements of the Desarguesian $(n-1)$-spread $\mathcal{D}$ in $\mathrm{PG}(2n-1, q)$.

A geometric characterisation of FR_{q^t}-sublines will be obtained in Chapter 6, Section 6.3.4.

Definition 2.2.2. Consider a field reduction map $\mathcal{F} = \mathcal{F}_{3,n,q}$ from $\mathrm{PG}(2, q^n)$ to $\mathrm{PG}(3n-1, q)$ and let $\mathcal{D}$ be the corresponding Desarguesian spread in $\mathrm{PG}(3n-1, q)$. An FR_{q^t}*-subplane in* $\mathcal{D}$ is a field reduced $\mathbb{F}_{q^t}$-subplane, i.e. it is a set of $q^{2t}+q^t+1$ $(n-1)$-spaces of $\mathrm{PG}(3n-1, q)$, obtained as the image of an $\mathbb{F}_{q^t}$-subplane of $\mathrm{PG}(2, q^n)$ under $\mathcal{F}$.

Note that for $t = 1$, an FR_q-subplane consists of the q^2+q+1 $(n-1)$-spaces forming one system of a Segre variety $\mathbf{S}_{2,n-1}$. For $t = n$, an FR_{q^n}-subplane is just the set of elements of the Desarguesian $(n-1)$-spread $\mathcal{D}$ in $\mathrm{PG}(3n-1, q)$.

It is well known that three disjoint $(n-1)$-spaces in $\mathrm{PG}(2n-1, q)$ are contained in a unique regulus. We will need a generalisation of this in terms of FR_{q^t}-sublines. However, the statement is not true for general t, that is, three disjoint $(n-1)$-spaces can be contained in different FR_{q^t}-sublines of different Desarguesian spreads. However, every Desarguesian spread containing the three elements has only one FR_{q^t}-subline through them.

Lemma 2.2.3. *Three disjoint $(n-1)$-spaces contained in a Desarguesian $(n-1)$-spread $\mathcal{D}$ in $\mathrm{PG}(2n-1,q)$ are contained in a unique FR_{q^t}-subline in $\mathcal{D}$, for all $t|n$.*

Proof. As these three disjoint $(n-1)$-spaces are contained in a Desarguesian $(n-1)$-spread, and all Desarguesian spreads are PGL-equivalent, they can be represented as the $(n-1)$-spaces $(1,0)_{\mathbb{F}_{q^n}}$, $(0,1)_{\mathbb{F}_{q^n}}$ and $(1,1)_{\mathbb{F}_{q^n}}$ of the Desarguesian spread $\mathcal{D} = \{(x,y)_{\mathbb{F}_{q^n}} \mid x,y \in \mathbb{F}_{q^n}\}$. Hence, they are contained in the FR_{q^t}-subline which is the field reduction of the subline $\{(x,y)_{\mathbb{F}_{q^n}} \mid x,y \in \mathbb{F}_{q^t}\}$. Any other FR_{q^t}-subline in $\mathcal{D}$ through these three elements would give rise to a different $\mathbb{F}_{q^t}$-subline through the three points $(1,0)_{\mathbb{F}_{q^n}}$, $(0,1)_{\mathbb{F}_{q^n}}$ and $(1,1)_{\mathbb{F}_{q^n}}$ in $\mathrm{PG}(1,q^n)$, a contradiction. □

We know by [76] that four $(n-1)$-spaces in $\mathrm{PG}(3n-1,q)$ in general position are contained in a unique Segre variety $\mathbf{S}_{2,n-1}$. Again, we need the generalisation for FR_{q^t}-subplanes of $\mathrm{PG}(3n-1,q)$. In exactly the same way as the previous lemma, we can also prove the following.

Lemma 2.2.4. *Four disjoint $(n-1)$-spaces in general position contained in a Desarguesian spread $\mathcal{D}$ in $\mathrm{PG}(3n-1,q)$ are contained in a unique FR_{q^t}-subplane in $\mathcal{D}$, for all $t|n$.*

We will need the following lemma on Desarguesian spreads which has a straightforward proof, but we include it for completeness.

Lemma 2.2.5. *Let $\mathcal{D}_1$ be a Desarguesian $(n-1)$-spread in a $(kn-1)$-dimensional subspace Π of $\mathrm{PG}((k+1)n-1,q)$, let μ be an element of $\mathcal{D}_1$ and let E_1 and E_2 be mutually disjoint $(n-1)$-spaces such that $\langle E_1, E_2\rangle$ meets Π exactly in the space μ. Then there exists a unique Desarguesian $(n-1)$-spread of $\mathrm{PG}((k+1)n-1,q)$ containing E_1, E_2 and all elements of $\mathcal{D}_1$.*

Proof. Since $\mathcal{D}_1$ is a Desarguesian spread in Π, we can choose coordinates for Π such that $\mathcal{D}_1 = \{(x_1,x_2,\dots,x_k)_{\mathbb{F}_{q^n}} \mid x_i \in \mathbb{F}_{q^n}\}$ and $\mu = (0,\dots,0,1)_{\mathbb{F}_{q^n}}$. We embed Π in $\mathrm{PG}((k+1)n-1,q)$ by mapping a point $(x_1,\dots,x_k)_{\mathbb{F}_q}$, $x_i \in \mathbb{F}_{q^n}$, of Π onto $(x_1,\dots,x_k,0)_{\mathbb{F}_q}$. Let ℓ_P denote the unique transversal line through a point P of μ to the regulus $\mathcal{R}(\mu,E_1,E_2)$.

We can still choose coordinates for $n+1$ points in general position in $\mathrm{PG}((k+1)n-1,q)\setminus\Pi$. We will choose these $n+1$ points such that n of them belong to E_1

and one of them belongs to E_2. Consider a set $\{y_i \mid i = 1, \ldots, n\}$ forming a basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. We may assume that the line ℓ_{P_i} through $P_i = (0, \ldots, 0, y_i, 0)_{\mathbb{F}_q} \in \mu$ meets E_1 in the point $(0, \ldots, 0, 0, y_i)_{\mathbb{F}_q}$. It follows that $E_1 = (0, \ldots, 0, 0, 1)_{\mathbb{F}_{q^n}}$. Moreover, we may assume that ℓ_Q with $Q = (0, \ldots, 0, \sum_{i=1}^n y_i, 0)_{\mathbb{F}_q} \in \mu$ meets E_2 in $(0, \ldots, 0, \sum_{i=1}^n y_i, \sum_{i=1}^n y_i)_{\mathbb{F}_q}$. As the point $(0, \ldots, 0, \sum_{i=1}^n y_i, \sum_{i=1}^n y_i)_{\mathbb{F}_q}$ has to be contained in the space spanned by the intersection points $R_i = \ell_{P_i} \cap E_2$, it follows that $R_i = (0, \ldots, 0, y_i, y_i)_{\mathbb{F}_q}$ and consequently, that $E_2 = (0, \ldots, 0, 1, 1)_{\mathbb{F}_{q^n}}$.

It is clear that the Desarguesian spread $\mathcal{D} = \{(x_1, \ldots, x_{k+1})_{\mathbb{F}_{q^n}} \mid x_i \in \mathbb{F}_{q^n}\}$ contains the spread $\mathcal{D}_1$ and the $(n-1)$-spaces E_1 and E_2. Moreover, since every element of $\mathcal{D}$, not in $\langle E_1, E_2\rangle$, is obtained as the intersection of $\langle E_1, X\rangle \cap \langle E_2, Y\rangle$, where $X, Y \in \mathcal{D}_1$, it is clear that $\mathcal{D}$ is the unique Desarguesian spread satisfying our hypothesis. $\square$

Theorem 2.2.6. *A point set $\mathcal{M}$ in $\mathrm{PG}(1, q^n)$, $q > 2$, containing at least three points, such that any three points of $\mathcal{M}$ determine a $\mathbb{F}_q$-subline entirely contained in $\mathcal{M}$, defines an $\mathbb{F}_{q^t}$-subline $\mathrm{PG}(1, q^t)$ for some $t|n$.*

Proof. Without loss of generality, we may assume that the points $(0,1)_{\mathbb{F}_{q^n}}$, $(1,0)_{\mathbb{F}_{q^n}}$ and $(1,1)_{\mathbb{F}_{q^n}}$ are contained in $\mathcal{M}$. Put $M = \{x \mid (1,x)_{\mathbb{F}_{q^n}} \in \mathcal{M}\}$, clearly $\mathbb{F}_q \subseteq M \subseteq \mathbb{F}_{q^n}$.

Consider $x, y \in M$, where $x \neq y$, then every point of the $\mathbb{F}_q$-subline through the distinct points $(0,1)_{\mathbb{F}_{q^n}}$, $(1,x)_{\mathbb{F}_{q^n}}$ and $(1,y)_{\mathbb{F}_{q^n}}$ has to be contained in $\mathcal{M}$. The points of this subline, different from $(0,1)_{\mathbb{F}_{q^n}}$ are given by $(1, x + (y-x)t)_{\mathbb{F}_{q^n}}$, where $t \in \mathbb{F}_q$. This implies that when x and y are in M, also $(1-t)x + ty \in M$, for all $t \in \mathbb{F}_q$. It easily follows that M is closed under taking linear combinations with elements of $\mathbb{F}_q$, hence, M forms an $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^n}$.

Now consider $x', y' \in M \setminus \{0\}$. We claim that (1) $x'^2/y' \in M$ and (2) $x'^2 \in M$. If $y'/x' \in \mathbb{F}_q$, our claim (1) immediately follows from the fact that M is an $\mathbb{F}_q$-subspace, so we may assume that $y'/x' \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$. Since $q > 2$, we can consider an element $t \in \mathbb{F}_q$ such that $t(t-1) \neq 0$. Put $z' := y' - (t-1)x'$. Since M is an $\mathbb{F}_q$-subspace, $z' \in M$. It is easy to check that $z' \notin \{0, x'\}$. Every point of the $\mathbb{F}_q$-subline containing distinct points $(1,0)_{\mathbb{F}_{q^n}}$, $(1,x')_{\mathbb{F}_{q^n}}$ and $(1,z')_{\mathbb{F}_{q^n}}$ has to be contained in M, and the points of this subline, different from $(1,z')_{\mathbb{F}_{q^n}}$, are given by $(z' - x' + t'x', t'x'z')_{\mathbb{F}_{q^n}}$, where $t' \in \mathbb{F}_q$. This implies that $(t'x'z')/(z' + (t'-1)x') \in M$, for every $t' \in \mathbb{F}_q$, so also for $t' = t$, which implies that $tx' - (t(t-1)x'^2)/y' \in M$. Since $tx' \in M$ and $t(t-1) \neq 0$, we conclude that $x'^2/y' \in M$ which proves claim (1). Claim (2) follows immediately from claim (1) by taking $y' = 1 \in \mathbb{F}_q \subseteq M$.

Note that from Claim (1) it follows that for any $y' \in M \setminus \{0\}$ its converse $1/y'$ is also contained in M.

Now let $v, w \in M$ and first suppose that q is odd, then $vw = \frac{1}{2}((v+w)^2 - v^2 - w^2)$, and since M is an $\mathbb{F}_q$-subspace and by claim (2), all terms on the right hand side are in M, hence, so is vw. If q is even, say $q^n = 2^h$, then $v = u^2$ for some $u \in \mathbb{F}_{q^n}$, but since $u = u^{2^h} = v^{2^{h-1}}$, we have that also u is contained in M. This implies that $\frac{v}{w} = \frac{u^2}{w} \in M$ by claim (1) and consequently, again by claim (1), $vw = \frac{v^2}{v/w} \in M$.

In both cases, we get that M is a subfield of $\mathbb{F}_{q^n}$ and the statement follows. □

We deduce the following lemma as a corollary.

Lemma 2.2.7. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two Desarguesian $(n-1)$-spreads in $\mathrm{PG}(2n-1, q)$, $q > 2$, with at least 3 elements in common, then $\mathcal{D}_1$ and $\mathcal{D}_2$ share exactly $q^t + 1$ elements for some $t|n$, forming an FR_{q^t}-subline in both $\mathcal{D}_1$ and $\mathcal{D}_2$. In particular, if n is prime, then $\mathcal{D}_1$ and $\mathcal{D}_2$ share a regulus or coincide.*

Proof. Let $\mathcal{X}$ be the set of common elements of $\mathcal{D}_1$ and $\mathcal{D}_2$. Since a Desarguesian spread $\mathcal{D}$ is regular, it has to contain the reguli defined by any three elements of $\mathcal{D}$, which, since $\mathcal{D}_1$ and $\mathcal{D}_2$ are Desarguesian, implies that the regulus through 3 elements of $\mathcal{X}$ is contained in $\mathcal{X}$. Since the set $\mathcal{X}$ is contained in a Desarguesian spread, $\mathcal{X}$ corresponds to a set of points $\mathcal{M}$ in $\mathrm{PG}(1, q^n)$ such that every $\mathbb{F}_q$-subline through 3 points of $\mathcal{M}$ is contained in $\mathcal{M}$. The first part of the statement now follows from Theorem 2.2.6. The second part follows from the fact that the only divisors of a prime n are 1 and n. □

2.3 Pseudo-ovals in even characteristic

In this section, we focus on *pseudo-ovals* in $\mathrm{PG}(3n-1, q)$. We explain the connection between dual pseudo-ovals and elation Laguerre planes and show that a pseudo-(hyper)oval in $\mathrm{PG}(3n-1, q)$, where q is even and n is prime, such that every element induces a Desarguesian spread, is elementary.

First, consider the following statement, which for $n = 1$, reduces to a well-known and easy to prove statement.

Theorem 2.3.1. [112] *A pseudo-oval in $\mathrm{PG}(3n-1, q)$, q even, is contained in a unique pseudo-hyperoval.*

Note that we call a pseudo-oval, obtained by applying field reduction to a conic in $\mathrm{PG}(2, q^n)$, a *pseudo-conic*. A pseudo-hyperoval (necessarily in even characteristic) obtained by applying field reduction to a conic, together with its nucleus, is called a *pseudo-hyperconic*.

Every element of a pseudo-oval induces a partial spread of $\mathrm{PG}(2n-1, q)$ of size q^n, i.e. we say it has *deficiency* 1. By [17, Theorem 4], we know that it can be extended to a spread in a unique way. This means that the set of points in $\mathrm{PG}(2n-1, q)$, not lying on an element of the partial spread of size q^n, forms an $(n-1)$-space. So by abuse of terminology, we say that an element of a pseudo-oval induces a spread instead of a partial spread.

A natural question to ask is whether we can characterise pseudo-ovals in terms of the induced spreads. Since an elementary pseudo-oval is contained in a Desarguesian spread and as a Desarguesian spread is normal, it follows that for every element of an elementary pseudo-oval the induced spread is Desarguesian. The following theorem shows that for q odd, a strong version of the converse also holds.

Theorem 2.3.2. [36] *If $\mathcal{O}$ is a pseudo-oval in $\mathrm{PG}(3n-1, q)$, q odd, such that for at least one element the induced spread is Desarguesian, then $\mathcal{O}$ is a pseudo-conic.*

In [94] this result was extended to large pseudo-arcs in $\mathrm{PG}(3n-1, q)$.

Theorem 2.3.3. [94] *If $\mathcal{K} = \{K_1, \ldots, K_s\}$ is a pseudo-arc in $\mathrm{PG}(3n-1, q)$, q odd, of size at least the size of the second largest complete arc in $\mathrm{PG}(2, q^n)$, where for one element K_i of $\mathcal{K}$, the partial spread $\mathcal{S} = \{K_1, \ldots, K_{i-1}, K_{i+1}, \ldots, K_s\}/K_i$ extends to a Desarguesian spread of the quotient space $\mathrm{PG}(2n-1, q) = \mathrm{PG}(3n-1, q)/K_i$, then $\mathcal{K}$ is contained in a pseudo-conic.*

The proof of Theorem 2.3.2 relies on a result of Chen and Kaerlein [38] for Laguerre planes in odd order, which in its turn relies on the theorem of Segre [101] characterising every oval in $\mathrm{PG}(2, q)$, q odd, as a conic. This clearly rules out a similar approach for even characteristic. The characterisation of pseudo-ovals in terms of the induced spreads for even characteristic was posed as an open problem in [115, Problem A.3.4].

In this section, we will prove that the following property holds:

Theorem 2.3.22. *If $\mathcal{O}$ is a pseudo-oval in $\mathrm{PG}(3n-1, q)$, $q = 2^h$, $h > 1$, n prime, such that the spread induced by every element of $\mathcal{O}$ is Desarguesian, then $\mathcal{O}$ is elementary.*

As a corollary, we prove a similar statement for pseudo-hyperovals.

Corollary 2.3.23. *Let $\mathcal{H}$ be a pseudo-hyperoval in* $\mathrm{PG}(3n-1, q)$*,* $q = 2^h$*,* $h > 1$*,* n *prime, such that the spread induced by at least* q^n+1 *elements of* $\mathcal{H}$ *is Desarguesian, then* $\mathcal{H}$ *is elementary.*

In Subsection 2.3.1, we will explain the connection between dual pseudo-ovals and elation Laguerre planes, meanwhile proving a theorem that characterises ovoidal Laguerre planes as those elation Laguerre planes obtained from an elementary dual pseudo-oval. In Subsection 2.3.2, we give a proof for our main theorem in $\mathrm{PG}(3n-1, q)$, n prime. We will prove this theorem in a setting for general n; we will formulate a conjecture on hyperovals in $\mathrm{PG}(2, q^n)$ which holds for n prime and which would have the statement for general n as a corollary. We end by stating a corollary of our main theorem in terms of ovoidal Laguerre planes in Subsection 2.3.3.

2.3.1 Laguerre planes

Definition 2.3.4. A *Laguerre plane* $\mathbb{L}$ is an incidence structure with point set $\mathcal{P}$, line set $\mathcal{L}$ and set of circles $\mathcal{C}$ such that $(\mathcal{P}, \mathcal{L} \cup \mathcal{C})$ satisfies the following four axioms:

AX1 Every point lies on a unique line.

AX2 A circle and a line meet in a unique point.

AX3 Through 3 points, no two on a line, there is a unique circle of $\mathcal{C}$.

AX4 If P is a point on a fixed circle C and Q is a point, not on the line through P and not on the circle C, then there is a unique circle C' through P and Q, meeting C only in the point P.

In a finite Laguerre plane, every circle contains $s+1$ points for some s; this constant s is called the *order* of the Laguerre plane.

Starting from a point P of a Laguerre plane $\mathbb{L} = (\mathcal{P}, \mathcal{L} \cup \mathcal{C})$, we obtain an affine plane $(\mathcal{P}', \mathcal{L}')$, where incidence is inherited from $\mathbb{L}$, as follows.

$\mathcal{P}'$: the points of $\mathcal{P}$, different from P and not collinear with P,

$\mathcal{L}'$: (a) the lines of $\mathcal{L}$, not through P,
(b) the circles of $\mathcal{C}$, through P.

The obtained affine plane is called the *derived affine plane* at P.

Definition 2.3.5. A finite *ovoidal* Laguerre plane $(\mathcal{P}, \mathcal{L} \cup \mathcal{C})$ with point set $\mathcal{P}$, line set $\mathcal{L}$ and set of circles $\mathcal{C}$ is a Laguerre plane that can be constructed from a cone $\mathcal{K}$ as follows. Consider a cone $\mathcal{K}$ in $\mathrm{PG}(3,q)$ with vertex a point V and base an oval O in a plane not containing V. Incidence is natural.

$\mathcal{P}:$ the points of $\mathcal{K} \setminus \{V\}$,

$\mathcal{L}:$ the *generators* of $\mathcal{K}$, i.e. the lines of $\mathrm{PG}(3,q)$, lying on $\mathcal{K}$,

$\mathcal{C}:$ the plane sections of $\mathcal{K}$, not containing V.

For later use, we will also consider the *dual model* in $\mathrm{PG}(3,q)$ of the ovoidal Laguerre plane obtained from the cone $\mathcal{K}$. Let H be a plane in $\mathrm{PG}(3,q)$ containing the dual oval O^D (which is the coordinate wise dual in $\mathrm{PG}(2,q)$ of the oval O). It is not hard to see that we find the following incidence structure $(\mathcal{P}, \mathcal{L} \cup \mathcal{C})$:

$\mathcal{P}:$ the planes that intersect H in a line of O^D,

$\mathcal{L}:$ the lines of H contained in O^D,

$\mathcal{C}:$ the points of $\mathrm{PG}(3,q)$ not contained in H, i.e. the affine points.

The ovoidal Laguerre plane obtained this way will be denoted by $L(O^D)$.

Definition 2.3.6. The *classical* Laguerre plane of order q is an ovoidal Laguerre plane obtained from a quadratic cone in $\mathrm{PG}(3,q)$, that is, a cone with base a conic.

Remark. Let $(PQRS)$ denote that four points P, Q, R, S are on a common circle. A Laguerre plane is called *Miquelian* if for each eight pairwise different points A, B, C, D, E, F, G, H, it follows from $(ABCD)$, $(ABEF)$, $(BCFG)$, $(CDGH)$, $(ADEH)$ that $(EFGH)$. However, by a theorem of van der Waerden and Smid, a Laguerre plane is Miquelian if and only if it is classical [117] and we, as well as many others, use the term 'Miquelian Laguerre plane' instead of 'classical Laguerre plane'.

It follows from Theorem 1.3.11 of Segre that an ovoidal Laguerre plane of odd order is necessarily Miquelian.

For later use, we will also introduce the *plane model* of the Miquelian Laguerre plane of even order q (for more information we refer to [16]). Consider a point N in $\mathrm{PG}(2,q)$, q even. Since three non-collinear points together with a nucleus determine a unique conic, one can easily count that there are exactly q^3-q^2 distinct conics in $\mathrm{PG}(2,q)$ having the same point N as their nucleus. The plane model of the Miquelian Laguerre plane is the following incidence structure $(\mathcal{P}, \mathcal{L} \cup \mathcal{C})$ embedded in $\mathrm{PG}(2,q)$, q even, with natural incidence.

$\mathcal{P}$: the points of $\mathrm{PG}(2,q)$ different from N,

$\mathcal{L}$: the lines of $\mathrm{PG}(2,q)$ containing N,

$\mathcal{C}$: the q^2 lines of $\mathrm{PG}(2,q)$ not containing N, and
the q^3-q^2 conics having N as their nucleus.

Remark. This model can be deduced from the standard cone model of the Miquelian Laguerre plane, i.e. the quadratic cone $\mathcal{K}$ with vertex V and base a conic O. That is, one obtains the plane model by projecting the cone $\mathcal{K}$ from a point (on the line through V and the nucleus of O) onto a plane not containing V.

Consider a Laguerre plane $\mathbb{L}$ and its automorphism group $\mathrm{Aut}(\mathbb{L})$. The *kernel* K of $\mathbb{L}$ is the subgroup of $\mathrm{Aut}(\mathbb{L})$ consisting of all automorphisms which map every point P of $\mathbb{L}$ onto a point collinear with P. In other words, K is the elementwise stabiliser of lines of $\mathbb{L}$.

Lemma 2.3.7. [106, Theorem 1] *The order of the kernel K of a Laguerre plane $\mathbb{L}$ of order s divides $s^3(s-1)$. Moreover, $|K| = s^3(s-1)$ if and only if $\mathbb{L}$ is ovoidal.*

Definition 2.3.8. A Laguerre plane $\mathbb{L}$ is an *elation Laguerre plane* if its kernel K acts transitively on the circles of $\mathbb{L}$.

We denote the dual of a pseudo-oval $\mathcal{O}$ of $\mathrm{PG}(3n-1,q)$ by $\mathcal{O}^D$. This is the coordinate wise dual.

Remark. Given a pseudo-oval $\mathcal{O}$ in $\pi = \mathrm{PG}(3n-1,q)$, for every element E of $\mathcal{O}$, there is a unique $(2n-1)$-space intersecting $\mathcal{O}$ only in E. This space is called the tangent space to $\mathcal{O}$ at E. When q is odd, the set of all tangent spaces to $\mathcal{O}$ forms a pseudo-oval in the dual space π^D. Sometimes, in other contents, this pseudo-oval is called the translation dual pseudo-oval of $\mathcal{O}$, written as $\mathcal{O}^*$ or $\mathcal{O}^D$. We will however never consider this type of dual or use this meaning or notation. Note that, when q is even, all tangent spaces intersect in a common $(n-1)$-space, this $(n-1)$-space extends the pseudo-oval to a pseudo-hyperoval.

A dual pseudo-oval $\mathcal{O}^D$ in $\mathrm{PG}(3n-1,q)$ gives rise to an elation Laguerre plane $L(\mathcal{O}^D)$ in the following way. Embed $H_\infty = \mathrm{PG}(3n-1,q)$ as a hyperplane at infinity of $\mathrm{PG}(3n,q)$ and define $L(\mathcal{O}^D)$ to be the incidence structure $(\mathcal{P}, \mathcal{L} \cup \mathcal{C})$ with natural incidence and with

$\mathcal{P}$: the $2n$-spaces meeting H_∞ in an element of $\mathcal{O}^D$,

$\mathcal{L}$: the elements of $\mathcal{O}^D$,

$\mathcal{C}$: the points of $\mathrm{PG}(3n, q)$, not in H_∞.

It is not hard to check that this incidence structure defines a Laguerre plane of order q^n and that the group $\mathrm{Persp}_q(H_\infty) \leq \mathrm{PGL}(3n, q)$ of perspectivities with axis H_∞ induces a subgroup of the kernel of $L(\mathcal{O}^D)$ that acts transitively on the circles of $L(\mathcal{O}^D)$. So $L(\mathcal{O}^D)$ is indeed an elation Laguerre plane.

In [106], Steinke showed the converse: every elation Laguerre plane can be constructed from a dual pseudo-oval.

Theorem 2.3.9. [106, Theorem 4] *A finite Laguerre plane $\mathbb{L}$ is an elation Laguerre plane if and only if $\mathbb{L} \cong L(\mathcal{O}^D)$ for some dual pseudo-oval $\mathcal{O}^D$.*

More explicitely, Steinke shows that a Laguerre plane of order q^n with kernel of order $q^{3n}(q-1)$ can be obtained from a dual pseudo-oval in $\mathrm{PG}(3n-1, q)$.

We will show in Theorem 2.3.11 that every elementary dual pseudo-oval gives rise to an ovoidal Laguerre plane and vice versa. In order to prove this, we need the following lemma.

Lemma 2.3.10. *Let $\mathbb{L}$ be an ovoidal Laguerre plane of order q^n, then there is a unique subgroup T of order q^{3n} in the kernel K of $\mathbb{L}$.*

Proof. Consider the dual model for an ovoidal Laguerre plane. Every perspectivity in $\mathrm{P\Gamma L}(4, q^n)$ with axis H induces an element of K. Since the group $\mathrm{Persp}_{q^n}(H)$ of perspectivities with axis H has order $q^{3n}(q^n - 1)$, which equals the order of K by Lemma 2.3.7, it follows that every element of K corresponds to a perspectivity. The group $\mathrm{Elat}_{q^n}(H)$ consisting of all elations in $\mathrm{PG}(3, q^n)$ with axis H is a normal subgroup of $\mathrm{Persp}_{q^n}(H)$ and has order q^{3n}.

Let S be a subgroup of K of order q^{3n}, $q = p^h$, p prime, then S is a Sylow p-subgroup and since all Sylow p-subgroups are conjugate and $\mathrm{Elat}_{q^n}(H)$ is normal in K, we obtain $S = \mathrm{Elat}_{q^n}(H)$. $\square$

Theorem 2.3.11. *A finite elation Laguerre plane $\mathbb{L}$ is ovoidal if and only if $\mathbb{L} \cong L(\mathcal{O}^D)$ where $\mathcal{O}^D$ is an elementary dual pseudo-oval in* $\mathrm{PG}(3n-1, q)$.

Proof. Let $\mathbb{L}$ be an elation Laguerre plane. By Theorem 2.3.9, $\mathbb{L}$ is isomorphic to $L(\mathcal{O}^D)$, where $\mathcal{O}^D$ is a dual pseudo-oval in $\mathrm{PG}(3n-1, q)$, for some q and n such that the order of $\mathbb{L}$ is q^n. So it remains to show that $L(\mathcal{O}^D)$ is ovoidal if and only if $\mathcal{O}^D$ is elementary. In view of the definition of an ovoidal Laguerre plane, using the dual setting, we will show that $L(\mathcal{O}^D)$ is isomorphic to $L(O^D)$ if and only if the dual pseudo-oval $\mathcal{O}^D$ in $\mathrm{PG}(3n-1, q)$ is obtained from the dual oval O^D in $\mathrm{PG}(2, q^n)$ by field reduction.

First suppose that the dual pseudo-oval $\mathcal{O}^D$ in $\mathrm{PG}(3n-1, q)$ is obtained from a dual oval, say O^D, in $\mathrm{PG}(2, q^n)$ by field reduction. Apply field reduction to the points, lines and circles of $L(O^D)$, then it is clear that the obtained incidence structure $\mathbb{L}^*$, contained in $\mathrm{PG}(4n-1, q)$ is isomorphic to $L(O^D)$. If we intersect the points, lines and circles of $\mathbb{L}^*$ with a fixed $3n$-dimensional subspace of $\mathrm{PG}(4n-1, q)$, through the $(3n-1)$-space containing the field reduced elements of O^D, then the obtained structure is clearly isomorphic to the points, lines and circles from $L(\mathcal{O}^D)$.

Now, let $\mathbb{L} = (\mathcal{P}, \mathcal{L} \cup \mathcal{C})$ be a Laguerre plane that on the one hand is isomorphic to $L(\mathcal{O}^D)$ (call this *model 1*) and on the other hand isomorphic to $L(O^D)$ (call this *model 2*). As before, the elementwise stabiliser of the lines in the automorphism group $\mathrm{Aut}(\mathbb{L})$ of $\mathbb{L}$ (the kernel of $\mathbb{L}$) is denoted by K.

From model 1, we know that the group of elations in $\mathrm{PG}(3n, q)$ with axis the hyperplane H_∞ (which contains the elements of $\mathcal{O}^D$) induces a subgroup of K of order q^{3n}, likewise, from model 2, we know that the group of elations in $\mathrm{PG}(3, q^n)$ with axis the hyperplane H (which contains the elements of O^D) induces a subgroup of K of order q^{3n}. By Lemma 2.3.10 these induced subgroups are the same, denote this group by T. Consider the stabiliser T_P of a point $P \in \mathcal{P}$ in T. From model 2, we have that T_P has order q^{2n}, that is, T_P corresponds to a subgroup of elations with axis H, fixing a plane of $\mathrm{PG}(3, q^n)$, different from H, through a line of O^D. In model 1, the elements of T_P correspond to elations of $\mathrm{PG}(3n-1, q)$ with axis H_∞, fixing a $2n$-space through an element of $\mathcal{O}^D$.

Since T corresponds to the group of elations in $\mathrm{PG}(3, q^n)$, T forms a 3-dimensional vector space V over $\mathbb{F}_{q^n}$. On the other hand, T also corresponds to the group of elations in $\mathrm{PG}(3n, q)$, hence T forms a $3n$-dimensional vector space V' over $\mathbb{F}_q$. In both cases, one can check that T_P is normalised by the group of perspectivities with axis H, respectively H_∞. Hence, by Theorem 1.3.5, we find that T_P forms a 2-dimensional vector subspace $W = V(2, q^n)$ (model 1), and a $2n$-dimensional vector subspace $W' = V(2n, q)$ (model 2). The projective space corresponding to V can be identified with $H \cong \mathrm{PG}(2, q^n)$, the projective space corresponding to V' can be identified with $H_\infty \cong \mathrm{PG}(3n-1, q)$. Clearly, since W and W' correspond to the same vector space, the projective subspace defined by W' is obtained from the projective subspace defined by W by field reduction from H to H_∞.

Now consider a circle $C \in \mathcal{C}$ and denote its $q^n + 1$ points by P_i, $i = 1, \ldots, q^n + 1$.

Every point P_i lies on a unique line ℓ_i of $\mathbb{L}$, so we can identify T_{P_i} with the line ℓ_i. Considering this projectively, we get that for all $i = 1, \ldots, q^n + 1$, the subgroup T_{P_i} is identified on the one hand to an element of $\mathcal{O}^D$ of H_∞ (model 1) and on the other hand to a line of O^D of H (model 2). This implies that $\mathcal{O}^D$ is obtained from O^D by field reduction. $\square$

From this we easily deduce the following corollaries.

Corollary 2.3.12. *A finite elation Laguerre plane* $\mathbb{L}$ *is Miquelian if and only if* $\mathbb{L} \cong L(\mathcal{O}^D)$ *where* $\mathcal{O}^D$ *is a dual pseudo-conic in* $\mathrm{PG}(3n-1, q)$.

Corollary 2.3.13. *Let* $\mathcal{H}^D$ *be a dual hyperoval such that there is an element* E^D *such that* $L(\mathcal{O}^D)$, *where* $\mathcal{O}^D = \mathcal{H}^D \setminus E^D$, *is Miquelian, then* $\mathcal{H}$ *is a pseudo-hyperconic with* E *as the field reduced nucleus.*

Proof. By Corollary 2.3.12, $\mathcal{O}^D$ is the field reduction of a dual conic O^D. The dual conic O^D in $\mathrm{PG}(2, q^n)$ uniquely extends to a dual hyperconic by adding the dual nucleus line N^D. This shows that $\mathcal{O}^D$ can be extended to a dual pseudo-hyperoval $\mathcal{H}^D$ by adding the $(2n-1)$-space E^D which is the field reduced line N^D. Since Theorem 2.3.1 shows that this extension is unique, we see that the element E is the $(n-1)$-space obtained by applying field reduction to the nucleus N of the conic O, and hence, $\mathcal{H}$ is a pseudo-hyperconic. $\square$

2.3.2 Towards the proof of the main theorem

Recall that we will prove the following:

Theorem 2.3.22. *If* $\mathcal{O}$ *is a pseudo-oval in* $\mathrm{PG}(3n-1, q)$, $q = 2^h$, $h > 1$, n *prime, such that the spread induced by every element of* $\mathcal{O}$ *is Desarguesian, then* $\mathcal{O}$ *is elementary.*

We know from Theorem 2.3.1 that a pseudo-oval in even characteristic extends in a unique way to a pseudo-hyperoval and hence, for the proof of our main theorem we will work with $\mathcal{H}$, that is the unique pseudo-hyperoval extending $\mathcal{O}$. As said before, we will only restrict ourselves to the case where n is a prime to finish our proof.

We will split the proof of our main theorem in two cases. In Case 1, we consider pseudo-hyperovals having a specific property (P1) and we prove that they are always elementary. In Case 2, we consider dual pseudo-hyperovals satisfying a property (P2), and again we show that they are elementary. Finally, we prove that if a pseudo-oval $\mathcal{O}$, such that every element induces a Desarguesian spread, extends to a pseudo-hyperoval $\mathcal{H}$ which does not meet property (P1), then its dual $\mathcal{H}^D$ necessarily meets (P2), which implies that $\mathcal{O}$ is elementary.

We need one more definition.

Definition 2.3.14. We will call an FR_{q^t}-subplane in a Desarguesian spread of $\mathrm{PG}(3n-1,q)$ *crowded* (w.r.t. a pseudo-hyperoval $\mathcal{H}$) if it contains q^t+2 elements of $\mathcal{H}$.

Note that for $t=n$, the existence of a crowded FR_{q^n}-subplane implies that $\mathcal{H}$ is contained in a Desarguesian spread, hence, is elementary.

Case 1

First, we will consider a pseudo-hyperoval $\mathcal{H}$ having the following property:

(P1): there exist four elements E_i, $i=1,\ldots,4$, of $\mathcal{H}$, such that

- (i) the induced spreads $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ are Desarguesian,
- (ii) for every $t<n$, the FR_{q^t}-subplane through $\{E_i\}_{i=1,\ldots,4}$ in the Desarguesian spread $\mathcal{D}$ determined by E_1, E_2 and $\mathcal{S}_1$ (seen in $\langle E_3,E_4\rangle$) is not crowded.

Remark. Note that the Desarguesian spread $\mathcal{D}$ in Property (P1) is uniquely determined by Lemma 2.2.5.

Theorem 2.3.15. *Consider a pseudo-hyperoval $\mathcal{H}$ in $\mathrm{PG}(3n-1,q)$, $q=2^h$, $h>1$, satisfying Property (P1), then $\mathcal{H}$ is elementary.*

Proof. Let E_1,E_2,E_3,E_4 be the four elements of $\mathcal{H}$ obtained from Property (P1). Denote the $(n-1)$-space $\langle E_1,E_2\rangle\cap\langle E_3,E_4\rangle$ by μ. The spreads $\mathcal{S}_1$ and $\mathcal{S}_2$ can be seen in $\langle E_3,E_4\rangle=\mathrm{PG}(2n-1,q)$. By the hypothesis, $\mathcal{S}_1$ and $\mathcal{S}_2$ are Desarguesian. Since by definition E_3,E_4 and μ are contained in $\mathcal{S}_1$ and $\mathcal{S}_2$, and $\mathcal{S}_1$ and $\mathcal{S}_2$ are Desarguesian and hence regular, the $q+1$ elements of the unique regulus $\mathcal{R}(\mu,E_3,E_4)$ through E_3,E_4 and μ are contained in $\mathcal{S}_1$ and $\mathcal{S}_2$. Consider the Desarguesian spread $\mathcal{D}$ in $\mathrm{PG}(3n-1,q)$ uniquely determined by $\mathcal{S}_1$ and E_1,E_2. We will prove that $\mathcal{S}_1=\mathcal{S}_2$ and/or $\mathcal{S}_3\subset\mathcal{D}$.

(Step 1)
Since μ,E_1,E_2 are elements of the Desarguesian spread $\mathcal{S}_3$ considered in $\langle E_1,E_2\rangle$,

every element of $\mathcal{R}(\mu, E_1, E_2)$ is contained in $\mathcal{S}_3$. Let X be an element of the regulus $\mathcal{R}(\mu, E_1, E_2)$, different from E_1, E_2 and μ (which exists since $q > 2$).

By construction, the space $\langle X, E_3\rangle$ contains an element E_5 of $\mathcal{H}$. The $(2n-1)$-space $\langle E_1, E_5\rangle$ meets $\langle E_3, E_4\rangle$ in an $(n-1)$-space Y, that is by construction contained in $\mathcal{S}_1$. Since $E_5 = \langle X, E_3\rangle \cap \langle Y, E_1\rangle$ and a Desarguesian spread is normal, we see that $E_5 \in \mathcal{D}$. This holds for every element $E_i \in \mathcal{H}$ on $\langle Z, E_3\rangle$ with $Z \in \mathcal{R}(\mu, E_1, E_2)$, let $E_5, \ldots, E_{q+2}$ be these elements of $\mathcal{H}$.

Now consider the $(n-1)$-spaces $T_i := \langle E_2, E_i\rangle \cap \langle E_3, E_4\rangle$, with $i = 5, \ldots, q+2$. The spaces T_i by definition belong to $\mathcal{S}_2$ (considered in $\langle E_3, E_4\rangle$). But since E_2, E_i, E_3, E_4 are elements of $\mathcal{D}$, T_i is an element of $\mathcal{D}$ and since $\mathcal{D} \cap \langle E_3, E_4\rangle = \mathcal{S}_1$, $T_i \in \mathcal{S}_1$.

So the spreads $\mathcal{S}_1$ and $\mathcal{S}_2$ contain $\mathcal{R}(\mu, E_3, E_4)$ and all elements T_i. If all elements T_i are contained in $\mathcal{R}(\mu, E_3, E_4)$, then the unique FR_q-subplane in $\mathcal{D}$ containing E_1, E_2, E_3 and E_4 is crowded (note that the FR_q-subplane is unique by Lemma 2.2.4). This is in clear contradiction with Property (P1). By Lemma 2.2.7, it now follows that $\mathcal{S}_1$ and $\mathcal{S}_2$ have an FR_{q^s}-subline in common for some $1 < s \leq n$, moreover, if $s = n$, then $\mathcal{S}_1 = \mathcal{S}_2$.

(Step 2)
Consider the smallest t, where $1 < t < n$, such that the unique FR_{q^t}-subline L of $\mathcal{D}$ through μ, E_3, E_4, contains all elements T_i, and thus is contained in both $\mathcal{S}_1$ and $\mathcal{S}_2$. Consider an element $U \in L \backslash \{\mu, E_3, E_4\}$. Since $U \in \mathcal{S}_2$, the space $\langle U, E_2\rangle$ contains an element $F_5 \in \mathcal{H}$. This holds for every element of $L \backslash \{\mu, E_3, E_4\}$, so let $F_5, \ldots, F_{q^t+2}$ be these elements of $\mathcal{H}$. Since a space F_i arises as the intersection $\langle E_2, U\rangle \cap \langle E_1, V\rangle$, for some $U \in L \subset \mathcal{D}$, $V \in \mathcal{S}_1 \subset \mathcal{D}$, all elements F_i belong to $\mathcal{D}$.

Now consider the $(n-1)$-spaces $T'_i := \langle E_3, F_i\rangle \cap \langle E_1, E_2\rangle$, with $i = 5, \ldots, q^t + 2$. The spaces T'_i by definition belong to $\mathcal{S}_3$. But since E_1, E_2, E_3, F_i are elements of $\mathcal{D}$, T'_i is an element of $\mathcal{D}$. By Lemma 2.2.7, this implies that $\mathcal{S}_3$ and $\mathcal{D} \cap \langle E_1, E_2\rangle$ have an FR_{q^s}-subline in common for some $1 < s \leq n$, moreover, if $s = n$, then $\mathcal{S}_3 \subset \mathcal{D}$.

(Step 3)
Consider the smallest t', where $1 < t' < n$, such that the unique $\mathrm{FR}_{q^{t'}}$-subline L' of $\mathcal{D}$ through μ, E_1, E_2, contains all elements T'_i and thus is also contained in $\mathcal{S}_3$. We will repeat the same argument as before in Step 1. Let X' be an element of $L' \setminus \{\mu, E_1, E_2\}$. By construction, the space $\langle X', E_3\rangle$ contains an element E'_5 of $\mathcal{H}$. The $(2n-1)$-space $\langle E_1, E'_5\rangle$ meets $\langle E_3, E_4\rangle$ in an $(n-1)$-space Y' that is by construction contained in $\mathcal{S}_1$. Since $E'_5 = \langle X', E_3\rangle \cap \langle Y', E_1\rangle$, we see that $E'_5 \in \mathcal{D}$.

This holds for every element $E'_i \in \mathcal{H}$ on $\langle E_3, Z' \rangle$ with $Z' \in L' \setminus \{\mu, E_1, E_2\}$, let $E'_5, \ldots, E'_{q^{t'}+2}$ be these elements of $\mathcal{H}$.

Now consider the $(n-1)$-spaces $T''_i := \langle E_2, E'_i \rangle \cap \langle E_3, E_4 \rangle$, with $i = 5, \ldots, q^{t'} + 2$. The spaces T''_i by definition belong to $\mathcal{S}_2$ (considered in $\langle E_3, E_4 \rangle$). But since E_2, E'_i, E_3, E_4 are elements of $\mathcal{D}$, T''_i is an element of $\mathcal{D}$ and since $\mathcal{D} \cap \langle E_3, E_4 \rangle = \mathcal{S}_1$, $T''_i \in \mathcal{S}_1$. This shows that $\mathcal{S}_1$ and $\mathcal{S}_2$ also have the elements T''_i in common. Now arguing as before, if all elements T''_i are contained in the unique $\mathrm{FR}_{q^{t'}}$-subline L'' through μ, E_3, E_4 in $\mathcal{D}$, then the $\mathrm{FR}_{q^{t'}}$-subplane through E_1, E_2, E_3, E_4 in $\mathcal{D}$ is crowded, a contradiction. This implies that $\mathcal{S}_1$ and $\mathcal{S}_2$ have elements outside of L'' in common.

(Step 4)
We can again consider the smallest value of s such that the unique FR_{q^s}-subline M in $\mathcal{D}$ through μ, E_3 and E_4 contains all elements T''_i, say this value is t''. Arguing as before, we either find (1) $t'' = n$, and then $\mathcal{S}_1 = \mathcal{S}_2$ which proves our claim, (2) $\mathcal{S}_3 \subset \mathcal{D}$ which proves our claim, (3) the $\mathrm{FR}_{q^{t''}}$-subplane through E_1, E_2, E_3, E_4 in $\mathcal{D}$ is crowded, which is a contradiction, or (4) we find elements in the intersection of $\mathcal{S}_1$ and $\mathcal{S}_2$ which are not contained in M. In the latter case we can repeat the same argument to find a larger value of s and eventually find that $\mathcal{S}_1 = \mathcal{S}_2$ or $\mathcal{S}_3 \subset \mathcal{D}$.

(Step 5)
When $\mathcal{S}_1 = \mathcal{S}_2$, every element E of $\mathcal{H}$, different from E_1, E_2, E_3, E_4 can be written as $\langle E_1, U \rangle \cap \langle E_2, V \rangle$, where U, V are elements of $\mathcal{S}_1 = \mathcal{S}_2$. It follows that $E \in \mathcal{D}$ for all $E \in \mathcal{H}$. Since $\mathcal{H}$ is contained in a Desarguesian spread, $\mathcal{H}$ is elementary.

When $\mathcal{S}_3 \subset \mathcal{D}$, every element E of $\mathcal{H}$, different from E_1, E_2, E_3, E_4 can be written as $\langle E_1, U \rangle \cap \langle E_3, V \rangle$, where $U \in \mathcal{S}_1 \subset \mathcal{D}$ and $V \in \mathcal{S}_3 \subset \mathcal{D}$. It follows that $E \in \mathcal{D}$ for all $E \in \mathcal{H}$. Since $\mathcal{H}$ is contained in a Desarguesian spread, $\mathcal{H}$ is elementary. □

From the final argument in the previous proof, it is clear that having two induced Desarguesian spreads that coincide is sufficient to say that the pseudo-hyperoval is elementary. This statement can be easily generalised for pseudo-arcs, hence we obtain the following theorem. Note that this statement (in terms of translation generalised quadrangles) was already proven in [114, Theorem 6.1] for pseudo-ovals by using the $\mathbb{F}_{q^n}$-extension of its elements.

Theorem 2.3.16. *Consider a pseudo-arc $\mathcal{K}$ in $\mathrm{PG}(3n-1, q)$. If $\mathcal{K}$ contains two elements E_1 and E_2 such that their induced partial spreads $\mathcal{S}_1$ and $\mathcal{S}_2$ (when viewed in the same $(2n-1)$-space) can both be extended to the same Desarguesian spread $\mathcal{S}$, then $\mathcal{K}$ is elementary.*

Case 2

We will proceed in this case explicitely assuming that the following conjecture holds true.

Conjecture 2.3.17. Consider an oval O in $\mathrm{PG}(2, q^n)$, $q > 2$ even, and let N be the unique point extending O to a hyperoval. Suppose that for every triple of distinct points P_1, P_2, P_3, there is a divisor $t < n$ of n such that the $\mathbb{F}_{q^t}$-subplane through P_1, P_2, P_3 and N contains $q^t + 1$ elements of O, then O is a conic with nucleus N.

Note that for different choices of triples P_1, P_2, P_3, the obtained value of t is allowed to vary. When the value of t is constant for all choices of triples P_1, P_2, P_3, then the conjecture follows from the following result.

Theorem 2.3.18. [91, Theorem 11, Remark 5] *Consider an oval O of* $\mathrm{PG}(2, q^n)$, *$q > 2$ even. Let N be the unique point extending O to a hyperoval. Then O is a conic if and only if every triple of distinct points of O together with N lies in a $\mathbb{F}_q$-subplane that meets O in $q + 1$ points.*

In particular, taking into account that the only divisor $t < n$ of a prime number n equals 1, Conjecture 2.3.17 holds for n prime.

In the proof of this case we will work in the dual setting, so we need the following lemma on dual pseudo-(hyper)ovals.

Lemma 2.3.19. *Let $\mathcal{O}$ be a pseudo-oval in* $\mathrm{PG}(3n-1, q)$ *such that every element $E_i \in \mathcal{O}$, $i = 1, \ldots, q^n + 1$ induces a Desarguesian spread $\mathcal{S}_i$, then the dual pseudo-oval $\mathcal{O}^D$ has the property that for every element E_i^D, the set of intersections $\{E_j^D \cap E_i^D \mid j \neq i\}$ forms a partial spread uniquely extending to a Desarguesian spread and vice versa. The analogous statement holds for pseudo-hyperovals.*

Proof. An element of $\mathcal{S}_i$, say E_1/E_i, equals $\langle E_1, E_i\rangle/E_i$. This $(n-1)$-space can be identified with $\langle E_1, E_i\rangle$ and its dual $\langle E_1, E_i\rangle^D$, which equals $E_1^D \cap E_i^D$.

This implies that the partial spread $\{E_1, \ldots, E_{i-1}, E_{i+1}, \ldots, E_{q^n+1}\}/E_i$ extends to a Desarguesian spread of $\mathrm{PG}(2n-1, q)$ if and only if $\{E_1^D \cap E_i^D, \ldots, E_{i-1}^D \cap E_i^D, E_{i+1}^D \cap E_i^D, \ldots, E_{q^n+1}^D \cap E_i^D\}$ extends to a Desarguesian spread. The same reasoning holds for pseudo-hyperovals. $\square$

Consider a pseudo-hyperoval $\mathcal{H}$ and its dual $\mathcal{H}^D = \{E_1^D, \ldots, E_{q^n+2}^D\}$. We say that an element E_i^D of $\mathcal{H}^D$ *induces* the $(n-1)$-spread $\mathcal{S}'_i := \{E_i^D \cap E_j^D \mid j \neq i\}$ of E_i^D. Now Lemma 2.3.19 states that $\mathcal{S}_i$ is Desarguesian if and only if $\mathcal{S}'_i$ is Desarguesian.

Take note of the following correspondence. There is a unique Desarguesian spread $\mathcal{D}$ containing E_1, E_2 and the elements of $\mathcal{S}_1$ (seen in $\langle E_3, E_4\rangle$). Hence, there is a unique dual Desarguesian spread containing E_1^D, E_2^D and all $(2n-1)$-space of $\mathcal{S}_1^D$, this is of course the dual spread $\mathcal{D}^D$. Two $(2n-1)$-spaces of $\mathcal{D}^D$ intersect each other in an $(n-1)$-space and the set of all these intersections forms a Desarguesian spread $\mathcal{D}'$ of $\mathrm{PG}(3n-1,q)$. The $(n-1)$-spaces arising as the intersection of E_1^D with the $(2n-1)$-spaces of $\mathcal{S}_1^D$ form exactly $\mathcal{S}_1'$. This means that the spread $\mathcal{D}'$ is the unique Desarguesian spread determined by $E_3^D \cap E_4^D, E_2^D \cap E_4^D$ and the elements of $\mathcal{S}_1'$.

We will also consider a dual FR_{q^t}-subplane contained in the dual Desarguesian spread $\mathcal{D}^D$. This is a set of $q^{2t}+q^t+1$ $(2n-1)$-spaces, and from Lemma 2.2.4, four disjoint $(2n-1)$-spaces of $\mathcal{D}^D$ in dual general position are contained in a unique dual FR_{q^t}-subplane of $\mathcal{D}^D$.

Suppose that the dual pseudo-hyperoval $\mathcal{H}^D$ has an element E_1^D such that $\mathcal{H}^D$ and E_1^D satisfy the following property:

(P2): (i) E_1^D induces a Desarguesian spread $\mathcal{S}_1'$,

(ii) for any three elements E_2^D, E_3^D, E_4^D of $\mathcal{H}^D \setminus \{E_1^D\}$ there exists a $t<n$ such that the dual FR_{q^t}-subplane through E_1^D, E_2^D, E_3^D and E_4^D in the dual spread $\mathcal{D}^D$ is crowded, where $\mathcal{D}$ is the Desarguesian spread determined by $\mathcal{S}_1$ and $\mathcal{R}(E_1, E_2, \langle E_1, E_2\rangle \cap \langle E_3, E_4\rangle)$.

Lemma 2.3.20. *Let $\mathcal{H}$ be a pseudo-hyperoval in* $\mathrm{PG}(3n-1,q)$*, $q=2^h$, $h>1$. Assume that*

- *the spread induced by a subset $\mathcal{T}$ of q^n+1 elements of $\mathcal{H}$ is Desarguesian,*
- *$\mathcal{H}^D$ satisfies Property* (P2) *for some element E_1^D of $\mathcal{T}^D$,*
- *Conjecture 2.3.17 holds,*

then the following statements hold:

(i) *the elation Laguerre plane $L(\mathcal{O}^D)$ where $\mathcal{O}^D = \mathcal{H}^D \backslash \{E_1^D\}$ is isomorphic to the Laguerre plane $(\mathcal{P}', \mathcal{L}' \cup \mathcal{C}')$ embedded in π, with natural incidence, given by*

$\mathcal{P}'$: *the lines of π different from ℓ_∞,*
$\mathcal{L}'$: *the points of ℓ_∞,*
$\mathcal{C}'$: *the q^{2n} point-pencils of π not containing ℓ_∞ and*
the $q^{3n}-q^{2n}$ dual ovals all having ℓ_∞ as their nucleus line,

where π is the Desarguesian projective plane from the André/Bruck-Bose representation obtained from the spread $\mathcal{S}_1'$ and where ℓ_∞ is the line at infinity of π corresponding to E_1^D.

(ii) *a dual oval A^D of the set $\mathcal{C}'$ is a dual conic with ℓ_∞ as its nucleus line.*

(iii) *$L(\mathcal{O}^D)$ is Miquelian.*

Proof. (i) Embed the space $\mathrm{PG}(3n-1,q)$, containing $\mathcal{O}^D$, as a hyperplane H_∞ in $\mathrm{PG}(3n,q)$. Recall that the elation Laguerre plane $L(\mathcal{O}^D)$ is the incidence structure $(\mathcal{P},\mathcal{L}\cup\mathcal{C})$ embedded in $\mathrm{PG}(3n,q)$ as follows:

$\mathcal{P}$: the $2n$-spaces meeting H_∞ in an element of $\mathcal{O}^D$,

$\mathcal{L}$: the elements of $\mathcal{O}^D$,

$\mathcal{C}$: the affine points of $\mathrm{PG}(3n,q)\setminus H_\infty$.

Consider a $2n$-space Π of $\mathrm{PG}(3n,q)$ intersecting H_∞ in E_1^D. The elements of $\mathcal{O}^D$ intersect E_1^D in the Desarguesian spread $\mathcal{S}_1'$. It follows that the (projective) André/Bruck-Bose representation in Π, using $\mathcal{S}_1'$, defines a Desarguesian projective plane $\pi\cong\mathrm{PG}(2,q^n)$. The elements of $\mathcal{S}_1'$ correspond to the points of the line ℓ_∞ at infinity of π. By intersecting the elements of $L(\mathcal{O}^D)$ with Π, we find the representation $(\mathcal{P}',\mathcal{L}'\cup\mathcal{C}')$ of the Laguerre plane $L(\mathcal{O}^D)$ in the Desarguesian plane π as given in the statement. For this, we identify every circle of $\mathcal{C}$ with the q^n+1 elements of $\mathcal{P}$ it contains and consider their intersection with Π. Then, an affine point contained in Π corresponds to a point-pencil of π not containing ℓ_∞. An affine point not contained in Π will also correspond to q^n+1 lines of π, different from ℓ_∞. However, since such an affine point does not belong to Π, any three of these lines will have empty intersection, hence they form a dual oval. Moreover, these q^n+1 lines are all concurrent in points not on ℓ_∞, therefore each dual oval extends uniquely to a dual hyperoval by adding the line ℓ_∞.

(ii) Consider the affine point P of $\mathrm{PG}(3n,q)\backslash\Pi$ corresponding to a dual oval A^D of $\mathcal{C}'$. Consider three lines ℓ_2,ℓ_3,ℓ_4 of A^D. These correspond to three elements of $\mathcal{H}^D$, say E_2^D, E_3^D and E_4^D. Now, since $\mathcal{H}^D$ satisfies Property (P2), we find a crowded dual FR_{q^t}-subplane $\mathcal{B}^D$ through the four $(2n-1)$-spaces E_1^D, E_2^D, E_3^D and E_4^D contained in $\mathcal{D}^D$, for some divisor $t<n$ of n. The element E_1^D is contained in $\mathcal{B}^D$, and the projection from P of the $q^{2t}+q^t$ $(2n-1)$-spaces in $\mathcal{B}^D$, different from E_1^D, onto the space Π (used in the André/Bruck-Bose representation) corresponds to $q^{2t}+q^t$ lines of the plane π. Every such projected line intersects ℓ_∞ in a point which corresponds to one of the q^t+1 elements of the unique FR_{q^t}-subline through

$E_1^D \cap E_2^D$, $E_1^D \cap E_3^D$ and $E_1^D \cap E_4^D$ contained in $\mathcal{S}_1'$. This implies that the set of $(2n-1)$-spaces of $\mathcal{B}^D$ corresponds to a dual $\mathbb{F}_{q^t}$-subplane in the Desarguesian plane π, which contains $\ell_\infty, \ell_2, \ell_3, \ell_4$ and $q^t - 2$ other lines of A^D. Since this is true for every choice of three distinct lines ℓ_2, ℓ_3, ℓ_4 of A^D, by Conjecture 2.3.17, A^D is a dual conic with ℓ_∞ as its nucleus line.

(*iii*) We consider the incidence structure $(\mathcal{P}'', \mathcal{L}'' \cup \mathcal{C}'')$ obtained from the incidence structure $(\mathcal{P}', \mathcal{L}' \cup \mathcal{C}')$ in the dual setting of $\mathrm{PG}(2, q^n)$. We use part (*ii*) which states that the dual ovals in $\mathcal{C}'$ are dual conics. Let the point N be the dual of the line ℓ_∞, then $(\mathcal{P}'', \mathcal{L}'' \cup \mathcal{C}'')$ is given by

$\mathcal{P}''$: the points of $\mathrm{PG}(2, q^n)$ different from N,

$\mathcal{L}''$: the lines of $\mathrm{PG}(2, q^n)$ containing N,

$\mathcal{C}''$: the q^{2n} lines of $\mathrm{PG}(2, q^n)$ not containing N, and
the $q^{3n} - q^{2n}$ conics in $\mathrm{PG}(2, q^n)$ having N as their nucleus.

This is the standard plane model for a Miquelian Laguerre plane of even order q^n. □

The proof of the main theorem

We now prove a lemma which gives the connection between Properties (P1) and (P2).

Lemma 2.3.21. *Let $\mathcal{H}$ be a pseudo-hyperoval in* $\mathrm{PG}(3n-1, q)$*, $q = 2^h$, $h > 1$, such that there is a subset $\mathcal{O}$ of $q^n + 1$ elements of $\mathcal{H}$ inducing a Desarguesian spread. If $\mathcal{H}$ does not satisfy Property* (P1)*, then $\mathcal{H}^D$ satisfies* (P2) *for every element of $\mathcal{O}^D$.*

Proof. Let $\mathcal{H}$ be a pseudo-hyperoval in $\mathrm{PG}(3n-1, q)$, $q = 2^h$, $h > 1$, such that there is a subset $\mathcal{O}$ of $q^n + 1$ elements of $\mathcal{H}$ inducing a Desarguesian spread.

Consider a fixed element E_1 of $\mathcal{O}$, and take three other elements E_2, E_3, E_4 of $\mathcal{H}$, where in the case that one of the chosen elements is the element in $\mathcal{H} \setminus \mathcal{O}$, we put $E_4 = \mathcal{H} \setminus \mathcal{O}$. If the hyperoval $\mathcal{H}$ does not satisfy property (P1), then clearly $\mathcal{H}$ does not satisfy property (P1)(*ii*). Hence, there is some $t < n$, for which the unique FR_{q^t}-subplane through E_1, E_2, E_3, E_4 in the unique Desarguesian spread $\mathcal{D}$ through $\mathcal{S}_1$ (seen in $\langle E_3, E_4 \rangle$) and $\mathcal{R}(E_1, E_2, \langle E_1, E_2 \rangle \cap \langle E_3, E_4 \rangle)$ is crowded. Dualising, we see that the element E_1^D satisfies (P2)(*i*) since $\mathcal{S}_1$ is Desarguesian if and only if $\mathcal{S}_1'$ is Desarguesian (by Lemma 2.3.19). Since the unique FR_{q^t}-subplane through E_1, E_2, E_3, E_4 in $\mathcal{D}$ is crowded, by dualising it follows that the unique dual FR_{q^t}-subplane through $E_1^D, E_2^D, E_3^D, E_4^D$ in the dual Desarguesian spread $\mathcal{D}^D$ is crowded. □

Theorem 2.3.22. *If $\mathcal{O}$ is a pseudo-oval in* $\mathrm{PG}(3n-1, q)$*, $q = 2^h$, $h > 1$, n prime, such that the spread induced by every element of $\mathcal{O}$ is Desarguesian, then $\mathcal{O}$ is elementary.*

Proof. By Theorem 2.3.1, consider the unique pseudo-hyperoval $\mathcal{H}$ extending $\mathcal{O}$. Clearly, $\mathcal{H}$ satisfies the conditions of Lemma 2.3.21. This implies that either $\mathcal{H}$ satisfies Property (P1), and then the statement follows from Theorem 2.3.15 (and the fact that a subset of an elementary set is elementary), or $\mathcal{H}$ satisfies Property (P2) for every element of $\mathcal{O}$. Moreover, if n is prime, Conjecture 2.3.17 is true by Theorem 2.3.18, so the assumptions of Lemma 2.3.20 hold for $\mathcal{H}$ and every element E of the subset $\mathcal{O}$ of $\mathcal{H}$. By Lemmas 2.3.20 and 2.3.13, $\mathcal{H}$ is a pseudo-hyperconic with E corresponding to the nucleus N of a conic O (hence $\mathcal{O}$ is elementary). Note that only for $q = 4$ this possibility can occur. When $q > 4$ it is impossible that the set $(O \cup \{N\}) \setminus \{P\}$, where P is a point of O, is again a conic. □

As a corollary, we state a similar statement for pseudo-hyperovals.

Corollary 2.3.23. *Let $\mathcal{H}$ be a pseudo-hyperoval in* $\mathrm{PG}(3n-1, q)$*, $q = 2^h$, $h > 1$, n prime, such that the spread induced by $q^n + 1$ elements of $\mathcal{H}$ is Desarguesian, then $\mathcal{H}$ is elementary.*

Proof. The subset $\mathcal{O}$ of elements inducing a Desarguesian spread is an elementary pseudo-oval by Theorem 2.3.22, suppose $\mathcal{O}$ is the field reduced oval O. There is a unique element extending $\mathcal{O}$ to a pseudo-hyperoval, so $\mathcal{H} \setminus \mathcal{O}$ must be the element corresponding to the unique point of $\mathrm{PG}(2, q^n)$ extending O to a hyperoval. □

2.3.3 A corollary in terms of Laguerre planes

Lemma 2.3.24. *A point P of an elation Laguerre plane $\mathbb{L} = L(\mathcal{O}^D)$, where $\mathcal{O}^D$ is a dual pseudo-oval in* $\mathrm{PG}(3n-1, q)$*, admits a Desarguesian derivation if and only if the spread $\mathcal{S}$, induced by the unique line of $L(\mathcal{O}^D)$ containing P, is Desarguesian.*

Proof. Let P be a point of $\mathbb{L}$, then P is a $2n$-space through an element E^D of $\mathcal{O}^D$. The derived affine plane of order q^n at the point P of $\mathbb{L}$ consists of point set $\mathcal{P}'$ and line set $\mathcal{L}'$ obtained as follows:

$\mathcal{P}'$: the $2n$-spaces in $\mathrm{PG}(3n, q)$ meeting H_∞ in an element of $\mathcal{O}^D \setminus \{E^D\}$,
$\mathcal{L}'$: the elements of $\mathcal{O}^D \setminus \{E^D\}$ and the affine points in P.

Suppose $\mathcal{S}_p$ is the partial $(n-1)$-spread in E^D obtained by intersections with the elements of $\mathcal{O}^D \setminus \{E^D\}$; let $\mathcal{S}$ be its unique extension to a spread. The derived affine plane can be represented in the $2n$-space P as follows:

$\mathcal{P}'$: the n-spaces in P intersecting E^D in an element of $\mathcal{S}_p$,
$\mathcal{L}'$: the elements of $\mathcal{S}_p$ and the affine points in P.

The affine plane $(\mathcal{P}', \mathcal{L}')$ clearly extends to a projective plane by adding the element in $\mathcal{S} \setminus \mathcal{S}_p$ as the line at infinity and adding E^D and the q^n n-spaces in P containing $\mathcal{S} \setminus \mathcal{S}_p$ as the points at infinity. This projective plane is the dual of the plane obtained from the (projective) André/Bruck-Bose representation starting from $\mathcal{S}$ and hence, is Desarguesian if and only if $\mathcal{S}$ is Desarguesian. □

If $\mathbb{L}$ is a Laguerre plane of odd order, then a result of Chen and Kaerlein [38] states that the existence of one point admitting a Desarguesian derivation forces $\mathbb{L}$ to be Miquelian. The following theorem, which is a consequence of our main theorem, gives a similar result in the case of even order Laguerre planes, which is of course not as strong as the result of [38].

Theorem 2.3.25. *Let $\mathbb{L}$ be a Laguerre plane of order q^n with kernel K, $|K| \geq q^{3n}(q-1)$, n prime, $q > 2$ even. Suppose that for every line of $\mathbb{L}$, there exists a point on that line that admits a Desarguesian derivation, then $\mathbb{L}$ is ovoidal and $|K| = q^{3n}(q^n - 1)$.*

Proof. From the hypothesis on the size of K and Lemma 2.3.7, we find that q^{3n} divides the order of T. Hence, by [106, Theorem 2], $\mathbb{L}$ is an elation Laguerre plane. By Theorem 2.3.9, $\mathbb{L}$ can be constructed from a dual pseudo-oval $\mathcal{O}^D$ in $\mathrm{PG}(3n-1, q)$, n prime. From Lemma 2.3.24, we obtain that for every element of $\mathcal{O}^D$ the induced spread is Desarguesian. By Theorem 2.3.22, $\mathcal{O}^D$ is elementary. By Theorem 2.3.11 this implies that $\mathbb{L}$ is ovoidal. Finally, this implies by Lemma 2.3.7 that $|K| = q^{3n}(q^n - 1)$. □

2.4 Characterisations of good (weak) eggs

In this section, we will use the theory of Desarguesian spreads to investigate elementary pseudo-caps and good (weak) eggs. Many previous proofs and characterisations of eggs rely on the connection with eggs and *translation generalised* *quadrangles* [115]. It is our aim to study eggs from a purely geometric perspective, without using neither this connection nor coordinates.

In Subsection 2.4.1 we obtain a connection between good eggs and Desarguesian spreads. This link will enable us to reprove, improve or extend known results in Subsections 2.4.2 and 2.4.3.

Lavrauw [73] characterises elementary eggs in odd characteristic as good eggs for which there exists a $(3n-1)$-space, that contains at least 5 elements of the egg, but is disjoint from the good element. In Subsection 2.4.2, we provide an adaptation of this characterisation for weak eggs in odd and even characteristic. As a corollary, we obtain a direct geometric proof for the theorem of Lavrauw.

Thas, Thas and Van Maldeghem [115] showed that an egg in $\mathrm{PG}(4n-1,q)$, q odd, with two good elements is elementary. By a short combinatorial argument, we show in Subsection 2.4.3 that a similar statement holds for large pseudo-caps, in odd and even characteristic. As a corollary, this improves and extends the result of [115] where one needs at least four good elements of an egg in even characteristic to obtain the same conclusion.

Note that an elementary pseudo-ovoid that arises from applying field reduction to an elliptic quadric is called *classical*.

2.4.1 Good eggs and Desarguesian spreads

A (weak) egg $\mathcal{E}$ in $\mathrm{PG}(2n+m-1,q)$, $m>n$, is *good* at an element $E\in\mathcal{E}$ if every $(3n-1)$-space containing E and at least two other elements of $\mathcal{E}$, contains exactly q^n+1 elements of $\mathcal{E}$. A (weak) egg that has at least one good element is called a *good (weak) egg*. If $\mathcal{E}$ is good at E, then for any two elements $E_1,E_2\in\mathcal{E}\backslash\{E\}$ the $(3n-1)$-space $\langle E,E_1,E_2\rangle$ intersects $\mathcal{E}$ in a pseudo-oval.

Lemma 2.4.1. *Good weak eggs in* $\mathrm{PG}(2n+m-1,q)$ *can only exist if* n *is a divisor of* m*. Good eggs only exist in* $\mathrm{PG}(4n-1,q)$*.*

Proof. Consider a weak egg $\mathcal{E}$ of $\mathrm{PG}(2n+m-1,q)$, $m>n$, good at an element $E\in\mathcal{E}$. Consider a second element $E_1\in\mathcal{E}\backslash\{E\}$. For every element $E_2\in\mathcal{E}\setminus\{E,E_1\}$, the $(3n-1)$-space $\langle E,E_1,E_2\rangle$ intersects $\mathcal{E}$ in a pseudo-oval. In this way we find a set $\mathcal{T}$ of $(3n-1)$-spaces containing $\langle E,E_1\rangle$, such that each space of $\mathcal{T}$ intersects $\mathcal{E}$ in a pseudo-oval. Every two spaces in $\mathcal{T}$ meet exactly in $\langle E,E_1\rangle$ and $\mathcal{E}$ is the union of the pseudo-ovals $\{T\cap\mathcal{E}\mid T\in\mathcal{T}\}$. The set $\mathcal{T}$ consists of $\frac{q^m-1}{q^n-1}$ $(3n-1)$-spaces; as q^n-1 has to be a divisor of q^m-1, it follows that n is a divisor of m.

Suppose $\mathcal{E}$ is an egg. For q even, by Theorem 2.1.3, eggs only exist in $\mathrm{PG}(4n-1,q)$ (or $\mathrm{PG}(3n-1,q)$). Consider now a good egg of $\mathrm{PG}(2n+m-1,q)$, q odd, where m is a multiple of n. By Theorem 2.1.3, $m=\frac{a+1}{a}n$, for some odd integer a, so we conclude that $m=2n$. $\square$

We will show that the good elements of an egg are exactly those inducing a partial spread which is extendable to a Desarguesian spread. Part (i) of the following theorem, for $\mathcal{E}$ an egg, is mentioned in [115, Remark 5.1.7].

Theorem 2.4.2.

(i) *If a weak egg $\mathcal{E}$ in $\mathrm{PG}(2n+m-1,q)$ is good at an element E, then E induces a partial spread which extends to a Desarguesian spread.*

(ii) *Let $\mathcal{E}$ be a weak egg in $\mathrm{PG}(2n+m-1,q)$ for q odd and an egg in $\mathrm{PG}(2n+m-1,q)$ for q even. If an element $E\in\mathcal{E}$ induces a partial spread extending to a Desarguesian spread, then $\mathcal{E}$ is good at E.*

Proof. (i) Suppose $\mathcal{E}$ is a weak egg which is good at E. Consider the partial spread $\mathcal{S}$ of $\mathrm{PG}(n+m-1,q)$ of size q^m induced by E. Because $\mathcal{E}$ is good at E, any two elements of $\mathcal{S}$ span a $(2n-1)$-space which contains a partial spread of q^n elements of $\mathcal{S}$. This partial spread has deficiency 1, so extends uniquely to a spread by one $(n-1)$-space (by [17, Theorem 4]).

Consider three elements $S_1,S_2,S_3\in\mathcal{S}$ not lying in the same $(2n-1)$-space, hence spanning a $(3n-1)$-space π. There are q^n elements of $\mathcal{S}$ contained in $\langle S_2,S_3\rangle$. For every element R of $\mathcal{S}\cap\langle S_2,S_3\rangle$, the $(2n-1)$-space $\langle S_1,R\rangle$ contains q^n elements of $\mathcal{S}$. Hence, there are q^n $(2n-1)$-spaces of π containing S_1 and q^n-1 other elements of $\mathcal{S}$. Similarly, there are q^n $(2n-1)$-spaces of π containing S_2 and q^n-1 other elements of $\mathcal{S}$. Since π has dimension $3n-1$, two such distinct $(2n-1)$-spaces, one containing S_1 and the other containing S_2, intersect in at least an $(n-1)$-space, hence, in exactly an $(n-1)$-space. This space is either an element of $\mathcal{S}$ or the $(n-1)$-space which extends both of them to a spread.

It follows that there are q^{2n} elements of $\mathcal{S}$ contained in π and if an element of $\mathcal{S}$ intersects π, then it is contained in π. Hence, if $\langle S_2,S_3\rangle$ meets a $(2n-1)$-space spanned by S_1 and an other element of $\mathcal{S}$, then they meet in an $(n-1)$-space.

As S_1,S_2,S_3 were chosen randomly, it follows in general that if two distinct $(2n-1)$-spaces spanned by elements of $\mathcal{S}$ intersect, then they meet in an $(n-1)$-space. They meet either in an $(n-1)$-space of $\mathcal{S}$ or in the $(n-1)$-space which extends the partial spreads of both $(2n-1)$-spaces to a spread. Since $\mathcal{S}$ has size q^m and spans $\mathrm{PG}(n+m-1,q)$, we see that $\mathcal{S}$ can be uniquely extended to a spread which is normal, thus Desarguesian.

(ii) Now let $\mathcal{E}$ be an egg if q is even and a weak egg if q is odd. Suppose E induces a partial spread $\mathcal{S}$ of size q^m which extends to a Desarguesian $(n-1)$-spread $\mathcal{D}$ of $\mathrm{PG}(n+m-1,q)$, hence $m = kn$ for some $k > 1$. There are $\frac{q^m-1}{q^n-1}$ elements of $\mathcal{D}$ not contained in $\mathcal{S}$.

When $\mathcal{E}$ is an egg, the elements of $\mathcal{D}\backslash\mathcal{S}$ span an $(m-1)$-space, corresponding to T_E. Hence, any $(2n-1)$-space spanned by two elements of $\mathcal{S}$ contains q^n elements of $\mathcal{S}$ and one element $\mathcal{D}\backslash\mathcal{S}$. So, $\mathcal{E}$ is good at E.

Suppose $\mathcal{E}$ is a weak egg, with q odd. As q is odd, no $(3n-1)$-space intersects $\mathcal{E}$ in a pseudo-hyperoval. Hence, any $(3n-1)$-space containing E intersects $\mathcal{E}$ in at most q^n+1 elements, so any $(2n-1)$-space spanned by two elements of $\mathcal{S}$ can contain at most q^n elements of $\mathcal{S}$. Consequently, any such space must contain at least one element of $\mathcal{D}\backslash\mathcal{S}$.

By field reduction, the elements of the Desarguesian spread $\mathcal{D}$ of $\mathrm{PG}(n+m-1,q)$ are in one-to-one correspondence with the points of $\mathrm{PG}(\frac{m}{n},q^n)$. Any $(2n-1)$-space spanned by two elements of $\mathcal{D}$ must contain at least one element of $\mathcal{D}\backslash\mathcal{S}$. Hence, the points corresponding to $\mathcal{D}\backslash\mathcal{S}$ form a line-blocking set of $\mathrm{PG}(\frac{m}{n},q^n)$. Since $|\mathcal{D}\backslash\mathcal{S}| = \frac{q^m-1}{q^n-1}$, from [22, Theorem 2] it follows that the points corresponding to $\mathcal{D}\backslash\mathcal{S}$ are the points of an $(\frac{m}{n}-1)$-space, hence the elements of $\mathcal{D}\backslash\mathcal{S}$ span an $(m-1)$-space. As before, it follows that $\mathcal{E}$ is good at E. $\square$

The following corollary, for $\mathcal{E}$ an egg, was also mentioned in [113, Theorem 4.3.4] in terms of translation generalised quadrangles.

Corollary 2.4.3. *If a weak egg $\mathcal{E}$, q odd, is good at an element E, then every pseudo-oval of $\mathcal{E}$ containing E is a pseudo-conic.*

Proof. Let Π be an $(n+m-1)$-space disjoint from E. By Theorem 2.4.2, the partial spread $\mathcal{E}/E$ in Π extends to a Desarguesian spread. Consider a pseudo-oval $\mathcal{O}$ of $\mathcal{E}$ containing E. The q^n elements of $\mathcal{O}/E$ are contained in $\mathcal{E}/E$ and thus extend to a Desarguesian spread of the $(2n-1)$-space $\langle\mathcal{O}\rangle \cap \Pi$.

The element E of the pseudo-oval $\mathcal{O}$ induces a partial spread $\mathcal{O}/E$ which extends to a Desarguesian spread, hence, by Theorem 2.3.2, the statement follows. $\square$

2.4.2 A geometric proof of a Theorem of Lavrauw

In this section, we obtain a characterisation of good weak eggs. We need the following lemma stating that every good element of a weak egg has a tangent space.

Lemma 2.4.4. *If a weak egg $\mathcal{E}$ in $\mathrm{PG}(2n+m-1,q)$ is good at an element E, then there exists a unique $(n+m-1)$-space T, such that $T \cap \mathcal{E} = \{E\}$.*

Proof. Consider an $(n+m-1)$-space Σ disjoint from E. If $\mathcal{E}$ is good at E, the element E induces a partial spread $\mathcal{S} = \mathcal{E}/E$ which extends to a Desarguesian spread $\mathcal{D}$ of Σ. As $\mathcal{E}$ is good at E, a $(3n-1)$-space containing E and two other elements intersects $\mathcal{E}$ in q^n+1 elements. Hence, for both q odd and q even, by following the proof of Theorem 2.4.2, part (ii), the elements of $\mathcal{D}\backslash\mathcal{S}$ span and cover an $(m-1)$-space. It is clear that the $(n+m-1)$-space $T = \langle E, \mathcal{D}\backslash\mathcal{S}\rangle$ satisfies $T \cap \mathcal{E} = E$. □

In [74] the authors proved that every egg of $\mathrm{PG}(7,2)$ arises from an elliptic quadric $Q^-(3,4)$ by field reduction. Hence, in the following characterisation, when $\mathcal{E}$ is an egg in $\mathrm{PG}(4n-1,q)$, the condition $q^n > 4$ is essentially not a restriction.

Theorem 2.4.5. *Suppose $n > 1$, $q^n > 4$, consider a weak egg $\mathcal{E}$ in $\mathrm{PG}(4n-1,q)$. Then $\mathcal{E}$ is elementary if and only if the following three properties hold:*

- *$\mathcal{E}$ is good at an element E,*
- *there exists a $(3n-1)$-space, disjoint from E, containing at least 5 elements E_1, E_2, E_3, E_4, E_5 of $\mathcal{E}$,*
- *all pseudo-ovals of $\mathcal{E}$ containing $\{E,E_1\}$, $\{E,E_2\}$ or $\{E,E_3\}$ are elementary.*

Proof. Clearly, if an egg is elementary, the statement is valid.

For the converse, consider the $(3n-1)$-space Π containing 5 elements E_1, E_2, E_3, E_4, E_5 of $\mathcal{E}$, but not the element E. As $\mathcal{E}$ is good at E, the element E induces a partial spread which extends to a Desarguesian $(n-1)$-spread $\mathcal{D}_0$ in Π, which contains E_i, $i = 1, \ldots, 5$.

By Lemma 2.4.4, there exists a unique $(3n-1)$-space T, such that $T \cap \mathcal{E} = \{E\}$. When $\mathcal{E}$ is an egg, this space corresponds to the tangent space T_E.

Consider the two $(n-1)$-spaces $F = \langle E_1, E_5\rangle \cap \langle E_2, E_4\rangle$ and $F' = \langle E_1, E_5\rangle \cap \langle E_3, E_4\rangle$. Both F and F' are contained in $\mathcal{D}_0$, but at most one of them can be contained in the $(2n-1)$-space $\Pi \cap T$. Suppose F is not contained in T (note that this choice has no further impact as E_2 and E_3 play the same role). This implies that the $(2n-1)$-space $\langle E, F\rangle$ contains an element $E_6 \in \mathcal{E}\setminus\{E\}$. By Theorem 2.2.5, there exists a unique Desarguesian spread $\mathcal{D}$ containing E, E_6 and all elements of $\mathcal{D}_0$. We will prove that $\mathcal{E}$ is contained in $\mathcal{D}$.

The $(3n-1)$-space $\langle E, E_1, E_5\rangle$ intersects $\mathcal{E}$ in a pseudo-oval $\mathcal{O}_1$, and the $(3n-1)$-space $\langle E, E_2, E_4\rangle$ intersects $\mathcal{E}$ in a pseudo-oval $\mathcal{O}_2$. Clearly, $\mathcal{O}_1$ and $\mathcal{O}_2$ both contain E_6.

By assumption, $\mathcal{O}_1$ and $\mathcal{O}_2$ are elementary pseudo-ovals. The Desarguesian $(n-1)$-spread in $\langle E, E_1, E_5\rangle$ containing $\mathcal{O}_1$ contains E, E_6 and the q^n+1 elements of $\mathcal{D}_0 \cap \langle E_1, E_5\rangle$. It follows that this Desarguesian spread is contained in $\mathcal{D}$, hence $\mathcal{O}_1$ is contained in $\mathcal{D}$. Analogously, the pseudo-oval $\mathcal{O}_2$ is also contained in $\mathcal{D}$.

There are q^n-2 pseudo-ovals $\mathcal{O}$ of $\mathcal{E}$, containing $\{E, E_3\}$, but not E_6, such that the $(3n-1)$-space $\langle\mathcal{O}\rangle$ does not contain the $(n-1)$-space $T\cap\langle\mathcal{O}_1\rangle$, nor the $(n-1)$-space $T \cap \langle\mathcal{O}_2\rangle$. Take such an oval $\mathcal{O}$, then there is an element E_7 of $\mathcal{E}\setminus\{E\}$ contained in $\langle\mathcal{O}\rangle \cap \langle\mathcal{O}_1\rangle$, hence, $E_7 \in \mathcal{O}\cap\mathcal{O}_1$. Likewise, there is an element E_8 of $\mathcal{E}\setminus\{E\}$ contained in $\mathcal{O}\cap\mathcal{O}_2$.

By assumption, $\mathcal{O}$ is elementary; let $\mathcal{S}_{\mathcal{O}}$ be the Desarguesian $(n-1)$-spread containing $\mathcal{O}$. As E_7 and E_8 are contained in $\mathcal{D}$, the Desarguesian spread $\mathcal{D}$ intersects $\langle E_7, E_8\rangle$ in a Desarguesian spread. Let P be an element of $\mathcal{D}\cap\langle E_7, E_8\rangle$, not contained in T, then $\langle E, P\rangle$ meets Π in an element of $\mathcal{D}$, and hence, $\langle E, P\rangle$ contains an element P' of $\mathcal{E}\setminus E$. As $\langle E, P\rangle$ is contained in $\langle\mathcal{O}\rangle$, P' is an element of $\mathcal{O}$, and hence also of $\mathcal{S}_{\mathcal{O}}$. Since P', E, E_7, E_8 are contained in $\mathcal{S}_{\mathcal{O}}$, the element $P = \langle E, P'\rangle \cap \langle E_7, E_8\rangle$ is an element of $\mathcal{S}_{\mathcal{O}}$. This implies that $\mathcal{D}\cap\langle E_7, E_8\rangle$ and $\mathcal{S}_{\mathcal{O}}\cap\langle E_7, E_8\rangle$ have at least q^n elements in common, which implies in turn that they have all their elements in common. We conclude that $\mathcal{S}_{\mathcal{O}}$ contains E, E_3 and the q^n+1 elements of $\mathcal{D}\cap\langle E_7, E_8\rangle$, hence $\mathcal{S}_{\mathcal{O}}$ and thus all elements of $\mathcal{O}$ are contained in $\mathcal{D}$.

Now, consider an element $E_9 \in \mathcal{E}$, not contained in $\mathcal{O}_1$, $\mathcal{O}_2$ or any of the previously considered q^n-2 pseudo-ovals $\mathcal{O}$. Look at the pseudo-oval $\mathcal{O}' = \langle E, E_1, E_9\rangle \cap \mathcal{E}$ and the pseudo-oval $\mathcal{O}'' = \langle E, E_2, E_9\rangle\cap\mathcal{E}$. At least one of them does not contain E_3. Suppose $\mathcal{O}'$ does not contain E_3 (the proof goes analogously if $\mathcal{O}''$ does not contain E_3). For at most one of the q^n-2 pseudo-ovals $\mathcal{O}$ containing $\{E, E_3\}$ we have $\langle\mathcal{O}\rangle\cap\langle\mathcal{O}'\rangle \in T$. Hence, since $(q^n-2)-1 \geq 2$, we can find at least two distinct elementary pseudo-ovals containing $\{E, E_3\}$ that are contained in $\mathcal{D}$ and have an element E_{10} and E_{11} respectively in common with $\mathcal{O}'$.

Let $\mathcal{S}_{\mathcal{O}'}$ be the Desarguesian $(n-1)$-spread containing $\mathcal{O}'$. As E_{10} and E_{11} are elements of $\mathcal{D}$ the same argument as before shows that all but one element of the

Desarguesian spread $\mathcal{D} \cap \langle E_{10}, E_{11} \rangle$ can be written as the intersection of $\langle E, P'' \rangle$ with $\langle E_{10}, E_{11} \rangle$ for some P'' in $\mathcal{O}'$. It follows that $\mathcal{S}_{\mathcal{O}'}$ contains E, E_1 and the $q^n + 1$ elements of $\mathcal{D} \cap \langle E_{10}, E_{11} \rangle$, hence, that $\mathcal{S}_{\mathcal{O}'}$ is contained in $\mathcal{D}$. In particular, the element E_9 is contained in $\mathcal{D}$, which implies that $\mathcal{E} \subset \mathcal{D}$. We obtain that $\mathcal{E}$ is elementary and more specifically, a field reduced ovoid. $\square$

When $\mathcal{E}$ is good at E and q is odd, by Corollary 2.4.3 all pseudo-ovals of $\mathcal{E}$ containing E are pseudo-conics. From this we obtain the following corollary. The same statement, where $\mathcal{E}$ is an egg, was proven in [73, Theorem 3.2] using coordinates. For $\mathcal{E}$ an egg, this was also shown in [115, Theorem 5.2.3] where a different proof was obtained independently, relying on a technical theorem concerning the $\mathbb{F}_{q^n}$-extension of the egg elements. We have now obtained a direct geometric proof.

Corollary 2.4.6. *A weak egg $\mathcal{E}$ of* $\mathrm{PG}(4n-1, q)$*, q odd, $n > 1$, is classical if and only if it is good at an element E and there exists a $(3n-1)$-space, not containing E, with at least 5 elements of $\mathcal{E}$.*

2.4.3 Eggs with two good elements

We recall the following theorem from [115].

Theorem 2.4.7. [115, Theorem 5.1.12] *If q is odd and an egg $\mathcal{E}$ in* $\mathrm{PG}(4n-1, q)$ *has at least two good elements, then $\mathcal{E}$ is classical. If q is even and an egg $\mathcal{E}$ in* $\mathrm{PG}(4n-1, q)$ *has at least four good elements, not contained in a common pseudo-oval on $\mathcal{E}$, then $\mathcal{E}$ is elementary.*

It was an open problem whether, for q even, being good at two elements is sufficient to be elementary, this was posed as Problem A.5.6 in [115]. We will give an affirmative answer to this question in a more general setting, namely in terms of pseudo-caps.

Lemma 2.4.8. *Consider a pseudo-cap $\mathcal{E}$ of* $\mathrm{PG}(4n-1, q)$ *containing an element E that induces a partial spread which extends to a Desarguesian spread. If Π is a $(3n-1)$-space spanned by E and two other elements of $\mathcal{E}$, then every element of $\mathcal{E}$ is either disjoint from Π or contained in Π.*

Proof. Let Σ be a $(3n-1)$-space skew from E and consider the induced partial spread $\mathcal{E}/E$ in Σ. If F is an element of $\mathcal{E}$ which meets Π, then the projection F/E of F from E onto Σ is an element of $\mathcal{E}/E$ which meets the space Π/E. By assumption, the space Π/E is spanned by spread elements of a partial spread extending to a Desarguesian spread. Hence, since a Desarguesian spread is normal, F/E is contained in Π/E. It follows that, since Π contains E, the element F is contained in Π. $\square$

Theorem 2.4.9. *Consider a pseudo-cap $\mathcal{E}$ in $\mathrm{PG}(4n-1,q)$, $q>2$, with $|\mathcal{E}| > q^{n+k}+q^n-q^k+1$, q odd, and $|\mathcal{E}| > q^{n+k}+q^n+2$, q even, where k is the largest divisor of n with $k \neq n$. The pseudo-cap $\mathcal{E}$ is elementary if and only if two of its elements induce a partial spread which extends to a Desarguesian spread.*

Proof. If $\mathcal{E}$ is elementary, then the elements of $\mathcal{E}$ are contained in a Desarguesian spread of $\mathrm{PG}(4n-1,q)$, so every element of $\mathcal{E}$ induces a partial spread which extends to a Desarguesian spread.

Now suppose that $\mathcal{E}$ contains two distinct elements E_1, E_2 that induce a partial spread which extends to a Desarguesian spread. Since $|\mathcal{E}| > q^n+2$, using Lemma 2.4.8, we can find two elements $E_3, E_4 \in \mathcal{E}$ such that $\langle E_1, E_2, E_3, E_4\rangle$ spans $\mathrm{PG}(4n-1,q)$.

The partial spread induced by E_1 in the space $\langle E_2, E_3, E_4\rangle$ can be extended to a Desarguesian spread $\mathcal{D}_1$. Analogously, the partial spread induced by E_2 in the space $\langle E_1, E_3, E_4\rangle$ can be extended to a Desarguesian spread $\mathcal{D}_2$. Since E_3 and E_4 are elements of the spreads $\mathcal{D}_1$ and $\mathcal{D}_2$, the Desarguesian spreads $\mathcal{D}_1$ and $\mathcal{D}_2$ intersect the $(2n-1)$-space $\langle E_3, E_4\rangle$ each in a Desarguesian spread, say $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively.

Take an element $E \in \mathcal{E}\backslash\{E_1, E_2\}$ and consider the $(3n-1)$-subspace $\langle E_1, E_2, E\rangle$. From Lemma 2.4.8 it follows that any element of $\mathcal{E}$ is either contained in or disjoint from $\langle E_1, E_2, E\rangle$. By considering the elements of $\mathcal{E} \setminus \{E_1, E_2\}$, we find a set $\mathcal{T}$ of $(3n-1)$-spaces containing $\langle E_1, E_2\rangle$, such that each space of $\mathcal{T}$ intersects $\mathcal{E}$ in a pseudo-cap. Every two spaces in $\mathcal{T}$ meet exactly in $\langle E_1, E_2\rangle$ and $\mathcal{E}$ is the union of the pseudo-caps $\{T \cap \mathcal{E} \mid T \in \mathcal{T}\}$. The set $\mathcal{T}$ intersects $\langle E_3, E_4\rangle$ in a partial $(n-1)$-spread $\mathcal{P}$.

Let P be an element of $\mathcal{P}$, then $\langle P, E_1, E_2\rangle$ is a $(3n-1)$-space containing at least one element E of $\mathcal{E}\backslash\{E_1, E_2\}$. The projection E' of E from E_1 onto $\langle E_2, E_3, E_4\rangle$ is contained in $\mathcal{D}_1$. We obtain that $P = \langle E', E_2\rangle \cap \langle E_3, E_4\rangle$, and since the elements E', E_2, E_3, E_4 are contained in $\mathcal{D}_1$, this implies that P is contained in $\mathcal{D}_1$. Moreover, since $P \subset \langle E_3, E_4\rangle$, the element P is contained in $\mathcal{S}_1$. Similarly, we obtain that P is contained in $\mathcal{S}_2$ and we conclude that every element of $\mathcal{P}$ must be contained in both $\mathcal{S}_1$ and $\mathcal{S}_2$.

Suppose that k is the largest divisor of n with $k \neq n$. The pseudo-cap $\mathcal{E}$ has size $|\mathcal{E}| > (q^n-\epsilon)(q^k+1)+2$ and every $(3n-1)$-space of $\mathcal{T}$ contains at most $q^n-\epsilon$

elements different from E_1, E_2, where $\epsilon = 1$ for q odd and $\epsilon = 0$ for q even. By the pigeonhole principle, it follows that $|\mathcal{P}| \geq q^k + 2$. Hence, the Desarguesian spreads $\mathcal{S}_1$ and $\mathcal{S}_2$ have at least $q^k + 2$ elements in common, where k is the largest divisor of n with $k \neq n$. As $q > 2$, by Lemma 2.2.7, we find that $\mathcal{S}_1 = \mathcal{S}_2$.

By Theorem 2.2.5, consider the unique Desarguesian spread $\mathcal{D}$ of $\mathrm{PG}(4n-1, q)$ containing all elements of $\mathcal{D}_1$ and two distinct elements of $\mathcal{D}_2 \backslash \mathcal{D}_1$. It is clear that, since $\mathcal{S}_1 = \mathcal{S}_2$, the spread $\mathcal{D}$ contains all elements of $\mathcal{D}_2$.

Every element of $\mathcal{E}$, not in $\mathcal{D}_1 \cup \mathcal{D}_2$, arises as the intersection $\langle E_1, X \rangle \cap \langle E_2, Y \rangle$ for some $X \in \mathcal{D}_1 \subset \mathcal{D}$ and $Y \in \mathcal{D}_2 \subset \mathcal{D}$, hence, since a Desarguesian spread is normal, every element of $\mathcal{E}$ belongs to $\mathcal{D}$. It follows that $\mathcal{E}$ is elementary. $\square$

We obtain the following corollary which improves [115, Theorem 5.1.12].

Corollary 2.4.10. *A weak egg in* $\mathrm{PG}(4n-1, q)$ *which is good at two distinct elements is elementary.*

Proof. A weak egg is a pseudo-cap of size $q^{2n} + 1$ in $\mathrm{PG}(4n-1, q)$. By Theorem 2.4.2, if the weak egg is good at two elements, these elements induce a partial spread which extends to a Desarguesian spread. We can repeat the proof of Theorem 2.4.9. Now the partial spread $\mathcal{P}$ has size $q^n + 1$, so the conclusion $\mathcal{S}_1 = \mathcal{S}_2$ follows immediately. We do not require Lemma 2.2.7, hence the restriction $q > 2$ can be dropped. $\square$

2.4.4 A corollary in terms of translation generalised quadrangles

Definition 2.4.11. A *generalised quadrangle* (GQ) of order (s, t), $s, t > 1$, is an incidence structure of points and lines satisfying the following axioms:

- every line has exactly $s + 1$ points,
- through every point, there are exactly $t + 1$ lines,
- if P is a point, not on the line L, then there is exactly one line through P which meets L non-trivially.

If a generalised quadrangle S has an abelian subgroup $T \subseteq \mathrm{Aut}(S)$ fixing all lines through a specific point P and acting regularly on all points not collinear with P, then we call S a *translation generalised quadrangle (TGQ) with base point* P. The *kernel* of a TGQ is a field with multiplicative group isomorphic to the subgroup of $\mathrm{Aut}(S)$ fixing all lines through the base point P and fixing all lines through a given point not collinear with P.

From every *egg* $\mathcal{E}$ in $\Sigma_\infty = \mathrm{PG}(2n+m-1,q)$ we can construct a generalised quadrangle $(\mathcal{P},\mathcal{L})$ as follows. Embed Σ_∞ as a hyperplane at infinity of $\mathrm{PG}(2n+m,q)$.

$\mathcal{P}$: (i) the *affine* points of $\mathrm{PG}(2n+m,q)$, i.e. not lying in Σ_∞,
(ii) the $(n+m)$-spaces meeting Σ_∞ in T_E for some $E \in \mathcal{E}$,
(iii) the symbol (∞).

$\mathcal{L}$: (a) the n-spaces meeting Σ_∞ in an element of $\mathcal{E}$,
(b) the elements of $\mathcal{E}$.

Incidence is defined as follows.

- A point of type (i) is incident with the lines of type (a) through it.
- A point of type (ii) is incident with the lines of type (a) it contains and the line of type (b) it contains.
- The point (∞) is incident with all lines of type (b).

The obtained generalised quadrangle is denoted as $T(\mathcal{E})$ and is a TGQ with base point (∞). Moreover, in [89, Theorem 8.7.1], it is proven that every TGQ of order (q^n,q^m), where $\mathbb{F}_q$ is a subfield of its kernel, is isomorphic to $T(\mathcal{E})$ for an egg $\mathcal{E}$ of $\mathrm{PG}(2n+m-1,q)$.

When $n=m=1$, the egg is an oval O of $\mathrm{PG}(2,q)$ and the TGQ is notated by $T_2(O)$. When $n=1$ and $m=2$, the egg is an ovoid $\mathcal{O}$ of $\mathrm{PG}(3,q)$ and the construction above is the construction $T_3(\mathcal{O})$ of Tits (see [115]).

Lemma 2.4.12. *Let $T=T(\mathcal{E})$ be a TGQ of order (q^n,q^{2n}) with base point (∞). Let m_1,m_2,m_3 be three distinct lines through (∞), and let E_1, E_2, E_3 denote the elements of $\mathcal{E}$ corresponding to m_1,m_2,m_3 respectively. Then there is a subquadrangle of order q^n through m_1,m_2,m_3 if and only if the $(3n-1)$-dimensional space $\langle E_1,E_2,E_3\rangle$ contains exactly q^n+1 elements of $\mathcal{E}$.*

Proof. Suppose that the $(3n-1)$-space $\Sigma=\langle E_1,E_2,E_3\rangle$ contains a set $\mathcal{O}$ of exactly q^n+1 elements of $\mathcal{E}$, then it is clear that $T(\mathcal{E})$ defines the incidence structure $T(\mathcal{O})$ in a $3n$-space through Σ. The structure $T(\mathcal{O})$ is a generalised quadrangle of order q^n, forming a subquadrangle of $T(\mathcal{E})$ and containing the lines m_1,m_2,m_3.

On the other hand, suppose that there is a subquadrangle T' of order q^n containing m_1,m_2,m_3, where the lines m_1,m_2,m_3 are incident with (∞). This implies that the point (∞) is in T', and since (∞) lies only on lines of type (b) (i.e. the lines corresponding to elements of $\mathcal{E}$), we deduce that T' contains exactly q^n+1 lines of type (b), among which the lines m_1, m_2 and m_3. Let $\{E_1, \dots, E_{q^n+1}\}$ be the egg elements corresponding to these lines. This means that there are $(q^n+1)q^{2n}$ lines in T' of type (a), containing in total $(q^n+1)q^{2n}(q^n)/(q^n+1) = q^{3n}$ points of type (i) (i.e. affine points).

Each $(n-1)$-space E_j is contained in q^{2n} n-spaces corresponding to a line of type (a) of T' and every affine point is contained in exactly one n-space containing E_j. Let P_j be a point of the space E_j, then we see that the q^{3n} affine points of T' lie on q^{2n} lines through P_j. As this holds for every $j \in \{1, \dots, q^n+1\}$, it is clear that the q^{3n} affine points of T' are contained in a $3n$-space. This in turn implies that the elements $E_1, \dots, E_{q^n+1}$ are contained in a $(3n-1)$-space, namely $\langle E_1, E_2, E_3 \rangle$. Hence, this space contains at least q^n+1 elements of $\mathcal{E}$. Since $\mathcal{E}$ is an egg, it is not possible that a $(3n-1)$-space contains more than q^n+1 elements of $\mathcal{E}$, which concludes the proof. $\square$

Lemma 2.4.13. *Let $T = T(\mathcal{E})$ be a TGQ of order (q^n, q^{2n}) with base point (∞). Let ℓ be a line through (∞) and E_ℓ the element of $\mathcal{E}$ corresponding to ℓ. The egg $\mathcal{E}$ is good at E_ℓ if and only if for every two distinct lines m_1, m_2 through (∞), where $m_1, m_2 \neq \ell$, there is a subquadrangle of order q^n through m_1, m_2, ℓ.*

Proof. This follows immediately from Lemma 2.4.12 and the definition of being good at an element. $\square$

We are now ready to state the promised characterisation of the translation generalised quadrangle $T_3(\mathcal{O})$, which follows from Corollary 2.4.10.

Theorem 2.4.14. *Let T be a TGQ of order (q^n, q^{2n}) with base point (∞). Suppose that T contains two distinct lines ℓ_i, $i = 1, 2$, through (∞), such that for every two distinct lines m_1, m_2 through (∞), where $m_1, m_2 \neq \ell_i$, $i = 1, 2$, there is a subquadrangle of order q^n through m_1, m_2, ℓ_i, $i = 1, 2$, then T is isomorphic to $T_3(\mathcal{O})$, where $\mathcal{O}$ is an ovoid of* $\mathrm{PG}(3, q^n)$.

3

A geometric characterisation of Desarguesian spreads

We provide a characterisation of $(n-1)$-spreads in $\mathrm{PG}(rn-1,q)$, $r>2$, that have r normal elements in general position. From this, we obtain a geometric characterisation of Desarguesian $(n-1)$-spreads in $\mathrm{PG}(rn-1,q)$, $r>2$.

These results were obtained in collaboration with J. Sheekey [96].

3.1 Introduction

In this chapter, we study spreads in finite projective spaces, specifically Desarguesian spreads.

Desarguesian spreads play an important role in finite geometries; for example in field reduction and linear (blocking) sets [75], eggs and TGQ's [3]. Even so, few geometric characterisations are known.

A geometric characterisation of Desarguesian $(n-1)$-spreads in $\mathrm{PG}(rn-1,q)$, $r>2$, was obtained by Beutelspacher and Ueberberg in [18, Corollary] by considering the intersection of spread elements with all $r(n-1)$-subspaces. Nevertheless, the two most famous and important characterisations of Desarguesian spreads arise from their correspondence with regular (for $q>2$) and normal spreads (for $r>2$).

We will focus on the normality of Desarguesian spreads, by introducing the notion of a *normal element* of a spread. We say that an element E of an $(n-1)$-spread $\mathcal{S}$ of $\mathrm{PG}(rn-1,q)$ is *normal* if $\mathcal{S}$ induces a spread in the $(2n-1)$-space spanned by E and any other element of $\mathcal{S}$. Clearly, by definition, a spread is normal if and only if all of its elements are normal. We consider the following questions:

> Can we characterise a spread given the configuration of its normal elements?
>
> How many normal elements does a spread need, to ensure that it is normal/Desarguesian?

This chapter is organised as follows.

We introduce the necessary preliminaries in Section 3.2.

In Section 3.3, we obtain a characterisation of $(n-1)$-spreads in $\mathrm{PG}(rn-1,q)$ having r normal elements in general position. We will see in Section 3.4 that, for some n and q, these spreads must be Desarguesian. Moreover, we obtain a characterisation of Desarguesian spreads as those having at least $r+1$ normal elements in general position.

Lastly, in Section 3.5, we consider spreads containing normal elements, not in general postion, but contained in the same $(2n-1)$-space.

3.2 Preliminaries

3.2.1 Choosing the coordinates

Definition 3.2.1. An element E of an $(n-1)$-spread $\mathcal{S}$ of $\mathrm{PG}(rn-1,q)$ is a *normal element* of $\mathcal{S}$ if, for every $F \in \mathcal{S} \setminus \{E\}$, the $(2n-1)$-space $\langle E, F\rangle$ is partitioned by elements of $\mathcal{S}$. Equivalently, this means that $\mathcal{S}/E$ defines an $(n-1)$-spread in the quotient space $\mathrm{PG}((r-1)n-1,q) \cong \mathrm{PG}(rn-1,q)/E$.

A spread is called *normal* when all its elements are normal elements. By [10], we know that an $(n-1)$-spread of $\mathrm{PG}(rn-1,q)$, $r>2$, is normal if and only if it is Desarguesian.

We will use the following coordinates. A point P of $\mathrm{PG}(r-1,q^n)$ is denoted by $(a_1,a_2,\ldots,a_r)_{\mathbb{F}_{q^n}}$, with $a_i \in \mathbb{F}_{q^n}$. Every point of $\mathrm{PG}(rn-1,q)$ is written as $(a_1,\ldots,a_r)_{\mathbb{F}_q}$, with $a_i \in \mathbb{F}_{q^n}$. When applying field reduction, a point $(a_1,\ldots,a_r)_{\mathbb{F}_{q^n}}$ in $\mathrm{PG}(r-1,q^n)$ corresponds to the $(n-1)$-space

$$\{(a_1x,\ldots,a_rx)_{\mathbb{F}_q} \mid x \in \mathbb{F}_{q^n}\}$$

of $\mathrm{PG}(rn-1,q)$.

Moreover, every $(n-1)$-space in $\mathrm{PG}(rn-1,q)$ can be represented in the following way. Let $a_1,\dots,a_r$ be $\mathbb{F}_q$-linear maps from $\mathbb{F}_{q^n}$ to itself. Then the set

$$\{(a_1(x),\dots,a_r(x))_{\mathbb{F}_q} \mid x \in \mathbb{F}_{q^n}\}$$

corresponds to an $(n-1)$-space of $\mathrm{PG}(rn-1,q)$. When choosing a basis for $\mathbb{F}_{q^n} \cong \mathbb{F}_q^n$ over $\mathbb{F}_q$, the $\mathbb{F}_q$-linear map $a_i, i=1,\dots,r$, is represented by an $n \times n$-matrix $A_i, i=1,\dots,r$, over $\mathbb{F}_q$ acting on row vectors of $\mathbb{F}_q^n$ from the right. We abuse notation and write the corresponding $(n-1)$-space of $\mathrm{PG}(rn-1,q)$ as

$$(A_1,\dots,A_r) := \{(xA_1,\dots,xA_r)_{\mathbb{F}_q} \mid x \in \mathbb{F}_q^n\}.$$

Recall that a set of $(n-1)$-spaces in $\mathrm{PG}(kn-1,q)$ such that any k span the full space, is called a set of $(n-1)$-spaces *in general position*.

Note that for any set $\mathcal{K}$ of $k+1$ $(n-1)$-spaces S_i, $i=0,\dots,k$, of $\mathrm{PG}(kn-1,q)$, $k>2$, in general position, there exists a field reduction map $\mathcal{F}$ from $\mathrm{PG}(k-1,q^n)$ to $\mathrm{PG}(kn-1,q)$, such that $\mathcal{K}$ is contained in the Desarguesian spread of $\mathrm{PG}(kn-1,q)$ defined by $\mathcal{F}$. This means that $\mathcal{K}$ is the field reduction of a frame in $\mathrm{PG}(k-1,q^n)$. This is equivalent with saying that we can choose coordinates for $\mathrm{PG}(kn-1,q)$ such that

$$\begin{aligned}\mathcal{K} = \{S_0 = (I,I,I,\dots,I,I), S_1 = (I,0,0,\dots,0,0),\\ S_2 = (0,I,0,\dots,0,0),\dots,S_k = (0,0,0,\dots,0,I)\},\end{aligned}$$

where I denotes the identity map or the identity matrix.

3.2.2 Quasifields and spread sets

André [2] and Bruck and Bose [24] obtained that finite translation planes and $(n-1)$-spreads of $\mathrm{PG}(2n-1,q)$ are equivalent objects. We will now also consider their correspondence with finite quasifields and matrix spread sets. For a more general overview than given here, we refer to [25, 47, 71].

Definition 3.2.2. A *finite (right) quasifield* $(Q,+,*)$ is a structure, where $+$ and $*$ are binary operations on Q, satisfying the following axioms:

(*i*) $(Q,+)$ is a group, with identity element 0,

(*ii*) $(Q_0 = Q\setminus\{0\},*)$ is a multiplicative loop with identity 1, i.e. $\forall a\in Q: 1*a = a*1 = a$ and $\forall a,b \in Q_0 : a*x = b$ and $y*a=b$ have unique solutions $x,y\in Q$, (*iii*) right distributivity: $\forall a, b, c \in Q : (a + b) * c = a * c + b * c$,

(*iv*) $\forall a, b, c \in Q, a \neq b : x * a = x * b + c$ has a unique solution $x \in Q$.

From now on, we will omit the term finite.

Definition 3.2.3. The *kernel* $K(Q)$ of a quasifield $(Q, +, *)$ is the set of all $k \in Q$ satisfying

$$\begin{aligned} &\forall x, y \in Q : k * (x * y) = (k * x) * y, \text{ and} \\ &\forall x, y \in Q : k * (x + y) = k * x + k * y. \end{aligned}$$

Note that the kernel $K(Q)$ of a quasifield Q is a field.

Definition 3.2.4. A *(matrix) spread set* is a family $\mathbf{M}$ of q^n $n \times n$-matrices over $\mathbb{F}_q$ such that, for every two distinct $A, B \in \mathbf{M}$, the matrix $A - B$ is non-singular.

Given a spread set $\mathbf{M}$, consider

$$\mathcal{S}(\mathbf{M}) = \{E_A \mid A \in \mathbf{M}\} \cup \{E_\infty\},$$

where

$$\begin{aligned} E_A &= (I, A) = \{(x, xA)_{\mathbb{F}_q} \mid x \in \mathbb{F}_q^n\}, \text{ and} \\ E_\infty &= (0, I) = \{(0, x)_{\mathbb{F}_q} \mid x \in \mathbb{F}_q^n\}. \end{aligned}$$

One can check that $\mathcal{S}(\mathbf{M})$ is an $(n-1)$-spread of $\mathrm{PG}(2n-1, q)$. Moreover, every $(n-1)$-spread of $\mathrm{PG}(2n-1, q)$ is $\mathrm{P\Gamma L}$-equivalent to a spread of the form $\mathcal{S}(\mathbf{M})$, for some spread set $\mathbf{M}$, such that the zero-matrix 0 and identity matrix I are both contained in $\mathbf{M}$ (see [24, Section 5]).

Consider now the vector space $V = V(n, q)$ and take a non-zero row vector e of V. For every vector $y \in V$, there exists a unique matrix $M_y \in \mathbf{M}$ such that $y = eM_y$. Define multiplication $*$ in V on row vectors by

$$x * y = xM_y.$$

By [24, Section 6], using this multiplication and the original addition, V becomes a right quasifield $Q_e(\mathbf{M}) = (V, +, *)$, with $\mathbb{F}_q$ in its kernel. Conversely, given a right quasifield Q with multiplication $*$ on V and $\mathbb{F}_q$ in its kernel, we can define a spread set $\mathbf{M}(Q) = \{M_y \mid y \in V\}$, where M_y is defined by $\forall x \in V : xM_y = x * y$. Clearly, $Q = Q_e(\mathbf{M}(Q))$.

Definition 3.2.5. A *semifield* is a right quasifield also satisfying left distributivity. A *(right) nearfield* Q is a (right) quasifield that satisfies associativity for multiplication, i.e. $\forall a,b,c \in Q : (a*b)*c = a*(b*c)$.

Note that a quasifield which is both a semifield and a nearfield is a *(finite) field.*

Theorem 3.2.6. [25, Section 11]
The quasifield $Q_e(\mathbf{M})$ is a nearfield if and only if $\mathbf{M}$ is closed under multiplication. The quasifield $Q_e(\mathbf{M})$ is a semifield if and only if $\mathbf{M}$ is closed under addition.

We conclude with the following connections (see [47, 71]).

$\mathcal{S}$ is a *nearfield spread*	$\Leftrightarrow$	$\mathcal{S} \cong \mathcal{S}(\mathbf{M})$ with $\mathbf{M}$ closed under multiplication;
$\mathcal{S}$ is a *semifield spread*	$\Leftrightarrow$	$\mathcal{S} \cong \mathcal{S}(\mathbf{M})$ with $\mathbf{M}$ closed under addition;
$\mathcal{S}$ is a *Desarguesian spread*	$\Leftrightarrow$	$\mathcal{S} \cong \mathcal{S}(\mathbf{M})$ with $\mathbf{M}$ closed under multiplication and under addition.

Equivalent to the previous, we say $\mathbf{M}$ is a *nearfield spread set*, respectively *semifield spread set* and *Desarguesian spread set.*

3.3 Spreads of $\mathrm{PG}(rn-1,q)$ containing r normal elements in general position

We denote the points of $\mathrm{PG}(rn-1,q)$ by $\{(x_1,\ldots,x_r)_{\mathbb{F}_q} \mid x_i \in \mathbb{F}_{q^n}\}$. Consider a spread set $\mathbf{M}$, containing 0 and I. We define the following $(n-1)$-spread

$$\mathcal{S}_r(\mathbf{M}) = \{(A_1, A_2, \ldots, A_r) \mid A_i \in \mathbf{M},\ \text{every first non-zero matrix } A_k = I\}$$

in $\mathrm{PG}(rn-1,q)$.

If $\mathbf{M}$ is a nearfield spread set, then one can check that $\mathcal{S}_r(\mathbf{M})$ contains r normal elements $S_i, i = 1,\ldots,r$, namely $S_1 = (I,0,0,\ldots,0)$, $S_2 = (0,I,0,\ldots,0), \ldots, S_r = (0,0,0,\ldots,0,I)$. This follows since the $(2n-1)$-space $\langle S_i, (A_1, A_2, \ldots, A_r)\rangle$ is partitioned by the elements

$$\{(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_r) \mid B \in \mathbf{M}\} \cup \{S_i\}.$$

Moreover, in this case, since $\mathbf{M}$ is closed under multiplication, we can simplify notation such that

$$\mathcal{S}_r(\mathbf{M}) = \{(A_1, A_2, \dots, A_r) \mid A_i \in \mathbf{M}\}.$$

Theorem 3.3.1. *An $(n-1)$-spread $\mathcal{S}$ in $\mathrm{PG}(rn-1,q), r > 2$, having r normal elements in general position is $\mathrm{P\Gamma L}$-equivalent to $\mathcal{S}_r(\mathbf{M})$, for some nearfield spread set $\mathbf{M}$.*

Proof. Suppose $r = 3$. Consider an $(n-1)$-spread $\mathcal{S}$ of $\mathrm{PG}(3n-1,q)$ having normal elements S_1, S_2, S_3 in general position. Without loss of generality, we may assume that $S_1 = (I,0,0), S_2 = (0,I,0)$ and $S_3 = (0,0,I)$. Moreover, we may also assume that the element $T = (I,I,I)$ is contained in $\mathcal{S}$.

As S_1, S_2 and S_3 are normal elements, the intersection of $\mathcal{S}$ with the $(2n-1)$-spaces $\langle S_2, S_3\rangle$, $\langle S_1, S_2\rangle$ and $\langle S_1, S_3\rangle$ are $(n-1)$-spreads. Moreover, since $T \in \mathcal{S}$, by considering its projection, we obtain that the $(n-1)$-spaces $(0,I,I), (I,I,0)$ and $(I,0,I)$ are all contained in $\mathcal{S}$. Hence, there exist spread sets $\mathbf{M_1}, \mathbf{M_2}$ and $\mathbf{M_3}$ (all containing 0 and I) such that

$$\begin{aligned}
\mathcal{S} \cap \langle S_2, S_3\rangle &= \{P_A = (0,A,I) \mid A \in \mathbf{M_1}\} \cup \{P_\infty = (0,I,0)\},\\
\mathcal{S} \cap \langle S_1, S_2\rangle &= \{Q_B = (I,B,0) \mid B \in \mathbf{M_2}\} \cup \{Q_\infty = (0,I,0)\},\\
\mathcal{S} \cap \langle S_1, S_3\rangle &= \{R_C = (C,0,I) \mid C \in \mathbf{M_3}\} \cup \{R_\infty = (I,0,0)\}.
\end{aligned}$$

Note that $S_2 = P_\infty = Q_\infty$ and $S_1 = R_\infty$. As S_2 and S_3 are normal elements of $\mathcal{S}$, we can obtain every element of $\mathcal{S}$, not contained in $\langle S_2, S_3\rangle$, as

$$(I, B, C^{-1}) = \langle S_2, R_C\rangle \cap \langle S_3, Q_B\rangle,$$

with $B \in \mathbf{M_2}$ and $C \in \mathbf{M_3}$. For any $B \in \mathbf{M_2}$ and $C \in \mathbf{M_3}$, consider the following projections of elements of $\mathcal{S}$ from S_1 onto $\langle S_2, S_3\rangle$:

$$\begin{aligned}
(0,B,I) &= \langle S_1, (I,B,I)\rangle \cap \langle S_2, S_3\rangle,\\
(0,C,I) &= \langle S_1, (I,I,C^{-1})\rangle \cap \langle S_2, S_3\rangle,\\
(0,BC,I) &= \langle S_1, (I,B,C^{-1})\rangle \cap \langle S_2, S_3\rangle.
\end{aligned}$$

As S_1 is a normal element, these subspaces are all contained in $\mathcal{S}$. From the first two, it follows that $\mathbf{M_2}$ and $\mathbf{M_3}$ are contained in $\mathbf{M_1}$, hence $\mathbf{M_1} = \mathbf{M_2} = \mathbf{M_3}$. Using the third, we find that $BC \in \mathbf{M_1}$, i.e. $\mathbf{M_1}$ is closed under multiplication. By Theorem 3.2.6, we conclude that the spread $\mathcal{S} \cap \langle S_2, S_3\rangle$ (and thus also $\mathcal{S} \cap \langle S_1, S_2\rangle$ and $\mathcal{S} \cap \langle S_1, S_3\rangle$) is a nearfield spread.

The result now follows for $r = 3$, since we have obtained that

$$\mathcal{S} = \mathcal{S}_3(\mathbf{M_1}) = \{(A_1, A_2, A_3) \mid A_i \in \mathbf{M_1}\},$$

where $\mathbf{M_1}$ is a nearfield spread set.

By induction, suppose the result is true for $r = t - 1 \geq 3$. We will now prove it is true for $r = t$. Consider an $(n-1)$-spread $\mathcal{S}$ of $\mathrm{PG}(tn-1, q)$ having t normal elements $S_1, \ldots, S_t$ in general position. Without loss of generality, we may assume

$$S_1 = (I, 0, 0, \ldots, 0), S_2 = (0, I, 0, \ldots, 0), \ldots, S_t = (0, 0, 0, \ldots, 0, I).$$

Consider the $((t-1)n-1)$-subspaces $\Pi_1 = \langle S_2, S_3, \ldots, S_t\rangle$ and $\Pi_2 = \langle S_1, S_3, \ldots, S_t\rangle$. Clearly, Π_1 corresponds to the points with coordinates $\{(0, x_2, \ldots, x_t)_{\mathbb{F}_q} \mid x_i \in \mathbb{F}_{q^n}\}$ and Π_2 corresponds to the points with coordinates $\{(x_1, 0, x_3, \ldots, x_r)_{\mathbb{F}_q} \mid x_i \in \mathbb{F}_{q^n}\}$. Since all S_j are normal elements, we have that for $i = 1, 2$, $\mathcal{S}_i = \mathcal{S} \cap \Pi_i$ is an $(n-1)$-spread of Π_i containing $t-1$ normal elements in general position. By the induction hypothesis, there exist nearfield spread sets $\mathbf{M_1}$ and $\mathbf{M_2}$ such that

$$\begin{aligned} \mathcal{S}_1 &= \{(0, A_2, A_3, \ldots, A_t) \mid A_i \in \mathbf{M_1}\}, \text{ and} \\ \mathcal{S}_2 &= \{(A_1, 0, A_3, \ldots, A_t) \mid A_i \in \mathbf{M_2}\}. \end{aligned}$$

The spreads $\mathcal{S}_1$ and $\mathcal{S}_2$ overlap in the $((t-2)n-1)$-space $\Pi_1 \cap \Pi_2$, hence we find that $\mathbf{M_1} = \mathbf{M_2}$.

All elements of $\mathcal{S}$, not contained in $\langle S_1, S_2\rangle$, are of the form $\langle S_1, U\rangle \cap \langle S_2, V\rangle$, for some $U \in \mathcal{S}_1, V \in \mathcal{S}_2$. To find the coordinates of the remaining elements of $\mathcal{S} \cap \langle S_1, S_2\rangle$, we can consider the projection of $\mathcal{S}$ from S_t onto $\langle S_1, S_2, \ldots, S_{t-1}\rangle$. We obtain that

$$\mathcal{S} = \{(A_1, A_2, \ldots, A_t) \mid A_i \in \mathbf{M_1}\}. \qquad \square$$

3.4 Characterising Desarguesian spreads

In this section, two characterisations of Desarguesian spreads are obtained, one dependent and one independent of n and q. For this, we first need the characterisation of finite nearfields.

Definition 3.4.1. A pair of positive integers (q, n), $n > 2$, is called a *Dickson number pair* if it satisfies the following relations:

(i) $q = p^h$ for some prime p,

(ii) each prime divisor of n divides $q - 1$,

(iii) if $q \equiv 3 \mod 4$, then $n \not\equiv 0 \mod 4$.

By [51] and [119], there is a uniform method for constructing a finite nearfield of order q^n, with kernel isomorphic to $\mathbb{F}_q$, whenever (q, n) is a Dickson number pair. Such a nearfield is called a *Dickson nearfield* or a *regular nearfield.* Moreover, by [119], apart from seven exceptions, every finite nearfield, which is not a field, is a Dickson nearfield. These seven nearfield exceptions have parameters $n = 2$ and $q \in \{5, 7, 11, 23, 29, 59\}$; note that there are two non-equivalent non-regular nearfields for $(q, n) = (11, 2)$.

Theorem 3.4.2. *Consider an $(n-1)$-spread $\mathcal{S}$ in $\mathrm{PG}(rn-1, q)$, $r > 2$, such that for every divisor $k|n$, we have that $(q^k, \frac{n}{k})$ is not a Dickson number pair and does not correspond to the parameters of one of the seven nearfield exceptions. If $\mathcal{S}$ contains r normal elements in general position, then $\mathcal{S}$ is a Desarguesian spread.*

Proof. By Theorem 3.3.1, the spread $\mathcal{S}$ is $\mathrm{P\Gamma L}$-equivalent to $\mathcal{S}_r(\mathbf{M})$, for a nearfield spread set $\mathbf{M}$. By assumption, for every divisor $k|n$, a nearfield of order q^n, having kernel $\mathbb{F}_{q^k}$, is a field, hence $\mathbf{M}$ is a Desarguesian spread set. It follows that $\mathcal{S}$ is a Desarguesian spread. □

Theorem 3.4.3. *Consider an $(n-1)$-spread $\mathcal{S}$ in $\mathrm{PG}(rn-1, q)$, $r > 2$. If $\mathcal{S}$ contains $r + 1$ normal elements in general position, then $\mathcal{S}$ is a Desarguesian spread.*

Proof. Suppose $r = 3$. Suppose $\mathcal{S}$ contains normal elements S_0, S_1, S_2, S_3 in general position. Without loss of generality, we may assume that $S_0 = (I, I, I)$, $S_1 = (I, 0, 0)$, $S_2 = (0, I, 0)$ and $S_3 = (0, 0, I)$.

As S_1, S_2, S_3 are normal elements, by following the proof of Theorem 3.3.1, we see that

$$\mathcal{S} = \mathcal{S}_3(\mathbf{M}) = \{(A_1, A_2, A_3) \mid A_i \in \mathbf{M}\},$$

for a nearfield spread set $\mathbf{M}$ (containing 0 and I).

Given $A \in \mathbf{M} \setminus \{0, I\}$, consider the spread element $R_A = (0, A, I) \in \mathcal{S} \cap \langle S_2, S_3 \rangle$, and look at the element

$$\langle S_0, R_A \rangle \cap \langle S_1, S_2 \rangle = (I, I - A, 0)\,.$$

As S_0 is a normal element for $\mathcal{S}$, this $(n-1)$-space is contained in $\mathcal{S} \cap \langle S_1, S_2 \rangle$. It follows that the matrix $I - A \in \mathbf{M}$.

As $\mathbf{M}$ is closed under multiplication, for all $A, B \in \mathbf{M}$, we have $B - BA \in \mathbf{M}$. Given matrices $A, C \in \mathbf{M}$, there exists a unique $B \in \mathbf{M}$ for which $BA = C$. Hence, for all $B, C \in \mathbf{M}$, we find that $B - C = B - BA$ is contained in $\mathbf{M}$. It follows that $\mathbf{M}$ is also closed under addition, hence $\mathbf{M}$ is a Desarguesian spread set. We conclude that $\mathcal{S}$ is a Desarguesian spread.

By induction, suppose the result is true for $r = t - 1$. We will now prove it is true for $r = t$. Consider an $(n-1)$-spread $\mathcal{S}$ of $\mathrm{PG}(tn - 1, q)$ having $t + 1$ normal elements $S_0, \ldots, S_t$ in general position. Consider the $((t-1)n-1)$-subspace $\Pi = \langle S_1, S_2, \ldots, S_{t-1} \rangle$. As all S_j are normal, $\mathcal{S} \cap \Pi$ is an $(n-1)$-spread containing $t - 1$ normal elements.

Consider the $(n-1)$-space $T = \langle S_0, S_t \rangle \cap \Pi$, clearly $T \in \mathcal{S} \cap \Pi$. Take an element $R \in \mathcal{S} \cap \Pi$ different from T and consider the $(3n-1)$-space $\pi = \langle S_0, S_t, R \rangle$. Note that the intersection $\Pi \cap \pi$ contains the elements T and R. Since S_0, S_t are normal elements, $\mathcal{S} \cap \pi$ is an $(n-1)$-spread. Both $\mathcal{S} \cap \Pi$ and $\mathcal{S} \cap \pi$ are $(n-1)$-spreads, hence $\mathcal{S} \cap (\Pi \cap \pi) = \mathcal{S} \cap \langle T, R \rangle$ is an $(n-1)$-spread. It follows that T is a normal element for the spread $\mathcal{S} \cap \Pi$.

Since the elements S_j lie in general position with respect to the full space $\mathrm{PG}(tn - 1, q)$, the elements $S_1, S_2, \ldots, S_{t-1}, T$ lie in general position with respect to Π. Hence, the spread $\mathcal{S} \cap \Pi$ contains t normal elements in general position, and thus, by the induction hypothesis, $\mathcal{S} \cap \Pi$ is a Desarguesian spread.

By Lemma 2.2.5, there exists a unique Desarguesian spread $\mathcal{D}$ of $\mathrm{PG}(tn - 1, q)$ containing S_0, S_t and all elements of $\mathcal{S} \cap \Pi$. Every element of $\mathcal{D}$, not contained in $\langle S_0, S_t \rangle$, can be obtained as $\langle S_0, U \rangle \cap \langle S_t, V \rangle$, for some $U, V \in \mathcal{S} \cap \Pi$. Since S_0 and S_t are normal elements of $\mathcal{S}$, all elements of $\mathcal{D} \setminus \langle S_0, S_t \rangle$ are elements of $\mathcal{S}$. Looking from the perspective from S_1, it is easy to see that the elements of $\mathcal{D} \cap \langle S_0, S_t \rangle$ are also elements of $\mathcal{S}$.

It follows that $\mathcal{S} = \mathcal{D}$ is a Desarguesian spread. $\square$

3.5 Spreads of $\mathrm{PG}(3n-1, q)$ containing three normal elements in a $(2n-1)$-space

In this section, we will characterise $(n-1)$-spreads of $\mathrm{PG}(3n-1, q)$ containing 3 normal elements contained in the same $(2n-1)$-space. First, we need to introduce some definitions and notations concerning the (restricted) closure of a point set and the field reduction of sublines and subplanes.

3.5.1 The (restricted) closure of a point set

Consider the field reduction map $\mathcal{F}$ from $\mathrm{PG}(1, q^n)$ to $\mathrm{PG}(2n-1, q)$. When we apply $\mathcal{F}$ to an $\mathbb{F}_{q_0}$-subline of $\mathrm{PG}(1, q^n)$, for a subfield $\mathbb{F}_{q_0} \leq \mathbb{F}_q$, we obtain a set $\mathcal{R}_{q_0}$ consisting of $q_0 + 1$ $(n-1)$-spaces, such that if a line meets 3 elements E_1, E_2, E_3 of $\mathcal{R}_{q_0}$, in the points P_1, P_2, P_3 respectively, then the unique $\mathbb{F}_{q_0}$-subline containing P_1, P_2, P_3 meets all elements of $\mathcal{R}_{q_0}$. When $\mathbb{F}_{q_0} = \mathbb{F}_q$, such a set $\mathcal{R}_q$ is called a regulus. It is well known that 3 mutually disjoint $(n-1)$-spaces E_1, E_2, E_3 in $\mathrm{PG}(2n-1, q)$ lie on a unique regulus, which we will denote by $\mathcal{R}(E_1, E_2, E_3) = \mathcal{R}_q(E_1, E_2, E_3)$. It now easily follows that 3 mutually disjoint $(n-1)$-spaces E_1, E_2 and E_3 in $\mathrm{PG}(2n-1, q)$ lie on a unique $\mathcal{R}_{q_0}$, which we will denote by $\mathcal{R}_{q_0}(E_1, E_2, E_3)$. Note that $\mathcal{R}_{q_0}(E_1, E_2, E_3) \subset \mathcal{R}_q(E_1, E_2, E_3)$.

Consider the field reduction map $\mathcal{F}$ from $\mathrm{PG}(2, q^n)$ to $\mathrm{PG}(3n-1, q)$. If we apply $\mathcal{F}$ to the point set of an $\mathbb{F}_{q_0}$-subplane, we find a set $\mathcal{V}_{q_0}$ of $q_0^2 + q_0 + 1$ elements of a Desarguesian spread $\mathcal{D}$. When $\mathbb{F}_{q_0} = \mathbb{F}_q$, the set $\mathcal{V}_q$ consists of one system of a Segre variety $\mathbf{S}_{2,n-1}$. By Theorem 1.4.3, we know that four $(n-1)$-spaces E_1, E_2, E_3, E_4 in $\mathrm{PG}(3n-1, q)$ in general position are contained in a unique Segre variety $\mathcal{V}_q$, which we will denote by $\mathcal{V}_q(E_1, E_2, E_3, E_4)$. As a corollary, we obtain that for any subfield $\mathbb{F}_{q_0} \leq \mathbb{F}_q$, four $(n-1)$-spaces E_1, E_2, E_3, E_4 in $\mathrm{PG}(3n-1, q)$ in general position are contained in a unique set $\mathcal{V}_{q_0}$, which we will denote by $\mathcal{V}_{q_0}(E_1, E_2, E_3, E_4)$. Note that for any three $(n-1)$-spaces S_1, S_2, S_3 in $\mathcal{V}_{q_0}(E_1, E_2, E_3, E_4)$, contained in the same $(2n-1)$-space, we have that $\mathcal{R}_{q_0}(S_1, S_2, S_3) \subset \mathcal{V}_{q_0}(E_1, E_2, E_3, E_4)$.

Inspired by the recursive construction of the closure of a point set (see Definition 1.3.8), we define the following. Consider a set S of points of $\mathrm{PG}(2, q)$ containing a specific subset $\{P_i\}_{i=1,\ldots,m}$. We define the *restricted closure* $\widetilde{S}$ *with respect to the points* $\{P_i\}$ to be the point set constructed recursively as follows:

(*i*) determine the set $\mathcal{A}$ of all lines of $\mathrm{PG}(2, q)$ of the form $\langle P_i, Q\rangle$, $i = 1, \ldots, m$, $Q \in S$;

(ii) determine the set $\widetilde{S}$ of points that occur as the exact intersection of two lines in $\mathcal{A}$, if $\widetilde{S} \neq S$ replace S by $\widetilde{S}$ and go to (i), otherwise stop.

Clearly the restricted closure $\widetilde{S}$ of S, with respect to all its points, is exactly its closure $\widehat{S}$.

Lemma 3.5.1. *Consider the point set S of $\mathrm{PG}(2,q)$, $q=p^h$, containing a frame $\{P_1,P_2,Q_1,Q_2\}$ and the point $P_3 = P_1P_2 \cap Q_1Q_2$. The points of the restricted closure $\widetilde{S}$ of S with respect to $\{P_1,P_2,P_3\}$, not on the line P_1P_2, are the points of the affine $\mathbb{F}_p$-subplane $\widehat{S} \setminus P_1P_2$.*

Proof. Clearly, the set $\widetilde{S}$ can contain no other points than the points of the $\mathbb{F}_p$-subplane $\widehat{S}$.

Without loss of generality, we may choose coordinates such that $P_1 = (1,0,0)_{\mathbb{F}_q}$, $P_2 = (0,0,1)_{\mathbb{F}_q}$, $Q_1 = (0,1,0)_{\mathbb{F}_q}$, $Q_2 = (1,1,1)_{\mathbb{F}_q}$ and hence $P_3 = (1,0,1)_{\mathbb{F}_q}$.

The point $U_1 = (0,1,1)_{\mathbb{F}_q} = P_1Q_2 \cap P_2Q_1$ is contained in $\widetilde{S}$. Hence, the point $T_2 = (1,1,2)_{\mathbb{F}_q} = P_3U_1 \cap P_2Q_2$ is contained in $\widetilde{S}$. Using $T_2 \in \widetilde{S}$, we see that the point $U_2 = (0,1,2)_{\mathbb{F}_q} = P_1T_2 \cap P_2Q_1$ is also contained in $\widetilde{S}$.

Continuing this process, we get that all points $U_a = (0,1,a)_{\mathbb{F}_q} \in P_2Q_1$ and $T_a = (1,1,a)_{\mathbb{F}_q} \in P_2Q_1$, with $a \in \mathbb{F}_p$, are contained in $\widetilde{S}$. All other points of the unique $\mathbb{F}_p$-subplane $\widehat{S}$ through P_1,P_2,Q_1,Q_2 and not on P_1P_2, can be written as the intersection point $\langle P_1,T_a\rangle \cap \langle P_2,U_b\rangle$, for some a and b in $\mathbb{F}_p$. Hence, they are all contained in $\widetilde{S}$. $\square$

We can translate the previous lemma to the higher dimensional case in the following way.

Lemma 3.5.2. *Consider an $(n-1)$-spread $\mathcal{S}$ in $\mathrm{PG}(3n-1,q)$, $q=p^h$, having 3 normal elements S_1,S_2,S_3 in the same $(2n-1)$-space Π_0. If two spread elements $R_1,R_2 \in \mathcal{S}$ satisfy $\langle R_1,R_2\rangle \cap \langle S_1,S_2\rangle = S_3$, then all $(n-1)$-spaces of $\mathcal{V}_p(S_1,S_2,R_1,R_2) \setminus \langle S_1,S_2\rangle$ are contained in $\mathcal{S}$.*

Proof. There exists a field reduction map $\mathcal{F}$ from $\mathrm{PG}(2,q^n)$ to $\mathrm{PG}(3n-1,q)$, such that S_1,S_2,S_3,R_1,R_2 are contained in the Desarguesian spread $\mathcal{D}$ of $\mathrm{PG}(3n-1,q)$ defined by $\mathcal{F}$. This means, there exist points P_1,P_2,P_3,Q_1,Q_2 such that their field reduction is equal to S_1,S_2,S_3,R_1,R_2 respectively.

Since S_1,S_2,S_3 are normal elements for $\mathcal{S}$, the field reduction of the points of the restricted closure of $\{P_1,P_2,P_3,Q_1,Q_2\}$ with respect to $\{P_1,P_2,P_3\}$ must all be contained in $\mathcal{S}$. By Lemma 3.5.1, the restricted closure contains all points of the $\mathbb{F}_p$-subplane defined by $\{P_1, P_2, Q_1, Q_2\}$, not on $\langle P_1, P_2\rangle$. Hence, its field reduction, i.e. the $(n-1)$-spaces of $\mathcal{V}_p(S_1, S_2, R_1, R_2) \setminus \langle S_1, S_2\rangle$ are all contained in $\mathcal{S}$. $\square$

3.5.2 Characterising spreads with 3 normal elements in the same $(2n-1)$-space

There exists a different but equivalent definition of a semifield spread than the one given in Subsection 3.2.2. Namely, an $(n-1)$-spread $\mathcal{S}$ of $\mathrm{PG}(2n-1, q)$ is a semifield spread if and only if it contains a special element E such that the stabiliser of $\mathcal{S}$ fixes E pointwise and acts transitively on the elements of $\mathcal{S} \setminus \{E\}$ (see [71, Corollary 5.60]). The element E is called the *shears* element of $\mathcal{S}$.

Consider a matrix spread set $\mathbf{M}$ containing 0 and closed under addition, and consider the corresponding semifield spread $\mathcal{S}(\mathbf{M}) = \{(I, A) \mid A \in \mathbf{M}\} \cup \{E = (0, I)\}$ of $\mathrm{PG}(2n-1, q)$. In this case, the element $E = (0, I)$ is the shears element of $\mathcal{S}(\mathbf{M})$.

Definition 3.5.3. The subsets of a semifield Q given as

$$\begin{aligned}
\mathbb{N}_r(Q) &= \{x \in Q \mid \forall a, b \in Q : (a * b) * x = a * (b * x)\}\\
\mathbb{N}_m(Q) &= \{x \in Q \mid \forall a, c \in Q : (a * x) * c = a * (x * c)\},\\
Z(Q) &= \{x \in \mathbb{N}_r(Q) \cap \mathbb{N}_m(Q) \mid \forall a \in Q : x * a = a * x\}
\end{aligned}$$

are all fields and are called, respectively, the *right nucleus*, *middle nucleus* and the *centre* of the semifield.

The parameters of the semifield $Q_e(\mathbf{M})$ can be translated to subsets of the associated spread set $\mathbf{M}$, that is $\mathcal{N}_r(\mathbf{M}) = \{M_x \in \mathbf{M} \mid x \in \mathbb{N}_r(Q_e(\mathbf{M}))\}$, $\mathcal{N}_m(\mathbf{M}) = \{M_x \in \mathbf{M} \mid x \in \mathbb{N}_m(Q_e(\mathbf{M}))\}$ and $\mathcal{Z}(\mathbf{M}) = \{M_x \in \mathbf{M} \mid x \in Z(Q_e(\mathbf{M}))\}$. As in [50, Theorem 2.1], but considering our conventions, we obtain that

$$\begin{aligned}
\mathbb{N}_r(Q_e(\mathbf{M})) &= \{x \in Q_e(\mathbf{M}) \mid \forall a, b \in Q_e(\mathbf{M}) : (a * b) * x = a * (b * x)\}\\
&= \{x \in Q_e(\mathbf{M}) \mid \forall a, b \in Q_e(\mathbf{M}) : aM_bM_x = aM_{b*x}\}\\
&= \{x \in Q_e(\mathbf{M}) \mid \forall b \in Q_e(\mathbf{M}) : M_bM_x = M_{b*x}\}\\
&= \{x \in Q_e(\mathbf{M}) \mid \forall b \in Q_e(\mathbf{M}) : M_bM_x \in \mathbf{M}\}\\
&= \{x \in Q_e(\mathbf{M}) \mid \forall M \in \mathbf{M} : MM_x \in \mathbf{M}\}.
\end{aligned}$$

Hence, it follows that

$$\mathcal{N}_r(\mathbf{M}) = \{X \in \mathbf{M} \mid \mathbf{M}X = \mathbf{M} \text{ or } X = 0.\}$$

Similarly, we obtain

$$\begin{aligned}\mathcal{N}_m(\mathbf{M}) &= \{X \in \mathbf{M} \mid X\mathbf{M} = \mathbf{M} \text{ or } X = 0\}, \text{ and}\\ \mathcal{Z}(\mathbf{M}) &= \{X \in \mathcal{N}_r(\mathbf{M}) \cap \mathcal{N}_m(\mathbf{M}) \mid \forall Y \in \mathbf{M} : YX = XY\}.\end{aligned}$$

Suppose $\mathbf{M}$ contains the identity I, then $\mathbf{M}$ is a subspace over a subfield $\mathbb{F}_{q_0} \leq \mathbb{F}_q$ if and only if $\{\lambda I \mid \lambda \in \mathbb{F}_{q_0}\} \subseteq \mathcal{Z}(\mathbf{M})$.

Recall that a spread $\mathcal{S}$ in $\mathrm{PG}(2n-1,q)$ is *regular* if and only if for any 3 elements $E_1, E_2, E_3 \in \mathcal{S}$, the elements of $\mathcal{R}_q(E_1,E_2,E_3)$ are all contained in $\mathcal{S}$. It is well known that every $(n-1)$-spread of $\mathrm{PG}(2n-1,q)$, $q>2$, is regular if and only if it is Desarguesian (see [25]). When $q=2$, every $(n-1)$-spread of $\mathrm{PG}(2n-1,2)$ is regular. More generally, for every three elements E_1,E_2,E_3 of an $(n-1)$-spread $\mathcal{S}$ of $\mathrm{PG}(2n-1,2^h)$, all elements of $\mathcal{R}_2(E_1,E_2,E_3) = \{E_1,E_2,E_3\}$ are contained in $\mathcal{S}$.

Loosening the concept of regularity of Desarguesian spreads, we obtain the following result for semifield spreads. For $\mathbb{F}_{q_0} = \mathbb{F}_q$, this result was already obtained in [79, Teorema 5].

Theorem 3.5.4. *Suppose $\mathcal{S}$ is an $(n-1)$-spread of $\mathrm{PG}(2n-1,q)$. Consider a subfield $\mathbb{F}_{q_0} \leq \mathbb{F}_q$, $q_0 > 2$, and an element $E \in \mathcal{S}$. The spread $\mathcal{S}$ is a semifield spread with $\mathbb{F}_{q_0}$ contained in its centre and having E as its shears element if and only if for all $E_1, E_2 \in \mathcal{S}$, we have $\mathcal{R}_{q_0}(E,E_1,E_2) \subset \mathcal{S}$.*

Proof. We may assume without loss of generality that $\mathcal{S} = \mathcal{S}(\mathbf{M}) = \{(I,A) \mid A \in \mathbf{M}\} \cup \{E = (0,I)\}$ for some spread set $\mathbf{M}$ containing 0 and I. Let $E_i = (I, A_i)$. Then the set $\mathcal{R}_{q_0}(E,E_1,E_2)$ consists of the spaces $(I,(1-\lambda)A_1+\lambda A_2)$, for $\lambda \in \mathbb{F}_{q_0}$, together with E. Hence, $(1-\lambda)A_1 + \lambda A_2 \in \mathbf{M}$, for all $A_1, A_2 \in \mathbf{M}$ and all $\lambda \in \mathbb{F}_{q_0}$.

Since $0 \in \mathbf{M}$, for all $A_2 \in \mathbf{M}$, we get that $\lambda A_2 \in \mathbf{M}$ for all $\lambda \in \mathbb{F}_{q_0}$. Therefore, for all $A_1, A_2 \in \mathbf{M}$, $\mu A_1 + \lambda A_2 \in \mathbf{M}$ for all $\lambda, \mu \in \mathbb{F}_{q_0}$, and so $\mathbf{M}$ is an $\mathbb{F}_{q_0}$-subspace, implying that $\mathbf{M}$ is a semifield spread set with centre containing $\mathbb{F}_{q_0}$. It follows that $\mathcal{S}$ is a semifield spread with shears element E. $\square$

Now, consider three disjoint $(n-1)$-spaces $S_1 = (I,0,0), S_2 = (I,I,0)$, $S_3 = (0,I,0)$ of $\mathrm{PG}(3n-1,q)$ contained in the same $(2n-1)$-space $\Pi_0 = \{(x,y,0)_{\mathbb{F}_q} \mid$

$x, y \in \mathbb{F}_{q^n}\}$. Consider two spread sets $\mathbf{M}$ and $\mathbf{M_0}$, both containing 0 and I, and define the following $(n-1)$-spread

$$\mathcal{T}_3(\mathbf{M}, \mathbf{M_0}) = \{(A, B, I) \mid A, B \in \mathbf{M}\} \cup \{(I, C, 0) \mid C \in \mathbf{M_0}\} \cup \{(0, I, 0)\}$$

in $\mathrm{PG}(3n-1, q)$. If $\mathbf{M}$ is a semifield spread set, then one can check that $\mathcal{T}_3(\mathbf{M}, \mathbf{M}_0)$ has at least 3 normal elements, namely S_1, S_2 and S_3.

Theorem 3.5.5. *Consider an $(n-1)$-spread $\mathcal{S}$ in $\mathrm{PG}(3n-1, q)$, q odd. If $\mathcal{S}$ contains 3 normal elements contained in the same $(2n-1)$-space Π_0, then $\mathcal{S}$ is $\mathrm{P\Gamma L}$-equivalent to $\mathcal{T}_3(\mathbf{M}, \mathbf{M_0})$, for some spread set $\mathbf{M_0}$ and a semifield spread set $\mathbf{M}$.*

Furthermore, the set of normal elements of $\mathcal{S}$ contained in $\mathcal{S} \cap \Pi_0$ is $\mathrm{P\Gamma L}$-equivalent to $\{(I, C, 0) \mid C \in \mathbf{M}_0 \cap \mathcal{N}_r(\mathbf{M})\} \cup \{(0, I, 0)\}$.

Proof. Consider an $(n-1)$-spread $\mathcal{S}$ containing normal elements S_1, S_2, S_3 contained in the same $(2n-1)$-space Π_0. Since S_3 is a normal element with respect to $\mathcal{S}$, we can consider a $(2n-1)$-space Π, meeting Π_0 in the space S_3, such that $\mathcal{S} \cap \Pi$ is an $(n-1)$-spread. Without loss of generality, we may assume that $S_1 = (I, 0, 0), S_2 = (I, I, 0), S_3 = (0, I, 0)$, $\Pi_0 = \{(x, y, 0)_{\mathbb{F}_q} \mid x, y \in \mathbb{F}_{q^n}\}$ and $\Pi = \{(0, y, z)_{\mathbb{F}_q} \mid y, z \in \mathbb{F}_{q^n}\}$.

Since $\mathcal{S} \cap \Pi$ is an $(n-1)$-spread, there exists a spread set $\mathbf{M}$ such that

$$\mathcal{S} \cap \Pi = \{(0, A, I) \mid A \in \mathbf{M}\} \cup \{(0, I, 0)\}.$$

Consider two elements R_1, R_2 of $\mathcal{S} \cap \Pi$, different from S_3. By Lemma 3.5.2, all elements of $\mathcal{V}_p(S_1, S_2, R_1, R_2) \setminus \Pi_0$ are contained in $\mathcal{S}$. It follows that the $p+1$ elements of $\mathcal{R}_p(S_3, R_1, R_2)$ are all contained in $\mathcal{S} \cap \Pi$. By Theorem 3.5.4, we see that $\mathcal{S} \cap \Pi$ is a semifield spread (with $\mathbb{F}_p$ contained in its centre and S_3 its shears element), hence $\mathbf{M}$ is a semifield spread set.

The elements S_1 and S_2 are normal elements, hence every element of $\mathcal{S} \setminus \langle S_1, S_2\rangle$ is of the form

$$(B - D, B, I) = \langle S_1, (0, B, I)\rangle \cap \langle S_2, (0, D, I)\rangle,$$

for some $B, D \in \mathbf{M}$. Since $\mathbf{M}$ is closed under addition, every element of $\mathcal{S} \setminus \langle S_1, S_2\rangle$ is of the form (A, B, I), with $A, B \in \mathbf{M}$. We conclude that

$$\mathcal{S} = \{(A, B, I) \mid A, B \in \mathbf{M}\} \cup \{(I, C, 0) \mid C \in \mathbf{M_0}\} \cup \{(0, I, 0)\},$$

for some spread set $\mathbf{M_0}$ (containing 0 and I) and a semifield spread set $\mathbf{M}$.

Now suppose that $(I, C, 0) \in \Pi_0$, $C \in \mathbf{M}_0$, is a normal element for $\mathcal{T}_3(\mathbf{M}, \mathbf{M_0})$. Given elements $A, B, D, E \in \mathbf{M}$, the spread element (D, E, I) is contained in $\langle (I, C, 0), (A, B, I) \rangle$ if and only if $(D - A)C = E - B$. Hence, $\langle (I, C, 0), (A, B, I) \rangle$ is partitioned by elements of $\mathcal{T}_3(\mathbf{M}, \mathbf{M_0})$ if and only if $(D - A)C + B \in \mathbf{M}$ for all $A, B, D \in \mathbf{M}$. As $\mathbf{M}$ is closed under addition, this occurs if and only if $\mathbf{M}C \subseteq \mathbf{M}$, i.e. $C \in \mathbf{M}_0 \cap \mathcal{N}_r(\mathbf{M})$. $\square$

II

Linear representations and their graphs

In Part II, we consider linear representations and their graphs. A linear representation of a point set is a point-line incidence structure embedded in a Desarguesian projective space. In Chapter 4, we study the isomorphism problem for linear representations. Using the incidence graph of these structures, we obtain infinite families of semisymmetric graphs in Chapter 5.

4 The isomorphism problem for linear representations

We study the isomorphism problem for *linear representations*. A linear representation $T_n^*(\mathcal{K})$ of a point set $\mathcal{K}$ is a point-line incidence structure, embedded in a projective space $\mathrm{PG}(n+1, q)$, where $\mathcal{K}$ is contained in a hyperplane.

This chapter combines the results published in [31] with P. Cara and G. Van de Voorde, and in [45] with S. De Winter and G. Van de Voorde.

4.1 Introduction

In finite geometry, one often considers geometries that are embedded in a projective or affine space. If G_1 and G_2 are two such embedded geometries, a natural question to ask is the following.

> (Q) Is every isomorphism between G_1 and G_2 induced by a collineation of the ambient projective space?

Of course, the answer to this question will strongly depend on the properties of the geometries G_1 and G_2, as well as on the type of embedding considered. For example, it is well known that this question has an affirmative answer for the standard embeddings of finite classical polar spaces and Segre varieties [67], and has been studied for various other types of geometries such as generalised quadrangles [89] and semipartial geometries [43].

In this chapter, we study problem (Q) for a particular, but broad, class of geometries, namely *linear representations*. The linear representation $T_2^*(\mathcal{O})$, $\mathcal{O}$ a hyperoval, was constructed by Ahrens and Szekeres [1] and independently by Hall [57]. This definition was extended to the linear representation of general point sets in [40].

Definition 4.1.1. Let $\mathcal{K}$ be a point set in a hyperplane $H_\infty \cong \mathrm{PG}(n,q)$ of $\mathrm{PG}(n+1,q)$. The *linear representation* $T_n^*(\mathcal{K})$ of $\mathcal{K}$ is a point-line incidence structure with natural incidence, point set $\mathcal{P}$ and line set $\mathcal{L}$ as follows:

$\mathcal{P}$: the *affine* points of $\mathrm{PG}(n+1,q)$, i.e. not contained in H_∞,

$\mathcal{L}$: the lines of $\mathrm{PG}(n+1,q)$ intersecting H_∞ exactly in a point of $\mathcal{K}$.

We see that a linear representation $T_n^*(\mathcal{K})$ in $\mathrm{PG}(n+1,q)$ is entirely determined by the point set $\mathcal{K}$ at infinity.

In some of the arguments, we also come across a generalisation of this incidence structure, called a generalised linear representation.

Definition 4.1.2. Let $\mathcal{K}$ be a set of disjoint $(t-1)$-dimensional subspaces in $\Pi_\infty \cong \mathrm{PG}(m,q)$. Embed Π_∞ as a hyperplane in $\mathrm{PG}(m+1,q)$. The *generalised linear representation* $T^*_{m,t-1}(\mathcal{K})$ of $\mathcal{K}$ is the incidence structure $(\mathcal{P}', \mathcal{L}')$ with natural incidence for which:

$\mathcal{P}'$: the *affine* points of $\mathrm{PG}(m+1,q)$, i.e. not contained in Π_∞,

$\mathcal{L}'$: the t-spaces of $\mathrm{PG}(m+1,q)$ intersecting Π_∞ in exactly a $(t-1)$-space of $\mathcal{K}$.

Clearly, when $t = 1$, this definition coincides with the definition of a linear representation.

If the answer to (Q) is affirmative, then two linear representations $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$ are isomorphic if and only if the point sets $\mathcal{K}$ and $\mathcal{K}'$ are PΓL-equivalent, which we denote by $\mathcal{K} \cong \mathcal{K}'$. Whether this is also true when the answer to (Q) is not affirmative, is what we call the *isomorphism problem for linear representations*.

Linear representations are mostly studied for point sets $\mathcal{K}$ that possess a lot of symmetry. For example, in the case $n = 2$ and $\mathcal{K}$ a hyperoval, $T_2^*(\mathcal{K})$ is a generalised quadrangle of order $(q-1, q+1)$. Bichara, Mazzocca and Somma showed in [19] that for $\mathcal{K}, \mathcal{K}'$ hyperovals, $T_2^*(\mathcal{K}) \cong T_2^*(\mathcal{K}')$ if and only if $\mathcal{K} \cong \mathcal{K}'$. The answer to (Q) is proven positive when $\mathcal{K}$ is a hyperconic in $\mathrm{PG}(2,q)$ [56]. When $\mathcal{K}$ is an ovoidal Buekenhout-Metz unital, question (Q) is answered affirmatively by De Winter [44]; in this case, the linear representation $T_2^*(\mathcal{K})$ is a semipartial geometry. It is worth noticing that our result includes these cases.

In Subsection 4.1.1, we clarify why we consider subsets $\mathcal{K}, \mathcal{K}'$ containing a frame of H_∞. In Section 4.2, we provide conditions on the point sets $\mathcal{K}$ and $\mathcal{K}'$ such that the answer to question (Q) is affirmative for $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$. This leads to the following theorem.

Theorem 4.2.10. *Let $q > 2$. Let $\mathcal{K}$ and $\mathcal{K}'$ denote point sets in $H_\infty = \mathrm{PG}(n, q)$ such that*

- *there is no plane of H_∞ intersecting $\mathcal{K}$ in two intersecting lines, or in two intersecting lines minus their intersection point,*
- *the closure $\widehat{\mathcal{K}}$ is equal to H_∞.*

If α is an isomorphism of incidence structures between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$, then α is induced by a collineation of the ambient space mapping $\mathcal{K}$ to $\mathcal{K}'$.

In Section 4.3, we investigate what happens when the first condition on $\mathcal{K}$ is not fulfilled. In Section 4.4, we will prove that when the second condition is not met, there always exist isomorphisms not induced by collineations of the ambient space and we explicitly describe them. Both conditions might have been silently assumed when the authors of [3, Corollary 7] answer the question (Q) affirmatively for all (generalised) linear representations; since we will see that the answer to question (Q) is in general not affirmative.

Theorem 4.2.10 handles the case where the closure $\widehat{\mathcal{K}}$ is equal to H_∞. The remaining case is when $\widehat{\mathcal{K}}$ is a subgeometry $\mathcal{S}$ of H_∞. In Section 4.4, we obtain the full automorphism group of a linear representation $T_n^*(\mathcal{S})$ of a subgeometry $\mathcal{S} \cong \mathrm{PG}(n, q)$ embedded in the projective space $\mathrm{PG}(n + 1, q^t)$.

Finally, in Section 4.5, we conclude with an answer to the isomorphism problem under a small condition.

Theorem 4.5.3. *Let $\mathcal{K}$ and $\mathcal{K}'$ be two point sets of $H_\infty = \mathrm{PG}(n, q)$, $q > 2$, $n > 1$, each containing a frame, such that $\langle\mathcal{K}\rangle = \langle\mathcal{K}'\rangle = H_\infty$. If $\widehat{\mathcal{K}} = H_\infty$, suppose furthermore that there is no plane of H_∞ intersecting $\mathcal{K}$ in two intersecting lines, or in two intersecting lines minus their intersection point. The linear representations $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$ are isomorphic if and only if the point sets $\mathcal{K}$ and $\mathcal{K}'$ are* $\mathrm{P\Gamma L}$-*equivalent.*

4.1.1 Restrictions on the point set $\mathcal{K}$

Consider a linear representation $T_n^*(\mathcal{K})$ of a point set $\mathcal{K}$ contained in $H_\infty = \mathrm{PG}(n, q)$. When $n = 1$, the linear representation $T_1^*(\mathcal{K})$ is a *net*, which is a well-studied object, introduced by Bruck [23], that has its own theory. This is a case we will not study.

Suppose the set $\mathcal{K}$ spans an m-dimensional subspace $\mathrm{PG}(m,q)$ of $H_\infty = \mathrm{PG}(n,q)$, $m < n$. The following result shows that in this case the corresponding point-line incidence graph of $T_n^*(\mathcal{K})$ is not connected.

Theorem 4.1.3. [39, Corollary 4.3] *The point-line incidence graph of $T_n^*(\mathcal{K})$ is connected if and only if the span $\langle\mathcal{K}\rangle$ has dimension n.*

Moreover, one can check that the linear representation $T_n^*(\mathcal{K})$ consists of q^{n-m} disjoint components, where each component is isomorphic to the linear representation $T_m^*(\mathcal{K})$. Points of different components are never collinear inside $T_n^*(\mathcal{K})$. We obtain that $T_n^*(\mathcal{K})$ is isomorphic to the cartesian product $T_m^*(\mathcal{K}) \times \{1, 2, \ldots, q^{n-m}\}$. In this case, the automorphism group of $T_n^*(\mathcal{K})$ is isomorphic to the *wreath product* $\mathrm{Aut}(T_m^*(\mathcal{K})) \wr S_{q^{n-m}}$. This clarifies why we will only consider linear representations $T_n^*(\mathcal{K})$, where $\langle\mathcal{K}\rangle = H_\infty$.

Remark. The action of $\mathrm{Aut}(T_m^*(\mathcal{K})) \wr S_{q^{n-m}}$ on $T_m^*(\mathcal{K}) \times \{1, 2, \ldots, q^{n-m}\}$ goes as follows. Consider A to be the direct product of q^{n-m} copies of $\mathrm{Aut}(T_m^*(\mathcal{K}))$, each one acting on one component $T_m^*(\mathcal{K})$ of $T_n^*(\mathcal{K})$. Let B be a copy of $S_{q^{n-m}}$ permuting the copies of $T_m^*(\mathcal{K})$. We have that B normalises A, i.e. $\forall b \in B : b^{-1}Ab = A$. The wreath product $\mathrm{Aut}(T_m^*(\mathcal{K})) \wr S_{q^{n-m}}$ corresponds to the semidirect product $A \rtimes B$ (for more information about the wreath product $\wr$, we refer to [59, page 81]).

Moreover, from now on, we will only consider linear representations $T_n^*(\mathcal{K})$ of a point set $\mathcal{K}$ containing a frame of $H_\infty = \mathrm{PG}(n,q)$, $n > 1$, even when we do not explicitly mention it. Note that the case we exclude, is when the points of $\mathcal{K}$ can be partitioned in k subsets $\mathcal{K}_1, \ldots, \mathcal{K}_k$, where the elements of $\mathcal{K}_i$ contain a frame for an n_i-dimensional subspace π_i, such that all spaces π_i are mutually disjoint.

4.2 Isomorphisms between linear representations

In this section, we will deal with question (Q) for linear representations. An isomorphism between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$ that is induced by a collineation of the ambient projective space $\mathrm{PG}(n+1,q)$ is called *geometric*. We will provide certain conditions on $\mathcal{K}$, to ensure that every isomorphism between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$ is geometric.

More generally, we prove that an isomorphism between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$ is induced by an isomorphism between two linear representations $T_n^*(\mathcal{S})$ and $T_n^*(\mathcal{S}')$ of the closures $\mathcal{S} = \widehat{\mathcal{K}}$ and $\mathcal{S}' = \widehat{\mathcal{K}'}$ of H_∞.

Recall that for a point set S, containing a frame of $\mathrm{PG}(n,q)$, its *closure* $\widehat{S}$ consists of the points of the smallest n-dimensional subgeometry of $\mathrm{PG}(n,q)$ containing all points of S (see Definition 1.3.8). The closure $\widehat{S}$ of a point set S can be constructed recursively as follows:

(i) determine the set $\mathcal{A}$ of all subspaces of $\mathrm{PG}(n,q)$ spanned by an arbitrary number of points of S;

(ii) determine the set $\widehat{S}$ of points that occur as the exact intersection of two subspaces in $\mathcal{A}$, if $\widehat{S} \neq S$ replace S by $\widehat{S}$ and go to (i), otherwise stop.

We also need the notion of *α-rigid* subspaces:

Definition 4.2.1. Let α be an isomorphism between the linear representations $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$. We will say that a k-subspace π_∞ of H_∞ is *α-rigid* if for every $(k+1)$-subspace π through π_∞, not contained in H_∞, the point set $\{\alpha(P) \mid P \in \pi \setminus \pi_\infty\}$ spans a $(k+1)$-subspace.

It follows from the definition of the linear representation $T_n^*(\mathcal{K})$ that every point of $\mathcal{K}$ is β-rigid, for any isomorphism β. Moreover, it is clear that if an isomorphism β between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$ is induced by a collineation, necessarily, every point of H_∞ is β-rigid.

Our way of approaching question (Q) is to find conditions on the point sets $\mathcal{K}$, $\mathcal{K}'$ that force every point of H_∞ to be β-rigid, for all isomorphisms β. Finally, these conditions will enable us to prove that every isomorphism between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$ is induced by a collineation.

We start off with an easy lemma on α-rigid subspaces.

Lemma 4.2.2. *If for some isomorphism α, two subspaces π_1 and π_2 are α-rigid subspaces meeting in at least one point, then $\pi_1 \cap \pi_2$ is an α-rigid subspace.*

Proof. Let R be an affine point and α an isomorphism between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$. Suppose $\dim(\pi_i) = k_i$ and $\dim(\pi_1 \cap \pi_2) = m \geq 0$. Since π_i is α-rigid, the affine points of $\langle R, \pi_i \rangle$ are mapped onto the affine points of a (k_i+1)-dimensional space μ_i. As μ_1 and μ_2 are subspaces of $\mathrm{PG}(n+1,q)$, they intersect in a subspace of $\mathrm{PG}(n+1,q)$. Moreover, as this subspace contains exactly the images under α of the points that are contained in $\langle R, \pi_1 \rangle \cap \langle R, \pi_2 \rangle$, it has dimension $m+1$. This implies that $\pi_1 \cap \pi_2$ is α-rigid. $\square$

In the previous lemma, we proved that for an isomorphism α, the intersection of α-rigid subspaces is an α-rigid space. The next step is to show that the span of α-rigid subspaces is again an α-rigid subspace. However, we need to impose certain restrictions since this is not always the case, as we will see later in Section 4.3.

Recall that the points of $\mathcal{K}$ are β-rigid, for all isomorphisms β. In the next lemma, we will give a condition that ensures that the span of two points of $\mathcal{K}$ is a β-rigid line. This result will be generalised in Lemma 4.2.7 to arbitrary β-rigid points and spaces of arbitrary dimension.

Definition 4.2.3. For a line L of $\mathrm{PG}(n+1,q)$ not in H_∞, we define $\infty(L)$ to be the point $L \cap H_\infty$.

A permutation β of the affine points extends naturally to a mapping on the lines L having a β-rigid point $\infty(L)$ at infinity, by defining $\beta(L)$ to be the line containing the points $\beta(R)$, $R \in L$.

Lemma 4.2.4. *Let $q > 2$. Suppose that no plane of H_∞ intersects $\mathcal{K}$ or $\mathcal{K}'$ in two intersecting lines or in two intersecting lines minus their intersection point. Let P_1 and P_2 be two points of $\mathcal{K}$, then P_1P_2 is α-rigid, for any isomorphism α between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$.*

Proof. First, assume that no plane of H_∞ intersects $\mathcal{K}'$ in two intersecting lines or in two intersecting lines minus their intersection point.

Let π be a plane through the line P_1P_2, not in H_∞. Let L_i, $i = 1, \ldots, q$, be the q lines in π through P_1, different from P_1P_2, and let M_j, $j = 1, \ldots, q$, be the q lines through P_2 in π, different from P_1P_2. It is clear that the lines L_1 and L_i, $i = 2, \ldots, q$, are not concurrent in $T_n^*(\mathcal{K})$.

Suppose that $\infty(\alpha(L_1))$ is different from $\infty(\alpha(L_i))$ for some $2 \leq i \leq q$. If for some $j \neq k$, $\alpha(M_j)$ and $\alpha(M_k)$ meet H_∞ in the same point, then $\alpha(L_1)$ and $\alpha(L_i)$ are lines in the same plane, hence they intersect, and thus by assumption they intersect in an affine point. This would mean that $\alpha(L_1)$ and $\alpha(L_i)$ are concurrent in $T_n^*(\mathcal{K}')$; a contradiction since α is an isomorphism.

This implies that if $\infty(\alpha(L_1))$ is different from $\infty(\alpha(L_i))$ for some i, then the points $\infty(\alpha(M_j))$, $1 \leq j \leq q$, are all distinct and hence also the points $\infty(\alpha(L_i))$, $1 \leq i \leq q$, are mutually different. Moreover, it is impossible that $\infty(\alpha(L_i)) = \infty(\alpha(M_j))$ for some $1 \leq i, j \leq q$, since L_i and M_j are concurrent and α is an isomorphism. Since the line $\alpha(L_i)$ meets $\alpha(M_j)$ in a point, for all $1 \leq i, j \leq q$, we see that the line sets $\{\alpha(L_1), \ldots, \alpha(L_q)\}$ and $\{\alpha(M_1), \ldots, \alpha(M_q)\}$ are each contained in a regulus of a hyperbolic quadric $Q^+(3,q)$.

Suppose that the line P_1P_2 contains an extra point $P_3 \in \mathcal{K}$. A line of $T_n^*(\mathcal{K})$ in π through P_3 is mapped by α to a line that contains q points of the quadric $Q^+(3,q)$, but is not contained in a regulus of this quadric, a contradiction, hence P_1P_2 is α-rigid.

Now suppose that the line P_1P_2 does not contain an extra point of $\mathcal{K}$, then we know that either P_1P_2 is α-rigid, or the lines $\alpha(L_1), \ldots, \alpha(L_q)$ and $\alpha(M_1), \ldots, \alpha(M_q)$ are each contained in a regulus of a hyperbolic quadric $Q^+(3,q)$. This quadric meets H_∞ in a plane η_∞ containing two lines N_1, N_2 and hence, the $2q$ points of N_1 and N_2, different from their intersection point, are necessarily points of $\mathcal{K}'$.

By our assumption, this is either not possible or there exists a point $P \in \mathcal{K}'$ on η_∞, not on the lines N_1 and N_2. Consider a line T through P intersecting N_1 in a point S_1 and intersecting the line N_2 in a point S_2, such that $S_1 \neq S_2$. Consider the plane π' through T, different from η_∞, intersecting the quadric $Q^+(3,q)$ in two lines, one line through S_1 and the other through S_2. By the first part of the proof, since T contains 3 points of $\mathcal{K}'$, T is β-rigid for any isomorphism β from $T_n^*(\mathcal{K}')$ to $T_n^*(\mathcal{K})$. Hence, the plane π' is mapped by α^{-1} to the plane π, so the lines of $T_n^*(\mathcal{K}')$ through S_1, S_2 respectively, are mapped to the lines through P_1, P_2 respectively. This is a contradiction since not the lines of π', but the lines of the reguli of the quadric are already mapped by α^{-1} to the lines through P_1 and P_2 in π. It follows that P_1P_2 is α-rigid.

Now suppose that $\mathcal{K}$ has the property that no plane of H_∞ intersects it in two intersecting lines or in two intersecting lines minus their intersection point. By repeating the proof, we obtain that for any two points Q_1, Q_2 of $\mathcal{K}'$, we have that the line Q_1Q_2 is α^{-1}-rigid. As the isomorphisms α and α^{-1} are each others inverse, it now follows that for any two points P_1, P_2 of $\mathcal{K}$, the line P_1P_2 is α-rigid. $\square$

Remark. For $q = 2$, a line of $\mathcal{L}$ contains only two points of $\mathcal{P}$. Hence, if $\mathcal{K} = H_\infty$, clearly any permutation of $\mathcal{P}$ induces an automorphism of $T_n^*(\mathcal{K})$. Moreover, we checked by computer that the linear representation $T_2^*(\mathcal{K})$, for any point set $\mathcal{K}$ in $H_\infty = \mathrm{PG}(2,2)$, has an automorphism group which is always larger than the automorphism group induced by collineations of $\mathrm{PG}(3,2)$.

From now on, in the remainder of this section, we **assume** that every point set $\mathcal{K}$ of H_∞ satisfies the following property.

Definition 4.2.5. We say that a point set $\mathcal{K}$ of H_∞ has *Property* $(*)$, if there is no plane of H_∞ intersecting $\mathcal{K}$ only in two intersecting lines, or in two intersecting lines minus their intersection point.

Lemma 4.2.6. *Let $q > 2$. Suppose $\mathcal{K}$ or $\mathcal{K}'$ satisfies Property $(*)$ and $\langle \mathcal{K} \rangle = \langle \mathcal{K}' \rangle = H_\infty$. Let α be an isomorphism between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$. We can define a mapping $\widetilde{\alpha}$ on the set of α-rigid points by putting $\widetilde{\alpha}(Q) = \infty(\alpha(L))$, where Q is an α-rigid point and L is any line for which $\infty(L) = Q$. This means, for two lines L_1 and L_2, if $\infty(L_1) = \infty(L_2) = Q$ is an α-rigid point, then $\infty(\alpha(L_1)) = \infty(\alpha(L_2)) = \widetilde{\alpha}(Q)$.*

Proof. We have to show that this is a mapping well defined, that is, if Q is an α-rigid point and L_1 and L_2 are two lines through Q, not contained in H_∞, then we will show that $\infty(\alpha(L_1)) = \infty(\alpha(L_2))$.

We first show that the mapping is well defined for points of $\mathcal{K}$. Consider two lines L_1, L_2 of $\mathcal{L}$ such that $\infty(L_1) = \infty(L_2) = P_1 \in \mathcal{K}$. Suppose that L_1 and L_2 are contained in a plane π intersecting H_∞ in a line P_1P_2, with points $P_i \in \mathcal{K}$. It follows by Lemma 4.2.4 that P_1P_2 is α-rigid. The lines of $T_n^*(\mathcal{K})$ through P_1 contained in π partition the affine points of π, and α is an isomorphism mapping lines of π through P_1 onto lines in a plane, hence $\infty(\alpha(L_1)) = \infty(\alpha(L_2))$ is the same point.

We proceed by induction. Suppose that for every line $L \in \mathcal{L}$, contained in a k-space π together with L_1, such that $\infty(L) = \infty(L_1) = P_1$ and such that π intersects H_∞ in a $(k-1)$-space $\langle P_1, \ldots, P_k \rangle$, $P_i \in \mathcal{K}$, $3 \leq k \leq n$, we have $\infty(\alpha(L)) = \infty(\alpha(L_1))$.

Consider a point $P_{k+1} \in \mathcal{K}$ not in $\langle P_1, \ldots, P_k \rangle$, and let $L_2 \in \mathcal{L}$ be a line through P_1, contained in a $(k+1)$-space π together with L_1 such that π intersects H_∞ in the k-space $\langle P_1, \ldots, P_{k+1} \rangle$. The plane $\langle L_2, P_{k+1} \rangle$ meets the k-space $\langle L_1, P_1, \ldots, P_k \rangle$ in a line M for which $\infty(M) = P_1$. By the induction hypothesis, $\infty(\alpha(M)) = \infty(\alpha(L_1))$. Moreover, from the case $k = 2$, we know that for the line L_2, it holds that $\infty(\alpha(L_2)) = \infty(\alpha(M)) = \infty(\alpha(L_1))$. Hence, proceeding by induction and using the fact that the points of $\mathcal{K}$ span H_∞, we have shown that the theorem is valid for all points of $\mathcal{K}$.

Now suppose that Q is an α-rigid point not contained in $\mathcal{K}$. Let L_1 be a line intersecting H_∞ in Q. Consider a point $P \in \mathcal{K}$ and the plane $\pi = \langle L_1, P \rangle$; let L_2 be a line through Q in π different from L_1. Let N_j, $j = 1, 2$, be two lines in π intersecting in the point P. We have shown in the previous part that $\infty(\alpha(N_1)) = \infty(\alpha(N_2))$. It follows that $\alpha(L_1)$ and $\alpha(L_2)$ lie in a plane (namely in $\langle \alpha(N_1), \alpha(N_2) \rangle$), hence, they meet in a point. Since α is an automorphism and L_1 and L_2 only meet in a point at infinity, $\alpha(L_1)$ and $\alpha(L_2)$ cannot meet in an affine point, hence $\infty(\alpha(L_1)) = \infty(\alpha(L_2))$. The lemma now follows by induction with the same argument as used above for points of $\mathcal{K}$. $\square$

The previous lemma shows that an isomorphism β between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$ can be extended to a mapping on the β-rigid points $Q \in H_\infty$, and we abuse notation by putting $\beta(Q) := \widetilde{\beta}(Q)$.

Lemma 4.2.7. *Let $q > 2$. Suppose $\mathcal{K}$ or $\mathcal{K}'$ satisfies Property $(*)$ and $\langle \mathcal{K} \rangle = \langle \mathcal{K}' \rangle = H_\infty$. Let α be an isomorphism between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$. If $P_1, \ldots, P_k$ are α-rigid points, then $\langle P_1, \ldots, P_k \rangle$ is an α-rigid space.*

Proof. We proceed by induction on the number k of considered α-rigid points. Suppose $k = 2$ and let π be a plane meeting H_∞ in the line P_1P_2, and let R be an affine point of π. By Lemma 4.2.6, since P_1 and P_2 are α-rigid, the points on the line RP_i, $i = 1, 2$, are mapped by α to the points on the line $\langle \alpha(R), \alpha(P_i) \rangle$. Let $S \neq R$ be a point of π, not on P_1P_2, RP_1 or RP_2 and let T_1 (respectively T_2) be the intersection point $SP_1 \cap RP_2$ (respectively $SP_2 \cap RP_1$). The point $\alpha(T_1)$ lies on $\langle \alpha(R), \alpha(P_2) \rangle$ and $\alpha(T_2)$ lies on $\langle \alpha(R), \alpha(P_1) \rangle$. It follows from Lemma 4.2.6 that $\alpha(S)$ lies on $\langle \alpha(T_1), \alpha(P_1) \rangle$ and $\langle \alpha(T_2), \alpha(P_2) \rangle$, hence, $\alpha(S)$ is contained in the plane $\langle \alpha(R), \alpha(P_1), \alpha(P_2) \rangle$. It follows that P_1P_2 is α-rigid.

Now suppose by induction that the statement is true for every set of $k-1$ α-rigid points, we will prove it is also true for a set of k α-rigid points.

Consider a space $\mu := \langle P_1, \ldots, P_k \rangle$ spanned by k α-rigid points $P_i, i = 1, \ldots, k$. The subspace $\pi := \langle P_1, \ldots, P_{k-1} \rangle$ is an α-rigid space by the induction hypothesis. If $P_k \in \pi$, then $\mu = \langle \pi, P_k \rangle = \pi$ is α-rigid.

Suppose $P_k \notin \pi$. Consider an affine point R of $\mathrm{PG}(n+1, q)$, then, since π is α-rigid and by Lemma 4.2.6, we have that every affine point of $\langle R, \pi \rangle$ is mapped by α to an affine point of $\langle \alpha(R), \alpha(P_1), \ldots, \alpha(P_{k-1}) \rangle$. Let S be an affine point of $\langle R, \mu \rangle$, not in $\langle R, \pi \rangle$. The line SP_k meets $\langle R, \pi \rangle$ in a point T. As $\alpha(T)$ lies in $\langle \alpha(R), \alpha(P_1), \ldots, \alpha(P_{k-1}) \rangle$ and $\alpha(S)$ lies on the line through $\alpha(P_k)$ and $\alpha(T)$ by Lemma 4.2.6, $\alpha(S)$ lies in $\langle \alpha(R), \alpha(P_1), \ldots, \alpha(P_{k-1}), \alpha(P_k) \rangle$. This proves our lemma. ☐

Theorem 4.2.8. *Let $q > 2$. Let $\mathcal{K}$ and $\mathcal{K}'$ be point sets in the hyperplane $H_\infty \cong \mathrm{PG}(n, q)$ such that $\mathcal{S} = \widehat{\mathcal{K}}$ and $\mathcal{S}' = \widehat{\mathcal{K}'}$ are n-dimensional subgeometries of H_∞. If $\mathcal{S} = H_\infty$, then furthermore suppose $\mathcal{K}$ satisfies Property $(*)$. Every isomorphism γ between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$ is induced by an isomorphism between $T_n^*(\mathcal{S})$ and $T_n^*(\mathcal{S}')$ mapping $\mathcal{S}$ onto $\mathcal{S}'$.*

Proof. If the subgeometry $\mathcal{S}$ is not the whole hyperplane H_∞, then the set $\mathcal{K}$ spans H_∞, but it does not contain full lines of H_∞, nor full lines minus one point, and thus it also satisfies Property $(*)$. Hence, the concept ‘rigid’ is well defined.

By the recursive construction of the closure of a set of points, we conclude, invoking Lemmas 4.2.2 and 4.2.7, that all points of $\mathcal{S}$ are γ-rigid. Hence, γ maps the affine points of a projective line intersecting H_∞ in exactly one point of $\mathcal{S}$, onto the affine points of a projective line intersecting H_∞ in exactly one point.

Making use of Lemma 4.2.6, we know that lines through the same γ-rigid point at infinity are mapped to lines intersecting each other in a point at infinity. As said before, we abuse notation and use γ for the extension of γ to all γ-rigid points of H_∞. Now, let P, Q, R be three points of $\mathcal{S}$ on a line L of H_∞ and let U be a point, not contained in H_∞. Since L is a γ-rigid line, γ maps the points of $\langle U, L\rangle$ onto points of a plane containing $\gamma(P), \gamma(Q)$, and $\gamma(R)$ at infinity. This implies that γ also maps collinear points of $\mathcal{S}$ to collinear points of H_∞. Since $\mathcal{K}$ is mapped to $\mathcal{K}'$, and collinearity needs to be preserved, clearly the points of $\mathcal{S}$ are mapped to the points of $\mathcal{S}'$ (keeping the recursive construction of $\mathcal{S}$ and $\mathcal{S}'$ in mind).

With the same argument, the points of $\mathcal{S}'$ are γ^{-1}-rigid, and collinear points of $\mathcal{S}'$ are mapped by γ^{-1} to collinear points of H_∞, thus belonging to $\mathcal{S}$. We conclude that γ is induced by an isomorphism between $T_n^*(\mathcal{S})$ and $T_n^*(\mathcal{S}')$ that maps $\mathcal{S}$ onto $\mathcal{S}'$ (preserving collinearity of points of $\mathcal{S}$). $\square$

Corollary 4.2.9. *Let $q > 2$. Let $\mathcal{K}$ denote a point set in $H_\infty = \mathrm{PG}(n, q)$ such that $\langle\mathcal{K}\rangle = H_\infty$. Consider the subgeometry $\mathcal{S} = \widehat{\mathcal{K}}$. When $\mathcal{S} = H_\infty$, furthermore suppose that $\mathcal{K}$ satisfies Property $(*)$. We obtain that $\mathrm{Aut}(T_n^*(\mathcal{K})) \leq \mathrm{Aut}(T_n^*(\mathcal{S}))$.*

Theorem 4.2.10. *Let $q > 2$. Let $\mathcal{K}$ and $\mathcal{K}'$ denote point sets satisfying Property $(*)$ in $H_\infty = \mathrm{PG}(n, q)$ such that $\widehat{\mathcal{K}} = H_\infty$. Let α be an isomorphism between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$. The map α is induced by an element of $\mathrm{P\Gamma L}(n+2, q)_{H_\infty}$ mapping $\mathcal{K}$ to $\mathcal{K}'$.*

If $\widehat{\mathcal{K}}$ is H_∞, clearly $\mathcal{K}$ spans H_∞ and hence, $(\mathrm{P\Gamma L}(n+2, q)_{H_\infty})_{\mathcal{K}} = \mathrm{P\Gamma L}(n+2, q)_{\mathcal{K}}$. Taking this into account, we obtain the following corollary.

Corollary 4.2.11. *Let $q > 2$. Let $\mathcal{K}$ denote a point set in $H_\infty = \mathrm{PG}(n, q)$ satisfying Property $(*)$ such that $\widehat{\mathcal{K}} = H_\infty$. The automorphism group $\mathrm{Aut}(T_n^*(\mathcal{K})) \cong \mathrm{P\Gamma L}(n+2, q)_{\mathcal{K}}$.*

4.3 Point sets not satisfying Property $(*)$

In Corollary 4.2.11, it is assumed that $\mathcal{K}$ satisfies Property $(*)$. It turns out that, if this condition is not satisfied, there exist counterexamples to this corollary. We give an explicit construction of such counterexamples and provide computer results that give more information on $\mathrm{Aut}(T_n^*(\mathcal{K}))$ for small n and q. All the mentioned computer results were obtained with GAP [54].

Recall that Property (∗) states that $\mathcal{K}$ is a point set such that there is no plane of H_∞ intersecting $\mathcal{K}$ only in two intersecting lines, or in two intersecting lines minus their intersection point. Now, let $\mathcal{K}$ be the point set of two intersecting lines in $\mathrm{PG}(2,q)$, we will show that there exist non-geometric automorphisms of $T_2^*(\mathcal{K})$. Moreover, we obtain the full automorphism group $\mathrm{Aut}(T_2^*(\mathcal{K}))$.

From the findings in the proof of Lemma 4.2.4, we deduce that a non-geometric automorphism ϕ of $T_2^*(\mathcal{K})$ acts such that the $2q$ lines of $\mathcal{L}$ contained in a plane intersecting $\mathcal{K}$ in exactly two points are mapped to $2q$ lines of a hyperbolic quadric $Q^+(3,q)$. It is easy to see that, if the lines of $\mathcal{L}$ in some plane that intersects $\mathcal{K}$ in exactly two points are mapped to the lines of a hyperbolic quadric $Q^+(3,q)$, then this is true for all planes that intersect $\mathcal{K}$ in exactly two points, just by looking at the intersection of two planes and the intersection of a plane and a hyperbolic quadric. We will construct such a mapping and show that if ψ and ψ' are non-geometric, then $\psi' = \chi_1\psi\chi_2$ with $\chi_i \in \mathrm{P\Gamma L}(4,q)_{\mathcal{K}}$.

Without loss of generality, let H_∞ be the plane of $\mathrm{PG}(3,q)$ with equation $X_0 = 0$, and let the set $\mathcal{K}$ consist of the points of two intersecting lines $L_1 : X_0 = X_1 = 0$ and $L_2 : X_0 = X_2 = 0$.

Theorem 4.3.1. *For the set of affine points $\mathcal{P}$ of $T_2^*(\mathcal{K})$, the mapping*

$$\phi_m : \mathcal{P} \to \mathcal{P} : (1,x,y,z)_{\mathbb{F}_q} \mapsto (1,x,y,z+mxy)_{\mathbb{F}_q}$$

induces a non-geometric automorphism of $T_2^*(\mathcal{K})$ when $m \in \mathbb{F}_q^*$.

Proof. The map ϕ_m is clearly a bijection.

We will describe the action of ϕ_m on all lines not in H_∞. Lines through $(0,0,0,1)_{\mathbb{F}_q}$ not in H_∞ are stabilised by ϕ_m. A line M of $T_2^*(\mathcal{K})$ through $(0,1,0,u)_{\mathbb{F}_q}$, $u \in \mathbb{F}_q$, such that $\langle M, L_2\rangle$ is the plane with equation $yX_0 - X_2 = 0$, with $y \in \mathbb{F}_q$, is mapped by ϕ_m to a line of $T_2^*(\mathcal{K})$ through $(0,1,0,u+my)_{\mathbb{F}_q}$. A line N of $T_2^*(\mathcal{K})$ through $(0,0,1,u')_{\mathbb{F}_q}$, $u' \in \mathbb{F}_q$, such that $\langle N, L_1\rangle$ is the plane with equation $xX_0 - X_1 = 0$, with $x \in \mathbb{F}_q$, is mapped by ϕ_m to a line of $T_2^*(\mathcal{K})$ through $(0,0,1,u'+mx)_{\mathbb{F}_q}$.

The affine points of a line through $(0,1,v,w)_{\mathbb{F}_q}$, $v,w \in \mathbb{F}_q$, $v \neq 0$, not in H_∞, are mapped by ϕ_m to the affine points of an irreducible conic containing the point $(0,0,0,1)_{\mathbb{F}_q}$. Specifically, the affine points of the line $\langle (1,x,y,z)_{\mathbb{F}_q}, (0,1,v,w)_{\mathbb{F}_q}\rangle$, $v \neq 0$, are mapped to the points of the conic with equation $(z-wx)X_0^2 + mvX_1^2 +$

$(w + my - mvx)X_0X_1 - X_0X_3 = 0$, different from $(0,0,0,1)_{\mathbb{F}_q}$. This conic is contained in the plane $X_2 = (y - vx)X_0 + vX_1$. The mapping ϕ_m induces an automorphism of $T_2^*(\mathcal{K})$, but is clearly not induced by a collineation. $\square$

Consider the group S of automorphisms of $T_2^*(\mathcal{K})$ induced by $\{\phi_m : \mathcal{P} \to \mathcal{P} : (1,x,y,z)_{\mathbb{F}_q} \mapsto (1,x,y,z+mxy)_{\mathbb{F}_q} \mid m \in \mathbb{F}_q\}$. It is clear that S is isomorphic to $(\mathbb{F}_q,+)$.

Theorem 4.3.2. *Let $q > 2$. The group $\langle \mathrm{P\Gamma L}(4,q)_{\mathcal{K}}, \phi_1\rangle = \mathrm{P\Gamma L}(4,q)_{\mathcal{K}} \rtimes S$ is the full automorphism group of $T_2^*(\mathcal{K})$ and is q times larger than the geometric automorphism group $\mathrm{P\Gamma L}(4,q)_{\mathcal{K}}$.*

Proof. Consider a non-geometric automorphism ψ of $T_2^*(\mathcal{K})$. We want to consider the action of ψ on (the affine points of) the planes that intersect H_∞ in the line N with equation $X_0 = X_3 = 0$. The lines of $T_2^*(\mathcal{K})$ in such a plane are mapped by ψ to lines of a hyperbolic quadric in $\mathrm{PG}(3,q)$ which contains the lines L_1 and L_2, that is, a quadric $Q^+(3,q)$ whose points satisfy the equation $X_0(aX_0 + bX_1 + cX_2 + X_3) + mX_1X_2 = 0$, for some $a,b,c,m \in \mathbb{F}_q$.

By multiplying with a well-chosen element of $\mathrm{P\Gamma L}(4,q)_{\mathcal{K}}$, we may assume that ψ fixes the two lines with equations $X_3 = X_1 = 0$ and $X_3 = X_2 = 0$ setwise. Hence, the planes $X_1 = 0$ and $X_2 = 0$ are also fixed setwise by ψ. The $2q$ lines of $T_2^*(\mathcal{K})$ in the plane $X_3 = 0$ are mapped by ψ to lines of a hyperbolic quadric with equation $X_0X_3 + mX_1X_2 = 0$, for some $m \in \mathbb{F}_q$. The set of planes $\{\pi_a : aX_0 + X_3 = 0 \mid a \in \mathbb{F}_q\}$, all containing the line N, provides a partition of the affine points in $\mathrm{PG}(3,q)$. Hence, the set of hyperbolic quadrics $\{\psi(\pi_a) \mid a \in \mathbb{F}_q\}$ must also provide a partition of the affine points. This means that for every quadric $\psi(\pi_a)$ there exists a unique plane through N intersecting it in two lines. Vice versa, every plane containing N, different from H_∞, contains two lines of exactly one quadric $\psi(\pi_a)$. Moreover, one of these lines is contained in the plane $X_1 = 0$, the other in $X_2 = 0$. Hence, we see that the set of planes $\{aX_0 + X_3 = 0 \mid a \in \mathbb{F}_q\}$ is mapped by ψ to the set of hyperbolic quadrics $\{X_0(a'X_0 + X_3) + mX_1X_2 = 0 \mid a' \in \mathbb{F}_q\}$ for some non-zero $m \in \mathbb{F}_q^*$.

By multiplying with a well-chosen element of $\mathrm{P\Gamma L}(4,q)_{\mathcal{K}}$, we may say that the set of planes $\{aX_0 + X_3 = 0 \mid a \in \mathbb{F}_q\}$ is mapped by ψ to the set of hyperbolic quadrics $\{X_0(a'X_0 + X_3) + X_1X_2 = 0 \mid a' \in \mathbb{F}_q\}$. It is clear that the sets of planes $\{bX_0 + X_1 = 0 \mid b \in \mathbb{F}_q\}$ and $\{cX_0 + X_2 = 0 \mid c \in \mathbb{F}_q\}$ are both fixed by ψ and not switched.

Recall that the mapping ϕ_1 maps every plane $aX_0 + X_3 = 0$ to the hyperbolic quadric $X_0(aX_0 + X_3) + X_1X_2 = 0$ and fixes all the planes of the form $bX_0 +$

$X_1 = 0$ and $cX_0 + X_1 = 0$. We now consider the mapping $\phi_1^{-1}\psi$, this map is an isomorphism of $T_n^*(\mathcal{K})$ which sends planes through N to planes through N, hence, as said before, sends all planes to planes. Thus, $\phi_1^{-1}\psi$ is induced by some collineation of $\mathrm{P\Gamma L}(4,q)_{\mathcal{K}}$. We see that $\psi = \chi_1\phi_1\chi_2$ with $\chi_i \in \mathrm{P\Gamma L}(4,q)_{\mathcal{K}}$ and hence $\langle \mathrm{P\Gamma L}(4,q)_{\mathcal{K}}, \phi_1\rangle$ is the full automorphism group of $T_2^*(\mathcal{K})$.

The group S has order q and $\mathrm{P\Gamma L}(4,q)_{\mathcal{K}}$ is a normal subgroup of $\langle \mathrm{P\Gamma L}(4,q)_{\mathcal{K}}, \phi_1\rangle$ such that $S \cap \mathrm{P\Gamma L}(4,q)_{\mathcal{K}}$ is trivial, hence the result follows. □

For $q = 3, 4$, we considered $\mathcal{K}$ to be the point set of two intersecting planes in $H_\infty = \mathrm{PG}(3,q)$. We checked by computer that the group $\mathrm{Aut}(T_3^*(\mathcal{K}))$ is q^2 times larger than $\mathrm{P\Gamma L}(5,q)_{\mathcal{K}}$. Hence, also in this case there exist automorphisms of $T_3^*(\mathcal{K})$ which are not induced by a collineation of the ambient projective space.

Remark. There are point sets $\mathcal{K}$ that do not satisfy Property $(*)$ but do have the property that $\mathrm{Aut}(T_3^*(\mathcal{K}))$ consists entirely of geometric automorphisms. E.g. let $\mathcal{K}$ be the point set of three lines L_1, L_2, L_3 in $H_\infty = \mathrm{PG}(3,q)$ such that $L_1 \cap L_2 = \emptyset$ and L_3 intersects L_1 and L_2, then, by going through the details of the proofs in the previous section, it is not too hard to check that $\mathrm{Aut}(T_3^*(\mathcal{K})) = \mathrm{P\Gamma L}(5,q)_{\mathcal{K}}$.

4.4 Linear representations of subgeometries

In Section 4.2 we proved that the automorphism group of the linear representation $T_n^*(\mathcal{K})$ is induced by a collineation group of the ambient space, provided certain conditions on $\mathcal{K}$ are fulfilled (see Corollary 4.2.9). In Section 4.3 we showed what could happen when $\mathcal{K}$ does not satisfy Property $(*)$. In this section, we consider what happens when the condition on the closure $\widehat{\mathcal{K}}$ is not met; more specifically, we will see that the automorphism group is always larger.

From Theorem 4.2.8 it follows that we only need to consider isomorphisms between two linear representations $T_n^*(\mathcal{S})$ and $T_n^*(\mathcal{S}')$ of subgeometries $\mathcal{S}$ and $\mathcal{S}'$, who are thus necessarily projectively equivalent. Hence, it is sufficient to focus on the automorphism group of $T_n^*(\mathcal{S})$, where $\mathcal{S}$ is a non-trivial n-dimensional subgeometry of H_∞. Let $\mathcal{S} = \mathrm{PG}(n,q)$ be a subgeometry of $H_\infty = \mathrm{PG}(n,q^t)$, q a prime power, $t \in \mathbb{N}_0 \backslash \{1\}$, and embed H_∞ in $\mathrm{PG}(n+1,q^t)$; consider the linear representation $T_n^*(\mathcal{S})$ of $\mathcal{S}$.

Remark. This case covers all possibilities of non-trivial subgeometries, since a subgeometry $\mathrm{PG}(n,q^i)$ of $\mathrm{PG}(n,q^t)$ can be written as the subgeometry $\mathrm{PG}(n,\tilde{q})$ of $\mathrm{PG}(n,\tilde{q}^{\frac{t}{i}})$ where $\tilde{q} = q^i$ since $i|t$.

In Subsection 4.4.1 we introduce the geometry $X(n,t,q)$; this is a generalisation of the semipartial geometry H_q^{n+2} that was introduced in [40]. We will explore the automorphism group of $X(n,t,q)$ and prove that this group consists solely of collineations of its ambient space $\mathrm{PG}(n+t+1,q)$.

In Subsection 4.4.2 we provide a geometric proof of the isomorphism between the geometry $X(n,t,q)$ and the linear representation $T_n^*(\mathcal{S})$ of a subgeometry $\mathcal{S} \cong \mathrm{PG}(n,q)$ of the hyperplane $H_\infty \cong \mathrm{PG}(n,q^t)$. We study $X(n,t,q)$ as a coset geometry in Subsection 4.4.3, by generalising the idea of translation semipartial geometries as introduced in [43]. In a group theoretical/algebraic way, we recover the isomorphism with $T_n^*(\mathcal{S})$.

In Subsection 4.4.4 we prove that this automorphism group is isomorphic to a specific collineation group of $\mathrm{PG}(t(n+1)+1,q)$, namely the collineation group stabilising the generalised linear representation isomorphic to $T_n^*(\mathcal{S})$. These results in particular show that the automorphism group of $T_n^*(\mathcal{S})$ is much larger than its automorphism group induced by $\mathrm{P\Gamma L}(n+2,q^t)$.

This allows us to correct a misconception that has appeared in the literature. More specifically, we can construct a non-geometric automorphism of $T_2^*(\mathcal{K})$, where $\mathcal{K}$ is a Baer subplane of $\mathrm{PG}(2,q^2)$. In this case, $T_2^*(\mathcal{K})$ is a semipartial geometry and the general belief was that for every $T_m^*(\mathcal{K})$ that is a semipartial geometry, every automorphism is geometric (see e.g. [42, Remark 7.3.13]).

4.4.1 The geometry $X(n,t,q)$

Definition 4.4.1. Consider an n-dimensional subspace π of the projective space $\mathrm{PG}(n+t,q)$. The geometry $X(n,t,q)$ is the incidence structure $(\mathcal{P},\mathcal{L})$, with natural incidence, where the point set $\mathcal{P}$ and line set $\mathcal{L}$ are defined as follows:

$\mathcal{P}$: the $(t-1)$-spaces of $\mathrm{PG}(n+t,q)$ skew to π,

$\mathcal{L}$: the t-spaces of $\mathrm{PG}(n+t,q)$ meeting π in exactly one point.

For $t=2$, this geometry was introduced in [40] as H_q^{n+2}. The geometry H_q^{n+2} was also given in [46] as an example of a semipartial geometry that is not a partial geometry for $n \geq 2$. There the author proved that for $n=2$ the geometry H_q^4 is isomorphic to $T_2^*(\mathcal{B})$, with $\mathcal{B}$ a Baer subplane of $\mathrm{PG}(2,q^2)$. We will show that this holds for general n and t, namely $X(n,t,q) \cong T_n^*(\mathcal{S})$, where $\mathcal{S}$ is an n-dimensional $\mathbb{F}_q$-subgeometry of $H_\infty = \mathrm{PG}(n,q^t)$; we provide both a geometric (Subsection 4.4.2) and algebraic proof (Subsection 4.4.3).

First, we will determine the full automorphism group of $X(n,t,q)$. We prove that an automorphism of $X(n,t,q)$ is induced by a collineation of its ambient space.

Theorem 4.4.2. *Every automorphism of $X(n,t,q)$ is induced by a collineation of its ambient space; more precisely,* $\mathrm{Aut}(X(n,t,q)) \cong \mathrm{P\Gamma L}(n+t+1,q)_\pi$.

Proof. For $t=1$, the statement is trivial since $X(n,1,q)$ corresponds to the affine space $\mathrm{AG}(n+1,q)$. From now on, assume $t>1$. We will prove that every automorphism of $X(n,t,q)$ is induced by a mapping on the points and lines of $\mathrm{PG}(n+t,q)$ which is incidence preserving, hence is a collineation.

Every automorphism ψ of $X(n,t,q)$ is a permutation of the $(t-1)$-dimensional subspaces of $\mathrm{PG}(n+t,q)$ disjoint from π. Consider a $(t-2)$-dimensional space μ disjoint from π, and take two elements $\nu_1,\nu_2 \in \mathcal{P}$ such that $\nu_1 \cap \nu_2 = \mu$. We clearly have $\langle \nu_1,\nu_2\rangle \in \mathcal{L}$. Since ψ preserves $\mathcal{L}$, ψ sends ν_1 and ν_2 to two elements ν_1' and ν_2' of $\mathcal{P}$ lying in an element of $\mathcal{L}$, thus intersecting in a $(t-2)$-dimensional space μ'.

Now consider $\nu_3 \in \mathcal{P}$ containing μ, but not lying in $\langle \nu_1,\nu_2\rangle$. As seen before, its image ν_3' intersects both ν_1' and ν_2' in a $(t-2)$-dimensional space. Hence, $\nu_3'\cap\nu_1' = \nu_3'\cap\nu_2' = \nu_1'\cap\nu_2' = \mu'$, since otherwise ν_1',ν_2' and ν_3' would lie in one and the same element of $\mathcal{L}$, a contradiction. It follows that ψ extends to a well-defined mapping on the $(t-2)$-dimensional subspaces μ of $\mathrm{PG}(n+t,q)$, disjoint from π, by putting $\psi(\mu) := \psi(\nu_1)\cap\psi(\nu_2)$ for $\nu_1,\nu_2\in\mathcal{P}$ with $\nu_1\cap\nu_2=\mu$. Furthermore, ψ preserves incidence between these $(t-2)$-spaces and the $(t-1)$-spaces of $\mathcal{P}$.

In other words, for $k=t-1$, the map ψ extends to a mapping which:

- (i) permutes the $(k+1)$-subspaces intersecting π in exactly one point, such that incidence with the k-subspaces disjoint from π is preserved,
- (ii) permutes the k-subspaces disjoint from π,
- (iii) permutes the $(k-1)$-subspaces disjoint from π, such that incidence with the k-subspaces disjoint from π is preserved and such that incidence with the $(k+1)$-subspaces intersecting π in one point is preserved.

We continue by induction. Suppose that for some $k \leq t-1$, the previous three properties are valid. We will prove that they are also valid for $k-1$. Clearly, property (ii) for $k-1$ follows immediately from property (iii) for k. First we prove property (i) and afterwards property (iii).

(Step 1: prove that ψ satisfies property (i) for $k-1$)
Consider a k-dimensional space α intersecting π in exactly one point P, and take

two $(k+1)$-dimensional spaces β_1, β_2 intersecting π only in P such that $\beta_1 \cap \beta_2 = \alpha$. The map ψ preserves the $(k-1)$-dimensional spaces of $\mathrm{PG}(n+t,q)$, disjoint from π, contained in α and their incidence with β_1 and β_2. The subspaces β_1 and β_2 are mapped to two $(k+1)$-dimensional spaces β'_1 and β'_2, intersecting in a subspace α'. The subspace α' needs to contain exactly q^k distinct $(k-1)$-dimensional spaces of $\mathrm{PG}(n+t,q)$ disjoint from π, hence α' is a k-dimensional subspace intersecting π in a point P'. It follows that this point P' is the point at infinity for both β'_1 and β'_2.

Consider a third $(k+1)$-dimensional space β_3, intersecting π in exactly P, and containing α. Its image β'_3 intersects both β'_1 and β'_2 in a k-dimensional space. Since ψ acts bijectively on $(k-1)$-dimensional subspaces disjoint from π and preserves their incidence with $(k+1)$-spaces intersecting π in one point, the intersections $\beta'_3 \cap \beta'_1$, $\beta'_3 \cap \beta'_2$ and $\beta'_1 \cap \beta'_2$ all contain the same $(k-1)$-dimensional subspaces and, again by counting, we see that they each coincide with the same k-dimensional subspace α' intersecting π in P'. It follows that ψ extends to a well-defined mapping on the k-dimensional subspaces of $\mathrm{PG}(n+t,q)$ intersecting π in one point, preserving incidence with the $(k-1)$-dimensional subspaces disjoint from π. That is, ψ satisfies property (i) for $k-1$.

(Step 2: prove that ψ satisfies property (iii) for $k-1$)
Consider a $(k-2)$-dimensional subspace γ of $\mathrm{PG}(n+t,q)$ disjoint from π. Take two $(k-1)$-dimensional subspaces δ_1 and δ_2, disjoint from π, such that $\delta_1 \cap \delta_2 = \gamma$ and such that $\langle \delta_1, \delta_2 \rangle = \epsilon$ is a k-dimensional space intersecting π in exactly one point Q. From property (i) it follows that their images δ'_1 and δ'_2 are contained in a k-space ϵ', which is the image of ϵ. Hence, they intersect in a $(k-2)$-dimensional subspace γ', disjoint from π.

Consider a third $(k-1)$-dimensional subspace δ_3 containing γ, disjoint from π and not contained in ϵ. This means $\langle \delta_1, \delta_2, \delta_3 \rangle$ is a $(k+1)$-dimensional space. From the above, the image δ'_3 intersects both δ'_1 and δ'_2 in a $(k-2)$-dimensional space. The intersection $\delta'_3 \cap \delta'_1$ equals $\delta'_3 \cap \delta'_2$, which must be γ', since otherwise $\langle \delta'_1, \delta'_2, \delta'_3 \rangle$ would be k-dimensional, that is, equal to ϵ'. This is a contradiction, because ψ preserves k-dimensional spaces intersecting π in exactly one point.

Hence, the map ψ extends to a well-defined mapping on the $(k-2)$-dimensional subspaces of $\mathrm{PG}(n+t,q)$ which are disjoint from π, such that incidence with the k-dimensional subspaces intersecting π in one point, and with the $(k-1)$-dimensional subspaces disjoint from π is preserved. That is, ψ satisfies property (iii) for $k-1$.

(Step 3: ψ extends to an incidence preserving map on π)
By induction we see that the map ψ extends to a well-defined incidence preserving

mapping on the m-dimensional spaces disjoint from π, $0 \leq m \leq t-1$, and on the l-dimensional spaces intersecting π in exactly one point, $1 \leq l \leq t$.

We now only need to show that ψ can be extended to an incidence preserving bijective mapping on the point set of π. Suppose that P is a point of π, and L_1 and L_2 are two distinct lines containing P, such that $\langle L_1, L_2 \rangle \cap \pi = \{P\}$. Since the plane $\langle L_1, L_2 \rangle$ is mapped to a plane intersecting π in one point P', both L_1 and L_2 are mapped to lines intersecting π in P'. Consider a line L_3 intersecting π in exactly P, such that $\langle L_1, L_3 \rangle$ intersects π in a line. The plane $\langle L_2, L_3 \rangle$ intersects π only in P, hence from the above, the line L_3 is also mapped to a line intersecting π in P'. So we can extend ψ to a mapping on the points of π by putting $\psi(P) := \psi(L_1) \cap \psi(L_2)$ for lines L_1, L_2 meeting π exactly in the point P.

Now consider a line L in π. Take a 3-dimensional space μ intersecting π in exactly L. Take two disjoint lines M and N in μ, both disjoint from L. The lines M and N are mapped to two disjoint lines spanning a 3-dimensional space μ' intersecting π in at most a line. The $q+1$ planes, spanned by M and a point of L, all intersect N in a point, hence their images lie in μ'. The images of the points of L are all different and lie in μ', hence they lie on a line L' which has to be the intersection of μ' with π. It follows that ψ also preserves the line set of π.

We have proved that every automorphism ψ of $X(n,t,q)$ is induced by a mapping on the points and lines of $\mathrm{PG}(n+t,q)$ which is incidence preserving, hence it is a collineation. From this, it is clear that the automorphism group of $X(n,t,q)$ is isomorphic to the stabiliser group $\mathrm{P\Gamma L}(n+t+1,q)_\pi$ of π. $\square$

4.4.2 A geometric isomorphism between $X(n,t,q)$ and $T_n^*(\mathcal{S})$

By field reduction, the points of $\mathrm{PG}(n+1,q^t)$ correspond to the elements of a Desarguesian $(t-1)$-spread $\mathcal{D}$ of $\mathrm{PG}(t(n+2)-1,q)$. In this way, the points of the embedded hyperplane $H_\infty = \mathrm{PG}(n,q^t)$ correspond to the $(t-1)$-spaces of $\mathcal{D}$ forming a Desarguesian spread $\mathcal{D}_\infty$ in a subspace $J_\infty = \mathrm{PG}(t(n+1)-1,q)$ of $\mathrm{PG}(t(n+2)-1,q)$. Say $\mathcal{F} = \mathcal{F}_{n+1,t,q}$ is the corresponding field reduction map, that is, denote the element of $\mathcal{D}$ corresponding to a point P of $\mathrm{PG}(n+1,q^t)$ by $\mathcal{F}(P)$, and define $\mathcal{F}(u) := \{\mathcal{F}(P) \mid P \in u\}$ for a subset u of $\mathrm{PG}(n+1,q^t)$.

If U is a subset of $\mathrm{PG}(t(n+2)-1,q)$, then we define $\mathcal{B}(U) := \{R \in \mathcal{D} \mid U \cap R \neq \emptyset\}$. We can identify the elements of $\mathcal{B}(U)$ with their corresponding points of $\mathrm{PG}(n+1,q^t)$, i.e. we will not always make the distinction between a point P and its corresponding spread element $\mathcal{F}(P)$.

If $\mathcal{S}$ is a subgeometry $\mathrm{PG}(n,q)$ of H_∞, then the set $\mathcal{F}(\mathcal{S})$ is a set of $\frac{q^{n+1}-1}{q-1}$ disjoint $(t-1)$-spaces in J_∞, forming one of the two systems of a *Segre variety* $\mathbf{S}_{n,t-1}$ (see Theorem 1.4.2).

We use the notation $\mathcal{F}(T_n^*(\mathcal{S}))$ for the field reduced geometry corresponding to the linear representation $T_n^*(\mathcal{S})$.

Definition 4.4.3. The geometry $\mathcal{F}(T_n^*(\mathcal{S}))$ is a point-line incidence structure embedded in $\mathrm{PG}(t(n+2)-1,q)$, with natural incidence, point set $\widetilde{\mathcal{P}} := \{\mathcal{F}(P) \mid P \in \mathcal{P}(T_n^*(\mathcal{K}))\}$ and line set $\widetilde{\mathcal{L}} := \{\langle \mathcal{F}(L)\rangle \mid L \in \mathcal{L}(T_n^*(\mathcal{K}))\}$, meaning:

- $\widetilde{\mathcal{P}}$: the $(t-1)$-spaces of the Desarguesian $(t-1)$-spread $\mathcal{D}$, not contained in $\mathcal{D}_\infty$,
- $\widetilde{\mathcal{L}}$: the $(2t-1)$-spaces spanned by elements of $\mathcal{D}$, meeting the space J_∞ in exactly one element of $\mathcal{F}(\mathcal{S})$.

Clearly, the geometry $\mathcal{F}(T_n^*(\mathcal{S}))$ is isomorphic to $T_n^*(\mathcal{S})$.

We will show that the geometry $X(n,t,q)$ is isomorphic to $\mathcal{F}(T_n^*(\mathcal{S}))$ by projecting $\mathcal{F}(T_n^*(\mathcal{S}))$ from a specific subspace π'. From this, we will conclude that $X(n,t,q)$ is isomorphic to $T_n^*(\mathcal{S})$.

Lemma 4.4.4. *Let $\mathcal{S}$ be an n-dimensional $\mathbb{F}_q$-subgeometry in $\mathrm{PG}(n,q^t)$. There exists an n-dimensional space π of J_∞ such that $\mathcal{S}$ corresponds to $\mathcal{B}(\pi)$ and a $(tn-n+t-2)$-dimensional subspace π' of J_∞ skew to π meeting all elements of $\mathcal{B}(\pi)$ in a $(t-2)$-space, such that π' does not contain a spread element of $\mathcal{D}_\infty$.*

Proof. As said before, $\mathcal{S}$ corresponds to one system of the Segre variety $\mathbf{S}_{n,t-1}$ in J_∞, denote this system consisting of $(t-1)$-spaces by system A and the system of n-spaces by system B. Now consider a subvariety $\mathbf{S}_{n,t-2}$, contained in $\mathbf{S}_{n,t-1}$, then $\mathbf{S}_{n,t-2}$ meets every element of system A in a $(t-2)$-space. Moreover, we know that $\mathbf{S}_{n,t-1}$ spans J_∞ and $\mathbf{S}_{n,t-2}$ spans a $(tn-n+t-2)$-space π', containing $\frac{q^{t-1}-1}{q-1}$ elements of system B. Now let π be an n-space of system B, not contained in $\mathbf{S}_{n,t-2}$. If π and π' would meet in a point P, this point would lie on two different elements of system B, namely π and the unique element of $\mathbf{S}_{n,t-2}$ of system B through P, a contradiction.

Now suppose that π' contains a spread element τ of $\mathcal{D}_\infty$, then τ corresponds to a point Q of $\mathrm{PG}(n,q^t)$, not contained in $\mathcal{S}$. Let μ be an $(n-2)$-dimensional subspace of π such that Q is not contained in the $(n-2)$-space of $\mathrm{PG}(n,q^t)$ spanned by the points of $\mathcal{S}$ corresponding to μ. This implies that the space $\langle \mathcal{B}(\mu),\tau\rangle$ corresponds to a hyperplane H of $\mathrm{PG}(n,q^t)$. However, $\langle \mathcal{B}(\mu),\tau\rangle$ meets π' in a space ρ of dimension $(tn-n)$. Since π' is $(tn-n+t-2)$-dimensional, this implies that

ρ meets all elements of $\mathcal{B}(\pi)$, hence, that the subgeometry $\mathcal{S}$ is contained in the hyperplane H of $\mathrm{PG}(n, q^t)$, a contradiction. □

Let π and π' be subspaces of J_∞ as described in Lemma 4.4.4. Note that $\langle \pi, \pi' \rangle = J_\infty$. For a subspace L of $\mathrm{PG}(t(n+2)-1, q)$, we denote $\infty(L)$ to be the subspace $L \cap J_\infty$. Let Π be an $(n+t)$-dimensional space of $\mathrm{PG}(t(n+2)-1, q)$ with $\infty(\Pi) = \pi$.

The following mapping ϕ will give us the projection of $\mathcal{F}(T_n^*(\mathcal{S}))$ from π' onto the embedding of the structure $X(n,t,q)$ in $\Pi = \mathrm{PG}(n+t, q)$.

$$\begin{array}{cccc} \phi: & \mathcal{F}(T_n^*(\mathcal{S})) & \to & \Pi; \\ & P \in \widetilde{\mathcal{P}} & \mapsto & \langle \pi', P \rangle \cap \Pi, \\ & L \in \widetilde{\mathcal{L}} & \mapsto & \langle \pi', L \rangle \cap \Pi. \end{array}$$

Theorem 4.4.5. *The mapping ϕ defines an isomorphism between $\mathcal{F}(T_n^*(\mathcal{S}))$ and $X(n,t,q)$.*

Proof. We need to prove that ϕ preserves incidence and defines a bijection between $\widetilde{\mathcal{P}}$ and $\mathcal{P}$, and between $\widetilde{\mathcal{L}}$ and $\mathcal{L}$.

(Step 1: ϕ is a bijection between $\widetilde{\mathcal{P}}$ and $\mathcal{P}$)
For $P \in \widetilde{\mathcal{P}}$, the space $\langle \pi', P \rangle$ is a $(tn+2t-n-2)$-dimensional subspace of $\mathrm{PG}(t(n+2)-1, q)$. This space intersects the $(n+t)$-dimensional space Π in a subspace P' of dimension at least $(tn+2t-n-2)+(n+t)-(t(n+2)-1) = t-1$. If the intersection would have dimension t or larger, then π' and Π would meet, a contradiction. It follows that $P \in \widetilde{\mathcal{P}}$ is mapped to a $(t-1)$-dimensional space of $\Pi \backslash \pi$, and thus belongs to the elements $\mathcal{P}$ of $X(n,t,q)$.

For $P, R \in \widetilde{\mathcal{P}}$, $P \neq R$, suppose $\phi(P) = \phi(R)$, then $\langle \pi', P \rangle \cap \Pi = \langle \pi', R \rangle \cap \Pi$. We see that $\langle \pi', P \rangle = \langle \langle \pi', P \rangle \cap \Pi, \pi' \rangle = \langle \langle \pi', R \rangle \cap \Pi, \pi' \rangle = \langle \pi', R \rangle$. The $(2t-1)$-dimensional space $\langle P, R \rangle$ lies in the space $\langle \pi', P \rangle$, so clearly $\langle P, R \rangle \cap J_\infty = \langle P, R \rangle \cap \pi'$. Since $P, R \in \mathcal{D}$, we have $\langle P, R \rangle \cap J_\infty \in \mathcal{D}_\infty$. This is not possible since there are no elements of $\mathcal{D}_\infty$ contained in $\pi' = \langle \pi', P \rangle \cap J_\infty$.

Since $|\widetilde{\mathcal{P}}| = |\mathcal{P}| = q^{t(n+1)}$, the map ϕ is a bijection between $\widetilde{\mathcal{P}}$ and $\mathcal{P}$.

(Step 2: ϕ is a bijection between $\widetilde{\mathcal{L}}$ and $\mathcal{L}$)
For $L \in \widetilde{\mathcal{L}}$, since $L \cap \pi'$ has dimension $t-2$, the space $\langle \pi', L \rangle$ is a $(tn+2t-n-1)$-dimensional subspace of $\mathrm{PG}(t(n+2)-1, q)$. This space intersects Π in a subspace L' of dimension at least t. If the intersection would have dimension $t+1$ or larger, then π' and Π would meet, a contradiction. It follows that $L \in \widetilde{\mathcal{L}}$ is mapped to a t-dimensional space of Π intersecting π in exactly one point, thus it belongs to the elements $\mathcal{L}$ of $X(n,t,q)$.

For $L, M \in \widetilde{\mathcal{L}}$, suppose $\phi(L) = \phi(M)$, then $\langle \pi', L\rangle \cap \Pi = \langle \pi', M\rangle \cap \Pi$. Intersection with J_∞ gives $\infty(L) \cap \pi = \infty(M) \cap \pi$, thus $\infty(L) = \infty(M)$ since they are both elements of $\mathcal{B}(\pi)$. The space $\langle \phi(L), \infty(L)\rangle$ has dimension $2t-1$ and contains L, thus is equal to L. It follows that $L = \langle \phi(L), \infty(L)\rangle = \langle \phi(M), \infty(M)\rangle = M$, so we conclude that the mapping is injective.

As $|\widetilde{\mathcal{L}}| = |\mathcal{L}| = q^{nt}\frac{q^{n+1}-1}{q-1}$, the map ϕ is a bijection between $\widetilde{\mathcal{L}}$ and $\mathcal{L}$.

(Step 3: ϕ preserves incidence)
First note that every line of $\widetilde{\mathcal{L}}$ contains q^t points of $\widetilde{\mathcal{P}}$, and also every line of $\mathcal{L}$ contains q^t points of $\mathcal{P}$. Now, suppose $P \in \widetilde{\mathcal{P}}$ is contained in $L \in \widetilde{\mathcal{L}}$, then clearly $\phi(P) = \langle \pi', P\rangle \cap \Pi$ is contained in $\phi(L) = \langle \pi', L\rangle \cap \Pi$, hence incidence is preserved. $\square$

Considering Theorem 4.4.2 and 4.4.5, we arrive to the following conclusion.

Theorem 4.4.6. *The geometries $X(n,t,q)$ and $T_n^*(\mathcal{S})$ are isomorphic and thus* $\mathrm{Aut}(T_n^*(\mathcal{S})) \cong \mathrm{Aut}(X(n,t,q)) \cong \mathrm{P\Gamma L}(n+t+1,q)_\pi$.

4.4.3 $X(n,t,q)$ as a coset geometry

In this section, we will see that $X(n,t,q)$ has a natural description as a coset geometry. We will prove that $X(n,t,q)$ is isomorphic to the linear representation $T_n^*(\mathcal{S})$ embedded in $\mathrm{PG}(n+1,q^t)$, where $\mathcal{S}$ is the subgeometry $\mathrm{PG}(n,q)$ of the hyperplane $H_\infty \cong \mathrm{PG}(n,q^t)$. This provides an elegant description of both the geometry and its automorphism group as determined in Subsection 4.4.1.

Without loss of generality, let π be the n-dimensional space of $\mathrm{PG}(n+t,q)$ with equation $X_0 = X_1 = \ldots = X_{t-1} = 0$. We consider the embedding of the geometry $X(n,t,q)$ in $\mathrm{PG}(n+t,q)$. Recall that the point set $\mathcal{P}$ consists of the $(t-1)$-subspaces of $\mathrm{PG}(n+t,q)$ disjoint from π, and the line set $\mathcal{L}$ consists of the t-subspaces of $\mathrm{PG}(n+t,q)$ intersecting π in exactly one point.

Consider the following $(n+1)$-dimensional spaces through π:

$$\begin{aligned}\Sigma_0 &: X_1 = \ldots = X_{t-1},\\ \Sigma_j &: X_0 = \ldots = X_{j-1} = X_{j+1} = \ldots = X_{t-1},\ \forall j = 1,\ldots,t-2,\\ \Sigma_{t-1} &: X_0 = \ldots = X_{t-2}.\end{aligned}$$

Every $(t-1)$-dimensional space $P \in \mathcal{P}$ is spanned by the set of t unique points $\{U_j = P \cap \Sigma_j\}_{j=0,\ldots,t-1}$, that is, for all $j = 0,\ldots,t-1$, the point U_j has coordinates $(0,\ldots,0,1,0,\ldots,0,a_{0j},\ldots,a_{nj})_{\mathbb{F}_q}$ with a 1 at position j and $a_{ij} \in \mathbb{F}_q$, for all $i = 0,\ldots,n$, and $j = 0,\ldots,t-1$.

For every $P \in \mathcal{P}$, we can consider an $(n+t+1) \times (n+t+1)$-matrix A'_P

$$A'_P = \begin{pmatrix} I_t & 0 \\ A_P & I_{n+1} \end{pmatrix},$$

where the $(n+1) \times t$-matrix A_P is defined by $(A_P)_{ij} = a_{ij}$, for all $0 \leq i \leq n$, $0 \leq j \leq t-1$, and where I_k denotes the $k \times k$-identity matrix. Conversely, every such matrix corresponds to a unique $P \in \mathcal{P}$.

Now let G be the set consisting of all these matrices,

$$G = \{A'_P \mid P \in \mathcal{P}\}.$$

This set G, under operation of multiplication, forms a group. Note that

$$A'_P A'_Q = \begin{pmatrix} I_t & 0 \\ A_P + A_Q & I_{n+1} \end{pmatrix}.$$

Hence, this group is elementary abelian of order $q^{(n+1)t}$. Furthermore, since every elementary abelian group corresponds to a vector space, this implies we will be able to interpret $X(n,t,q)$ as a geometry “embedded” in an $\mathbb{F}_p$-vector space, or equivalently, in an affine space over $\mathbb{F}_p$.

We define the action of G on $\mathcal{P}$ by $A'_R(P) = Q$ if and only if $A'_R A'_P = A'_Q$. It is obvious that this is well defined and that the group G acts sharply transitively on the point set $\mathcal{P}$ of $X(n,t,q)$.

One could also envision G as a subgroup of $\mathrm{PGL}(n+t+1,q)$ and consider the action on the points of $\mathrm{PG}(n+t,q)$ by left-multiplication, that is we let the matrix act on column vectors from the left. The induced action on the elements of $\mathcal{P}$ is exactly the same as the one we defined earlier. Note that all the points of π are fixed by the group G.

Next we will describe the lines of $X(n,t,q)$ in an algebraic way. Choose for every point $R \in \pi$ a corresponding vector $(0,\ldots,0,b_0,\ldots,b_n)$ (determined up to a scalar multiple). We define the subgroup $G(R)$ of G as follows:

$$G(R) = \left\{ \begin{pmatrix} I_t & 0 \\ B_a & I_{n+1} \end{pmatrix} \,\middle|\, a \in \mathbb{F}_q^t \right\},$$

with the $(n+1) \times t$-matrix B_a defined by $(B_a)_{ij} = b_i a_j$, $0 \leq i \leq n$, $0 \leq j \leq t-1$, for $a = (a_0, a_1, \dots, a_{t-1}) \in \mathbb{F}_q^t$. Clearly the group $G(R)$ is independent of the chosen coordinates for R.

For $0 \le i \le n+t$, consider the point Q_i of $\mathrm{PG}(n+t, q)$ with corresponding vector w_i, where $w_0 = (1, 0, \dots, 0), w_1 = (0, 1, 0, \dots, 0), \dots, w_{n+t} = (0, \dots, 0, 1)$. The group $G(R)$ has size q^t and stabilises the t-space

$$L_R = \langle R, Q_0, Q_1, \dots, Q_{t-1} \rangle.$$

Note that this space is an element of $\mathcal{L}$. The group $G(R)$ fixes the point R and acts transitively on the t-tuples $(R_0, \dots, R_{t-1})$, where R_i is a point of $\langle R, Q_i \rangle \setminus \{R\}$. Hence, it acts transitively on the $(t-1)$-spaces of L_R not through R. These are exactly the elements of $\mathcal{P}$ contained in this line L_R of $X(n, t, q)$. Furthermore, since this group has size q^t, this action is sharply transitive.

The space $I = \langle Q_0, \dots, Q_{t-1} \rangle \in \mathcal{P}$ corresponds to the identity matrix of G. From the above, we learned that the lines of $X(n, t, q)$ through I correspond to the subgroups $G(R)$, $R \in \pi$. Furthermore, since G (interpreted as a subgroup of $\mathrm{PGL}(n+t+1, q)$) fixes all points of π and acts (sharply) transitively on the $(t-1)$-spaces disjoint from π, we deduce that the elements of $\mathcal{L}$ (the lines of $X(n, t, q)$) are in one-to-one correspondence with the cosets of $G(R)$ in G. However, at this point we can simplify notation if we take into account that all important properties of G and the subgroups $G(R)$ are determined by the $(n+1) \times t$-submatrices in the lower left corner of the elements of G. Let $\mathbf{M}$ be the group of all $(n+1) \times t$-matrices over $\mathbb{F}_q$ under matrix addition. We have obtained the following description of $X(n, t, q)$ as a *coset geometry* $\mathcal{M} = (\mathcal{P}_{\mathcal{M}}, \mathcal{L}_{\mathcal{M}})$ with natural incidence (containment), point set $\mathcal{P}_{\mathcal{M}}$ and line set $\mathcal{L}_{\mathcal{M}}$ as follows:

$\mathcal{P}_{\mathcal{M}}$: the elements of $\mathbf{M}$, that is, the $(n+1) \times t$-matrices over $\mathbb{F}_q$,

$\mathcal{L}_{\mathcal{M}}$: the cosets in $\mathbf{M}$ of the subgroups $L_b := \left\{ b^T a \mid a \in \mathbb{F}_q^t \right\}$, for all $b \in \mathbb{F}_q^{n+1} \setminus \{(0, \dots, 0)\}$.

There are exactly $\frac{q^n-1}{q-1}$ lines of type L_b, since $L_b = L_c$, when b and c are scalar multiples of each other.

Note that this also provides a nice description of the adjacency graph of our geometry as a *Cayley graph*: the vertices are the elements of $\mathbf{M}$, and two vertices are adjacent if and only if their difference (in $\mathbf{M}$) is of the form $b^T a$, for some $a \in \mathbb{F}_q^t$ and $b \in \mathbb{F}_q^{n+1} \setminus \{(0, \dots, 0)\}$.

Next we will find the lowest dimensional affine space in which $X(n, t, q)$ naturally embeds, and establish the isomorphism with $T_n^*(\mathcal{S})$ in an algebraic way. Recall the representation of the Singer group of Subsection 1.3.3. That is, let $f(x) = x^t - m_{t-1}x^{t-1} - \cdots - m_1 x - m_0$ be an irreducible monic polynomial of degree t over $\mathbb{F}_q$ used to construct $\mathbb{F}_{q^t}$. Let M be the companion matrix of $f(x)$, that is,

$$M = \begin{pmatrix} 0 & I_{t-1} \\ m_0 & m \end{pmatrix},$$

with $m = (m_1, \ldots, m_{t-1})$. When we define

$$H = \left\{ a_0 I_t + a_1 M + \cdots + a_{t-1} M^{t-1} \mid a_i \in \mathbb{F}_q \right\},$$

then H has the structure of $\mathbb{F}_{q^t}$ under usual matrix addition and multiplication. It follows that $H \setminus \{0\}$ acts sharply transitively on the points of $\mathrm{AG}(t, q)$, different from $(0, \ldots, 0)_{\mathbb{F}_q}$.

Now define the action of H on $\mathbf{M}$ by right-multiplication, i.e. in the following way:

$$(H, \mathbf{M}) \to \mathbf{M} : (C, A_P) \mapsto A_P C.$$

It is readily checked that this makes $\mathbf{M}$ into an H-vector space, that is, an $\mathbb{F}_{q^t}$-vector space. However, if we consider this action restricted to the subgroups L_b of $\mathbf{M}$, then we see that we for every $C \in H$ obtain an action

$$(C, L_b) \mapsto L_b.$$

This makes the subgroups L_b into H-vector subspaces of the H-vector space $\mathbf{M}$.

From the above it now follows that we can view our geometry $X(n, t, q)$ as a geometry embedded in an $(n+1)$-dimensional vector space over $\mathbb{F}_{q^t}$, where the lines through I correspond to certain vector subspaces and the other lines to cosets of these subspaces, which are parallel subspaces when seen in $\mathrm{AG}(n+1, q^t)$. Hence we obtain a representation of $X(n, t, q)$ as a generalised linear representation. Since the lines of $X(n, t, q)$ have size $q^t = |H| = |\mathbb{F}_{q^t}|$, this is in fact a linear representation, and we obtain $X(n, t, q)$ as a linear representation of a point set in $\mathrm{PG}(n, q^t)$.

Define the following set

$$\mathcal{A} = \{(A, B, C, l) \mid A \in \mathbf{M}, B \in \mathrm{GL}(n+1, q), C \in \mathrm{GL}(t, q), l \in \mathbb{Z}_h\},$$

and a binary operation $\circ$ on $\mathcal{A}$ as follows

$$\begin{aligned}(A_2, B_2, C_2, l_2) &\circ (A_1, B_1, C_1, l_1)\\ &= (B_2^{p^{-l_1}} A_1 C_2^{p^{-l_1}} + A_2^{p^{-l_1}}, B_2^{p^{-l_1}} B_1, C_1 C_2^{p^{-l_1}}, l_1 + l_2).\end{aligned}$$

Then $\mathcal{A}, \circ$ is easily checked to be a group.

Next define an action of $\mathcal{A}$ on the points of $\mathcal{M}$ as follows

$$((A, B, C, l), A_P) \mapsto (BA_P C + A)^{p^l}.$$

A simple verification now shows that this makes $\mathcal{A}$ into a group of automorphisms of $\mathcal{M}$. The kernel of the described action clearly is $K = \{(0, \lambda I_{n+1}, \lambda^{-1} I_t, 0) \mid \lambda \in \mathbb{F}_q^*\}$.

Elements of the form $(A, I_{n+1}, I_t, 0)$ map every element of $\mathcal{L}_{\mathcal{M}}$ to one of its cosets, meaning it fixes the point set at infinity of the linear representation while permuting the lines that go through a given point at infinity. An element $(0, I_{n+1}, C, 0)$ fixes every subgroup L_b and permutes the group elements of every subgroup, meaning it fixes the point set at infinity and the affine point corresponding to the zero matrix (meaning the space I), while permuting the points of every line of $\mathcal{P}$ through this point. Elements of the form $(0, I_{n+1}, I_t, l)$ provide the semilinear maps corresponding to the elements of $\mathrm{Aut}(\mathbb{F}_q)$. The action of the subgroup $\{(0, B, I_t, 0) \mid B \in \mathrm{GL}(n+1, q)\} \leq \mathcal{A}$ fixes the point corresponding to the zero matrix (that is, the origin in the corresponding linear representation) while permuting the lines that go through it. Hence this subgroup stabilises the point set at infinity of the linear representation.

We still need to uncover this point set at infinity. With every vector $b \in \mathbb{F}_q^{n+1} \setminus \{(0, \ldots, 0)\}$, up to scalar multiple, there is a corresponding point at infinity. Define a point-line incidence structure with as points the subgroups L_b and as lines the set of subgroups $\{L_{b_i} \mid i = 0, \ldots, q\}$ where the vectors b_i belong to a plane, meaning, the corresponding projective points are contained in a projective line. This means the lines through the affine point corresponding to I naturally correspond to the structure $\mathrm{PG}(n, q)$. Since this structure is stabilised by the subgroup $\{(0, B, I_t, 0) \mid B \in \mathrm{GL}(n+1, q)\}$ of the automorphism group of $\mathcal{M}$, we know that our point set at infinity is a subgeometry isomorphic to $\mathrm{PG}(n, q)$ and hence that $\mathcal{M} \cong T_n^*(\mathcal{S})$.

Finally, since this group has the size of the full automorphism group

$$\frac{|\mathcal{A}|}{|K|} = q^{(n+1)t} \left|\mathrm{GL}(n+1, q)\right| \left|\mathrm{GL}(t, q)\right| \left|\mathrm{Aut}(\mathbb{F}_q)\right| / (q-1) = \left|\mathrm{P\Gamma L}(n+t+1, q)_\pi\right|,$$

we see that $\mathcal{A}$ provides a natural description of the full automorphism group of $T_n^*(\mathcal{S})$.

4.4.4 The automorphism group

In this section, by some easy counting arguments, we will show that while the automorphism group of $T_n^*(\mathcal{S})$ is not induced by collineations of its ambient space, there is another setting where we can see the automorphism group as a collineation group, namely when we consider a generalised linear representation isomorphic to $T_n^*(\mathcal{S})$.

Recall the definition of a generalised linear representation.

Definition 4.4.7. Let $\mathcal{K}$ be a set of disjoint $(t-1)$-dimensional subspaces in $\Pi_\infty \cong \mathrm{PG}(m, q)$. Embed Π_∞ as a hyperplane in $\mathrm{PG}(m+1, q)$. The *generalised linear representation* $T^*_{m,t-1}(\mathcal{K})$ of $\mathcal{K}$ is the incidence structure $(\mathcal{P}', \mathcal{L}')$ with natural incidence for which:

$\mathcal{P}'$: the *affine* points of $\mathrm{PG}(m+1, q)$, i.e. not contained in Π_∞,

$\mathcal{L}'$: the t-spaces of $\mathrm{PG}(m+1, q)$ intersecting Π_∞ in exactly a $(t-1)$-space of $\mathcal{K}$.

Recall the *Barlotti-Cofman representation* [10] of $\mathrm{PG}(n+1, q^t)$ inside $\mathrm{PG}(t(n+1), q)$ (see Section 1.5). Here, the points of the hyperplane $H_\infty \cong \mathrm{PG}(n, q^t)$ in $\mathrm{PG}(n+1, q^t)$ are represented as $(t-1)$-dimensional spaces of a Desarguesian spread $\mathcal{D}_\infty$ in $J_\infty \cong \mathrm{PG}(t(n+1)-1, q)$. The affine points of $\mathrm{PG}(n+1, q^t)$ with respect to H_∞ can be identified with the affine points of the space $\mathrm{PG}(t(n+1), q) \setminus J_\infty$. The lines of $\mathrm{PG}(n+1, q^t)$ intersecting H_∞ in a point correspond to the t-dimensional spaces of $\mathrm{PG}(t(n+1), q)$ meeting J_∞ in an element of $\mathcal{D}_\infty$.

We consider the Barlotti-Cofman representation of the points and lines of the linear representation $T_n^*(\mathcal{S})$. The points of the hyperplane H_∞ are mapped to elements of a Desarguesian $(t-1)$-spread $\mathcal{D}_\infty$, the points of $\mathcal{S}$ are mapped to the $(t-1)$-spaces of $\mathcal{F}(\mathcal{S})$, where $\mathcal{F}$ is the field reduction map corresponding to $\mathcal{D}_\infty$. The affine points go to affine points, the lines of $T_n^*(\mathcal{S})$ go to t-spaces intersecting J_∞ in an element of $\mathcal{F}(\mathcal{S})$. In this way we obtain the generalised linear representation $T^*_{t(n+1)-1,t-1}(\mathcal{F}(\mathcal{S}))$, which is thus clearly isomorphic to $T_n^*(\mathcal{S})$.

Note that this representation corresponds to the intersection of the 'points' and 'lines' of $\mathcal{F}(T_n^*(\mathcal{S}))$ with a space $\mathrm{PG}(t(n+1), q)$ through J_∞, hence $T_n^*(\mathcal{S}) \cong T^*_{t(n+1)-1,t-1}(\mathcal{F}(\mathcal{S})) \cong \mathcal{F}(T_n^*(\mathcal{S})) \cong X(n, t, q)$.

It is clear that the group $\mathrm{P\Gamma L}(t(n+1)+1,q)_{\mathcal{F}(\mathcal{S})}$ stabilises the generalised linear representation $T^*_{t(n+1)-1,t-1}(\mathcal{F}(\mathcal{S}))$ and hence is isomorphic to a subgroup of $\mathrm{Aut}(X(n,t,q)) \cong \mathrm{P\Gamma L}(n+t+1,q)_\pi$. We will see by counting that the groups are in fact isomorphic.

Recall that the group of all perspectivities of $\mathrm{PG}(m,q)$ with a fixed axis H is a subgroup of $\mathrm{PGL}(m+1,q)$ and is denoted by $\mathrm{Persp}_q(H)$.

Lemma 4.4.8. *Suppose $\mathcal{K}$ is a subset of a hyperplane H of $\mathrm{PG}(m,q)$ such that $\langle\mathcal{K}\rangle = H$. The group $\mathrm{P\Gamma L}(m+1,q)_\mathcal{K}$ is an extension of $\mathrm{Persp}_q(H)$ by $\mathrm{P\Gamma L}(m,q)_\mathcal{K}$ and $\mathrm{PGL}(m+1,q)_\mathcal{K}$ is an extension of $\mathrm{Persp}_q(H)$ by $\mathrm{PGL}(m,q)_\mathcal{K}$.*

Proof. The kernel of the action of $(\mathrm{P\Gamma L}(m+1,q)_H)_\mathcal{K}$, $(\mathrm{PGL}(m+1,q)_H)_\mathcal{K}$ respectively, on H is clearly $\mathrm{Persp}_q(H)$. The image of the action is isomorphic to $\mathrm{P\Gamma L}(m,q)_\mathcal{K}$, $\mathrm{PGL}(m,q)_\mathcal{K}$ respectively, showing that $(\mathrm{P\Gamma L}(m+1,q)_H)_\mathcal{K}$, $(\mathrm{PGL}(m+1,q)_H)_\mathcal{K}$ respectively, is an extension of $\mathrm{Persp}_q(H)$ by $\mathrm{P\Gamma L}(m,q)_\mathcal{K}$, $\mathrm{PGL}(m,q)_\mathcal{K}$ respectively. $\square$

Theorem 4.4.9. [67, Theorem 25.5.13] *The projective automorphism group of a Segre variety $\mathbf{S}_{l,k}$ of $\mathrm{PG}((l+1)(k+1)-1,q)$ is either isomorphic to $\mathrm{PGL}(l+1,q)\times\mathrm{PGL}(k+1,q)$ when $l \neq k$, or is isomorphic to $(\mathrm{PGL}(l+1,q)\times\mathrm{PGL}(k+1,q))\rtimes C_2$ when $l=k$.*

Theorem 4.4.10.

$$\mathrm{Aut}(T^*_n(\mathcal{S})) \cong \mathrm{P\Gamma L}(n+t+1,q)_\pi \cong \mathrm{P\Gamma L}(t(n+1)+1,q)_{\mathcal{F}(\mathcal{S})}$$

Proof. The full automorphism group of $T^*_n(\mathcal{S})$ is isomorphic to $\mathrm{P\Gamma L}(n+t+1,q)_\pi$, see Theorem 4.4.6. Since $\mathrm{P\Gamma L}(n+t+1,q)$ acts transitively on the n-spaces of $\mathrm{PG}(n+t,q)$, we find the following:

$$\begin{aligned}|\mathrm{P\Gamma L}(n+t+1,q)_\pi| &= \frac{|\mathrm{P\Gamma L}(n+t+1,q)|}{\begin{bmatrix} n+t+1 \\ n+1\end{bmatrix}_q} \\ &= q^{t(n+1)}q^{\frac{t(t-1)}{2}}(q^t-1)\cdots(q-1)|\mathrm{P\Gamma L}(n+1,q)|.\end{aligned}$$

Now we calculate the size of $\mathrm{P\Gamma L}(t(n+1)+1,q)_{\mathcal{F}(\mathcal{S})}$. By Lemma 4.4.8, since $\mathcal{F}(\mathcal{S})$ spans J_∞, we find:

$$|\mathrm{P\Gamma L}(t(n+1)+1,q)_{\mathcal{F}(\mathcal{S})}| = |\mathrm{Persp}_q(J_\infty)||\mathrm{P\Gamma L}(t(n+1),q)_{\mathcal{F}(\mathcal{S})}|.$$

As seen before, the set of points contained in $\mathcal{F}(\mathcal{S})$ forms a Segre variety $\mathbf{S}_{n,t-1}$. Hence the stabiliser of $\mathcal{F}(\mathcal{S})$ is the stabiliser of the Segre variety that in the case $t = n+1$ does not switch the two systems. Thus, by Theorem 4.4.9, we find $\mathrm{PGL}(t(n+1),q)_{\mathcal{F}(\mathcal{S})} = \mathrm{PGL}(n+1,q) \times \mathrm{PGL}(t,q)$. The semilinear automorphisms stabilising any one of the systems of the Segre variety naturally extend to elements of $\mathrm{P\Gamma L}(t(n+1),q)$. Hence we obtain $\left|\mathrm{P\Gamma L}(t(n+1),q)_{\mathcal{F}(\mathcal{S})}\right| = |\mathrm{PGL}(n+1,q)|\,|\mathrm{PGL}(t,q)|\,|\mathrm{Aut}(\mathbb{F}_q)|$.

We conclude that $\mathrm{P\Gamma L}(t(n+1)+1,q)_{\mathcal{F}(\mathcal{S})}$ has the same size as $\mathrm{P\Gamma L}(n+t+1,q)_\pi$:

$$\begin{aligned}
&|\mathrm{P\Gamma L}(t(n+1)+1,q)_{\mathcal{F}(\mathcal{S})}|\\
&\qquad = |\mathrm{Persp}_q(J_\infty)||\mathrm{PGL}(n+1,q)||\mathrm{PGL}(t,q)|\,|\mathrm{Aut}(\mathbb{F}_q)|\\
&\qquad = |\mathrm{Persp}_q(J_\infty)||\mathrm{P\Gamma L}(n+1,q)||\mathrm{PGL}(t,q)|\\
&\qquad = q^{t(n+1)}(q-1)q^{\frac{t(t-1)}{2}}(q^t-1)\cdots(q^2-1)|\mathrm{P\Gamma L}(n+1,q)|.
\end{aligned}$$

Clearly, every collineation of $\mathrm{P\Gamma L}(t(n+1)+1,q)_{\mathcal{F}(\mathcal{S})}$ is a non-trivial element of $\mathrm{Aut}(T_n^*(\mathcal{S}))$. Both groups have the same size, thus we obtain that $\mathrm{Aut}(T_n^*(\mathcal{S})) \cong \mathrm{P\Gamma L}(t(n+1)+1,q)_{\mathcal{F}(\mathcal{S})}$. □

Recall that the subgroup of $\mathrm{P\Gamma L}(t(n+1),q)$ stabilising the Desarguesian spread $\mathcal{D}_\infty$ elementwise, is isomorphic to the Singer group $\mathrm{SG}(t,q)$. This means that the group elements of $\mathrm{Persp}_{q^t}(H_\infty)$ stabilising the hyperplane H_∞ of $\mathrm{PG}(n+1,q^t)$ pointwise correspond to elements of a group isomorphic to $\mathrm{Persp}_q(J_\infty).\mathrm{SG}(t,q)$ stabilising the Desarguesian spread $\mathcal{D}_\infty$ in the hyperplane J_∞ of $\mathrm{PG}(t(n+1),q)$ elementwise.

The subgroup of the automorphism group of the linear representation $T_n^*(\mathcal{S})$ for which the elements are induced by collineations of the space $\mathrm{PG}(n+1,q^t)$ will be called the *geometric automorphism group*.

Theorem 4.4.11. *The full automorphism group of $T_n^*(\mathcal{S})$ is*

$$\frac{|\mathrm{PGL}(t,q)|}{t|\mathrm{SG}(t,q)|} = \frac{1}{t}q^{\frac{t(t-1)}{2}}(q^{t-1}-1)\cdots(q^2-1)(q-1)$$

times larger than the geometric automorphism group of $T_n^*(\mathcal{S})$.

Proof. Since $\mathcal{S}$ spans H_∞, the geometric automorphism group of $T_n^*(\mathcal{S})$ is isomorphic to

$$
\begin{aligned}
&\mathrm{P\Gamma L}(n{+}2, q^t)_{\mathcal{S}} \\
&\qquad \cong \mathrm{Persp}_{q^t}(H_\infty).\mathrm{P\Gamma L}(n+1, q^t)_{\mathcal{S}} \\
&\qquad \cong \mathrm{Persp}_{q^t}(H_\infty).\left(\mathrm{PGL}(n+1, q) \rtimes \mathrm{Aut}(\mathbb{F}_{q^t})\right) \\
&\qquad \cong \mathrm{Persp}_{q^t}(H_\infty).\left(\mathrm{PGL}(n+1, q) \rtimes (\mathrm{Aut}(\mathbb{F}_{q^t}/\mathbb{F}_q).\mathrm{Aut}(\mathbb{F}_q))\right).
\end{aligned}
$$

The full automorphism group is isomorphic to

$$
\begin{aligned}
&\mathrm{P\Gamma L}((n+1)t+1, q)_{\mathcal{F}(\mathcal{S})} \\
&\qquad \cong \mathrm{Persp}_q(J_\infty).\mathrm{P\Gamma L}((n+1)t, q)_{\mathcal{F}(\mathcal{S})} \\
&\qquad \cong \mathrm{Persp}_q(J_\infty).\left((\mathrm{PGL}(n+1, q) \times \mathrm{PGL}(t, q)) \rtimes \mathrm{Aut}(\mathbb{F}_q)\right).
\end{aligned}
$$

□

4.5 Conclusion

We can now give a full answer to the isomorphism problem for linear representations $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$ of point sets $\mathcal{K}$, $\mathcal{K}'$.

Theorem 4.5.1. *Let $\mathcal{K}$ and $\mathcal{K}'$ denote point sets in $H_\infty \cong \mathrm{PG}(n, q^t)$, $n > 1$, $t > 1$, such that the closures $\widehat{\mathcal{K}}$ and $\widehat{\mathcal{K}'}$ are non-trivial n-dimensional subgeometries of H_∞. Suppose $\widehat{\mathcal{K}} \cong \mathrm{PG}(n, q)$ and let α be an isomorphism between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$. Then α is induced by an element of $\mathrm{P\Gamma L}(t(n{+}1){+}1, q)_{J_\infty}$ mapping $\mathcal{F}(\mathcal{K})$ onto $\mathcal{F}(\mathcal{K}')$, for the field reduction map $\mathcal{F} = \mathcal{F}_{n+1,t,q}$.*

Proof. Suppose $\mathcal{S} = \widehat{\mathcal{K}}$ and $\mathcal{S}' = \widehat{\mathcal{K}'}$ are n-dimensional subgeometries of H_∞. From Theorem 4.2.8, we know that every isomorphism between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$ is induced by an isomorphism between $T_n^*(\mathcal{S})$ and $T_n^*(\mathcal{S}')$ mapping $\mathcal{S}$ onto $\mathcal{S}'$, that is $\mathcal{S} \cong \mathcal{S}' \cong \mathrm{PG}(n, q)$.

From the previous section we know that every isomorphism between $T_n^*(\mathcal{S})$ and $T_n^*(\mathcal{S}')$ is induced by an element of $\mathrm{P\Gamma L}(t(n+1)+1, q)_{J_\infty}$ mapping $\mathcal{F}(\mathcal{S})$ onto $\mathcal{F}(\mathcal{S}')$. It is clear that an isomorphism between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$ corresponds to an element of $\mathrm{P\Gamma L}(t(n+1)+1, q)_{J_\infty}$ mapping $\mathcal{F}(\mathcal{K})$ onto $\mathcal{F}(\mathcal{K}')$. □

Theorem 4.5.2. *Let $\mathcal{K}$ be a point set of $H_\infty = \mathrm{PG}(n, q^t)$, $n > 1$, $t > 1$, such that*

$\widehat{\mathcal{K}} \cong \mathcal{S} \cong \mathrm{PG}(n, q)$. *The full automorphism group of* $T_n^*(\mathcal{K})$ *is*

$$\frac{|\mathrm{PGL}(t, q)|}{t|\mathrm{SG}(t, q)|} = \frac{1}{t} q^{\frac{t(t-1)}{2}} (q^{t-1} - 1) \cdots (q^2 - 1)(q - 1)$$

times larger than the geometric automorphism group of $T_n^*(\mathcal{K})$.

Proof. Since $\mathcal{K}$ contains a frame, the set $\mathcal{F}(\mathcal{K})$ contains a set of $n+2$ $(t-1)$-spaces in general position. Hence, by Theorem 1.4.3, the set $\mathcal{F}(\mathcal{K})$ is contained in a unique Segre variety $\mathbf{S}_{n,t-1}$, which necessarily corresponds to $\mathcal{F}(\mathcal{S})$. It follows that the stabiliser of $\mathcal{F}(\mathcal{K})$ in $\mathrm{PGL}((n+1)t - 1, q)$ necessarily stabilises $\mathcal{F}(\mathcal{S})$, hence

$$\mathrm{PGL}((n+1)t, q)_{\mathcal{F}(\mathcal{K})} \cong \mathrm{PGL}(n+1, q)_{\mathcal{K}} \times \mathrm{PGL}(t, q).$$

The result now follows as in the proof of Theorem 4.4.11. □

Theorem 4.5.3. *Let* $\mathcal{K}$ *and* $\mathcal{K}'$ *be two point sets of* $H_\infty = \mathrm{PG}(n, q^t)$, $n > 1$, $q^t > 2$, *such that* $\langle \mathcal{K} \rangle = \langle \mathcal{K}' \rangle = H_\infty$. *If* $\widehat{\mathcal{K}} = H_\infty$, *suppose furthermore that* $\mathcal{K}$ *satisfies Property* $(*)$. *The linear representations* $T_n^*(\mathcal{K})$ *and* $T_n^*(\mathcal{K}')$ *are isomorphic if and only if the point sets* $\mathcal{K}$ *and* $\mathcal{K}'$ *are* PΓL*-equivalent.*

Proof. When $\widehat{\mathcal{K}} = H_\infty$ such that $\mathcal{K}$ satisfies Property $(*)$, we know by Theorem 4.2.10 that every isomorphism is induced by a collineation of the ambient space, hence the result follows.

Suppose $\widehat{\mathcal{K}}$ and $\widehat{\mathcal{K}'}$ are non-trivial isomorphic subgeometries $\mathcal{S}$ and $\mathcal{S}'$ of H_∞. Say $\mathcal{S} \cong \mathcal{S}' \cong \mathrm{PG}(n, q)$, then it follows from the previous theorem that given an isomorphism α between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$, there is a corresponding map in $\mathrm{P\Gamma L}(t(n+1), q)$ mapping $\mathcal{F}(\mathcal{S})$ onto $\mathcal{F}(\mathcal{S}')$ such that $\mathcal{F}(\mathcal{K})$ is mapped to $\mathcal{F}(\mathcal{K}')$.

The induced action of this map in $\mathrm{PG}(n+1, q^t)$ is not necessarily geometric, however, it always preserves the inherited incidence structure of the subgeometries $\mathcal{S}$ and $\mathcal{S}'$. It follows that the point sets $\mathcal{K}$ and $\mathcal{K}'$ must be PΓL-equivalent. □

5 Infinite families of semisymmetric graphs

In this chapter, we present a general construction leading to several non-isomorphic families of connected q-regular semisymmetric graphs $\Gamma_{n,q}(\mathcal{K})$ of order $2q^{n+1}$, embeddable in $\mathrm{PG}(n+1,q)$ by considering the *linear representation* $T_n^*(\mathcal{K})$ of a particular point set $\mathcal{K}$ of size q.

By varying the point set $\mathcal{K}$, we obtain new examples of semisymmetric graphs. Moreover, by using the results of the previous chapter, we obtain in almost all cases the full automorphism group. When $\mathcal{K}$ is a *normal rational curve* with one point removed, the graphs $\Gamma_{n,q}(\mathcal{K})$ are isomorphic to the graphs constructed for $q = p^h$ in [78] and to the graphs constructed for q prime in [48]. These graphs were known to be semisymmetric, but the authors did not determine their full automorphism group.

These results were published in [30] and are joint work with P. Cara and G. Van de Voorde.

5.1 Introduction

All graphs are assumed to be finite and simple, i.e. they are undirected graphs which contain no loops or multiple edges.

Definition 5.1.1. We say that a graph is *vertex-transitive* if its automorphism group acts transitively on the vertices. Similarly, a graph is *edge-transitive* if its automorphism group acts transitively on the edges. A graph is *semisymmetric* if it is regular and edge-transitive but not vertex-transitive (see [53]).

One can easily prove that a semisymmetric graph must be bipartite with equal partition sizes. Moreover, the automorphism group must be transitive on both partition sets. General constructions of semisymmetric graphs are quite rare. We will construct several infinite families $\Gamma_{n,q}(\mathcal{K})$ of semisymmetric graphs using the linear representation $T_n^*(\mathcal{K})$ of the point set $\mathcal{K}$.

The chapter is organised as follows. In Section 5.2, we introduce the incidence graph $\Gamma_{n,q}(\mathcal{K})$ and we obtain a condition on $\mathcal{K}$ to ensure that the graph is not vertex-transitive. In Section 5.3, we will explicitly describe the geometric automorphism group of the constructed graphs and provide an easy condition on $\mathcal{K}$ to ensure that the graph is edge-transitive. In Section 5.4, we will consider point sets $\mathcal{K}$ such that the closure $\widehat{\mathcal{K}}$ is equal to H_∞. In Section 5.5, we will look for point sets $\mathcal{K}$ spanning H_∞ such that the closure $\widehat{\mathcal{K}}$ is equal to a non-trivial subgeometry of H_∞.

We give a brief overview of all constructions to come in Table 5.1. Note that we use the abbreviation NRC for a normal rational curve. If $q = p$ is prime and $\mathcal{K}$ contains a frame and satisfies Property $(*)$ (see Definition 4.2.5) then every automorphism of $\Gamma_{n,q}(\mathcal{K})$ is geometric, that is, every automorphism is induced by a collineation of the ambient space. When $q = p^h$ is not prime, in Theorem 4.5.2 we saw that, when the closure $\widehat{\mathcal{K}}$ is isomorphic to a subgeometry $\mathrm{PG}(n, q_0)$, for some $q = q_0^k$, then one can obtain an automorphism group which is

$$\mathrm{ng}(q_0, k) := \frac{1}{k} q_0^{\frac{k(k-1)}{2}} \prod_{i=1}^{k-1} (q_0^i - 1)$$

times larger than the geometric one. When $\mathcal{K}$ contains a frame, this is the full automorphism group.

$\mathcal{K}$	Condition	$\lvert\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K}))\rvert$	Ref.
basis	$q = n+1$	$> hq^{n+1}(q-1)q!\mathrm{ng}(p,h)$	§5.4.1
frame	$q = n+2$	$hq^{n+1}(q-1)^n q!\mathrm{ng}(p,h)$	§5.4.1
$\subset$ NRC	$q \geq n+3$	$hq^{n+2}(q-1)^2$	§5.4.2
$\subset$ q-arc	$q > 4$ even	$hq^5(q-1)^2$	§5.4.3
$\subset$ Glynn-arc	$q = 9$	$9^6 8^2$	§5.4.4
$\subset Q^-(3,q)$	$q > 4$ square	$2hq^5(q-1)^2\mathrm{ng}(\sqrt{q},2)$	§5.5.1
$\subset$ Tits ovoid	$q = 2^{2(2e+1)}$	$hq^5(q-1)(\sqrt{q}-1)\mathrm{ng}(\sqrt{q},2)$	§5.5.2
$\subset Q^+(3,q)$	$q > 4$ square	$2hq^5(q-1)(\sqrt{q}-1)^2\mathrm{ng}(\sqrt{q},2)$	§5.5.3
$\subset$ cone $V\mathcal{O}$	$q = q_0^k$	$kq^{2n+1}(q-1)^2\lvert\mathrm{P\Gamma L}(n,q_0)_{\mathcal{O}}\rvert\mathrm{ng}(q_0,k)$	§5.5.4

Table 5.1: Overview of all constructions of Chapter 5

5.2 The graph $\Gamma_{n,q}(\mathcal{K})$

We introduce the concept of the incidence graph $\Gamma_{n,q}(\mathcal{K})$ of the linear representation $T_n^*(\mathcal{K})$ of a point set $\mathcal{K}$.

Definition 5.2.1. We denote the *point-line incidence graph* of $T_n^*(\mathcal{K})$ by $\Gamma_{n,q}(\mathcal{K})$. It is the bipartite graph with as classes the point set $\mathcal{P}$ and the line set $\mathcal{L}$ of $T_n^*(\mathcal{K})$, and adjacency corresponding to the natural incidence of the structure $T_n^*(\mathcal{K})$.

The main goal of this chapter is the construction of infinite families of semisymmetric graphs. Note that, since a semisymmetric graph is regular, any graph $\Gamma_{n,q}(\mathcal{K})$ that is semisymmetric, necessarily has $|\mathcal{K}| = q$. For this reason, we will investigate point sets of size q in $\mathrm{PG}(n,q)$. Moreover, as in the previous chapter, we will only consider point sets $\mathcal{K}$ such that $\langle\mathcal{K}\rangle = H_\infty$, since otherwise, the graph $\Gamma_{n,q}(\mathcal{K})$ would not be connected (see Theorem 4.1.3).

Whenever we consider the incidence graph $\Gamma_{n,q}(\mathcal{K})$, we still regard the set of vertices as a set of points and lines in $\mathrm{PG}(n+1,q)$. In this way we can use the inherited properties of this space and borrow expressions such as the span of points, a subspace, incidence, and others.

It is easy to see that the problem of describing the automorphism group of $\Gamma_{n,q}(\mathcal{K})$ is essentially the same as dealing with the problem for $T_n^*(\mathcal{K})$, as long as there is no automorphism of $\Gamma_{n,q}(\mathcal{K})$ mapping a vertex corresponding to a point onto a vertex corresponding to a line. In Subsection 5.2.1, we give a condition on $\mathcal{K}$ to ensure that every automorphism of $\Gamma_{n,q}(\mathcal{K})$ preserves the set of points $\mathcal{P}$. After that we can use the results of Chapter 4 whenever this condition is met.

5.2.1 A property of $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K}))$

Note that an automorphism of $T_n^*(\mathcal{K})$, viewed as an incidence structure, always maps points onto points and lines onto lines, whereas an automorphism of $\Gamma_{n,q}(\mathcal{K})$ might map vertices corresponding to points onto vertices corresponding to lines. Of course, in this latter case, the sets $\mathcal{P}$ and $\mathcal{L}$ must have equal size, that is, $|\mathcal{K}| = q$.

Consider a graph Γ and a positive integer i, for a vertex v of Γ, we write $\Gamma_i(v)$ for the set of vertices at distance i from v.

The following lemma provides a condition which forces the neighbourhood of a vertex in the set $\mathcal{P}$ to be essentially different from the neighbourhood of a vertex in the set $\mathcal{L}$.

Lemma 5.2.2. *Let $\mathcal{K}$ be a set of points of H_∞ such that every point of $H_\infty \backslash \mathcal{K}$ lies on at least one tangent line to $\mathcal{K}$, then $\forall P \in \mathcal{P}, \forall L \in \mathcal{L} : \Gamma_{n,q}(\mathcal{K})_4(P) \not\cong \Gamma_{n,q}(\mathcal{K})_4(L)$.*

Proof. We will prove that, for every line $L \in \mathcal{L}$, the set of vertices $\Gamma_{n,q}(\mathcal{K})_4(L)$ contains at least one vertex that has all its neighbours in $\Gamma_{n,q}(\mathcal{K})_3(L)$, while for every point $P \in \mathcal{P}$, a vertex in the set $\Gamma_{n,q}(\mathcal{K})_4(P)$ cannot have all its neighbours in $\Gamma_{n,q}(\mathcal{K})_3(P)$.

To prove the first claim, consider a line $L \in \mathcal{L}$ with $L \cap H_\infty = P_1 \in \mathcal{K}$. Choose an affine point Q on L and a point $P_2 \in \mathcal{K}$ different from P_1. Take a point R on QP_2, not equal to Q or P_2, then clearly the line $RP_1 \in \Gamma_{n,q}(\mathcal{K})_4(L)$. We will show that RP_1 has all its neighbours in $\Gamma_{n,q}(\mathcal{K})_3(L)$. Consider a neighbour S of RP_1, i.e a point $S \in RP_1 \setminus \{P_1\}$. The line SP_2 meets L in a point T. We find a minimal path $S \sim SP_2 \sim T \sim L$, it follows that $S \in \Gamma_{n,q}(\mathcal{K})_3(L)$.

Consider now a point $P \in \mathcal{P}$ and a point $T \in \Gamma_{n,q}(\mathcal{K})_4(P)$. Consider a minimal path of length 4 from T to P, that is $T \sim QP_1 \sim Q \sim QP_2 \sim P$, for some affine point Q and distinct points $P_1, P_2 \in \mathcal{K}$, such that $T \in QP_1$ and $P \in QP_2$. Consider the point $R = PT \cap H_\infty$, then R must lie on the line P_1P_2. Moreover R is different from P_1 and P_2, and hence, since $PR = PT \notin \Gamma_{n,q}(\mathcal{K})_1(P)$, we have R not in $\mathcal{K}$. By assumption, there is a tangent line of $\mathcal{K}$ through R, say RP_3, with $P_3 \in \mathcal{K}$. The line TP_3 is a neighbour of T. Suppose that TP_3 belongs to $\Gamma_{n,q}(\mathcal{K})_3(P)$, then there exists a line PT' through a point $P_4 \in \mathcal{K}$, with T' on TP_3, which implies that RP_3 contains the point $P_4 \in \mathcal{K}$, a contradiction. $\square$

Theorem 5.2.3. *Let $|\mathcal{K}| \neq q$ or let $\mathcal{K}$ be a set of points of H_∞ such that every point of $H_\infty \backslash \mathcal{K}$ lies on at least one tangent line to $\mathcal{K}$. Suppose α is an automorphism of $\Gamma_{n,q}(\mathcal{K})$, then α stabilises $\mathcal{P}$.*

Proof. Since any graph automorphism preserves distance and hence neighbourhoods, it follows from Lemma 5.2.2 that no automorphism of $\Gamma_{n,q}(\mathcal{K})$ maps a vertex in $\mathcal{P}$ to a vertex in $\mathcal{L}$. $\square$

Corollary 5.2.4. *If $\mathcal{K}$ is a set of q points of H_∞ such that every point of H_∞ lies on at least one tangent line to $\mathcal{K}$, then $\Gamma_{n,q}(\mathcal{K})$ is not vertex-transitive.*

Remark. If $\mathcal{K}$ does not satisfy the conditions of the previous corollary, then sometimes we can find an automorphism switching the sets $\mathcal{P}$ and $\mathcal{L}$ of the graph $\Gamma_{n,q}(\mathcal{K})$. That is, let $\mathcal{K}$ be the q-arc $\{(0,1,x,x^2)_{\mathbb{F}_q} \mid x \in \mathbb{F}_q\}$, q even, embedded in the plane H_∞ of $\mathrm{PG}(3,q)$ with equation $X_0 = 0$. Consider the map ϕ, such that for all $a, b, c \in \mathbb{F}_q$, the affine point $(1, a, b, c)_{\mathbb{F}_q}$ is mapped to the line $\langle (0, 1, a, a^2)_{\mathbb{F}_q}, (1, 0, c, b^2)_{\mathbb{F}_q} \rangle$. One can check that this map preserves the edges of the graph $\Gamma_{2,q}(\mathcal{K})$ but switches the sets $\mathcal{P}$ and $\mathcal{L}$.

From Theorems 4.2.11, 4.5.2 and 4.5.3, we easily deduce the following corollaries in terms of the incidence graphs.

Corollary 5.2.5. *Let $q > 2$ and consider two point sets $\mathcal{K}$ and $\mathcal{K}'$ in $H_\infty = \mathrm{PG}(n, q)$. Let $\mathcal{K}$ be a set of q points such that $\langle \mathcal{K} \rangle = H_\infty$ and such that every point of H_∞ lies on at least one tangent line to $\mathcal{K}$. The graphs $\Gamma_{n,q}(\mathcal{K})$ and $\Gamma_{n,q}(\mathcal{K}')$ are isomorphic if and only if the point sets $\mathcal{K}$ and $\mathcal{K}'$ are $\mathrm{P\Gamma L}$-equivalent.*

Corollary 5.2.6. *Let $q > 2$ and let $\mathcal{K}$ be a set of q points such that its closure $\widehat{\mathcal{K}}$ is equal to H_∞ and such that every point of H_∞ lies on at least one tangent line to $\mathcal{K}$. Then $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K})) \cong \mathrm{P\Gamma L}(n + 2, q)_{\mathcal{K}}$.*

Corollary 5.2.7. *Let $\mathcal{K}$ be a point set of $H_\infty = \mathrm{PG}(n, q)$, $n > 1$, $q > 2$, of size q such that $\widehat{\mathcal{K}} \cong \mathrm{PG}(n, q_0)$, where $q = q_0^k$. Suppose that every point of H_∞ lies on at least one tangent line to $\mathcal{K}$. The full automorphism group of $\Gamma_{n,q}(\mathcal{K})$ is*

$$\mathrm{ng}(q_0, k) := \frac{1}{k} q_0^{\frac{k(k-1)}{2}} \prod_{i=1}^{k-1} (q_0^i - 1)$$

times larger than its geometric automorphism group $\mathrm{P\Gamma L}(n + 2, q)_{\mathcal{K}}$.

5.3 Describing the automorphism group

An element of $(\mathrm{P\Gamma L}(n+2, q)_{H_\infty})_{\mathcal{K}}$ induces a geometric automorphism of $\Gamma_{n,q}(\mathcal{K})$; it defines an element of $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K}))$. In the previous chapter, we have shown that, under certain conditions, $(\mathrm{P\Gamma L}(n + 2, q)_{H_\infty})_{\mathcal{K}}$ is the full automorphism group of $\Gamma_{n,q}(\mathcal{K})$. In this section, it is our goal to provide a more explicit description of $(\mathrm{P\Gamma L}(n + 2, q)_{H_\infty})_{\mathcal{K}}$.

Recall that the subgroup of $\mathrm{P\Gamma L}(n+2, q)$ consisting of all perspectivities with axis H_∞ is denoted as $\mathrm{Persp}_q(H_\infty)$. In the previous chapter we obtained the following lemma.

Lemma 4.4.8. *Consider a point set $\mathcal{K}$ spanning H_∞. The group $(\mathrm{P\Gamma L}(n + 2, q)_{H_\infty})_{\mathcal{K}}$ is an extension of $\mathrm{Persp}_q(H_\infty)$ by $\mathrm{P\Gamma L}(n+1, q)_{\mathcal{K}}$ and $(\mathrm{PGL}(n+2, q)_{H_\infty})_{\mathcal{K}}$ is an extension of $\mathrm{Persp}_q(H_\infty)$ by $\mathrm{PGL}(n + 1, q)_{\mathcal{K}}$.*

Remark. In general, $\mathrm{P\Gamma L}(n+2,q)_{H_\infty}$ is an extension of $\mathrm{Persp}_q(H_\infty)$ by $\mathrm{P\Gamma L}(n+1,q)$. However, this extension does not necessarily split since $\mathrm{P\Gamma L}(n+1,q)$ is not necessarily embeddable in $\mathrm{P\Gamma L}(n+2,q)$. For example, $\mathrm{PGL}(4,4)$ has no subgroup isomorphic to $\mathrm{PGL}(3,4)$. Depending on the choice of $\mathcal{K}$, we can investigate whether $(\mathrm{P\Gamma L}(n+2,q)_{H_\infty})_{\mathcal{K}}$ does split over $\mathrm{Persp}_q(H_\infty)$. To show that this extension splits, we need to embed the group $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ in $(\mathrm{P\Gamma L}(n+2,q)_{H_\infty})_{\mathcal{K}}$, and different groups $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ may require a different proof. In the next theorem, we give a general condition on $\mathcal{K}$ that is sufficient to show that the extension splits. The condition is not necessary: in Subsection 5.4.1, the result is shown to hold when $\mathcal{K}$ is a basis or frame in H_∞ and in [44], it is shown that the same holds for ovoidal Buekenhout-Metz unitals, in particular the classical unital, which does not satisfy our condition. However, our theorem obtains the result simultaneously for a lot of different point sets $\mathcal{K}$.

We start with an easy lemma.

Lemma 5.3.1. *If there is an element of* $\mathrm{P\Gamma L}(n+1,q)$ *mapping* $\mathcal{K}$ *to a point set* $\mathcal{K}'$*, where* $\mathcal{K}'$ *is stabilised under the Frobenius automorphism, then* $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}} \cong \mathrm{PGL}(n+1,q)_{\mathcal{K}} \rtimes \mathrm{Aut}(\mathbb{F}_q)$*.*

Proof. Since all automorphisms of $\mathbb{F}_q$ are generated by the Frobenius automorphism, every automorphism of $\mathbb{F}_q$ stabilises $\mathcal{K}'$. From this, if there is an element of $\mathrm{P\Gamma L}(n+1,q)$ mapping $\mathcal{K}$ to $\mathcal{K}'$, we can also find an element of $\mathrm{PGL}(n+1,q)$ mapping $\mathcal{K}$ to $\mathcal{K}'$. Since $\mathcal{K}'$ is contained in the orbit of $\mathcal{K}$ under $\mathrm{PGL}(n+1,q)$, we find that $\mathrm{PGL}(n+1,q)_{\mathcal{K}} \cong \mathrm{PGL}(n+1,q)_{\mathcal{K}'}$ and $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}} \cong \mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}'}$. Since $\mathrm{Aut}(\mathbb{F}_q)$ stabilises $\mathcal{K}'$, we can restrict the well-known isomorphism $\mathrm{P\Gamma L}(n+1,q) \cong \mathrm{PGL}(n+1,q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$ to elements of $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}'}$, and the lemma follows. $\square$

Theorem 5.3.2. *If the setwise stabiliser* $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$*, respectively* $\mathrm{PGL}(n+1,q)_{\mathcal{K}}$*, of a point set* $\mathcal{K}$ *spanning* $H_\infty = \mathrm{PG}(n,q)$*, fixes a point of* H_∞*, then* $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}} \cong \mathrm{Persp}_q(H_\infty) \rtimes \mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$*, respectively* $\mathrm{PGL}(n+2,q)_{\mathcal{K}} \cong \mathrm{Persp}_q(H_\infty) \rtimes \mathrm{PGL}(n+1,q)_{\mathcal{K}}$*.*

Proof. Since $\mathcal{K}$ spans H_∞, the group $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$ is contained in $\mathrm{P\Gamma L}(n+2,q)_{H_\infty}$. By Lemma 4.4.8, we see that $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$ is an extension of $\mathrm{Persp}_q(H_\infty)$ by $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$. This extension splits if and only if $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ can be embedded in $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$ in such a way that it intersects trivially with $\mathrm{Persp}_q(H_\infty)$. By assumption, $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ fixes a point $P \in H_\infty$. Suppose that P has corresponding vector $(0, c_1, c_2, \ldots, c_{n+1})$, where the first non-zero coordinate equals one. This implies that for each $\beta \in \mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$, there exists a unique $(n+1)\times(n+1)$

matrix $B = (b_{ij})$, $1 \leq i, j \leq n+1$, and an element $\theta \in \mathrm{Aut}(\mathbb{F}_q)$ corresponding to β, such that $(c_1, \ldots, c_n)^\theta B = (c_1, \ldots, c_n)$. Moreover, the obtained semi-linear maps (B, θ) form a subgroup G of $\Gamma\mathrm{L}(n+1, q)$. Let $A_B = (a_{ij})$, $0 \leq i, j \leq n+1$, be the $(n+2) \times (n+2)$ matrix with $a_{00} = 1$, $a_{i0} = a_{0j} = 0$ for $i, j \geq 1$, and $a_{ij} = b_{ij}$ for $1 \leq i, j \leq n+1$. It is clear that the set of all maps (A_B, θ) forms a subgroup of $\mathrm{P\Gamma L}(n+2, q)_{H_\infty}$, where every map (A_B, θ) corresponds to a collineation α acting in the same way as β on H_∞. If $\theta \neq \mathbb{1}$, then α is not a perspectivity. If $\theta = \mathbb{1}$, then α fixes every point on the line through P and $(1, 0, \ldots, 0)$, thus fixes at least two affine points, and hence is not a perspectivity. This implies that we have found a subgroup of $\mathrm{P\Gamma L}(n+2, q)_\mathcal{K}$ isomorphic to $\mathrm{P\Gamma L}(n+1, q)_\mathcal{K}$ and intersecting $\mathrm{Persp}_q(H_\infty)$ trivially.

The second claim can be proved in the same way. ☐

Corollary 5.3.3. *If the setwise stabiliser* $\mathrm{PGL}(n+1, q)_\mathcal{K}$ *of a point set* $\mathcal{K}$ *spanning* $H_\infty = \mathrm{PG}(n, q)$ *fixes a point of* H_∞*, and* $\mathrm{P\Gamma L}(n+1, q)_\mathcal{K} \cong \mathrm{PGL}(n+1, q)_\mathcal{K} \rtimes \mathrm{Aut}(\mathbb{F}_q/\mathbb{F}_{q_0})$*, for* $q_0 = p^{h_0}$*,* $h_0|h$*, or* $\mathrm{P\Gamma L}(n+1, q)_\mathcal{K} \cong \mathrm{PGL}(n+1, q)_\mathcal{K}$*, then* $\mathrm{P\Gamma L}(n+2, q)_\mathcal{K} \cong \mathrm{Persp}_q(H_\infty) \rtimes \mathrm{P\Gamma L}(n+1, q)_\mathcal{K}$*.*

Proof. It is clear that $\mathrm{Aut}(\mathbb{F}_q)$ can be embedded in $\mathrm{P\Gamma L}(n+2, q)$ by mapping $\theta \in \mathrm{Aut}(\mathbb{F}_q)$ to the semi-linear map $(I_{n+2}, \theta) \in \mathrm{P\Gamma L}(n+2, q)$ where I_{n+2} is the $(n+2) \times (n+2)$ identity matrix. Since $\mathrm{Persp}_q(H_\infty)$ intersects $\mathrm{Aut}(\mathbb{F}_q)$ trivially, the corollary follows. ☐

Examples of point sets satisfying the conditions of Theorem 5.3.2 are ubiquitous; the case where $\mathcal{K}$ is a q-arc in H_∞ will be studied in detail in Section 5.4.

The following theorem is easy to prove. We will use it to show edge-transitivity of the constructed graphs.

Theorem 5.3.4. *Consider a point set* $\mathcal{K}$ *of size* q *in* $H_\infty = \mathrm{PG}(n, q)$ *such that* $\langle \mathcal{K} \rangle = H_\infty$*. The graph* $\Gamma_{n,q}(\mathcal{K})$ *is an edge-transitive if and only if the subgroup* $\mathrm{P\Gamma L}(n+1, q)_\mathcal{K} \leq \mathrm{P\Gamma L}(n+1, q)$ *acting on* H_∞ *stabilises* $\mathcal{K}$ *and acts transitively on the points of* $\mathcal{K}$*.*

Proof. Consider two edges (R_i, L_i), $i = 1, 2$, where $R_i \in \mathcal{P}$, $L_i \in \mathcal{L}$, $R_i \in L_i$, we will construct a mapping from one edge to the other. Let P_i be $L_i \cap H_\infty$. Since $\mathrm{P\Gamma L}(n+1, q)_\mathcal{K}$ acts transitively on $\mathcal{K}$, we may take an element β of $\mathrm{P\Gamma L}(n+1, q)_\mathcal{K}$ such that $\beta(P_1) = P_2$. This element extends to an element β' of $(\mathrm{P\Gamma L}(n+2, q)_{H_\infty})_\mathcal{K}$ mapping P_1 onto P_2.

If $\beta'(R_1) = R_2$, then $\beta'(L_1) = L_2$, hence the statement follows. If $\beta'(R_1) \neq R_2$, then let S be the point at infinity of the line $\beta'(R_1)R_2$. There is a (unique) elation γ with centre S and axis H_∞ mapping $\beta'(R_1)$ to R_2. This elation maps $\beta'(L_1)$ onto L_2. Since $\gamma \circ \beta'$ is an element of $(\mathrm{P\Gamma L}(n+2,q)_{H_\infty})_{\mathcal{K}}$ mapping (R_1, L_1) onto (R_2, L_2), the statement follows.

Moreover, by Corollary 5.2.5, every isomorphism of $\Gamma_{n,q}(\mathcal{K})$ (which does not switch the classes $\mathcal{P}$ and $\mathcal{L}$) induces on action on the point set $\mathcal{K}$ which is also induced by an element of $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ stabilising $\Gamma_{n,q}(\mathcal{K})$. Hence, the graph $\Gamma_{n,q}(\mathcal{K})$ is edge-transitive only if the group $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ acts transitively on the points of $\mathcal{K}$. $\square$

5.4 Semisymmetric graphs arising from arcs

We are in search of point sets $\mathcal{K}$ such that the closure $\widehat{\mathcal{K}}$ is equal to H_∞ and such that $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ acts transitively on the points of $\mathcal{K}$. An arc of size q turns out to be an excellent choice.

If $\mathcal{A}$ is a k-arc in $\mathrm{PG}(n,q)$, then $k \geq n+1$, hence, we will only consider the case where $q \geq n+1$. If $q = n+1$, then it is easy to see that an arc of size q in $\mathrm{PG}(n,q)$ is a basis. If $q = n+2$, then every arc of size q is a frame. Hence, when $q = n+1$ or $q = n+2$, all arcs of size q in $\mathrm{PG}(n,q)$ are $\mathrm{P\Gamma L}$-equivalent. Because of the isomorphism of the graph $\Gamma_{n,q}(\mathcal{K})$ with other graphs (see Section 5.6), we will explicitly investigate these cases, but the more interesting examples occur when $q \geq n+3$. Recall that the classical example of an arc of size $q+1$ is given by the normal rational curve. A *normal rational curve* in $\mathrm{PG}(n,q)$, $2 \leq n \leq q-2$, is a $(q+1)$-arc PGL-equivalent to the $(q+1)$-arc $\{(0,\dots,0,1)_{\mathbb{F}_q}\} \cup \{(1,t,t^2,t^3,\dots,t^n)_{\mathbb{F}_q} \mid t \in \mathbb{F}_q\}$.

Remark. In [65] an overview can be found of several results concerning normal rational curves and the extendability of arcs. There are results showing that an arc of size q in $\mathrm{PG}(n,q)$ can be extended to an arc of size $q+1$, see [65, Table 3.2] for q sufficiently large w.r.t. n, and [65, Table 3.4] for q close to n. Moreover, other results show that for many values of q and n, all $(q+1)$-arcs in $\mathrm{PG}(n,q)$ are normal rational curves. The combination of these results leads to the understanding why there are not many known examples of q-arcs in $\mathrm{PG}(n,q)$ that are not contained in a normal rational curve.

We will construct different families of graphs, arising from non-$\mathrm{P\Gamma L}$-equivalent arcs of size q. Since these arcs satisfy the conditions of Corollary 5.2.5, we see that the obtained graphs are non-isomorphic.

In view of Corollary 5.2.6, our first goal is to show that the closure $\widehat{\mathcal{K}}$ of a set $\mathcal{K}$ of q points of an arc in $\mathrm{PG}(n,q)$, $q \geq n+3$ or $q = p = n+2$ prime, is H_∞. When $n = 2$, this follows immediately, since an arc that is contained in a non-trivial subplane of $\mathrm{PG}(2,q)$ can have size at most $\sqrt{q}+2$. In the following lemmas, we deal with the case $n \geq 3$.

Lemma 5.4.1. *Let $\mathcal{K}$ be an arc of size q in $\mathrm{PG}(n,q)$, $n \geq 3$. Let P_1 and P_2 be any two points of $\mathcal{K}$;*

- *if $q = n+2$, then the line P_1P_2 contains at least one additional point of $\widehat{\mathcal{K}}$,*
- *if $q \geq n+3$, then the line P_1P_2 contains at least $q/2$ additional points of $\widehat{\mathcal{K}}$.*

Proof. Note that a k-space π, $k \leq n-2$, with $k+1$ points of $\mathcal{K}$, different from P_1 and P_2, does not intersect P_1P_2, since otherwise $\langle \pi, P_1P_2 \rangle$ would be a $(k+1)$-space containing $k+3$ points of $\mathcal{K}$, contradicting the arc condition.

Consider n points $P_3, \ldots, P_{n+2}$ of $\mathcal{K}$, all different from P_1 and P_2. The space $\langle P_3, \ldots, P_{n+2} \rangle$ is a hyperplane of H_∞, hence, it meets the line P_1P_2 in a point Q. This point Q is contained in $\widehat{\mathcal{K}}$ but not contained in $\mathcal{K}$ since $\mathcal{K}$ is an arc. If $q = n+2$, there is exactly one set $\{P_3, \ldots, P_{n+2}\}$ of n points of $\mathcal{K}$, different from P_1 and P_2, yielding an extra point in $\widehat{\mathcal{K}}$ on P_1P_2.

If $n+3 \leq q \leq 2n+2$, then let $\{P_3, \ldots, P_{n+3}\}$ be a set of $n+1$ points of $\mathcal{K}$, different from P_1 and P_2. Any subset with n points of $\{P_3, \ldots, P_{n+3}\}$ defines a hyperplane intersecting P_1P_2 in a point $Q \neq P_1, P_2$ contained in $\widehat{\mathcal{K}}$. These points Q are all different since any two considered hyperplanes intersect in an $(n-2)$-space with $n-1$ points of $\mathcal{K}$, and hence this space does not intersect P_1P_2. There are $n+1$ such subsets, so the line P_1P_2 contains $q/2 \leq n+1 \leq q-2$ additional points in $\widehat{\mathcal{K}}$ different from P_1 and P_2.

If $q \geq 2n+2$, then let $P_3, \ldots, P_{n+1}$ be $n-1$ points of $\mathcal{K}$, different from P_1 and P_2. Clearly, $\langle P_3, \ldots, P_{n+1} \rangle$ is disjoint from P_1P_2. There are $q-n-1$ points of $\mathcal{K}$ different from all P_i, $i = 1, \ldots, n+1$. For every such point $R \in \mathcal{K} \setminus \{P_1, \ldots, P_{n+1}\}$, the hyperplane $\langle P_3, \ldots, P_{n+1}, R \rangle$ intersects P_1P_2 in a point of $\widehat{\mathcal{K}}$ different from P_1 and P_2. Again, all these points are different since two such hyperplanes intersect in $\langle P_3, \ldots, P_{n+1} \rangle$. The line P_1P_2 contains $q-n-1 \geq q/2$ points of $\widehat{\mathcal{K}}$ different from P_1 and P_2. $\square$

Lemma 5.4.2. *Let $\mathcal{K}$ be an arc of size q in $\mathrm{PG}(n,q)$, $n \geq 2$, with $q \geq n+3$ or $q = p = n+2$. If μ_∞ is a plane containing 3 points of $\mathcal{K}$, then every point of μ_∞ is contained in $\widehat{\mathcal{K}}$.*

Proof. Consider three points P_1, P_2, P_3 of $\mathcal{K}$ and let μ_∞ be the plane $\langle P_1, P_2, P_3 \rangle$. Consider $q \geq n+3$. By Lemma 5.4.1, we know that there exist at least $q/2$ points in $\widehat{\mathcal{K}}$ on each of the lines P_2P_3, P_1P_3 and P_1P_2, different from P_1, P_2 and P_3. Consider the set S containing all these points and the points P_1, P_2 and P_3. Its closure $\widehat{S}$ forms a subplane π of μ_∞ consisting of only points of $\widehat{\mathcal{K}}$. Since a proper subplane of $\mathrm{PG}(2,q)$ contains at most $\sqrt{q}+1 < q/2+2$ points of the line P_1P_2, we see that π must be μ_∞.

If $q = n+2$ is prime, by Lemma 5.4.1, we find an extra point $Q_i \in \widehat{\mathcal{K}}$, $i = 2,3$, on the line P_1P_i. The closure of $\{P_1, P_2, P_3, Q_2, Q_3\}$ forms a subplane with all points in $\widehat{\mathcal{K}}$. By the fact that q is prime, this subplane equals $\mu_\infty = \mathrm{PG}(2,q)$. □

Lemma 5.4.3. *Let L be a line such that every point is in $\widehat{\mathcal{K}}$. If π_∞ is a plane of H_∞ containing L and at least two points R_1 and R_2 of $\widehat{\mathcal{K}} \setminus L$, then every point in the plane π_∞ belongs to $\widehat{\mathcal{K}}$.*

Proof. The closure of the set of points of $\widehat{\mathcal{K}}$ on the line L, together with the points R_1 and R_2 is clearly the plane π_∞ itself. □

Lemma 5.4.4. *For $n \geq 2$, let $q \geq n+3$ or $q = p = n+2$, and let $\mathcal{K}$ be an arc of size q in $\mathrm{PG}(n,q)$, then $\widehat{\mathcal{K}} = \mathrm{PG}(n,q)$.*

Proof. For $n = 2$, this follows from before. Let $P_1, \ldots, P_q$ be the points of $\mathcal{K}$. By Lemma 5.4.2, we know that every point of $\langle P_1, P_2, P_3 \rangle$ belongs to $\widehat{\mathcal{K}}$. Suppose, by induction on $k \leq n$ that every point in $\langle P_1, \ldots, P_k \rangle$ belongs to $\widehat{\mathcal{K}}$.

The point P_{k+1} is not contained in $\langle P_1, \ldots, P_k \rangle$. Let S be a point of $\langle P_1, \ldots, P_{k+1} \rangle$, not on the line P_1P_{k+1}, and let R be the intersection of the line SP_{k+1} with $\langle P_1, \ldots, P_k \rangle$. By induction, every point on the line RP_1 belongs to $\widehat{\mathcal{K}}$.

By Lemma 5.4.1, there exists an additional point Q in $\widehat{\mathcal{K}}$ on the line P_1P_{k+1}. Since the plane $\langle RP_1, P_{k+1} \rangle$ contains $RP_1 \subset \mathcal{K}$ and the points Q and P_{k+1} of $\widehat{\mathcal{K}}$, Lemma 5.4.3 implies that the point S belongs to $\widehat{\mathcal{K}}$, as do all the points of P_1P_{k+1}. This shows that every point in $\langle P_1, \ldots, P_{k+1} \rangle$ is in $\widehat{\mathcal{K}}$. The lemma follows by induction and the fact that $H_\infty = \langle P_1, \ldots, P_{n+1} \rangle$. □

Theorem 5.4.5. *For $n = 2$, suppose q odd, for $n \geq 3$, suppose $q \geq n+3$ or $q = p = n+2$. If $\mathcal{K}$ is an arc in $\mathrm{PG}(n,q)$, then $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K})) \cong \mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$.*

Proof. It is clear that if $n = 2$, q odd or $n \geq 3$, then every point of H_∞ lies on a tangent line to the arc. By Lemma 5.4.4, $\widehat{\mathcal{K}}$ equals $\mathrm{PG}(n, q)$. The theorem follows from Corollary 5.2.6. $\square$

5.4.1 $\mathcal{K}$ is a q-arc in PG(n, q) with $q = n + 1$ or $q = n + 2$

As noted before, a q-arc in $\mathrm{PG}(n, q)$, with $q = n+1$, is a basis, a q-arc in $\mathrm{PG}(n, q)$, with $q = n + 2$, is a frame. In these cases, the linear representation of a q-arc gives rise to a semisymmetric graph, however, the description of the automorphism group is different from the case $q \geq n+3$. Sadly, we cannot use the same techniques as in the proof of Lemma 4.4.8 to show that $\mathrm{P\Gamma L}(n+2, q)_{\mathcal{K}}$ splits over $\mathrm{Persp}_q(H_\infty)$.

We introduce some definitions.

Definition 5.4.6. A *permutation matrix* is a square binary matrix that has exactly one entry 1 in each row and each column, and 0's elsewhere. A *monomial matrix* or *generalised permutation matrix* has exactly one nonzero entry in each row and each column.

The $(n+1) \times (n+1)$-monomial matrices over $\mathbb{F}_q$ form a subgroup $\mathrm{Mon}(n+1, q)$ of $\mathrm{GL}(n + 1, q)$. Let $\mathrm{PMon}(n + 1, q)$ denote the corresponding subgroup of $\mathrm{PGL}(n + 1, q)$. Let S_k denote the *symmetric group* of degree k, meaning the group of all permutations of $\{1, 2, \ldots, k\}$.

Theorem 5.4.7. *If $\mathcal{K}$ is a q-arc in* $\mathrm{PG}(n, q)$*, $n \geq 2$, $q = n + 1$ or $q = n + 2$, with $(n, q) \neq (2, 4)$, then $\Gamma_{n,q}(\mathcal{K})$ is a semisymmetric graph. The group* $\mathrm{P\Gamma L}(n + 2, q)_{\mathcal{K}}$ *is a subgroup of* $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K}))$ *and is isomorphic to* $\mathrm{Persp}_q(H_\infty) \rtimes \mathrm{P\Gamma L}(n+1, q)_{\mathcal{K}}$*, where* $\mathrm{P\Gamma L}(n + 1, q)_{\mathcal{K}}$ *is isomorphic to*

(*i*) $\mathrm{S}_q \rtimes \mathrm{Aut}(\mathbb{F}_q)$ *if $q = n + 2$, having size $hq^{n+1}(q - 1)q!$;*

(*ii*) $\mathrm{PMon}(n + 1, q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$ *if $q = n + 1$, having size $hq^{n+1}(q - 1)^n q!$.*

If $q = n + 2$ and q is prime, then $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K}))$ *is geometric and isomorphic to* $\mathrm{P\Gamma L}(n + 2, q)_{\mathcal{K}}$.

If $q = n + 2$ and $q = p^h$ is not prime, then $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K}))$ *is not geometric and* $|\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K}))| = |\mathrm{P\Gamma L}(n + 2, q)_{\mathcal{K}}|\mathrm{ng}(p, h) = hq^{n+1}(q - 1)^n q!\mathrm{ng}(p, h)$*, where* $\mathrm{ng}(p, h) := \frac{1}{h} p^{\frac{h(h-1)}{2}} \prod_{i=1}^{h-1}(p^i - 1)$.

Proof. (*i*) If $q = n + 2$, then $\mathcal{K}$ is PGL-equivalent to the frame $\mathcal{K}'$ of $H_\infty = \mathrm{PG}(n, q)$ with points $P_1, \ldots, P_{n+2}$, where P_i has corresponding vector v_i, and $v_1 =$

$(1, 0, \ldots, 0), v_2 = (0, 1, 0, \ldots, 0), \ldots, v_{n+1} = (0, \ldots, 0, 1), v_{n+2} = (-1, -1, \ldots, -1)$. For all $1 \leq k \leq n+1$, consider $B_k = (b_{ij})$ to be the matrix with $b_{ii} = 1$, $i \neq k$, $1 \leq i \leq n+1$, $b_{ki} = -1$, $1 \leq i \leq n+1$, and $b_{ij} = 0$ for all other i, j. The considered action of B_k on the points of $\mathrm{PG}(n, q)$ is by right-multiplication on the row vector of their coordinates. Let G_{per} denote the subgroup of permutation matrices of $\mathrm{GL}(n+1, q)$, and consider the subgroup G of $\mathrm{GL}(n+1, q)$, generated by the elements of G_{per} and the matrices B_k, $1 \leq k \leq n+1$.

Let H_∞ correspond to the hyperplane $X_0 = 0$ of $\mathrm{PG}(n+1, q)$, that is, every point of $H_\infty = \mathrm{PG}(n, q)$ with vector $(c_1, \ldots, c_{n+1})$ corresponds to the point of $\mathrm{PG}(n+1, q)$ with vector $(0, c_1, \ldots, c_{n+1})$. For every matrix $B = (b_{ij})$, $1 \leq i, j \leq n+1$, in G, we can define a matrix $A_B = (a_{ij})$, $0 \leq i, j \leq n+1$, as the $(n+2) \times (n+2)$ matrix with $a_{00} = 1$, $a_{i0} = a_{0j} = 0$ for $i, j \geq 1$ and $a_{ij} = b_{ij}$ for $1 \leq i, j \leq n+1$. Let $\widetilde{G}$ be the subgroup of $\mathrm{PGL}(n+2, q)$ obtained as the set of all matrices A_B with corresponding matrix $B \in G$. It is clear that the elements of G are exactly the permutations of the elements of $\{v_1, \ldots, v_{n+2}\}$ and hence that $\widetilde{G}$ is isomorphic to $\mathrm{PGL}(n+1, q)_{\mathcal{K}} \cong \mathrm{S}_q$.

It follows that the only element of $\widetilde{G}$ fixing $\mathcal{K}$ pointwise corresponds to the identity matrix, which implies that any element of $\mathrm{Persp}_q(H_\infty)$ contained in $\widetilde{G}$ is trivial. Hence, $\mathrm{PGL}(n+2, q)_{\mathcal{K}}$ is isomorphic to $\mathrm{Persp}_q(H_\infty) \rtimes \mathrm{PGL}(n+1, q)_{\mathcal{K}}$. Clearly, $\mathrm{PGL}(n+1, q)_{\mathcal{K}}$ acts transitively on the points of $\mathcal{K}$, hence by Theorem 5.3.4 the graph $\Gamma_{n,q}(\mathcal{K})$ is edge-transitive.

(ii) Now suppose $q = n+1$. The group $\mathrm{PSL}(n+1, q)$ is a subgroup of $\mathrm{PGL}(n+1, q)$ and a quotient of $\mathrm{SL}(n+1, q)$. When $q = n+1$, all three groups have the same order and thus are all isomorphic. Hence, $\mathrm{PGL}(n+1, q)$ can be embedded in $\mathrm{PGL}(n+2, q)_{H_\infty}$ by taking all matrices $B = (b_{ij})$, $1 \leq i, j \leq n+1$, of $\mathrm{SL}(n+1, q)$ and, as before, defining $A_B = (a_{ij})$, $0 \leq i, j \leq n+1$, with $a_{00} = 1$, $a_{i0} = a_{0j} = 0$ for $i, j \geq 1$ and $a_{ij} = b_{ij}$ for $1 \leq i, j \leq n+1$. An element of $\mathrm{Persp}_q(H_\infty)$ corresponds to a matrix of the form $D = (d_{ij})$, $0 \leq i, j \leq n+1$, with $d_{0j} = \lambda_j$, $0 \leq j \leq n+1$, $d_{ii} = \mu$, $1 \leq i \leq n+1$, for some $\lambda_j, \mu \in \mathbb{F}_q$, $\mu \neq 0$ and $d_{ij} = 0$ otherwise. This implies that the group $\widetilde{G}$ of matrices A_B defined in this way meets $\mathrm{Persp}_q(H_\infty)$ trivially. Hence, $\mathrm{PGL}(n+2, q)_{\mathcal{K}}$ is isomorphic to $\mathrm{Persp}_q(H_\infty) \rtimes \mathrm{PGL}(n+1, q)_{\mathcal{K}}$.

Since $q = n+1$, the curve $\mathcal{K}$ is PGL-equivalent to the set $\mathcal{K}'$ consisting of points $P_1, \ldots, P_{n+1}$ in $\mathrm{PG}(n, q)$, where P_i has coordinates v_i, and $v_1 = (1, 0, \ldots, 0)$, $v_2 = (0, 1, 0, \ldots, 0), \ldots, v_{n+1} = (0, \ldots, 0, 1)$. Using this, it is clear that $\mathrm{PGL}(n+1, q)_{\mathcal{K}}$ is isomorphic to $\mathrm{PMon}(n+1, q)$ and that $\mathrm{PGL}(n+1, q)_{\mathcal{K}}$ acts transitively on $\mathcal{K}$. Hence, $\Gamma_{n,q}(\mathcal{K})$ is an edge-transitive graph.

In both cases, it is clear that $\mathcal{K}'$ is stabilised by the Frobenius automorphism, hence, using Theorem 5.3.2, it also follows that $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}} \cong \mathrm{Persp}_q(H_\infty) \rtimes (\mathrm{PGL}(n+1,q)_{\mathcal{K}} \rtimes \mathrm{Aut}(\mathbb{F}_q))$. The observation on the sizes follows from $|\mathrm{S}_q| = q!$ and $|\mathrm{PMon}(n+1,q)| = |\mathrm{S}_q|\frac{|(\mathbb{F}_q^*)^n|}{(q-1)} = q!(q-1)^{n-1}$.

Since through every point of H_∞, there is a tangent line to $\mathcal{K}$, Corollary 5.2.4 shows that $\Gamma_{n,q}(\mathcal{K})$ is not vertex-transitive. Since $\mathcal{K}$ spans H_∞ and $|\mathcal{K}| = q$, we get that $\Gamma_{n,q}(\mathcal{K})$ is connected and semisymmetric.

The last part of the statement follows from Theorem 5.4.5 and Corollary 5.2.7. $\square$

Remark. By using the computer program GAP [54], we obtained that when $\mathcal{K}$ a basis of $\mathrm{PG}(2,3)$ (i.e. $n=2, q=3$) every automorphism of $\Gamma_{2,3}(\mathcal{K})$ is induced by a collineation of $\mathrm{PG}(3,3)$. Hence, the automorphism group of $\Gamma_{2,3}(\mathcal{K})$ is isomorphic to $\mathrm{P\Gamma L}(4,3)_{\mathcal{K}}$. However, for $n=3, q=4$, we found that $[\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K})) : \mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}] = 8$. This implies that there exist automorphisms of the graph $\Gamma_{3,4}(\mathcal{K})$ that are not collineations of $\mathrm{PG}(4,4)$. For $n=4, q=5$, this index is already 7776. This might indicate that the general description of the full automorphism group of $\Gamma_{n,q}(\mathcal{K})$, with $n+1=q$ and $\mathcal{K}$ a basis, is a hard problem.

5.4.2 $\mathcal{K}$ is contained in a normal rational curve and $q \geq n+3$

We will use the following theorem by Segre.

Theorem 5.4.8. [102] *If $q \geq n+2$ and S is a set of $n+3$ points in $\mathrm{PG}(n,q)$, no $n+1$ of which lie in a hyperplane, then there is a unique normal rational curve in $\mathrm{PG}(n,q)$ containing the points of S.*

Corollary 5.4.9. *If $\mathcal{K}$ is a set of q points of a normal rational curve $\mathcal{N}$ in $\mathrm{PG}(n,q)$, $q \geq n+3$, then $\mathcal{N}$ is the unique normal rational curve containing the points of $\mathcal{K}$.*

The following theorem is well known; a proof can be found in e.g. [67, Theorem 27.5.3].

Theorem 5.4.10. *If $q \geq n+2$ and $\mathcal{N}$ is a normal rational curve in $\mathrm{PG}(n,q)$, then the stabiliser of $\mathcal{N}$ in $\mathrm{P\Gamma L}(n+1,q)$ is isomorphic to $\mathrm{P\Gamma L}(2,q)$ (in its faithful action on $q+1$ points).*

These results enable us to give a construction for the following infinite two-parameter family of semisymmetric graphs.

Note that the subgroup of $\mathrm{P\Gamma L}(2,q)$ fixing one point in its natural action, is isomorphic to the affine semilinear group $\mathrm{A\Gamma L}(1,q)$.

Theorem 5.4.11. *If $\mathcal{K}$ is a set of q points, contained in a normal rational curve of* $\mathrm{PG}(n,q)$*,* $q = p^h$*,* $n \geq 3$*,* $q \geq n+3$*, or* $n = 2$*,* q *odd, then* $\Gamma_{n,q}(\mathcal{K})$ *is a semisymmetric graph.*

Moreover, $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K}))$ *is isomorphic to* $\mathrm{Persp}_q(H_\infty) \rtimes \mathrm{A\Gamma L}(1,q)$ *and has size* $hq^{n+2}(q-1)^2$*.*

Proof. Since $|\mathcal{K}| = q$, the graph $\Gamma_{n,q}(\mathcal{K})$ is q-regular. The set $\mathcal{K}$ is an arc spanning the space $\mathrm{PG}(n,q)$. It is clear that if $n \geq 3$, or if q is odd, every point of $\mathrm{PG}(n,q)$ lies on at least one tangent line to $\mathcal{K}$. Hence, by Theorem 4.1.3, Corollary 5.2.4 and Theorem 5.4.5, $\Gamma_{n,q}(\mathcal{K})$ is a connected not vertex-transitive graph for which $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K})) \cong \mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$. By Corollary 5.4.9, $\mathcal{K}$ extends by a point P to a unique normal rational curve $\mathcal{N}$. Since P must be fixed by the stabiliser of $\mathcal{K}$ and $\mathrm{P\Gamma L}(2,q)_P \cong \mathrm{A\Gamma L}(1,q)$, we get $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}} \cong \mathrm{Persp}_q(H_\infty) \rtimes \mathrm{A\Gamma L}(1,q)$, by Theorem 5.3.2. The size of this group follows when considering that $|\mathrm{Persp}_q(H_\infty)| = q^{n+1}(q-1)$ and $|\mathrm{A\Gamma L}(1,q)| = hq(q-1)$. By Theorem 5.3.4, the graph $\Gamma_{n,q}(\mathcal{K})$ is edge-transitive and thus semisymmetric. $\square$

5.4.3 $\mathcal{K}$ is contained in an arc in PG$(3,q)$, q even

The $(q+1)$-arcs in $\mathrm{PG}(3,q)$, q even, have been classified, each of them has the same stabiliser group as the normal rational curve.

Theorem 5.4.12. [33] *In* $\mathrm{PG}(3,q)$*,* $q \geq 8$ *even, every* $(q+1)$*-arc is* PGL*-equivalent to a* $(q+1)$*-arc* $\mathcal{C}(\sigma) = \{(1,x,x^\sigma,x^{\sigma+1}) \mid x \in \mathbb{F}_q\} \cup \{(0,0,0,1)\}$*, for some generator* σ *of* $\mathrm{Aut}(\mathbb{F}_q)$*.*

Theorem 5.4.13. [80] *In* $\mathrm{PG}(3,q)$*,* $q \geq 8$ *even, the stabiliser of* $\mathcal{C}(\sigma)$ *in* $\mathrm{P\Gamma L}(4,q)$ *is isomorphic to* $\mathrm{P\Gamma L}(2,q)$ *(in its faithful action on* $q+1$ *points).*

The case $q = 4$ is already discussed in Section 5.4.1.

Theorem 5.4.14. [32] *For a* k*-arc of* $\mathrm{PG}(3,q)$*,* $q \geq 4$ *even, we have* $k \leq q+1$*.*

Theorem 5.4.15. [26] *Let* $\mathcal{K}$ *be a* k*-arc in* $\mathrm{PG}(3,q)$*,* q *even. If* $k > (q+4)/2$*, then* $\mathcal{K}$ *is contained in a unique complete arc.*

Corollary 5.4.16. *Consider a* $(q+1)$*-arc* $\mathcal{C}(\sigma)$ *of* $\mathrm{PG}(3,q)$*,* $q \geq 8$ *even. If* $\mathcal{K}$ *is a set of* q *points contained in* $\mathcal{C}(\sigma)$*, then there is a unique* $(q+1)$*-arc through the points of* $\mathcal{K}$*, namely* $\mathcal{C}(\sigma)$*.*

Proof. Using Theorem 5.4.15, since $q > (q+4)/2$ when $q > 4$, we find a unique complete arc through $\mathcal{K}$. This arc has size at most $q+1$ by Theorem 5.4.14 and thus is equal to $\mathcal{C}(\sigma)$. □

Theorem 5.4.17. *If $\mathcal{K}$ is a set of q points contained in any $(q+1)$-arc of* $\mathrm{PG}(3,q)$*, $q = 2^h \geq 8$ even, then $\Gamma_{3,q}(\mathcal{K})$ is a semisymmetric graph.*

Moreover, $\mathrm{Aut}(\Gamma_{3,q}(\mathcal{K}))$ *is isomorphic to* $\mathrm{Persp}_q(H_\infty) \rtimes \mathrm{A\Gamma L}(1,q)$ *and has size* $hq^5(q-1)^2$*.*

Proof. The proof goes in exactly the same way as the proof of Theorem 5.4.11, by making use of Corollary 5.4.16, Theorem 5.4.12 and Theorem 5.4.13. □

5.4.4 $\mathcal{K}$ is contained in the Glynn arc in PG(4, 9)

In [55], Glynn constructed an example of an arc of size 10 in $\mathrm{PG}(4,9)$, which is not a normal rational curve. We call this 10-arc the *Glynn arc (of size 10).* He also shows that an arc in $\mathrm{PG}(4,9)$ of size 10 is either a normal rational curve or a Glynn arc.

Theorem 5.4.18. [55] *The stabiliser in* $\mathrm{P\Gamma L}(5,9)$ *of the Glynn arc of size* 10 *in* $\mathrm{PG}(4,9)$ *is isomorphic to* $\mathrm{PGL}(2,9)$*.*

Theorem 5.4.19. [20] *A k-arc in* $\mathrm{PG}(n,q)$*, $n \geq 3$, q odd and $k \geq \frac{2}{3}(q-1)+n$, is contained in a unique complete arc of* $\mathrm{PG}(n,q)$*.*

Corollary 5.4.20. *If $\mathcal{K}$ is a set of* 9 *points contained in a Glynn* 10*-arc $\mathcal{C}$ of* $\mathrm{PG}(4,9)$*, then $\mathcal{K}$ is contained in a unique* 10*-arc, namely $\mathcal{C}$.*

Theorem 5.4.21. *If $\mathcal{K}$ is a* 9*-arc contained in a Glynn* 10*-arc of* $\mathrm{PG}(4,9)$*, then $\Gamma_{4,9}(\mathcal{K})$ is a semisymmetric graph.*

Moreover, $\mathrm{Aut}(\Gamma_{4,9}(\mathcal{K}))$ *is isomorphic to* $\mathrm{Persp}_q(H_\infty) \rtimes \mathrm{AGL}(1,9)$ *and has size* $9^6 8^2$*.*

Proof. Since $|\mathcal{K}| = 9$, $\Gamma_{4,9}(\mathcal{K})$ is a 9-regular graph. The set $\mathcal{K}$ is an arc spanning the space $\mathrm{PG}(4,9)$. It is clear that every point of $\mathrm{PG}(4,9)$ lies on at least one tangent line to $\mathcal{K}$. Hence, by Theorem 4.1.3, Corollary 5.2.4 and Theorem 5.4.5, $\Gamma_{4,9}(\mathcal{K})$ is a connected not vertex-transitive graph for which $\mathrm{Aut}(\Gamma_{4,9}(\mathcal{K})) \cong \mathrm{P\Gamma L}(6,9)_{\mathcal{K}}$. By Corollary 5.4.20, $\mathcal{K}$ extends by a point P to a unique Glynn 10-arc $\mathcal{C}$. By Theorem 5.3.2, we have $\mathrm{P\Gamma L}(6,9)_{\mathcal{K}} \cong \mathrm{Persp}_q(H_\infty) \rtimes \mathrm{P\Gamma L}(5,9)_{\mathcal{K}}$. Since $\mathrm{PGL}(2,9)_P \cong \mathrm{AGL}(1,9)$, we find $\mathrm{P\Gamma L}(6,9)_{\mathcal{K}} \cong \mathrm{Persp}_q(H_\infty) \rtimes \mathrm{AGL}(1,9)$. As before, the size easily follows. By Theorem 5.3.4, the graph $\Gamma_{4,9}(\mathcal{K})$ is edge-transitive and thus semisymmetric. $\square$

5.4.5 Using the MDS dual arc construction

Let $\mathcal{K} = \{P_1, \ldots, P_k\}$ be a k-arc in $\mathrm{PG}(n,q)$, $k \geq n+4$. Consider corresponding vectors $(a_{0j}, \ldots, a_{nj})$ of P_j, $1 \leq j \leq k$, then the rows of the $(n+1) \times k$-matrix $A = (a_{ij})$ determine a vector subspace $V_1 = V(n+1,q)$ of $V(k,q)$ (i.e. an *MDS code*). The space V_1 has a unique orthogonal complement $V_2 = V(k-n-1,q)$ in $V(k,q)$. Then V_2 also is an MDS code [82, p. 319]. A k-arc $\ddot{\mathcal{K}} = \{Q_1, \ldots, Q_k\}$ of $\mathrm{PG}(k-n-2,q)$, with respective vectors $(b_{0j}, \ldots, b_{k-n-2,j})$ defining Q_j, $1 \leq j \leq k$, such that the $(k-n-1) \times k$-matrix $B = (b_{ij})$ generates V_2, is called an *MDS dual k-arc* $\ddot{\mathcal{K}}$ of the k-arc $\mathcal{K}$ [111].

It should be noted that MDS duality for arcs is a one-to-one correspondence between *equivalence classes* of arcs, rather than a correspondence between arcs. Choosing other orderings of the points of $\mathcal{K}$ or choosing other coordinates for the points of $\mathcal{K}$, could give distinct arcs, however, all these MDS dual k-arcs of the arc $\mathcal{K}$ are $\mathrm{P\Gamma L}$-equivalent.

Theorem 5.4.22. [109, Theorem 2.1] *A k-arc $\mathcal{K}$ in $\mathrm{PG}(n,q)$, $k \geq n+4$, and a MDS dual k-arc $\ddot{\mathcal{K}}$ of $\mathcal{K}$ in $\mathrm{PG}(k-n-2,q)$ have isomorphic collineation groups and isomorphic projective groups.*

The duality transformation maps normal rational curves to normal rational curves and non-classical arcs to non-classical arcs. This implies that the arcs in Sections 5.4.3 and 5.4.4 give rise to a different family of semisymmetric graphs. This follows from the following theorem.

Theorem 5.4.23. *Let $\mathcal{K}$ be a q-arc in $H_\infty = \mathrm{PG}(n,q)$, $q \geq n+4$, and let $\ddot{\mathcal{K}}$ be a MDS dual arc of $\mathcal{K}$ in $\ddot{H}_\infty = \mathrm{PG}(q-n-2,q)$. Suppose that one of the groups $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ or $\mathrm{P\Gamma L}(q-n-1,q)_{\ddot{\mathcal{K}}}$ fixes a point outside of $\mathcal{K}$, $\ddot{\mathcal{K}}$ respectively, and acts transitively on the points of $\mathcal{K}$, $\ddot{\mathcal{K}}$ respectively, then $\Gamma_{n,q}(\mathcal{K})$ and $\Gamma_{q-n-2,q}(\ddot{\mathcal{K}})$ are semisymmetric, $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K})) \cong \mathrm{Persp}_q(H_\infty) \rtimes \mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ and $\mathrm{Aut}(\Gamma_{q-n-2,q}(\ddot{\mathcal{K}})) \cong \mathrm{Persp}_q(\ddot{H}_\infty) \rtimes \mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$.*

Proof. In the same way as before, using Theorem 4.1.3, Corollary 5.2.4 and Theorem 5.4.5, we see that $\Gamma_{n,q}(\mathcal{K})$ and $\Gamma_{q-n-2,q}(\ddot{\mathcal{K}})$ are connected not vertex-transitive graphs for which $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K})) \cong \mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$ and $\mathrm{Aut}(\Gamma_{q-n-2,q}(\ddot{\mathcal{K}})) \cong \mathrm{P\Gamma L}(q-n,q)_{\ddot{\mathcal{K}}}$.

Suppose w.l.o.g. that $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ fixes a point outside of $\mathcal{K}$, then by Theorem 5.3.2, $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}} \cong \mathrm{Persp}_q(H_\infty) \rtimes \mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$. The embedding of $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ in $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$ used to show this result was constructed by adding a 1 at the upper left corner of every matrix B corresponding to an element (B,θ) of $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$, for some $\theta \in \mathrm{Aut}(\mathbb{F}_q)$ to obtain a matrix B' corresponding to an element (B',θ) of $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$. This subgroup meets $\mathrm{Persp}_q(H_\infty)$ trivially, which implies that in the group of matrices defining elements of $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$, no proper scalar multiple of the identity matrix occurs.

Now, from the isomorphism of Theorem 5.4.22, it follows that the group $\mathrm{PGL}(q-n-1,q)_{\ddot{\mathcal{K}}}$, which is isomorphic to $\mathrm{PGL}(n+1,q)_{\mathcal{K}}$, also contains no proper scalar multiple of the identity matrix. Hence, by embedding $\mathrm{P\Gamma L}(q-n-1,q)_{\ddot{\mathcal{K}}}$ in $\mathrm{P\Gamma L}(q-n,q)$ in the same way (by adding a 1 at the upper left corner), we see that it meets $\mathrm{Persp}_q(\ddot{H}_\infty)$ trivially. This implies that $\mathrm{P\Gamma L}(q-n,q)_{\ddot{\mathcal{K}}} \cong \mathrm{Persp}_q(\ddot{H}_\infty) \rtimes \mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$.

We know that $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ and $\mathrm{P\Gamma L}(q-n-1,q)_{\ddot{\mathcal{K}}}$ are permutation isomorphic, hence, if one of them acts transitively on the points of $\mathcal{K}$ or $\ddot{\mathcal{K}}$, so does the other. By Theorem 5.3.4, the graphs $\Gamma_{n,q}(\mathcal{K})$ and $\Gamma_{q-n-2,q}(\ddot{\mathcal{K}})$ are edge-transitive and hence semisymmetric. □

If we restrict ourselves in the previous theorem to elements of the projective groups, using Theorem 5.3.3, we get the following corollary.

Corollary 5.4.24. *Let $\mathcal{K}$ be a q-arc in $H_\infty = \mathrm{PG}(n,q)$, $q \geq n+4$, and let $\ddot{\mathcal{K}}$ be a MDS dual arc of $\mathcal{K}$ in $\ddot{H}_\infty = \mathrm{PG}(q-n-2,q)$. Suppose that one of the groups $\mathrm{PGL}(n+1,q)_{\mathcal{K}}$ or $\mathrm{PGL}(q-n-1,q)_{\ddot{\mathcal{K}}}$ fixes a point outside of $\mathcal{K}$, $\ddot{\mathcal{K}}$ respectively, and acts transitively on the points of $\mathcal{K}$, $\ddot{\mathcal{K}}$ respectively. Suppose $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}} \cong \mathrm{PGL}(n+1,q)_{\mathcal{K}} \rtimes \mathrm{Aut}(\mathbb{F}_q/\mathbb{F}_{q_0})$ or $\mathrm{P\Gamma L}(q-n-1,q)_{\ddot{\mathcal{K}}} \cong \mathrm{PGL}(q-n-1,q)_{\ddot{\mathcal{K}}} \rtimes \mathrm{Aut}(\mathbb{F}_q/\mathbb{F}_{q_0})$ respectively, for $q_0 = p^{h_0}$, $h_0|h$, or $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}} \cong \mathrm{PGL}(n+1,q)_{\mathcal{K}}$, $\mathrm{P\Gamma L}(q-n-1,q)_{\ddot{\mathcal{K}}} \cong \mathrm{PGL}(q-n-1,q)_{\ddot{\mathcal{K}}}$ respectively.*

Then $\Gamma_{n,q}(\mathcal{K})$ and $\Gamma_{q-n-2,q}(\ddot{\mathcal{K}})$ are semisymmetric, $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K})) \cong \mathrm{Persp}_q(H_\infty) \rtimes \mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ and $\mathrm{Aut}(\Gamma_{q-n-2,q}(\ddot{\mathcal{K}})) \cong \mathrm{Persp}_q(\ddot{H}_\infty) \rtimes \mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$.

Consider the Glynn 10-arc contained in $\mathrm{PG}(4,9)$ and take any point P of this 10-arc; if we project the arc from P onto a $\mathrm{PG}(3,9)$ skew to P, then we obtain a complete 9-arc of $\mathrm{PG}(3,9)$. In [55] the author also shows that all complete 9-arcs in $\mathrm{PG}(3,9)$ can be obtained in this way, i.e. all complete 9-arcs of $\mathrm{PG}(3,9)$ are $\mathrm{P\Gamma L}$-equivalent. It follows from [108] that the complete 9-arc in $\mathrm{PG}(3,9)$ is the MDS dual of a 9-arc that is contained in the Glynn arc in $\mathrm{PG}(4,9)$. If we apply

Theorem 5.4.23 to the Glynn 10-arc, we obtain the following corollary. The size of the automorphism group follows as before.

Corollary 5.4.25. *If* $\mathcal{K}$ *is a complete* 9*-arc of* $\mathrm{PG}(3,9)$*, then* $\Gamma_{3,9}(\mathcal{K})$ *is a semisymmetric graph.*

Moreover, $\mathrm{Aut}(\Gamma_{3,9}(\mathcal{K}))$ *is isomorphic to* $\mathrm{Persp}_q(H_\infty) \rtimes \mathrm{AGL}(1,9)$ *and has size* $9^5 8^2$*.*

We can also apply Theorem 5.4.23 to the arcs of Section 5.4.3.

Corollary 5.4.26. *Let* $\mathcal{K}$ *be an arc of size* q *contained in any* $(q+1)$*-arc of* $\mathrm{PG}(q-4,q)$*,* $q=2^h>8$*, then* $\Gamma_{q-4,q}(\mathcal{K})$ *is a semisymmetric graph.*

Moreover, $\mathrm{Aut}(\Gamma_{q-4,q}(\mathcal{K}))$ *is isomorphic to* $\mathrm{Persp}_q(H_\infty) \rtimes \mathrm{A\Gamma L}(1,q)$ *and has size* $hq^{q-2}(q-1)^2$*.*

5.5 Semisymmetric graphs arising from other point sets

By Corollary 5.2.6, if $\mathcal{K}$ is a set of points such that its closure $\widehat{\mathcal{K}}$ is the whole space H_∞, then every automorphism of the graph $\Gamma_{n,q}(\mathcal{K})$ is induced by a collineation of its ambient space $\mathrm{PG}(n+1,q)$. However, we do not need this property for the construction of semisymmetric graphs. From the results of Section 5.2, the following theorem clearly follows.

Theorem 5.5.1. *Let* $\mathcal{K}$ *be a point set of* $H_\infty = \mathrm{PG}(n,q)$ *of size* q *spanning* H_∞ *such that every point of* $H_\infty \backslash \mathcal{K}$ *belongs to at least one tangent line to* $\mathcal{K}$*, and such that* $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ *acts transitively on the points of* $\mathcal{K}$*. Then the graph* $\Gamma_{n,q}(\mathcal{K})$ *is a connected semisymmetric graph.*

We now give some examples of semisymmetric graphs for which $\widehat{\mathcal{K}}$ is a subgeometry of H_∞. In the first three examples, $\widehat{\mathcal{K}}$ is a Baer subgeometry, obviously this only works if we look at a projective space over a field of square order. We will also construct their geometric automorphism group.

5.5.1 $\mathcal{K}$ is contained in an elliptic quadric

Let π be a Baer subgeometry $\mathrm{PG}(3,\sqrt{q})$ embedded in $H_\infty = \mathrm{PG}(3,q)$, q a square. Let $\mathcal{K}$ denote the set of points of an elliptic quadric $Q^-(3,\sqrt{q})$ in π with one point removed. This set $\mathcal{K}$ has q points and clearly every point not in $\mathcal{K}$ lies on at least one tangent line to $\mathcal{K}$.

Theorem 5.5.2. [9] *A q-cap in $\mathrm{PG}(3, \sqrt{q})$, q an odd square, is uniquely extendable to an elliptic quadric $Q^-(3, \sqrt{q})$.*

Theorem 5.5.3. [104, Chapter IV] *In $\mathrm{PG}(3, \sqrt{q})$, $q > 4$ an even square, a k-cap, with $q - \sqrt[4]{q}/2 + 1 < k < q + 1$, is contained in a unique complete $(q + 1)$-cap.*

Theorem 5.5.4. [63, Section 15.3] *The stabiliser in $\mathrm{P\Gamma L}(4, \sqrt{q})$ of an elliptic quadric in $\mathrm{PG}(3, \sqrt{q})$ is $\mathrm{P\Gamma O}^-(4, \sqrt{q})$, which is isomorphic to $\mathrm{P\Gamma L}(2, q)$ (in its faithful action on $q + 1$ points).*

Theorem 5.5.5. *Let $\mathcal{K}$ denote the set of points of an elliptic quadric $Q^-(3, \sqrt{q})$ in $H_\infty = \mathrm{PG}(3, q)$, with one point removed. The graph $\Gamma_{3,q}(\mathcal{K})$, $q > 4$ square, is semisymmetric. Moreover, the geometric automorphism group is isomorphic to $\mathrm{Persp}_q(H_\infty) \rtimes (\mathrm{A\Gamma L}(1, q) \rtimes 2)$ and has size $2hq^5(q-1)^2$. The size of $\mathrm{Aut}(\Gamma_{3,q}(\mathcal{K}))$ is equal to $2hq^5(q - 1)^2\mathrm{ng}(\sqrt{q}, 2)$.*

Proof. As $\mathcal{K}$ consists of q points spanning $\mathrm{PG}(3, q)$, the graph $\Gamma_{3,q}(\mathcal{K})$ is q-regular and it is connected by Theorem 4.1.3. The graph $\Gamma_{3,q}(\mathcal{K})$ is not vertex-transitive by Corollary 5.2.4. The geometric automorphism group of $\Gamma_{3,q}(\mathcal{K})$ is $\mathrm{P\Gamma L}(5, q)_\mathcal{K}$. By Theorem 5.5.2 ($q$ odd) and 5.5.3 (q even), the cap $\mathcal{K}$ extends uniquely to an elliptic quadric in $\mathrm{PG}(3, \sqrt{q})$ by a point P. This point is obviously fixed by the stabiliser of $\mathcal{K}$ and hence, by Theorem 5.3.2, we find $\mathrm{P\Gamma L}(5, q)_\mathcal{K} \cong \mathrm{Persp}_q(H_\infty) \rtimes \mathrm{P\Gamma L}(4, q)_\mathcal{K}$.

The group stabilising $\mathcal{K}$ also stabilises the subgeometry $\widehat{\mathcal{K}}$, hence $\mathrm{P\Gamma L}(4, q)_\mathcal{K} \cong \mathrm{P\Gamma L}(4, \sqrt{q})_\mathcal{K} \rtimes (\mathrm{Aut}(\mathbb{F}_q/\mathbb{F}_{\sqrt{q}})) \cong \mathrm{P\Gamma L}(4, \sqrt{q})_\mathcal{K} \rtimes 2$. The stabiliser of $\mathcal{K}$ stabilises the elliptic quadric and fixes its point P, hence we find $\mathrm{P\Gamma L}(4, \sqrt{q})_\mathcal{K} \cong \mathrm{P\Gamma O}^-(4, \sqrt{q})_P \cong \mathrm{P\Gamma L}(2, q)_P \cong \mathrm{A\Gamma L}(1, q)$. Since $\mathrm{A\Gamma L}(1, q)$ acts transitively on the points of $\mathcal{K}$, the graph is semisymmetric. The size of this group follows from $|\mathrm{Persp}_q(H_\infty)| = q^4(q - 1)$ and $|\mathrm{A\Gamma L}(1, q)| = hq(q - 1)$. The size of the full automorphism group follows from Corollary 5.2.7. $\square$

5.5.2 $\mathcal{K}$ is contained in a Tits ovoid

Let π be a Baer subgeometry $\mathrm{PG}(3, \sqrt{q})$ embedded in $H_\infty = \mathrm{PG}(3, q)$, $q = 2^{2(2e+1)}$, $e > 0$. Let $\mathcal{K}$ denote the set of points of a Tits ovoid in π, with one point removed. This set $\mathcal{K}$ has q points and forms a cap in $\mathrm{PG}(3, q)$.

The canonical form of a Tits ovoid in $\mathrm{PG}(3,\sqrt{q})$, $\sqrt{q}=2^{2e+1}$, is

$$\{(1,s,t,st+s^{\sigma+2}+t^{\sigma})_{\mathbb{F}_q} \mid s,t\in\mathbb{F}_{\sqrt{q}}\}\cup\{(0,0,0,1)_{\mathbb{F}_q}\},$$

where $\sigma:\mathbb{F}_{\sqrt{q}}\to\mathbb{F}_{\sqrt{q}}: x\mapsto x^{2^{e+1}}$. Let the set $\mathcal{K}$ correspond to the points of this ovoid minus the point $(0,0,0,1)_{\mathbb{F}_q}$, then $\mathcal{K}$ is clearly stabilised by $\mathrm{Aut}(\mathbb{F}_q)$.

Theorem 5.5.6. [116] *The stabiliser of* $\mathcal{K}$ *in* $\mathrm{PGL}(4,\sqrt{q})$ *is the* 2*-transitive Suzuki simple group* $\mathrm{Sz}(\sqrt{q})$*.*

Following the notation of [70, Chapter 11], the point stabiliser of $\mathrm{Sz}(\sqrt{q})$ will be denoted by $\mathfrak{F}\mathfrak{H}$. Since $\mathrm{Sz}(\sqrt{q})$ is 2-transitive, the group $\mathfrak{F}\mathfrak{H}$ is transitive.

Theorem 5.5.7. *Let* $\mathcal{K}$ *denote the set of points of a Tits ovoid in an* $\mathbb{F}_{\sqrt{q}}$*-subgeometry of* H_∞*, with one point removed. The graph* $\Gamma_{3,q}(\mathcal{K})$*,* $q=2^{2(2e+1)}$*,* $e>0$*, is semisymmetric. Moreover, the geometric automorphism group is isomorphic to* $\mathrm{Persp}_q(H_\infty)\rtimes(\mathfrak{F}\mathfrak{H}\rtimes\mathrm{Aut}(\mathbb{F}_q))$ *and has size* $hq^5(q-1)(\sqrt{q}-1)$*. The size of* $\mathrm{Aut}(\Gamma_{3,q}(\mathcal{K}))$ *is equal to* $hq^5(q-1)(\sqrt{q}-1)\mathrm{ng}(\sqrt{q},2)$*.*

Proof. The proof works in almost the same way as for the elliptic quadric. The size of the group follows when considering that $|\mathfrak{F}\mathfrak{H}|=q(\sqrt{q}-1)$. □

5.5.3 $\mathcal{K}$ is contained in a hyperbolic quadric

Let π be a Baer subgeometry $\mathrm{PG}(3,\sqrt{q})$ embedded in $H_\infty=\mathrm{PG}(3,q)$, $q>4$ square. Let $\mathcal{K}$ denote the set of points of a hyperbolic quadric $Q^+(3,\sqrt{q})$ in π with two lines of different reguli removed. This set $\mathcal{K}$ has q points.

Theorem 5.5.8. [63, Section 15.3] *The stabiliser in* $\mathrm{P\Gamma L}(4,\sqrt{q})$ *of a hyperbolic quadric in* $\mathrm{PG}(3,\sqrt{q})$ *is* $\mathrm{P\Gamma O}^+(4,\sqrt{q})$*, which is isomorphic to* $((\mathrm{PGL}(2,\sqrt{q})\times\mathrm{PGL}(2,\sqrt{q}))\rtimes 2)\rtimes\mathrm{Aut}(\mathbb{F}_{\sqrt{q}})$ *for* $\sqrt{q}>2$*.*

Corollary 5.5.9. *For* $\sqrt{q}>2$*, the stabiliser in* $\mathrm{P\Gamma L}(4,\sqrt{q})$ *of a hyperbolic quadric in* $\mathrm{PG}(3,\sqrt{q})$ *fixing two lines of different reguli is isomorphic to* $((\mathrm{AGL}(1,\sqrt{q})\times\mathrm{AGL}(1,\sqrt{q}))\rtimes 2)\rtimes\mathrm{Aut}(\mathbb{F}_{\sqrt{q}})$*.*

Theorem 5.5.10. *Let* $\mathcal{K}$ *denote the set of points of a hyperbolic quadric* $Q^+(3,\sqrt{q})$ *of an* $\mathbb{F}_{\sqrt{q}}$*-subgeometry of* H_∞ *with two lines of different reguli removed. The graph* $\Gamma_{3,q}(\mathcal{K})$*,* $q=p^h>4$ *square, is semisymmetric. Moreover, the geometric automorphism group is isomorphic to* $\mathrm{Persp}_q(H_\infty)\rtimes((\mathrm{AGL}(1,\sqrt{q})\times\mathrm{AGL}(1,\sqrt{q}))\rtimes 2)\rtimes\mathrm{Aut}(\mathbb{F}_q)$ *and has size* $2hq^5(q-1)(\sqrt{q}-1)^2$*. The size of* $\mathrm{Aut}(\Gamma_{3,q}(\mathcal{K}))$ *is equal to* $2hq^5(q-1)(\sqrt{q}-1)^2\mathrm{ng}(\sqrt{q},2)$*.*

Proof. Since $\mathcal{K}$ consists of q points spanning $\mathrm{PG}(3,q)$, $\Gamma_{3,q}(\mathcal{K})$ is q-regular and is connected by Theorem 4.1.3. Clearly every point of $\mathrm{PG}(3,q)$ not in $\mathcal{K}$ belongs to at least one tangent to $\mathcal{K}$, hence $\Gamma_{3,q}(\mathcal{K})$ is not vertex-transitive by Corollary 5.2.4. The geometric automorphism group is $\mathrm{P\Gamma L}(5,q)_{\mathcal{K}}$. Clearly $\mathcal{K}$ extends uniquely to a hyperbolic quadric in $\mathrm{PG}(3,\sqrt{q})$ by adding the missing line of each regulus. Since the intersection point of these lines will be fixed by the stabiliser of $\mathcal{K}$, we find by Theorem 5.3.2 that $\mathrm{P\Gamma L}(5,q)_{\mathcal{K}} \cong \mathrm{Persp}_q(H_\infty) \rtimes \mathrm{P\Gamma L}(4,q)_{\mathcal{K}}$. Since the group stabilising the hyperbolic quadric also stabilises the subgeometry $\widehat{\mathcal{K}} = \mathrm{PG}(3,\sqrt{q})$ and the canonical form of $Q^+(3,\sqrt{q})$ is fixed by $\mathrm{Aut}(\mathbb{F}_q)$, we find $\mathrm{P\Gamma L}(4,q)_{\mathcal{K}} \cong \mathrm{PGL}(4,\sqrt{q})_{\mathcal{K}} \rtimes \mathrm{Aut}(\mathbb{F}_q) \cong ((\mathrm{AGL}(1,\sqrt{q}) \times \mathrm{AGL}(1,\sqrt{q})) \rtimes 2) \rtimes \mathrm{Aut}(\mathbb{F}_q)$, by Theorem 5.3.3. Since $(\mathrm{AGL}(1,\sqrt{q}) \times \mathrm{AGL}(1,\sqrt{q})) \rtimes 2$ acts transitively on the points of $\mathcal{K}$, the graph is semisymmetric. The size of the full automorphism group follows from Corollary 5.2.7. □

5.5.4 $\mathcal{K}$ is contained in a cone

Let Π be a subgeometry $\mathrm{PG}(n,q_0)$ embedded in $H_\infty = \mathrm{PG}(n,q)$, $q = q_0^k$. Let π be a hyperplane of Π. Consider a set $\mathcal{O}$ of q_0^{k-1} points of π. Let V be a point of $\Pi\backslash\pi$ and let $V\mathcal{O}$ denote the set of points of the cone in Π with vertex V and base $\mathcal{O}$. Let $\mathcal{K}$ be the point set of $V\mathcal{O}$ minus its vertex V, clearly $\mathcal{K}$ contains q points.

Lemma 5.5.11. *Let $\mathcal{K}$ be the cone $V\mathcal{O}$ of Π minus its vertex V, such that every point of $\pi\backslash\mathcal{O}$ belongs to at least one tangent line to $\mathcal{O}$, then $\forall P \in \mathcal{P}, \forall L \in \mathcal{L}$: $\Gamma_{n,q}(\mathcal{K})_4(P) \ncong \Gamma_{n,q}(\mathcal{K})_4(L)$.*

Proof. Let $\Gamma = \Gamma_{n,q}(\mathcal{K})$. We will prove that, for every line $L \in \mathcal{L}$, the set of vertices $\Gamma_4(L)$ contains more than $q-1$ vertices that have all their neighbours in $\Gamma_3(L)$, while for every point $P \in \mathcal{P}$, there are exactly $q-1$ vertices in the set $\Gamma_4(P)$ that have all their neighbours in $\Gamma_3(P)$.

To prove the first claim, consider a line $L \in \mathcal{L}$ with $L \cap H_\infty = P_1 \in \mathcal{K}$. Choose an affine point Q on L and a point $P_2 \in \mathcal{K}$ different from P_1. Take a point R on QP_2, not equal to Q or P_2, then clearly the line $RP_1 \in \Gamma_4(L)$. We will show that RP_1 has all its neighbours in $\Gamma_3(L)$. Consider a neighbour S of RP_1, i.e $S \in RP_1 \backslash \{P_1\}$. The line SP_2 meets L in a point T. Since $T \in \Gamma_1(L)$ and $TP_2 \in \Gamma_2(L)$, it follows that $S \in \Gamma_3(L)$. Clearly any line $M \in \mathcal{L}$ through P_1, such that $\langle M, L\rangle \cap H_\infty$ contains at least two points in $\mathcal{K}$, belongs to $\Gamma_4(L)$ and has all its neighbours in $\Gamma_3(L)$. Since the points of $\mathcal{K}$ are not contained in one line, there are more than $q-1$ such lines M.

Consider now a point $P \in \mathcal{P}$ and a point $T \in \Gamma_4(P)$. Consider a minimal path of length 4 from T to P: $T \sim QP_1 \sim Q \sim QP_2 \sim P$, for some affine point Q and distinct points $P_1, P_2 \in \mathcal{K}$, such that $T \in QP_1$ and $P \in QP_2$. Consider the point $R = PT \cap H_\infty$, then R belongs to the line P_1P_2. Since $PR \notin \Gamma_1(P)$, we have R not in $\mathcal{K}$. First, suppose there is a tangent line of $\mathcal{K}$ through R, say RP_3, with $P_3 \in \mathcal{K}$. The line TP_3 is a neighbour of T. If TP_3 belongs to $\Gamma_3(P)$, then there exists a line PT' through a point $P_4 \in \mathcal{K}$, with T' on TP_3, which implies that RP_3 contains the point $P_4 \in \mathcal{K}$, a contradiction. Hence in this case there are neighbours of T that do not belong to $\Gamma_3(P)$. Now suppose there is no tangent line of $\mathcal{K}$ through R, then by construction, R is the vertex V of the cone. A line through V either contains 0 or q_0 points of $\mathcal{K}$, so in this case, any neighbour of T belongs to $\Gamma_3(P)$. There are exactly $q-1$ points on the line VP different from P and V. □

Corollary 5.5.12. *The graph $\Gamma_{n,q}(\mathcal{K})$ is not vertex-transitive.*

Proof. Since any graph automorphism preserves distance and hence neighbourhoods, no automorphism of $\Gamma_{n,q}(\mathcal{K})$ can map a vertex in $\mathcal{P}$ to a vertex in $\mathcal{L}$. □

Recall that the subgroup of $\mathrm{P\Gamma L}(n+1,q)$ consisting of the perspectivities with centre V is denoted by $\mathrm{Persp}_q(V)$.

Lemma 5.5.13. *Consider $\mathcal{K}$, the point set of the cone $V\mathcal{O}$ in $\mathrm{PG}(n,q_0)$, minus its vertex V, where $\mathcal{O}$ spans π. If $\mathrm{P\Gamma L}(n,q_0)_{\mathcal{O}}$ and $\mathrm{PGL}(n,q_0)_{\mathcal{O}}$, respectively, fix a point of π, then*

$$\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}} \cong \mathrm{Persp}_q(V) \rtimes (\mathrm{P\Gamma L}(n,q_0)_{\mathcal{O}} \rtimes \mathrm{Aut}(\mathbb{F}_q/\mathbb{F}_{q_0}))$$

and

$$\mathrm{PGL}(n+1,q)_{\mathcal{K}} \cong \mathrm{Persp}_q(V) \rtimes \mathrm{PGL}(n,q_0)_{\mathcal{O}},$$

respectively.

Proof. First, it should be noted that the $\mathbb{F}_{q_0}$-span of $\mathcal{O}$ is π, the $\mathbb{F}_{q_0}$-span of $\mathcal{K}$ is Π, so $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ and $\mathrm{PGL}(n+1,q)_{\mathcal{K}}$ stabilise the subgeometry Π of H_∞. This implies that $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}} \cong (\mathrm{P\Gamma L}(n+1,q)_\Pi)_{\mathcal{K}}$, and $\mathrm{PGL}(n+1,q)_{\mathcal{K}} \cong (\mathrm{PGL}(n+1,q)_\Pi)_{\mathcal{K}}$ respectively. Since $\mathrm{P\Gamma L}(n+1,q)_\Pi$ is clearly isomorphic to $\mathrm{PGL}(n+1,q_0) \rtimes \mathrm{Aut}(\mathbb{F}_q/\mathbb{F}_{q_0})$, we have that $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}} \cong \mathrm{P\Gamma L}(n+1,q_0)_{\mathcal{K}} \rtimes \mathrm{Aut}(\mathbb{F}_q/\mathbb{F}_{q_0})$. Also, since $\mathrm{PGL}(n+1,q)_\Pi$ is isomorphic to $\mathrm{PGL}(n+1,q_0)$, we have that $\mathrm{PGL}(n+1,q)_{\mathcal{K}} \cong \mathrm{PGL}(n+1,q_0)_{\mathcal{K}}$.

Let ϕ be an element of $\mathrm{P\Gamma L}(n+1, q_0)_{\mathcal{K}}$, then ϕ preserves the lines through V. Define the action of ϕ on the lines L of $V\mathcal{O}$ to be the mapping taking $L \cap \pi$ to $\phi(L) \cap \pi$. The kernel of this action of $\mathrm{P\Gamma L}(n+1, q_0)_{\mathcal{K}}$ on π is clearly isomorphic to $\mathrm{Persp}_q(V)$, as it consists of all collineations fixing the lines through V. The image of the action is isomorphic to $\mathrm{P\Gamma L}(n, q_0)_{\mathcal{O}}$, showing that $\mathrm{P\Gamma L}(n+1, q_0)_{\mathcal{K}}$ is an extension of $\mathrm{Persp}_q(V)$ by $\mathrm{P\Gamma L}(n, q_0)_{\mathcal{O}}$. To show that this extension splits, we embed $\mathrm{P\Gamma L}(n, q_0)_{\mathcal{O}}$ in $\mathrm{P\Gamma L}(n+1, q_0)_{\mathcal{K}}$ in such a way that it intersects trivially with $\mathrm{Persp}_q(V)$. By assumption, $\mathrm{P\Gamma L}(n, q_0)_{\mathcal{O}}$ fixes a point $P \in \pi$. W.l.o.g. let π be the hyperplane with equation $X_0 = 0$ and let V be the point $(1, 0, \ldots, 0)_{\mathbb{F}_q}$. Suppose that P has corresponding vector $(0, c_1, c_2, \ldots, c_n)$, where the first non-zero coordinate equals one. This implies that for each $\beta \in \mathrm{P\Gamma L}(n, q_0)_{\mathcal{O}}$, there exists a unique $n \times n$ matrix $B = (b_{ij})$, $1 \leq i, j \leq n$, and $\theta \in \mathrm{Aut}(\mathbb{F}_{q_0})$ corresponding to β, such that $(c_1, \ldots, c_n)^\theta B = (c_1, \ldots, c_n)$. Moreover, the obtained maps (B, θ) form a subgroup of $\Gamma\mathrm{L}(n, q_0)$. Let $A_B = (a_{ij})$, $0 \leq i, j \leq n$, be the $(n+1) \times (n+1)$ matrix with $a_{00} = 1$, $a_{i0} = a_{0j} = 0$ for $i, j \geq 1$ and $a_{ij} = b_{ij}$ for $1 \leq i, j \leq n$. It is clear that the semi-linear map (A_B, θ) defines an element of $\mathrm{P\Gamma L}(n+1, q_0)_{\mathcal{K}}$, corresponding to a collineation α acting in the same way as β on H_∞. If θ is not the identity $\mathbb{1}$, then α is not a perspectivity. If $\theta = \mathbb{1}$, then α fixes every point on the line through P and V, thus fixes at least two affine points and hence is not a perspectivity. This implies that the elements α form a subgroup of $\mathrm{P\Gamma L}(n+1, q)_{\mathcal{K}}$ isomorphic to $\mathrm{P\Gamma L}(n, q_0)_{\mathcal{O}}$ and intersecting $\mathrm{Persp}_q(V)$ trivially. This implies that $\mathrm{P\Gamma L}(n+1, q_0)_{\mathcal{K}} \cong \mathrm{Persp}_q(V) \rtimes \mathrm{P\Gamma L}(n, q_0)_{\mathcal{O}}$, and we have seen before that $\mathrm{P\Gamma L}(n+1, q)_{\mathcal{K}} \cong \mathrm{P\Gamma L}(n+1, q_0)_{\mathcal{K}} \rtimes \mathrm{Aut}(\mathbb{F}_q/\mathbb{F}_{q_0})$. Since $\mathrm{Persp}_q(V)$ intersects trivially with the standard embedding of $\mathrm{Aut}(\mathbb{F}_q/\mathbb{F}_{q_0})$, the claim follows.

The claim for $\mathrm{PGL}(n+1, q)_{\mathcal{K}}$ can be proved in the same way. $\square$

Since $\mathrm{Persp}_q(V)$ acts transitively on the points of each line through V, we obtain the following corollary.

Corollary 5.5.14. *If the stabiliser* $\mathrm{P\Gamma L}(n, q_0)_{\mathcal{O}}$ *acts transitively on* $\mathcal{O}$*, then* $\mathrm{P\Gamma L}(n+1, q)_{\mathcal{K}}$ *acts transitively on* $\mathcal{K}$*.*

Theorem 5.5.15. *Let* $\mathcal{K}$ *denote the points of a cone* $V\mathcal{O}$ *in an* $\mathbb{F}_{q_0}$*-subgeometry of* $H_\infty = \mathrm{PG}(n, q)$, $q = q_0^k$*, minus its vertex* V*. Suppose that* $\mathcal{O}$ *spans* π*, that every point of* $\pi \backslash \mathcal{O}$ *belongs to a tangent line to* $\mathcal{O}$ *and that* $\mathrm{P\Gamma L}(n, q_0)_{\mathcal{O}}$ *acts transitively on* $\mathcal{O}$*. Then the graph* $\Gamma_{n,q}(\mathcal{K})$ *is semisymmetric.*

The geometric automorphism group of $\Gamma_{n,q}(\mathcal{K})$ *is isomorphic to* $\mathrm{Persp}_q(H_\infty) \rtimes \big(\mathrm{Persp}_q(V) \rtimes (\mathrm{P\Gamma L}(n, q_0)_{\mathcal{O}} \rtimes \mathrm{Aut}(\mathbb{F}_q/\mathbb{F}_{q_0}))\big)$*. The group* $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K}))$ *has size* $kq^{2n+1}(q-1)^2|\mathrm{P\Gamma L}(n, q_0)_{\mathcal{O}}|\mathrm{ng}(q_0, k)$*.*

Proof. Since $\mathcal{K}$ consists of q points spanning $\mathrm{PG}(n,q)$, $\Gamma_{n,q}(\mathcal{K})$ is q-regular and is connected by Theorem 4.1.3. The graph $\Gamma_{n,q}(\mathcal{K})$ is not vertex-transitive by Corollary 5.5.12. Clearly, $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ stabilises the point V, so we find by Theorem 5.3.2 that $\mathrm{PGL}(n+2,q)_{\mathcal{K}} \cong \mathrm{Persp}_q(H_\infty) \rtimes \mathrm{PGL}(n+1,q)_{\mathcal{K}}$. The expression for the geometric automorphism group follows from Lemma 5.5.13. Since $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ acts transitively on the points of $\mathcal{K}$, by Theorem 5.3.4, the graph is edge-transitive, and hence semisymmetric. The size of the full automorphism group follows from Corollary 5.2.7. □

5.6 Isomorphisms with other graphs

In this section, we will show that the graphs constructed by Du, Wang and Zhang [48], and the graphs of Lazebnik and Viglione [78] belong to the family $\Gamma_{n,q}(\mathcal{K})$, where $\mathcal{K}$ is a q-arc contained in a normal rational curve (see Section 5.4.2).

5.6.1 The graph of Du, Wang and Zhang

If $q = p$ prime, then the point set $\mathcal{N}$ of $\mathrm{PG}(n,p)$ defined as

$$\mathcal{N} = \left\{(0,\dots,0,1)_{\mathbb{F}_q}\right\} \cup \left\{\phi^i(P) \,\middle|\, P = (1,0,\dots,0)_{\mathbb{F}_q}, i = 0,1,\dots,p-1\right\},$$

where the element $\phi \in \mathrm{PGL}(n+1,p)$ has order p and is defined by the matrix A_ϕ (under right-multiplication on row vectors) forms a normal rational curve in $\mathrm{PG}(n,p)$ (see [107]):

$$A_\phi = \begin{pmatrix} 1 & 0 & & \cdots & & 0 \\ 1 & 1 & & & & \\ 0 & 1 & \ddots & & & \vdots \\ & & \ddots & & & \\ \vdots & & & 1 & 1 & 0 \\ 0 & \cdots & & 0 & 1 & 1 \end{pmatrix}.$$

When we consider the orbit of P under ϕ as the point set $\mathcal{K}$ at infinity, we obtain a reformulation of the construction of the semisymmetric graphs found by Du, Wang and Zhang in [48]. This shows that our construction of the graph $\Gamma_{n,p}(\mathcal{K})$, with $\mathcal{K}$ a set of p points, contained in a normal rational curve, contains their family (and extends their construction to the case where q is not a prime).

Moreover, the edge-transitive group of automorphisms described by the authors is not the full automorphism group of the graph: they only consider automorphisms induced by the group $\langle \phi \rangle$ of order p acting on the points of $\mathcal{K}$, together with $\mathrm{Persp}_p(H_\infty)$.

5.6.2 The graph of Lazebnik and Viglione

In [78], the authors define the graph $\Lambda_{n,q}$ as follows. Let $\mathcal{P}_n$ and $\mathcal{L}_n$ be two $(n+1)$-dimensional vector spaces over $\mathbb{F}_q$, $q = p^h$. The vertex set of $\Lambda_{n,q}$ is $\mathcal{P}_n \cup \mathcal{L}_n$, and we declare a point $(p) = (p_1, p_2, \dots, p_{n+1})$ adjacent to a line $[l] = [l_1, l_2, \dots, l_{n+1}]$ if and only if the following n relations on their coordinates hold:

$$\begin{aligned} l_2 + p_2 &= p_1 l_1, \\ l_3 + p_3 &= p_1 l_2, \\ &\vdots \\ l_{n+1} + p_{n+1} &= p_1 l_n. \end{aligned}$$

In the following theorem, we will show that the graph $\Lambda_{n,q}$ is isomorphic to the graph $\Gamma_{n,q}(\mathcal{K})$, where $\mathcal{K}$ is contained in a normal rational curve; hence, $\Gamma_{n,q}(\mathcal{K})$ provides an embedding of the Lazebnik-Viglione graph in $\mathrm{PG}(n+1, q)$. It should be noted that in [78], the authors provide some automorphisms, acting on the graph $\Lambda_{n,q}$, to show that this graph is semisymmetric. From the isomorphism with $\Gamma_{n,q}(\mathcal{K})$, it follows that $\mathrm{P\Gamma L}(n+2, q)_{\mathcal{K}}$ is also the full automorphism group of the Lazebnik-Viglione graph when $q \geq n+3$ or $q = p = n+2$.

Theorem 5.6.1. *We have $\Lambda_{n,q} \cong \Gamma_{n,q}(\mathcal{K})$, where $\mathcal{K}$ is a q-arc contained in a normal rational curve.*

Proof. The graph $\Lambda_{n,q}$ is isomorphic to the graph $\Lambda'_{n,q}$ obtained by reversing the role of the points and the lines in the definition of $\Lambda_{n,q}$. So, $\Lambda'_{n,q}$ is the bipartite graph with parts $\mathcal{P}_n$ and $\mathcal{L}_n$, where $(p_1, \dots, p_{n+1}) \in \mathcal{P}_n$ is incident with $[l_1, \dots, l_{n+1}] \in \mathcal{L}_n$ if and only if $p_{i+1} + l_{i+1} = l_1 p_i$ for all $1 \leq i \leq n$. Let $[l] = [l_1, \dots, l_{n+1}]$ be a vertex of $\Lambda'_{n,q}$, then the points, incident with $[l]$, form a line of $\mathrm{AG}(n+1, q)$: suppose $(p_1, \dots, p_{n+1})$ and $(p'_1, \dots, p'_{n+1})$ are vertices, adjacent with $[l]$, then so is the vertex $(p_1 + \lambda(p'_1 - p_1), \dots, p_{n+1} + \lambda(p'_{n+1} - p_{n+1}))$, for any $\lambda \in \mathbb{F}_q$.

Now let $(p_1, \dots, p_{n+1})$ and $(p'_1, \dots, p'_{n+1})$ be vertices of $\Lambda'_{n,q}$ and consider their corresponding affine points in $\mathrm{PG}(n+1, q)$, by identifying $(p_1, \dots, p_{n+1})$ with

$(1, p_1, \ldots, p_{n+1})_{\mathbb{F}_q}$. The line L determined by these points meets the hyperplane at infinity with equation $X_0 = 0$ of $\mathrm{AG}(n+1, q)$ in the point $P_\infty = (0, p_1 - p'_1, \ldots, p_{n+1} - p'_{n+1})_{\mathbb{F}_q}$. Now the affine point set of L is a vertex of $\Lambda'_{n,q}$ if and only if there is an element $[l_1, \ldots, l_{n+1}] \in \mathcal{L}_n$ such that for all $1 \leq i \leq n$:

$$\begin{aligned} p_{i+1} + l_{i+1} &= l_1 p_i, \\ p'_{i+1} + l_{i+1} &= l_1 p'_i. \end{aligned}$$

This implies that there exists some $l_1 \in \mathbb{F}_q$ such that $p_{i+1} - p'_{i+1} = l_1(p_i - p'_i)$ for all $1 \leq i \leq n$. Hence, the point P_∞ has coordinates $(0, 1, l_1, l_1^2, \ldots, l_1^n)_{\mathbb{F}_q}$, which implies that all the vertices $[l_1, \ldots, l_{n+1}]$ of $\Lambda'_{n,q}$ define a line in $\mathrm{PG}(n+1, q)$ through a point of the standard normal rational curve $\mathcal{K}$, minus the point $(0, \ldots, 0, 1)_{\mathbb{F}_q}$. This is exactly the description of the graph $\Gamma_{n,q}(\mathcal{K})$. $\square$

Corollary 5.6.2. *The automorphism group* $\mathrm{Aut}(\Lambda_{n,q})$ *of the graph* $\Lambda_{n,q}$ *is isomorphic to the edge-transitive group* $\mathrm{P\Gamma L}(n+2, q)_{\mathcal{K}}$. *Moreover*

- *if* $q \geq n+3$, $q = p^h$, $n \geq 3$ *(or* $n = 2$ *and* q *odd), then* $\mathrm{Aut}(\Lambda_{n,q})$ *has size* $hq^{n+2}(q-1)^2$;
- *if* $q = p = n+2$, *then* $\mathrm{Aut}(\Lambda_{n,q})$ *has size* $q^{n+1}(q-1)q!$, *and, if* $q = p^h = n+2$, $h > 1$, *then* $\mathrm{Aut}(\Lambda_{n,q})$ *has size* $q^{n+1}(q-1)q!\mathrm{ng}(p, h)$.

III

The André/Bruck-Bose representation

In Part III, we consider substructures in the André/Bruck-Bose representation of $\mathrm{PG}(2, q^n)$ in $\mathrm{PG}(2n, q)$. We investigate the André/Bruck-Bose representation of $\mathbb{F}_{q^k}$-sublines and $\mathbb{F}_{q^k}$-subplanes of $\mathrm{PG}(2, q^n)$ in Chapter 6. In Chapter 7, we obtain a characterisation of ovoidal Buekenhout-Metz unitals in $\mathrm{PG}(2, q^2)$ by considering their corresponding point set in $\mathrm{PG}(4, q)$.

6
Subgeometries in the André/Bruck-Bose representation

We consider the André/Bruck-Bose representation of the projective plane $\mathrm{PG}(2, q^n)$ in $\mathrm{PG}(2n, q)$. We investigate the representation of $\mathbb{F}_{q^k}$-sublines and $\mathbb{F}_{q^k}$-subplanes of $\mathrm{PG}(2, q^n)$, extending the results of [11, 13, 14]. More precisely, we characterise the representation of $\mathbb{F}_{q^k}$-sublines tangent to or contained in the line at infinity, $\mathbb{F}_q$-sublines external to the line at infinity, $\mathbb{F}_q$-subplanes tangent to and $\mathbb{F}_{q^k}$-subplanes secant to the line at infinity.

The results in this chapter are joint work with J. Sheekey and G. Van de Voorde, and were published in [97].

6.1 Introduction

The *André/Bruck-Bose representation*, or *ABB-representation*, is the well-known representation of the plane $\mathrm{PG}(2, q^n)$ in $\mathrm{PG}(2n, q)$ (for details, see Section 1.5). The aim of this chapter is to characterise the ABB-representation of $\mathbb{F}_{q^k}$-sublines and $\mathbb{F}_{q^k}$-subplanes of $\mathrm{PG}(2, q^n)$ in $\mathrm{PG}(2n, q)$. For this, we need to make the distinction between sublines that are contained in, tangent to or external to the line at infinity, as well as the distinction between subplanes that are secant to, tangent to or external to the line at infinity.

For $k = 1$ and $n = 2$, these problems are thoroughly studied and solved (see [12, Section 3.4.2]). For sublines tangent to and subplanes secant to the line at infinity where $k = 2^i$ and $n = 2^r$, this problem was solved in [11]. Recently, the case $k = 1$ and $n = 3$, for tangent and external sublines as well as for secant and tangent subplanes, was settled in [13, 14]. In [13, Theorem 3.5], the authors extended their proof for sublines with $n = 3$ to general n. However, we note in Theorem

6.3.5 that this extension is not entirely correct for the case of an external subline (unless n is a prime).

We will settle the characterisation of the following cases:

- $\mathbb{F}_{q^k}$-sublines tangent to or contained in the line at infinity:
  for general k and n (Theorem 6.3.3 and Theorem 6.3.8),
- $\mathbb{F}_q$-sublines external to the line at infinity:
  for general n (Theorem 6.3.5),
- $\mathbb{F}_{q^k}$-subplanes secant to the line at infinity:
  for general k and n (Theorem 6.4.1),
- $\mathbb{F}_q$-subplanes tangent to the line at infinity:
  for general n (Theorem 6.4.5).

The chapter is organised as follows. Indicator sets and subspreads of Desarguesian spreads are considered in Subsection 6.2.1. We will use explicit coordinates for the ABB-representation, which are introduced in Subsection 6.2.2. This enables us to determine the indicator set of the Desarguesian spread in an explicit way in Subsection 6.2.3. To allow us to use coordinates in the most convenient form in the calculations of Sections 6.3 and 6.4, we determine in Subsection 6.2.4 the induced action of the stabiliser of the line at infinity of $\mathrm{PG}(2, q^n)$ on the points of the ABB-representation in $\mathrm{PG}(2n, q)$. Sections 6.3 and 6.4 are devoted to the proofs of the characterisation theorems for sublines and subplanes.

Remark. We wish to remark the following correspondence between field reduction and the ABB-representation. Apply the field reduction map $\mathcal{F}_{3,n,q}$ to the points and lines of a projective plane $\pi \cong \mathrm{PG}(2, q^n)$. The points of π correspond to $(n-1)$-spaces of a Desarguesian $(n-1)$-spread $\mathcal{D}$ in a space $\Pi \cong \mathrm{PG}(3n-1, q)$; the lines of π correspond to $(2n-1)$-spaces in Π partitioned by elements of $\mathcal{D}$. Take a line l_∞ of π and its corresponding $(2n-1)$-space H_∞ of Π. Consider now a $2n$-space Σ in Π containing H_∞. The space Σ intersects an element of $\mathcal{D} \setminus H_\infty$ in a point and intersects a $(2n-1)$-space corresponding to a line of π (different from l_∞) in an n-space containing an element of $\mathcal{D} \cap H_\infty$. It is clear that the incidence structure of these points and n-spaces in Σ corresponds to the ABB-representation of π with respect to l_∞. The image under the field reduction map $\mathcal{F}_{3,n,q}$ of an $\mathbb{F}_q$-subline, respectively $\mathbb{F}_q$-subplane, of π is a regulus, respectively Segre variety, contained in $\mathcal{D}$. Hence, characterising the ABB-representation of such $\mathbb{F}_q$-sublines and $\mathbb{F}_q$-subplanes corresponds to characterising the intersection of a regulus and a Segre variety with an appropriate subspace. This was an approach used in [77] to describe the ABB-representation of an external $\mathbb{F}_q$-subline. However, obtaining a one-to-one correspondence this way seems unfeasible, since the indicator sets play a crucial role in the characterisation. Moreover, since there is not (yet) a nice description of the field reduction of $\mathbb{F}_{q^k}$-sublines and $\mathbb{F}_{q^k}$-subplanes for $k > 1$, their ABB-representation can not be obtained this way. Hence, we will not be using this approach.

6.2 Preliminaries

6.2.1 The indicator set of a Desarguesian spread and its subspreads

As seen in Section 1.4, a Desarguesian spread of $\mathrm{PG}(rn-1, q)$ can be obtained by applying field reduction to the points of $\mathrm{PG}(r-1, q^n)$. However, by Segre [103], a Desarguesian spread can also be constructed as follows. Embed $\Lambda = \mathrm{PG}(rn-1, q)$ as a subgeometry of $\Lambda^* = \mathrm{PG}(rn-1, q^n)$. The subgroup of $\mathrm{P\Gamma L}(rn, q^n)$ fixing Λ pointwise is isomorphic to $\mathrm{Aut}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. Consider a generator σ of this group. One can prove that there exists an $(r-1)$-space ν skew to the subgeometry Λ such that $\langle \nu, \nu^\sigma, \dots, \nu^{\sigma^{n-1}} \rangle = \mathrm{PG}(rn-1, q^n)$. Moreover, a subspace of $\mathrm{PG}(rn-1, q^n)$ of dimension s is fixed by σ if and only if it intersects the subgeometry Λ in a subspace of dimension s (see [34]). Let P be a point of ν and let $L(P)$ denote the $(n-1)$-dimensional subspace generated by the *conjugates* of P, i.e., $L(P) = \langle P, P^\sigma, \dots, P^{\sigma^{n-1}} \rangle$. Then $L(P)$ is fixed by σ and hence it intersects $\mathrm{PG}(rn-1, q)$ in an $(n-1)$-dimensional subspace. Repeating this for every point of ν, one obtains a set $\mathcal{D}$ of $(n-1)$-spaces of the subgeometry Γ forming a spread. This spread $\mathcal{D}$ is a Desarguesian spread and $\{\nu, \nu^\sigma, \dots, \nu^{\sigma^{n-1}}\}$ is called the *indicator set* of $\mathcal{D}$. An indicator set is sometimes also called a set of *director spaces* [103] or a set of *transversal spaces* [13]. It is known from [34] that for any Desarguesian $(n-1)$-spread of $\mathrm{PG}(rn-1, q)$ there exists a unique indicator set in $\mathrm{PG}(rn-1, q^n)$.

Definition 6.2.1. A $(k-1)$-*subspread* of an $(n-1)$-spread $\mathcal{S}$ of $\mathrm{PG}(rn-1, q)$, $k|n$, is a $(k-1)$-spread of $\mathrm{PG}(rn-1, q)$ that induces a $(k-1)$-spread in each element of $\mathcal{S}$.

We can construct a Desarguesian $(k-1)$-subspread of the Desarguesian spread $\mathcal{D}$ as follows. For every divisor $k|n$, consider the $(rn-1)$-dimensional subgeometry $\Lambda_k := \mathrm{Fix}(\sigma^k) = \mathrm{PG}(rn-1, q^k)$ of Λ^*. Obviously, Λ is contained in Λ_k. Consider the $(\frac{rn}{k}-1)$-dimensional subgeometry $\Pi = \langle \nu, \nu^{\sigma^k}, \ldots, \nu^{\sigma^{n-k}} \rangle \cap \Lambda_k$, this space is disjoint from Λ. One can see that the set $\{\Pi, \Pi^{\sigma}, \ldots, \Pi^{\sigma^{k-1}}\}$ is the indicator set of a $(k-1)$-spread $\mathcal{D}_k$ in Λ. By construction, $\mathcal{D}_k$ is Desarguesian.

Consider a spread element $E \in \mathcal{D}$ and its $\mathbb{F}_{q^n}$-extension E^* in Λ^*; there exists a unique point $P \in \nu$ such that $E^* = \langle P, P^{\sigma}, \ldots, P^{\sigma^{n-1}} \rangle$. Consider the $(\frac{n}{k}-1)$-dimensional subgeometry

$$\pi = \langle P, P^{\sigma^k}, \ldots, P^{\sigma^{n-k}} \rangle \cap \Lambda_k$$

in E^*; this is a subspace of Π. The set $\{\pi, \pi^{\sigma}, \ldots, \pi^{\sigma^{k-1}}\}$ is the indicator set of a $(k-1)$-spread of E and each of these $(k-1)$-spaces is a spread element of $\mathcal{D}_k$. Hence, the spread $\mathcal{D}_k$ induces a $(k-1)$-spread in each $(n-1)$-space of $\mathcal{D}$. It follows that $\mathcal{D}_k$ is a Desarguesian subspread of $\mathcal{D}$.

In [11, Theorem 2.4], the authors proved that there is a unique Desarguesian 1-subspread of a Desarguesian 3-spread in $\mathrm{PG}(7,q)$. This is true in general, we will prove the following corollary in Subsection 6.3.4.

Corollary 6.3.9. *A Desarguesian $(n-1)$-spread of $\mathrm{PG}(rn-1,q)$ has a unique Desarguesian $(k-1)$-subspread for each $k|n$.*

Remark. The Desarguesian subspreads can also be obtained by the connection with field reduction. Consider a field reduction map $\mathcal{F}$ from subspaces of $\mathrm{PG}(r-1,q^n)$ to subspaces of $\mathrm{PG}(rn-1,q)$:

$$\mathcal{F} : \mathrm{PG}(r-1,q^n) \to \mathrm{PG}(rn-1,q).$$

For a divisor $k|n$, this map can be written as the composition of two other field reduction maps $\mathcal{F} = \mathcal{F}_2 \circ \mathcal{F}_1$:

$$\begin{aligned} &\mathrm{PG}(r-1,q^n) \underset{\mathcal{F}}{\longrightarrow} \mathrm{PG}(rn-1,q) \\ = \ &\mathrm{PG}(r-1,q^n) \underset{\mathcal{F}_1}{\longrightarrow} \mathrm{PG}(\frac{rn}{k}-1,q^k) \underset{\mathcal{F}_2}{\longrightarrow} \mathrm{PG}(rn-1,q). \end{aligned}$$

If $\mathcal{D}$ is the Desarguesian $(n-1)$-spread in $\mathrm{PG}(rn-1,q)$ obtained by applying the field reduction map $\mathcal{F}$ to the points of $\mathrm{PG}(r-1,q^n)$, then its subspread $\mathcal{D}_k$ is the Desarguesian $(k-1)$-spread in $\mathrm{PG}(rn-1,q)$ obtained by applying the field reduction map $\mathcal{F}_2$ to the points of $\mathrm{PG}(\frac{rn}{k}-1,q^k)$.

6.2.2 The André/Bruck-Bose representation

Consider the ABB-representation of $\mathrm{PG}(2,q^n)$ in $\mathrm{PG}(2n,q)$ with respect to a line l_∞. Let $\mathcal{D}$ be the Desarguesian $(n-1)$-spread in the hyperplane H_∞ in $\mathrm{PG}(2n,q)$, corresponding to the points of the line l_∞. We will consider a specific way of giving coordinates to $\mathrm{PG}(2,q^n)$ and $\mathrm{PG}(2n,q)$, and using this we will define an explicit map $\phi : \mathrm{PG}(2,q^n) \to \mathrm{PG}(2n,q)$ mapping each point to its corresponding element in the ABB-representation.

Recall that a point P of $\mathrm{PG}(2,q^n)$ defined by a vector $(a,b,c) \in (\mathbb{F}_{q^n})^3$ is denoted by $(a,b,c)_{\mathbb{F}_{q^n}}$. We fix a line at infinity of $\mathrm{PG}(2,q^n)$, say l_∞, such that

$$l_\infty = \{(a,b,0)_{\mathbb{F}_{q^n}} \mid a,b \in \mathbb{F}_{q^n}, (a,b) \neq (0,0)\}.$$

The affine points are the points of $\mathrm{PG}(2,q^n)\backslash l_\infty$ and clearly every affine point can be written as $(a,b,1)_{\mathbb{F}_{q^n}}$, $a,b \in \mathbb{F}_{q^n}$.

On the other hand, each point of $\mathrm{PG}(2n,q)$ can be denoted by $(a,b,c)_{\mathbb{F}_q}$, $a,b \in \mathbb{F}_{q^n}, c \in \mathbb{F}_q$. We consider the hyperplane H_∞ of $\mathrm{PG}(2n,q)$ with the following coordinates

$$H_\infty = \{(a,b,0)_{\mathbb{F}_q} \mid a,b \in \mathbb{F}_{q^n}, (a,b) \neq (0,0)\}.$$

Furthermore, H_∞ contains the Desarguesian $(n-1)$-spread $\mathcal{D}$ defined by

$$\mathcal{D} = \left\{ \{(ax,bx,0)_{\mathbb{F}_q} \mid x \in \mathbb{F}_{q^n}^*\} \,\middle|\, a,b \in \mathbb{F}_{q^n}, (a,b) \neq (0,0) \right\}.$$

Note that for all $\rho \in \mathbb{F}_{q^n}^*$, the tuples (a,b) and $(\rho a, \rho b)$ in $\mathbb{F}_{q^n}^2$ give rise to the same spread element of $\mathcal{D}$.

It is clear that the following map ϕ, for $a,b \in \mathbb{F}_{q^n}, (a,b) \neq (0,0)$, corresponds to the ABB-representation.

$$\begin{aligned}
\phi : \mathrm{PG}(2,q^n) &\to \mathrm{PG}(2n,q) : \\
(a,b,0)_{\mathbb{F}_{q^n}} &\mapsto \{(ax,bx,0)_{\mathbb{F}_q} \mid x \in \mathbb{F}_{q^n}^*\} \\
(a,b,1)_{\mathbb{F}_{q^n}} &\mapsto (a,b,1)_{\mathbb{F}_q}.
\end{aligned}$$

The map ϕ will also be called the *ABB-map*.

6.2.3 Choosing the appropriate coordinates

Now consider the Desarguesian $(n-1)$-spread $\mathcal{D}$ in $H_\infty = \mathrm{PG}(2n-1,q)$ constructed in the previous section and the embedding of H_∞ as a hyperplane in

$\Sigma = \mathrm{PG}(2n, q)$.

We wish to consider an embedding of Σ as a subgeometry in $\Sigma^* = \mathrm{PG}(2n, q^n)$, such that the induced embedding of $H_\infty \subset \Sigma$ in $H^*_\infty \subset \Sigma^*$ provides us with a convenient description of the indicator set of $\mathcal{D}$ consisting of lines.

We can denote points of $\Sigma^* = \mathrm{PG}(2n, q^n)$ by

$$(a_0, a_1, \ldots, a_{n-1}; b_0, b_1, \ldots, b_{n-1}; c)_{\mathbb{F}_{q^n}}, \text{ for } a_i, b_i, c \in \mathbb{F}_{q^n}.$$

Define the hyperplane $H^*_\infty = \mathrm{PG}(2n-1, q^n)$ to consist of the points of Σ^* for which $c = 0$.

We define a collineation σ of Σ^* by

$$\begin{aligned}\sigma : &\mathrm{PG}(2n, q^n) \to \mathrm{PG}(2n, q^n) : \\ &(a_0, a_1, \ldots, a_{n-1}; b_0, b_1, \ldots, b_{n-1}; c)_{\mathbb{F}_{q^n}} \\ &\qquad \mapsto (a^q_{n-1}, a^q_0, \ldots, a^q_{n-2}; b^q_{n-1}, b^q_0, \ldots, b^q_{n-2}; c^q)_{\mathbb{F}_{q^n}}.\end{aligned}$$

The corresponding map on the vector defining a point of $\mathrm{PG}(2n, q^n)$ will also be denoted by σ. The points of Σ^* fixed by σ form a subgeometry isomorphic to $\mathrm{PG}(2n, q)$; this subgeometry is the following:

$$\left\{(a, a^q, \ldots, a^{q^{n-1}}; b, b^q, \ldots, b^{q^{n-1}}; c)_{\mathbb{F}_{q^n}} \,\middle|\, (a, b, c) \in (\mathbb{F}_{q^n} \times \mathbb{F}_{q^n} \times \mathbb{F}_q)^* \right\}.$$

Hence, we can see the embedding of $\Sigma = \mathrm{PG}(2n, q)$ in $\Sigma^* = \mathrm{PG}(2n, q^n)$ via the following map ι, for $a, b \in \mathbb{F}_{q^n}$, $c \in \mathbb{F}_q$.

$$\begin{aligned}\iota : &\mathrm{PG}(2n, q) \to \mathrm{PG}(2n, q^n) : \\ &(a, b, c)_{\mathbb{F}_q} \mapsto (a, a^q, \ldots, a^{q^{n-1}}; b, b^q, \ldots, b^{q^{n-1}}; c)_{\mathbb{F}_{q^n}}.\end{aligned}$$

Clearly,

$$\iota(H_\infty) = \left\{(a, a^q, \ldots, a^{q^{n-1}}; b, b^q, \ldots, b^{q^{n-1}}; 0)_{\mathbb{F}_{q^n}} \,\middle|\, (a, b) \in (\mathbb{F}_{q^n} \times \mathbb{F}_{q^n})^* \right\}$$

forms a $(2n-1)$-dimensional $\mathbb{F}_q$-subgeometry of H^*_∞.

Let us now consider the line ν in H^*_∞, disjoint from $\iota(H_\infty)$, defined as

$$\nu = \left\langle (1, 0, \ldots, 0; 0, 0, \ldots, 0; 0)_{\mathbb{F}_{q^n}}, (0, 0, \ldots, 0; 1, 0, \ldots, 0; 0)_{\mathbb{F}_{q^n}} \right\rangle.$$

Then the set $\{\nu, \nu^\sigma, \ldots, \nu^{\sigma^{n-1}}\}$ is an indicator set defining a Desarguesian spread of $\iota(H_\infty)$ consisting of the $(n-1)$-spaces

$$\left\{(ax,(ax)^q,\dots,(ax)^{q^{n-1}};bx,(bx)^q,\dots,(bx)^{q^{n-1}};0)_{\mathbb{F}_{q^n}}\,\middle|\, x\in\mathbb{F}_{q^n}^*\right\},$$

for $a,b\in\mathbb{F}_{q^n}$, $(a,b)\neq(0,0)$. It is easy to see that this is precisely $\iota(\mathcal{D})$. By abuse of notation, from now on, we will denote $\iota(H_\infty)$ and $\iota(\mathcal{D})$ again by H_∞ and $\mathcal{D}$.

One can check that the $(k-1)$-subspread $\mathcal{D}_k$ of $\mathcal{D}$ in its original setting of H_∞ (see Subsection 6.2.1) corresponds to the set

$$\mathcal{D}_k=\left\{\{(ax,bx,0)_{\mathbb{F}_q}\mid x\in\mathbb{F}_{q^k}^*\}\,\middle|\, a,b\in\mathbb{F}_{q^n},(a,b)\neq(0,0)\right\}.$$

This implies that $\iota(\mathcal{D}_k)$, which we also denote by $\mathcal{D}_k$, corresponds to the set of $(k-1)$-spaces of the form

$$\left\{(ax,(ax)^q,\dots,(ax)^{q^{n-1}};bx,(bx)^q,\dots,(bx)^{q^{n-1}};0)_{\mathbb{F}_{q^n}}\,\middle|\, x\in\mathbb{F}_{q^k}^*\right\},$$

for $a,b\in\mathbb{F}_{q^n}$, $(a,b)\neq(0,0)$.

6.2.4 The induced action of the stabiliser of l_∞

We use the notations introduced in the previous subsections. Recall that points in $\mathrm{PG}(2,q^n)$ have coordinates of the form $(a,b,c)_{\mathbb{F}_{q^n}}$ and that the line l_∞ has equation $c=0$. We will now consider how an element of the stabiliser of l_∞ in $\mathrm{PGL}(3,q^n)$, say G, induces an action on $\Sigma=\mathrm{PG}(2n,q)$ and we describe its extension to an element of $\mathrm{P\Gamma L}(2n+1,q^n)$ acting on $\Sigma^*=\mathrm{PG}(2n,q^n)$. This will be of use later since it will allow us to study the representation of a representative of orbits of sublines or subplanes under G.

Every element χ_0 of the stabiliser G of l_∞ corresponds to a matrix of the form

$$X=\begin{pmatrix}x_{11}&x_{12}&0\\x_{21}&x_{22}&0\\x_{31}&x_{32}&1\end{pmatrix},$$

with $x_{ij}\in\mathbb{F}_{q^n}$ and $x_{11}x_{22}-x_{12}x_{21}\neq0$, where we let the matrix act on row vectors from the right, hence for $(a,b,c)\in\mathbb{F}_{q^n}^3\setminus\{(0,0,0)\}$:

$$\begin{aligned}\chi_0:\mathrm{PG}(2,q^n)&\to\mathrm{PG}(2,q^n):\\ (a,b,c)_{\mathbb{F}_{q^n}}&\mapsto((a,b,c)X)_{\mathbb{F}_{q^n}}.\end{aligned}$$

The map χ_0 induces a natural action χ on the points $(a,b,c)_{\mathbb{F}_q}$ of Σ, $a,b \in \mathbb{F}_{q^n}$, $c \in \mathbb{F}_q$, in the following way:

$$\begin{aligned} \chi : \mathrm{PG}(2n,q) &\to \mathrm{PG}(2n,q) : \\ (a,b,c)_{\mathbb{F}_q} &\mapsto ((a,b,c)X)_{\mathbb{F}_q}. \end{aligned}$$

Recall that we chose coordinates such that the points of $\Sigma^* = \mathrm{PG}(2n,q^n)$ were denoted by $(a_0, a_1, \ldots, a_{n-1}; b_0, b_1, \ldots, b_{n-1}; c)_{\mathbb{F}_{q^n}}$, for $a_i, b_i, c \in \mathbb{F}_{q^n}$. For shorthand, we will write these now as $((a_i);(b_i);c)_{\mathbb{F}_{q^n}}$, where the index i is assumed to range from 0 to $n-1$.

We define the *extension* of χ, denoted by χ^*, to be the collineation of Σ^* which acts as follows on a generic point:

$$\begin{aligned} \chi^* : \mathrm{PG}(2n,q^n) &\to \mathrm{PG}(2n,q^n) : \\ &((a_i);(b_i);c)_{\mathbb{F}_{q^n}} \\ &\quad\mapsto ((x_{11}^{q^i} a_i + x_{21}^{q^i} b_i + x_{31}^{q^i} c); (x_{12}^{q^i} a_i + x_{22}^{q^i} b_i + x_{32}^{q^i} c); c)_{\mathbb{F}_{q^n}}. \end{aligned}$$

Lemma 6.2.2. *Using the notations from above, we have the following.*

(i) The map χ satisfies the following properties for every divisor k of n:

- *χ stabilises the Desarguesian spread $\mathcal{D}_k$ of H_∞,*
- *χ stabilises the set of k-spaces of $\mathrm{PG}(2n,q)$ meeting H_∞ in an element of $\mathcal{D}_k$.*

Moreover, let $\phi : \mathrm{PG}(2,q^n) \to \mathrm{PG}(2n,q)$ be the ABB-map as defined in Subsection 6.2.2. For every point P of $\mathrm{PG}(2,q^n)$, we have that

$$\phi\chi_0(P) = \chi\phi(P).$$

(ii) The extension χ^* of χ satisfies the following properties:

- *$\chi^*(P^\sigma) = (\chi^*(P))^\sigma$ for all $P \in \Sigma^*$, i.e. χ^* maps conjugate points to conjugate points,*
- *χ^* stabilises the indicator set of $\mathcal{D}_k$ for each divisor k of n.*

Moreover, let $\iota : \mathrm{PG}(2n,q) \to \mathrm{PG}(2n,q^n)$ be the embedding as defined in Subsection 6.2.3. For every point P of $\mathrm{PG}(2n,q)$, we have that

$$\iota\chi(P) = \chi^*\iota(P).$$

Proof. The proof follows from straightforward, but tedious calculations.

(i) Notice that the image of an element $\{(ax, bx, 0)_{\mathbb{F}_q} \mid x \in \mathbb{F}^*_{q^k}\}$ of $\mathcal{D}_k$ under χ is given by $\{((ax_{11} + bx_{21})x, (ax_{12} + bx_{22})x, 0)_{\mathbb{F}_q} \mid x \in \mathbb{F}^*_{q^k}\} \in \mathcal{D}_k$. From this the two properties of χ follow.

For the second statement of (i), we consider a point $P \in \mathrm{PG}(2, q^n)$ with coordinates $(a, b, c)_{\mathbb{F}_{q^n}}$. We have that $\chi_0(P) = ((a, b, c)X)_{\mathbb{F}_{q^n}} = (ax_{11} + bx_{21} + cx_{31}, ax_{12} + bx_{22} + cx_{32}, c)_{\mathbb{F}_{q^n}}$. So, when $c \neq 0$, we obtain $\phi\chi_0(P) = (ax_{11} + bx_{21} + cx_{31}, ax_{12} + bx_{22} + cx_{32}, c)_{\mathbb{F}_q}$ and we get $\chi\phi(P) = \chi(a, b, c)_{\mathbb{F}_q} = (ax_{11} + bx_{21} + cx_{31}, ax_{12} + bx_{22} + cx_{32}, c)_{\mathbb{F}_q}$. When $c = 0$, we have that both $\phi\chi_0(P)$ and $\chi\phi(P)$ correspond to the $(n-1)$-space $\{((ax_{11}+bx_{21}+cx_{31})x, (ax_{12}+bx_{22}+cx_{32})x, 0)_{\mathbb{F}_q} \mid x \in \mathbb{F}^*_{q^n}\}$.

(ii) Consider a point P with coordinates $((a_i); (b_i); c)_{\mathbb{F}_{q^n}}$ in Σ^*. If we put $a_{-1} = a_{n-1}$ and $b_{-1} = b_{n-1}$, then for $i \in \{0, \ldots, n-1\}$, the $(i+1)$-th coordinate of $\chi^*(P)$ is $x_{11}^{q^i} a_i + x_{21}^{q^i} b_i + x_{31}^{q^i} c$, which implies that the $(i+1)$-th coordinate of $(\chi^* P)^\sigma$ is $x_{11}^{q^{i+1}} a_{i-1}^q + x_{21}^{q^{i+1}} b_{i-1}^q + x_{31}^{q^{i+1}} c$. This is clearly equal to the $(i+1)$-th coordinate of $\chi^*(P^\sigma)$. The same argument holds for the other coordinate positions. This also implies that χ^* stabilises the indicator sets.

For the last statement, consider a point P in $\mathrm{PG}(2n, q)$ with coordinates $(a, b, c)_{\mathbb{F}_{q^n}}$, then $\iota(P) = ((a^{q^i}); (b^{q^i}); c)_{\mathbb{F}_{q^n}}$ and

$$\begin{aligned}
\chi^*\iota(P) &= \Big((x_{11}^{q^i} a^{q^i} + x_{21}^{q^i} b^{q^i} + x_{31}^{q^i} c); (x_{12}^{q^i} a^{q^i} + x_{22}^{q^i} b^{q^i} + x_{32}^{q^i} c); c\Big)_{\mathbb{F}_{q^n}} \\
&= \Big(((x_{11}a + x_{21}b + x_{31}c)^{q^i}); ((x_{12}a + x_{22}b + x_{32}c)^{q^i}); c\Big)_{\mathbb{F}_{q^n}} \\
&= \iota\chi(P).
\end{aligned}$$

□

6.3 The ABB-representation of sublines

In this section, we determine the ABB-representation of $\mathbb{F}_{q^k}$-sublines of $\mathrm{PG}(2, q^n)$. We need to make a distinction between sublines that are tangent to, external to, or contained in l_∞.

We start with the first two cases, which will be handled by use of coordinates. We end with the case of $\mathbb{F}_{q^k}$-sublines contained in l_∞, for which no coordinates are needed.

6.3.1 Equivalent sublines under the stabiliser of the line at infinity

In the case of tangent and external sublines, we will consider sublines with specific coordinates. As before, we consider the plane $\mathrm{PG}(2,q^n) = \{(a,b,c)_{\mathbb{F}_{q^n}} \mid a,b,c \in \mathbb{F}_{q^n}\}$ and the line l_∞ with equation $c = 0$. Let l be the line with equation $a = 0$. We will show in Lemma 6.3.2 that any subline tangent or external to l_∞ is equivalent to a particular subline contained in l. Note that an $\mathbb{F}_{q^k}$-subline of $\mathrm{PG}(2,q^n)$, $k|n$, is uniquely determined by two distinct vectors of $\mathbb{F}_{q^n}^3$, or, equivalently, by three collinear projective points.

Given $\omega \in \mathbb{F}_{q^n}$ and a divisor $k|n$, denote by $l_{\omega,k}$ the $\mathbb{F}_{q^k}$-subline uniquely determined by the vectors $(0,1,\omega)$, $(0,0,1)$, i.e. $l_{\omega,k}$ consists of the points corresponding to the vectors $\{(0,1,\omega) + t(0,0,1) \mid t \in \mathbb{F}_{q^k}\} \cup \{(0,0,1)\}$.

Alternatively, the $\mathbb{F}_{q^k}$-subline $l_{\omega,k}$ is uniquely determined by the points $(0,1,\omega)_{\mathbb{F}_{q^n}}$, $(0,0,1)_{\mathbb{F}_{q^n}}$, $(0,1,\omega+1)_{\mathbb{F}_{q^n}}$, i.e.

$$l_{\omega,k} = \left\{(0,1,\omega+t)_{\mathbb{F}_{q^n}} \mid t \in \mathbb{F}_{q^k}\right\} \cup \left\{(0,0,1)_{\mathbb{F}_{q^n}}\right\}.$$

Consider a set of sublines; we refer to a subline being the *smallest* of the set, if it has the smallest number of points, or equivalently, if it is the subline over the smallest field.

Lemma 6.3.1. *Given $\omega \in \mathbb{F}_{q^n}$, if $\mathbb{F}_{q^k} = \mathbb{F}_q(\omega)$ is the smallest subfield of $\mathbb{F}_{q^n}$ containing ω (and $\mathbb{F}_q$), then the subline $l_{\omega,k}$ is the smallest subline tangent to l_∞ and containing $l_{\omega,1}$.*

Proof. For all $k|n$, there is a unique $\mathbb{F}_{q^k}$-subline containing the points $(0,1,\omega)_{\mathbb{F}_{q^n}}$, $(0,0,1)_{\mathbb{F}_{q^n}}$ and $(0,1,\omega+1)_{\mathbb{F}_{q^n}}$. Hence, the $\mathbb{F}_{q^k}$-subline containing the points of $l_{\omega,1}$ is of the form $l_{\omega,k}$. The subline $l_{\omega,k}$ contains the point $(0,1,0)_{\mathbb{F}_{q^n}}$ of l_∞ if and only if $-\omega \in \mathbb{F}_{q^k}$, and the statement follows. □

Lemma 6.3.2. *Every $\mathbb{F}_{q^k}$-subline external to l_∞ is equivalent to $l_{\omega,k}$ for some $\omega \notin \mathbb{F}_{q^k}$, under the action of the stabiliser of l_∞ in $\mathrm{PGL}(3,q^n)$.*

Every $\mathbb{F}_{q^k}$-subline tangent to l_∞ is equivalent to $l_{0,k}$, under the action of the stabiliser of l_∞ in $\mathrm{P\Gamma L}(3,q^n)$.

Proof. Consider an $\mathbb{F}_{q^k}$-subline m not contained in l_∞. Then m is determined by two vectors in $\mathbb{F}_{q^n}^3$, which we may take to be $(\alpha,\beta,1)$ and (γ,δ,ω). Consider the matrix

$$\begin{pmatrix} u & v & 0 \\ \gamma - \omega\alpha & \delta - \omega\beta & 0 \\ \alpha & \beta & 1 \end{pmatrix},$$

where u, v are chosen so that this matrix is invertible. This matrix, acting on row vectors from the right, maps $(0,0,1)$ to $(\alpha,\beta,1)$ and $(0,1,\omega)$ to (γ,δ,ω), and hence defines a collineation ψ in the stabiliser of l_∞ which maps $l_{\omega,k}$ to m.

As ψ stabilises l_∞, if m is tangent to l_∞, then $l_{\omega,k}$ is as well, and vice versa. The subline $l_{\omega,k}$ meets l_∞ if and only if $\omega \in \mathbb{F}_{q^k}$ by Lemma 6.3.1, in which case $l_{\omega,k} = l_{0,k}$, proving the claim. □

6.3.2 Sublines tangent to l_∞

Theorem 6.3.3.

(*i*) *The affine points of an $\mathbb{F}_{q^k}$-subline m in $\mathrm{PG}(2,q^n)$ tangent to l_∞ correspond to the points of a k-dimensional affine space π in the ABB-representation, such that $\overline{\pi} \cap H_\infty$ is an element of $\mathcal{D}_k$.*

(*ii*) *Conversely, let π be a k-dimensional affine space of Σ such that $\overline{\pi}$ intersects H_∞ in a spread element of $\mathcal{D}_k$. Then the points of π correspond to the affine points of an $\mathbb{F}_{q^k}$-subline m tangent to l_∞.*

Proof. (*i*) From Lemma 6.3.2, the tangent $\mathbb{F}_{q^k}$-subline m is equivalent to $l_{0,k}$ under an element of the stabiliser G of l_∞ in $\mathrm{PGL}(3,q^n)$, say $\chi_0(l_{0,k}) = m$ for $\chi_0 \in G$. Note that $l_{0,k}$ consists of the following points:

$$l_{0,k} = \left\{(0,b,1)_{\mathbb{F}_{q^n}} \mid b \in \mathbb{F}_{q^k}\right\} \cup \left\{(0,1,0)_{\mathbb{F}_{q^n}}\right\}.$$

In the ABB-representation, the point $(0,1,0)_{\mathbb{F}_{q^n}} \in l_{0,k}$ corresponds to the spread element $\{(0,x,0)_{\mathbb{F}_q} \mid x \in \mathbb{F}^*_{q^n}\} \in \mathcal{D}$. By definition of the ABB-map ϕ, the affine points of $l_{0,k}$ in the ABB-representation form an affine k-space π defined as follows:

$$\pi = \left\{(0,b,1)_{\mathbb{F}_q} \,\middle|\, b \in \mathbb{F}_{q^k}\right\}.$$

Using the embedding ι of π in $\mathrm{PG}(2n,q^n)$, we obtain the set of points

$$\left\{(0,\ldots,0;b,b^q,\ldots,b^{q^{n-1}};1)_{\mathbb{F}_{q^n}} \,\middle|\, b \in \mathbb{F}_{q^k}\right\}.$$

Since $b^{q^{k+j}} = b^{q^j}$ for all $b \in \mathbb{F}_{q^k}$, it is clear that the projective completion $\overline{\pi}$ of π intersects H_∞ in an element of $\mathcal{D}_k$, more specifically in $\langle Q, Q^\sigma, \ldots, Q^{\sigma^{k-1}} \rangle \cap H_\infty$, where

$$Q = (v + v^{\sigma^k} + v^{\sigma^{2k}} + \cdots + v^{\sigma^{n-k}})_{\mathbb{F}_{q^n}},$$
$$v = (0,0,\ldots,0;1,0,\ldots,0;0), \quad \text{thus } (v^{\sigma^i})_{\mathbb{F}_{q^n}} \in \nu^{\sigma^i}.$$

We know that $\phi(m) = \phi(\chi_0(l_{0,k}))$, and that the affine points of $\phi(l_{0,k})$ form the point set of an affine space π such that its projective completion $\overline{\pi}$ intersects H_∞ in an element of $\mathcal{D}_k$. By Lemma 6.2.2, we know that $\phi\chi_0 = \chi\phi$ and that χ stabilises the set of k-spaces meeting H_∞ in an element of $\mathcal{D}_k$. This implies that also the affine points of $\phi(m)$ form the point set of an affine space such that its projective completion intersects H_∞ in an element of $\mathcal{D}_k$.

(ii) To prove that the converse also holds, it is sufficient to use a counting argument.

The number of affine points of $\mathrm{PG}(2,q^n) \setminus l_\infty$ is equal to the number of affine points of $\mathrm{PG}(2n,q) \setminus H_\infty$. Hence, the number of choices for any two distinct affine points is the same in both cases.

For any k, two affine points A_1, A_2 in $\mathrm{PG}(2,q^n)$ define a unique $\mathbb{F}_{q^k}$-subline tangent to l_∞. Because of (i), this subline corresponds to a k-dimensional space intersecting H_∞ in an element of $\mathcal{D}_k$.

The two affine points A_1, A_2 correspond in the ABB-representation to two affine points B_1, B_2 in $\mathrm{PG}(2n,q)$. There is a unique element of $\mathcal{D}_k$ containing the point $\langle B_1, B_2 \rangle \cap H_\infty$, hence there is a unique k-dimensional space containing B_1, B_2 and intersecting H_∞ in an element of $\mathcal{D}_k$.

From this we see that the number of $\mathbb{F}_{q^k}$-sublines tangent to l_∞ is equal to the number of k-spaces intersecting H_∞ in an element of $\mathcal{D}_k$. Hence, the statement follows. □

6.3.3 Sublines disjoint from l_∞

Recall that a *normal rational curve* in $\mathrm{PG}(l,q)$, $2 \le l \le q-2$, is a $(q+1)$-arc PGL-equivalent to the $(q+1)$-arc

$$\{(0,\ldots,0,1)_{\mathbb{F}_q}\} \cup \{(1,t,t^2,t^3,\ldots,t^l)_{\mathbb{F}_q} \mid t \in \mathbb{F}_q\}.$$

Note that when $l = 1$, the corresponding point set would consist of all points of $\mathrm{PG}(1, q)$.

We say that a set $\mathcal{C}$ in $\mathrm{PG}(N, q)$ is a normal rational curve of *degree* (or *order*) k if and only if it is a normal rational curve in a k-dimensional subspace of $\mathrm{PG}(N, q)$, or equivalently, if and only if there exist linearly independent vectors $e_0, e_1, \dots, e_k$ in $V(N+1, q)$ such that

$$\mathcal{C} = \left\{ (s^k e_0 + s^{k-1} r e_1 + \cdots + s r^{k-1} e_{k-1} + r^k e_k)_{\mathbb{F}_q} \,\middle|\, s, r \in \mathbb{F}_q \right\}.$$

Note that, by abuse of phrasing, a normal rational curve of degree 1 corresponds to a projective line.

Consider the following theorem of [102] introduced in Chapter 5.

Theorem 5.4.8. *Consider a $(k+3)$-arc $\mathcal{A}$ in $\mathrm{PG}(k, q)$, $k \leq q-2$, then there exists a unique normal rational curve containing all points of $\mathcal{A}$.*

To illustrate the previous result, consider $k+3$ points in general position defined by vectors $u_0, \dots, u_k, a, b$, where $a = \sum_{i=0}^{k} a_i u_i$ and $b = \sum_{i=0}^{k} b_i u_i$. Note that since the points form an arc, we have that $\forall i = 0, \dots, k : a_i \neq 0, b_i \neq 0$. The unique normal rational curve through these points may be parametrised as

$$\left\{ \left(\sum_{i=0}^{k} \prod_{j=0, j \neq i}^{k} (a_j^{-1} s - b_j^{-1} r) u_i \right)_{\mathbb{F}_q} \,\middle|\, s, r \in \mathbb{F}_q \right\}.$$

The point $(u_j)_{\mathbb{F}_q}$ corresponds to $(s, r) = (b_j^{-1}, a_j^{-1})$, the point $(a)_{\mathbb{F}_q}$ to $(1, 0)$ and the point $(b)_{\mathbb{F}_q}$ to $(0, 1)$.

Consider a normal rational curve $\mathcal{C}$ of $\mathrm{PG}(k, q)$, $2 \leq k \leq q-2$, and the embedding of $\mathrm{PG}(k, q)$ as a subgeometry of $\mathrm{PG}(k, q^n)$. Because of the previous result, a normal rational curve $\mathcal{C}^*$ in $\mathrm{PG}(k, q^n)$ containing the points of $\mathcal{C}$ is unique and we call this the *$\mathbb{F}_{q^n}$-extension* $\mathcal{C}^*$ of $\mathcal{C}$.

As we have seen, a normal rational curve $\mathcal{C}$ is the point set of an algebraic variety in $\mathrm{PG}(k, q)$ defined by the parameter $t \in \mathbb{F}_q$. The extension $\mathcal{C}^*$ of $\mathcal{C}$ can be obtained as the point set from the variety which is obtained by allowing the parameter t to range over $\mathbb{F}_{q^n}$.

Before we can consider the characterisation of disjoint sublines, we first need the following lemma.

Lemma 6.3.4. *Consider three non-collinear affine points A, B, C of* $\mathrm{PG}(2n, q)$ *contained in an n-space intersecting H_∞ in an element E of $\mathcal{D}$, and consider the line $l := \langle A, B, C\rangle \cap H_\infty$. Consider the set S of values i for which some element E_i of $\mathcal{D}_i$ contains the line l. For every $m \neq \min(S)$ in S, the points A, B, C and the m conjugate points generating the element E_m are not in general position.*

Proof. Let k be the minimum of S. It follows from the definition of $\mathcal{D}_i$ that S is the set of all values tk where $tk|n$.

Let $m = tk$ with $t > 1$. Suppose $L(Q) = \langle Q, Q^\sigma, \ldots, Q^{\sigma^{m-1}}\rangle$ generates E_m. The $(k-1)$-space E_k is part of the $(k-1)$-spread $\mathcal{S} = \mathcal{D}_k \cap E_m$. Consider the $(t-1)$-space $\Pi = \langle Q, Q^{\sigma^k}, \ldots, Q^{\sigma^{(t-1)k}}\rangle$. The set $\{\Pi, \Pi^\sigma, \ldots, \Pi^{\sigma^{k-1}}\}$ is the indicator set of the spread $\mathcal{S}$ of E_m. Consider the space $L(P) = \langle P, P^\sigma, \ldots, P^{\sigma^{k-1}}\rangle$ spanned by conjugate points generating E_k, where the point P^{σ^i} is contained in Π^{σ^i} for all i.

We prove that the $m + 3$ points of the set $\{Q, Q^\sigma, \ldots, Q^{\sigma^{m-1}}, A, B, C\}$ (that is contained in an m-dimensional space) are not in general position, by constructing $u + 2$ points that are contained in a u-space, where $u \leq m - 1$.

The space $\langle L(P), A, B, C\rangle$ has dimension k, and every space Π^{σ^i} intersects it in exactly one point, namely the point P^{σ^i}. Consider the $(k-1)t + 3$ points contained in $\{Q, Q^\sigma, \ldots, Q^{\sigma^{m-1}}, A, B, C\}$, but not contained in the space $\Pi^{\sigma^{k-1}} = \langle Q^{\sigma^{k-1}}, Q^{\sigma^{2k-1}}, \ldots, Q^{\sigma^{tk-1}}\rangle$. These points are contained in the space spanned by $\langle L(P), A, B, C, \Pi, \Pi^\sigma, \ldots, \Pi^{\sigma^{k-2}}\rangle$ which has dimension at most $k + (k-1)(t-1) = (k-1)t + 1$. Since $t > 1$, we have that $u = (k-1)t + 1 \leq m - 1 = kt - 1$ and our claim follows. $\square$

Remark. In fact, for $m = \min(S)$, the points A, B, C and the m conjugate points generating the element E_m are in general position. This follows from the proof of Theorem 6.3.5.

Recall that the *norm map* of $\mathbb{F}_{q^k}$ over $\mathbb{F}_q$ is denoted as follows: $N_k(\alpha) = \prod_{i=0}^{k-1} \alpha^{q^i} \in \mathbb{F}_q$.

Note that in [77, Theorem 4.2], it was proven that the ABB-representation of an $\mathbb{F}_q$-subline external to l_∞ satisfies part (i) of the following theorem.

Theorem 6.3.5. *A set of points $\mathcal{C}$ in* $\mathrm{PG}(2n, q)$*, $n \leq q-2$, is the ABB-representation of an $\mathbb{F}_q$-subline m of* $\mathrm{PG}(2, q^n)$ *external to l_∞ if and only if*

(i) *$\mathcal{C}$ is a normal rational curve of degree k contained in a k-space intersecting H_∞ in an element of $\mathcal{D}_k$,* (*ii*) *its extension* $\mathcal{C}^*$ *to* $\mathrm{PG}(2n, q^n)$ *intersects the indicator set* $\{\Pi, \Pi^\sigma, \ldots, \Pi^{\sigma^{k-1}}\}$ *of* $\mathcal{D}_k$ *in* k *conjugate points.*

Moreover, the smallest subline containing m *and tangent to* l_∞ *is an* $\mathbb{F}_{q^k}$*-subline.*

Proof. We include a proof using coordinates for part (*i*), as it will be necessary for proving part (*ii*).

(*i*) Consider the smallest k, such that m is contained in a tangent $\mathbb{F}_{q^k}$-subline. It follows from Theorem 6.3.3 that the $q+1$ points corresponding to m are contained in a k-space intersecting H_∞ in an element of $\mathcal{D}_k$.

By Lemmas 6.3.1 and 6.3.2, we know that m is G-equivalent to the $\mathbb{F}_q$-subline $l_{\omega,1}$, say $m = \chi_0(l_{\omega,1})$, where $\omega \in \mathbb{F}_{q^n}$ such that $\mathbb{F}_q(\omega) = \mathbb{F}_{q^k}$ and χ_0 in the stabiliser G of l_∞ in $\mathrm{PGL}(3, q^n)$.

By definition, $\phi(l_{\omega,1})$, where ϕ is the ABB-map, is a set $\mathcal{C}_\omega$ defined as follows:

$$\mathcal{C}_\omega = \left\{ \left(0, \frac{1}{\omega + t}, 1\right)_{\mathbb{F}_q} \middle| \; t \in \mathbb{F}_q \right\} \cup \left\{ (0,0,1)_{\mathbb{F}_q} \right\}.$$

Now $N_k(\omega + t) = \prod_{i=0}^{k-1}(\omega^{q^i} + t) \in \mathbb{F}_q$ for all $t \in \mathbb{F}_q$, and is never zero, and so

$$\mathcal{C}_\omega = \left\{ \left(0, \prod_{i=1}^{k-1}(\omega^{q^i} + t), \prod_{i=0}^{k-1}(\omega^{q^i} + t)\right)_{\mathbb{F}_q} \middle| \; t \in \mathbb{F}_q \right\} \cup \left\{ (0,0,1)_{\mathbb{F}_q} \right\}.$$

Expanding the products, we find $k+1$ non-zero vectors $v_i \in \mathbb{F}_{q^n} \times \mathbb{F}_{q^n} \times \mathbb{F}_q$ (which depend only on ω) such that

$$\mathcal{C}_\omega = \{(v_0 + tv_1 + \cdots + t^k v_k)_{\mathbb{F}_q} \mid t \in \mathbb{F}_q\} \cup \{(v_k)_{\mathbb{F}_q}\}.$$

The vectors v_i span $0 \times \mathbb{F}_{q^k} \times \mathbb{F}_q$, hence $\mathcal{C}_\omega$ is a normal rational curve of degree k, contained in a projective k-space meeting H_∞ in an element of $\mathcal{D}_k$, as claimed.

(*ii*) Using the embedding ι to embed $\mathcal{C}_\omega$ in $\mathrm{PG}(2n, q^n)$ gives the points

$$\left(0, \ldots, 0; \prod_{i=1}^{k-1}(\omega^{q^i} + t), \ldots, \prod_{i=1}^{k-1}(\omega^{q^{i+n-1}} + t); \prod_{i=0}^{k-1}(\omega^{q^i} + t)\right)_{\mathbb{F}_{q^n}},$$

for $t \in \mathbb{F}_q$. Since $\omega \in \mathbb{F}_{q^k}$, the $(n+j)$-th entry $\prod_{i=1}^{k-1}(\omega^{q^{i+j}} + t)$ is equal to the $(n+j+k)$-th entry $\prod_{i=1}^{k-1}(\omega^{q^{i+j+k}} + t)$ for all $0 \leq j \leq n-1$. The extension

$\mathcal{C}_\omega^*$ of $\mathcal{C}_\omega$ is the normal rational curve obtained by allowing t to range over $\mathbb{F}_{q^n}$. The intersection of $\mathcal{C}_\omega^*$ with H_∞^* can be obtained by finding the elements $t \in \mathbb{F}_{q^n}$ such that $\prod_{i=0}^{k-1}(\omega^{q^i}+t) = 0$. Clearly these are precisely $t = -\omega, -\omega^q, \ldots, -\omega^{q^{k-1}}$. When we consider the value $t = -\omega^{q^j}$, then all coordinates become zero, excluding those in position $(n+j+rk)$ for $r = 0, 1, \ldots, n/k-1$. Hence we get that the points of $\mathcal{C}_\omega^* \cap H_\infty^*$ are precisely the k conjugate points given by

$$Q^{\sigma^j} = (v^{\sigma^j} + v^{\sigma^{j+k}} + v^{\sigma^{j+2k}} + \cdots + v^{\sigma^{j+(n-k)}})_{\mathbb{F}_{q^n}}, j = 0, 1, \ldots, k-1,$$

where $P = (v)_{\mathbb{F}_{q^n}}$ with $v = (0, \ldots, 0; 1, 0, \ldots, 0; 0)$, so $P \in \nu$. If $k = n$, i.e. if ω is not contained in any proper subfield of $\mathbb{F}_{q^n}$, then $\mathcal{C}_\omega^*$ contains n conjugate points, one on each of the lines of the indicator set $\{\nu, \nu^\sigma, \ldots, \nu^{\sigma^{n-1}}\}$. However if $k < n$, each point belongs to one of the spaces of the set $\{\pi, \pi^\sigma, \ldots, \pi^{\sigma^{k-1}}\}$, where $\pi = \langle P, P^{\sigma^k}, \ldots, P^{\sigma^{n-k}}\rangle$. It is clear that $\pi \subset \Pi$, where $\{\Pi, \Pi^\sigma, \ldots, \Pi^{\sigma^{k-1}}\}$ is the indicator set of $\mathcal{D}_k$.

Now $m = \chi_0(l_{\omega,1})$ and $\phi(m) = \phi(\chi_0(l_{\omega,1})) = \chi(\phi(l_{\omega,1}))$ by Lemma 6.2.2. Since χ is a collineation, this implies that $\phi(m)$ is also a normal rational curve of degree k, and since χ stabilises the elements of $\mathcal{D}_k$, this normal rational curve lies in a k-space intersecting H_∞ in an element of $\mathcal{D}_k$. Now, again by Lemma 6.2.2, we have that $\iota\phi(m) = \iota\chi\phi(l_{\omega,1}) = \chi^*\iota\phi(l_{\omega,1})$. Since χ^* stabilises the indicator sets, the unique $\mathbb{F}_{q^n}$-extension of $\iota\phi(m)$ is also a normal rational curve which intersects the indicator set in conjugate points. This proves the first part of the claim.

To prove that the converse also holds, it is sufficient to use a counting argument. As three points on a line of $\mathrm{PG}(2, q^n)$ uniquely determine a subline, the number of choices for three points defining an external $\mathbb{F}_q$-subline of a fixed line is equal to $\binom{q^n}{3} - \frac{q^n(q^n-1)(q-2)}{3\cdot 2} = \frac{q^n(q^n-1)(q^n-q)}{6}$.

Three such affine points in $\mathrm{PG}(2, q^n)$ correspond to three non-collinear affine points in $\mathrm{PG}(2n, q)$ contained in an n-space containing an element of $\mathcal{D}$. By the first part of the proof, we know that these three non-collinear affine points are contained in a normal rational curve $\mathcal{C}$ satisfying (i) and (ii) for some value k. By Lemma 6.3.4, we know that through three non-collinear affine points, there is at most one value of k (namely, $\min(S)$) such that there is a normal rational curve of degree k satisfying (i) and (ii), so the obtained normal rational curve $\mathcal{C}$ is unique. Hence it suffices to note that the number of triples of non-collinear affine points in a fixed n-space is $\frac{q^n(q^n-1)(q^n-q)}{6}$, which equals the number of triples uniquely defining external sublines of a fixed line, completing the proof. □

Remark. In the statement of [13, Theorem 3.5], characterising the external $\mathbb{F}_q$-

sublines, the authors state that $\mathcal{C}^*$ meets each transversal line ν^{σ^j}. This is due to the fact that the authors choose one representative for a subline, equivalent to our choice of $l_{\omega,1}$, where ω is a primitive element of $\mathbb{F}_{q^n}$. With this choice, the unique subline tangent to l_∞ and containing $l_{\omega,1}$ is always the full line and in that case, we have indeed seen in Theorem 6.3.5 that $\mathcal{C}^*$ meets each transversal line ν^{σ^j}. But as follows from Lemma 6.3.1, unless n is prime, there are many choices for a subline $l_{\omega,1}$ in $\mathrm{PG}(2,q^n)$ contradicting this statement.

Corollary 6.3.6. *An $\mathbb{F}_{q^k}$-subline m in $\mathrm{PG}(2,q^n)$, $n \leq q-2$, external to l_∞, such that the smallest subline containing m and tangent to l_∞ is an $\mathbb{F}_{q^r}$-subline, corresponds in the ABB-representation to a set M of q^k+1 affine points no three collinear, contained in an r-space intersecting H_∞ in an element of $\mathcal{D}_r$. Every three points of M determine an affine normal rational curve whose points are all contained in M.*

Consider an $\mathbb{F}_{q^k}$-subline m of $\mathrm{PG}(1,q^n)$. The unique $\mathbb{F}_q$-subline defined by any three points of m is completely contained in m. Moreover, this property gives a characterisation of an $\mathbb{F}_{q^k}$-subline, as seen in the following result from Chapter 2.

Theorem 2.2.6. *A set $\mathcal{M}$ of at least 3 points in $\mathrm{PG}(1,q^n)$, $q>2$, such that the $\mathbb{F}_q$-subline determined by any three points of $\mathcal{M}$ is entirely contained in $\mathcal{M}$, defines an $\mathbb{F}_{q^k}$-subline of $\mathrm{PG}(1,q^n)$ for some $k|n$.*

Corollary 6.3.7. *An $\mathbb{F}_{q^k}$-subline m in $\mathrm{PG}(2,q^n)$, n a prime power, $2<n\leq q-2$, external to l_∞, such that the smallest subline containing m and tangent to l_∞ is an $\mathbb{F}_{q^r}$-subline, corresponds in the ABB-representation to a set M of q^k+1 affine points of $\mathrm{PG}(2n,q)$ if and only if*

- *(i) the set M spans an r-space intersecting H_∞ in an element of $\mathcal{D}_r$,*
- *(ii) every three points of M define a normal rational curve $\mathcal{C}$ of degree r in $\mathrm{PG}(2n,q)$ whose points are all contained in M, such that its $\mathbb{F}_{q^n}$-extension $\mathcal{C}^*$ intersects the indicator set $\{\Pi,\Pi^\sigma,\dots,\Pi^{\sigma^{r-1}}\}$ of $\mathcal{D}_r$ in r conjugate points.*

Proof. Let m' be the $\mathbb{F}_{q^r}$-subline containing m and tangent to l_∞. Any three points of m define an $\mathbb{F}_q$-subline m_0 completely contained in m. We claim that, because n is a prime power, any subline containing m_0 is either contained in m or contains m. Hence, the smallest subline containing m_0 and tangent to l_∞ equals m'. This is valid for the following reason: consider $\mathrm{PG}(1,q^n)$, where m_0 and m are the canonical $\mathbb{F}_q$- and $\mathbb{F}_{q^k}$-subline respectively, hence m_0 is contained in m. Take a point $(1,a)_{\mathbb{F}_{q^n}}$ not in m, that is $a\in\mathbb{F}_{q^n}\setminus\mathbb{F}_{q^k}$ and consider r such that

$\mathbb{F}_{q^r} = \mathbb{F}_{q^k}(a)$. The subline containing m and the point $(1,a)_{\mathbb{F}_{q^n}}$ is an $\mathbb{F}_{q^r}$-subline, say m'. The smallest subline containing m_0 and the point $(1,a)_{\mathbb{F}_{q^n}}$ also equals m', since $\mathbb{F}_q(a) = \mathbb{F}_{q^k}(a) = \mathbb{F}_{q^r}$, because n is a prime power and $a \notin \mathbb{F}_{q^k}$.

The statement now follows immediately from Theorem 6.3.5. □

6.3.4 Sublines contained in l_∞

Note that the following theorem provides a characterisation of FR_{q^k}-sublines of the Desarguesian spread $\mathcal{D}$ (as defined in Chapter 2, Section 2.2).

Theorem 6.3.8. *Let $\mathcal{S}$ be a set of q^k+1 elements of the Desarguesian spread $\mathcal{D}$ of $H_\infty = \mathrm{PG}(2n-1,q)$, $q>2$. Then the following statements are equivalent:*

- (i) *$\mathcal{S}$ is the ABB-representation of an $\mathbb{F}_{q^k}$-subline of l_∞,*
- (ii) *for any three elements of $\mathcal{S}$, the unique regulus through them is contained in $\mathcal{S}$,*
- (iii) *there exists a $(2k-1)$-dimensional subspace of H_∞ intersecting each element of $\mathcal{S}$ in a $(k-1)$-dimensional space,*
- (iv) *there exists a $(2k-1)$-dimensional subspace of H_∞ intersecting each element of $\mathcal{S}$ in a $(k-1)$-dimensional space of $\mathcal{D}_k$.*

Proof. $(i) \iff (ii)$
The ABB-map ϕ, when restricted to the points of the line l_∞, clearly corresponds to applying field reduction to the points of $\mathrm{PG}(1,q^n)$. By field reduction, an $\mathbb{F}_q$-subline contained in l_∞ corresponds to a regulus of $\mathcal{D}$, and vice versa.

A set $\mathcal{S}$ of q^k+1 elements of $\mathcal{D}$ is the image of a set m of q^k+1 points of l_∞. By the previous paragraph, we get that, for any three elements of $\mathcal{S}$, the unique regulus through them is contained in $\mathcal{S}$ if and only if for any three points of m the unique $\mathbb{F}_q$-subline through them is contained in m. Because of Theorem 2.2.6, this is true if and only if m is an $\mathbb{F}_{q^k}$-subline contained in l_∞.

$(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (iv)$
Consider a $(2k-1)$-dimensional subspace π of H_∞ intersecting each of the q^k+1 elements of $\mathcal{S}$ in a $(k-1)$-space. We see that $\mathcal{S} \cap \pi$ is a $(k-1)$-spread of π. Take a $2k$-space Π of $\mathrm{PG}(2n,q)$ intersecting H_∞ in π. Clearly, the ABB-representation $\overline{A(\mathcal{S}\cap\pi)}$ contained in Π is a projective plane of order q^k. This subplane is embedded in the original plane $\mathrm{PG}(2,q^n)$. Every $\mathbb{F}_{q^k}$-subline of this plane is contained in a line of $\mathrm{PG}(2,q^n)$, hence the subplane is a subgeometry

isomorphic to $\mathrm{PG}(2, q^k)$. It follows that $\mathcal{S}$ is the image of an $\mathbb{F}_{q^k}$-subline contained in l_∞.

Moreover, since every tangent $\mathbb{F}_{q^k}$-subline of this subplane corresponds to a k-space intersecting H_∞ in an element of $\mathcal{D}_k$ (Theorem 6.3.3), we know that all $(k-1)$-spaces of $\mathcal{S} \cap \pi$ belong to $\mathcal{D}_k$.

$(i) \Rightarrow (iv)$
Suppose $\mathcal{S}$ is the image of an $\mathbb{F}_{q^k}$-subline m contained in l_∞. The subline m together with an $\mathbb{F}_{q^k}$-subline tangent to l_∞ in a point of m, defines a unique $\mathbb{F}_{q^k}$-subplane μ of $\mathrm{PG}(2, q^n)$. The image of the q^{2k} affine points of μ in the ABB-representation is a set M of q^{2k} affine points of $\mathrm{PG}(2n, q)$.

Every two affine points of μ are contained in an $\mathbb{F}_{q^k}$-subline of μ that is tangent to l_∞. Hence, by Theorem 6.3.3, every two affine points of the set M are contained in an affine k-space completely contained in M. This means that the affine line through any two points of M is contained in M, hence M is an affine subspace. Since M contains q^{2k} points, it is a $2k$-dimensional affine subspace. Its projective completion intersects H_∞ in a $(2k-1)$-space π, and clearly this space π can intersect $\mathcal{D}$ only in elements of $\mathcal{S}$.

Consider any affine point P of μ and its image $\phi(P)$ of M, under the ABB-map ϕ. There are q^k+1 distinct $\mathbb{F}_{q^k}$-sublines of μ tangent to l_∞ and containing P. So, there are q^k+1 affine k-spaces through $\phi(P)$ contained in M and, because of Theorem 6.3.3, their projective completion intersects H_∞ in an element of $\mathcal{D}_k$. Hence, π intersects each element of $\mathcal{S}$ in an element of $\mathcal{D}_k$.

$(iv) \Rightarrow (iii)$
Obvious. □

Corollary 6.3.9. *A Desarguesian $(n-1)$-spread of $\mathrm{PG}(rn-1, q)$ has a unique Desarguesian $(k-1)$-subspread for each $k|n$.*

Proof. In Subsection 6.2.1, we constructed a Desarguesian $(k-1)$-subspread of a Desarguesian $(n-1)$-spread of $\mathrm{PG}(rn-1, q)$.

To prove that such a spread is unique, first consider the case $r = 2$. Consider the Desarguesian $(n-1)$-spread $\mathcal{D}$ of $\mathrm{PG}(2n-1, q)$ and any Desarguesian $(k-1)$-subspread $\mathcal{S}_k$ of $\mathcal{D}$. Take two elements of $\mathcal{S}_k$ contained in different elements of $\mathcal{D}$. These span a $(2k-1)$-space containing q^k+1 elements of $\mathcal{S}_k$, all contained in different elements of $\mathcal{D}$. From the equivalence of statements (iii) and (iv) in Theorem 6.3.8, it follows that all elements of $\mathcal{S}_k$ are elements of $\mathcal{D}_k$.

Now consider $r > 2$ and a Desarguesian $(n-1)$-spread $\mathcal{D}'$ of $\mathrm{PG}(rn-1, q)$. The spread $\mathcal{D}'$ induces a Desarguesian spread $\mathcal{D}$ in any $(2n-1)$-space spanned by two elements of $\mathcal{D}'$. By the previous part this spread $\mathcal{D}$ has a unique Desarguesian $(k-1)$-subspread, and the statement follows. □

6.4 The ABB-representation of subplanes

An $\mathbb{F}_{q^k}$-subplane is said to be *secant*, *tangent*, or *external* if it is secant, tangent, or external to l_∞. In this section we will characterise secant $\mathbb{F}_{q^k}$-subplanes and tangent $\mathbb{F}_q$-subplanes.

6.4.1 Secant subplanes

Theorem 6.4.1. *A set Π of affine points in $\mathrm{PG}(2n, q)$, $q > 2$, is the ABB-representation of the affine points of an $\mathbb{F}_{q^k}$-subplane in $\mathrm{PG}(2, q^n)$ secant to l_∞ if and only if*

(i) Π is a $2k$-dimensional affine space,

(ii) its projective completion $\overline{\Pi}$ intersects H_∞ in a $(2k-1)$-space which intersects $q^k + 1$ elements of $\mathcal{D}$ in exactly a $(k-1)$-space.

Moreover, this $(2k-1)$-space intersects each of the $q^k + 1$ spread elements of $\mathcal{D}$ in a $(k-1)$-space of $\mathcal{D}_k$.

Proof. Follows from the proof of Theorem 6.3.8. □

6.4.2 Equivalent subplanes under the stabiliser of the line at infinity

An $\mathbb{F}_{q^k}$-subplane of $\mathrm{PG}(2, q^n)$ is uniquely determined by four projective points in general position, or alternatively, by three independent vectors in $V(3, q^n)$. Given $\omega, \lambda \in \mathbb{F}_{q^n}$, denote by $\pi_{\omega,\lambda}$ the $\mathbb{F}_q$-subplane determined by the vectors $(1, 0, \lambda), (0, 1, \omega), (0, 0, 1)$, i.e.

$$\begin{aligned}\pi_{\omega,\lambda} &= \{(s, u, s\lambda + u\omega + t)_{\mathbb{F}_{q^n}} \mid s, t, u \in \mathbb{F}_q\} \\ &= \{(s, 1, s\lambda + \omega + t)_{\mathbb{F}_{q^n}} \mid s, t \in \mathbb{F}_q\} \cup \{(1, 0, \lambda + t)_{\mathbb{F}_{q^n}} \mid t \in \mathbb{F}_q\} \\ &\quad \cup \{(0, 0, 1)_{\mathbb{F}_{q^n}}\}.\end{aligned}$$

We see that the plane $\pi_{\omega,\lambda}$ is an external subplane if and only if $\{1, \omega, \lambda\}$ are linearly independent over $\mathbb{F}_q$.

Lemma 6.4.2. *Every $\mathbb{F}_q$-subplane external to l_∞ is equivalent, under the action of the stabiliser of l_∞ in $\mathrm{PGL}(3, q^n)$, to $\pi_{\omega,\lambda}$ for some $\omega, \lambda \in \mathbb{F}_{q^n}$ such that $\{1, \omega, \lambda\}$ are linearly independent over $\mathbb{F}_q$.*

Every $\mathbb{F}_q$-subplane tangent to l_∞ is equivalent, under the action of the stabiliser of l_∞ in $\mathrm{PGL}(3, q^n)$, to $\pi_{\omega,0}$ where $\omega \notin \mathbb{F}_q$.

Proof. Consider an $\mathbb{F}_q$-subplane μ. Then μ is determined by three distinct independent vectors in $\mathbb{F}_{q^n}^3$, which we may take to be (α, β, ω), $(\gamma, \delta, \lambda)$ and $(\epsilon, \zeta, 1)$. The matrix

$$\begin{pmatrix} \gamma - \lambda\epsilon & \delta - \lambda\zeta & 0 \\ \alpha - \omega\epsilon & \beta - \omega\zeta & 0 \\ \epsilon & \zeta & 1 \end{pmatrix}$$

is non-singular and, when acting on row vectors from the right, it maps $(0, 1, \omega)$ to (α, β, ω), $(1, 0, \lambda)$ to $(\gamma, \delta, \lambda)$ and $(0, 0, 1)$ to $(\epsilon, \zeta, 1)$. Hence, the corresponding collineation $\psi \in \mathrm{PGL}(3, q^n)$ maps $\pi_{\omega,\lambda}$ to μ.

As ψ stabilises l_∞, if μ is disjoint from, tangent to, or external to l_∞, then $\pi_{\omega,\lambda}$ is also disjoint from, tangent to, or external to l_∞. If the subplane μ is tangent to l_∞, we may choose our first vector such that $\lambda = 0$, and hence μ is equivalent to the subplane $\pi_{\omega,0}$. If $\omega \in \mathbb{F}_q$, clearly the subplane is secant to l_∞. $\square$

Lemma 6.4.3. *Given $\omega \in \mathbb{F}_{q^n}$, if $\mathbb{F}_{q^k} = \mathbb{F}_q(\omega)$ is the smallest subfield of $\mathbb{F}_{q^n}$ containing ω, then the smallest subplane secant to l_∞ and containing $\pi_{\omega,0}$ is an $\mathbb{F}_{q^k}$-subplane.*

Proof. For all $k|n$, there is a unique $\mathbb{F}_{q^k}$-subplane containing the points $(1, 0, 0)_{\mathbb{F}_{q^n}}$, $(0, 1, \omega)_{\mathbb{F}_{q^n}}$, $(0, 0, 1)_{\mathbb{F}_{q^n}}$, $(1, 1, \omega{+}1)_{\mathbb{F}_{q^n}}$, and every such subplane contains the points of $\pi_{\omega,0}$. Such an $\mathbb{F}_{q^k}$-subplane is secant to l_∞ if and only if it contains the point $(0, 1, 0)_{\mathbb{F}_{q^n}} \in l_\infty$, if and only if $-\omega \in \mathbb{F}_{q^k}$, and the statement follows. $\square$

6.4.3 Tangent subplanes

Consider two normal rational curves $\mathcal{C}_1$ and $\mathcal{C}_2$ of degree k and l respectively. Embed both curves in $\mathrm{PG}(N, q)$, $N \geq k+l+1$, such that the subspaces they span, of dimension k and l respectively, are disjoint.

Let ρ_1, ρ_2 be maps from $\mathrm{PG}(1,q) \to \mathrm{PG}(N,q)$ defined by

$$\rho_1 : (s,t)_{\mathbb{F}_q} \mapsto \left(\sum_{i=0}^{k} s^{k-i} t^i e_i \right)_{\mathbb{F}_q} ,$$

$$\rho_2 : (s,t)_{\mathbb{F}_q} \mapsto \left(\sum_{i=0}^{l} s^{l-i} t^i f_i \right)_{\mathbb{F}_q} ,$$

for defining vectors e_i, f_i such that $\mathcal{C}_1 = \mathrm{Im}(\rho_1)$ and $\mathcal{C}_2 = \mathrm{Im}(\rho_2)$.

A *normal rational scroll of bidegree* $\{k,l\}$ defined by $\{\mathcal{C}_1, \mathcal{C}_2\}$ consists of the set of lines of $\mathrm{PG}(N,q)$ defined as follows

$$\{\langle \rho_1(P), \rho_2(\psi(P))\rangle \mid P \in \mathrm{PG}(1,q)\} ,$$

where ψ is an element of $\mathrm{PGL}(2,q)$.

The stabiliser in $\mathrm{PGL}(N+1,q)$ of a normal rational curve in $\mathrm{PG}(N,q)$ is isomorphic to $\mathrm{PGL}(2,q)$ [67, Theorem 27.5.3]. So, if we take different choices ρ_i' defining $\mathcal{C}_i$, then $\rho_i' = \rho_i \psi_i$ for some $\psi_i \in \mathrm{PGL}(2,q)$, and the normal rational scroll defined by ρ_1, ρ_2 and ψ is equal to the one defined by ρ_1', ρ_2', and $\psi_2^{-1}\psi\psi_1$. Hence, the set of all normal rational scrolls defined by $\{\mathcal{C}_1, \mathcal{C}_2\}$ does not depend on the choice of $\rho_i, i = 1,2$.

When $\mathcal{C}_1$ and $\mathcal{C}_2$ are general curves, such a set of lines is often called a *ruled surface*. The curve of the smallest degree is often called the *directrix*.

Consider the embedding of $\mathrm{PG}(N,q)$ as a subgeometry of $\mathrm{PG}(N,q^n)$. The $\mathbb{F}_{q^n}$-extensions of the curves $\mathcal{C}_i$ are unique and we can consider the canonical extension $\rho_i^* : \mathrm{PG}(1,q^n) \to \mathrm{PG}(N,q^n)$ of ρ_i and $\psi^* \in \mathrm{PGL}(2,q^n)$ of ψ. Clearly these define the $\mathbb{F}_{q^n}$*-extension* of a normal rational scroll and such an extension is unique.

Before considering the characterisation of external subplanes, we introduce a lemma on the existence of normal rational scrolls.

Lemma 6.4.4. *Let $\mathcal{C}_1, \mathcal{C}_2$ be two normal rational curves in Σ, and $\mathcal{C}_1^*, \mathcal{C}_2^*$ their respective extensions to Σ^*. Consider points $P \in \mathcal{C}_1^* \backslash \mathcal{C}_1$, $Q \in \mathcal{C}_2^* \backslash \mathcal{C}_2$ such that P is not contained in the $\mathbb{F}_{q^2}$-extension of $\mathcal{C}_1$. Then there exists at most one normal rational scroll S^* in Σ^*, containing $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$, such that S^* contains the line $\langle P, Q\rangle$, and such that S^* meets Σ in a normal rational scroll S containing $\mathcal{C}_1$ and $\mathcal{C}_2$. Moreover, if such a scroll exists, then it contains the lines $\langle P^{\sigma^i}, Q^{\sigma^i}\rangle$ for each i, where $\Sigma = \mathrm{Fix}(\sigma)$.*

Proof. Let $\mathcal{C}_i^*$ be defined by the map ρ_i^* from $\mathrm{PG}(1,q^n)$ into Σ^* such that $\mathcal{C}_i$ is the image under ρ_i^* of the canonical $\mathbb{F}_q$-subline $l = \{(s,t)_{\mathbb{F}_{q^n}} \mid s,t \in \mathbb{F}_q, (s,t) \neq (0,0)\}$ in $\mathrm{PG}(1,q^n)$. Now $P = \rho_1^*((s,t)_{\mathbb{F}_{q^n}})$ and $Q = \rho_2^*((u,v)_{\mathbb{F}_{q^n}})$ for some $s,t,u,v \in \mathbb{F}_{q^n}^*$, such that $s/t, u/v \in \mathbb{F}_{q^n} \backslash \mathbb{F}_q$. Note that $s/t \notin \mathbb{F}_{q^2}$, for otherwise P would be contained in the $\mathbb{F}_{q^2}$-extension of $\mathcal{C}_1$.

Hence a normal rational scroll exists if and only if there exists a $\psi \in \mathrm{PGL}(2,q^n)$ fixing l (hence $\psi \in \mathrm{PGL}(2,q)$) and mapping $(s,t)_{\mathbb{F}_{q^n}}$ to $(u,v)_{\mathbb{F}_{q^n}}$. We will show that the stabiliser of the point $(s,t)_{\mathbb{F}_{q^n}}$ under the action of $\mathrm{PGL}(2,q)$ is trivial unless $s/t \in \mathbb{F}_{q^2}$, and hence the result follows. This statement is obtained, by considering the following equality

$$\begin{pmatrix} s & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ks & k \end{pmatrix}.$$

By eliminating k we obtain the equation $bs^2 + (d-a)s - c = 0$. One can obtain a solution for this equation, different from $(a,b,c,d) = (k,0,0,k)$, only if $s \in \mathbb{F}_{q^2}$.

The map σ fixes Σ pointwise and hence also fixes $\mathcal{C}_1$ pointwise. As $\mathcal{C}_1^*$ is the unique extension of $\mathcal{C}_1$, the curve $\mathcal{C}_1^*$ is stabilised by σ. This means σ induces a collineation σ' on $\mathrm{PG}(1,q^n)$ such that $(\rho_1^*((s,t)_{\mathbb{F}_{q^n}}))^\sigma = \rho_1^*((s,t)_{\mathbb{F}_{q^n}}^{\sigma'})$. As $\mathrm{PG}(1,q) \subset \mathrm{PG}(1,q^n)$ must be fixed pointwise by σ', we have that σ' corresponds to an element of the naturally embedded group $\mathrm{Aut}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ in $\mathrm{P\Gamma L}(2,q)$. Hence, σ' commutes with an element ψ mapping $(s,t)_{\mathbb{F}_{q^n}}$ to $(u,v)_{\mathbb{F}_{q^n}}$, that is $\psi((s,t)_{\mathbb{F}_{q^n}}^{\sigma'}) = (u,v)_{\mathbb{F}_{q^n}}^{\sigma'}$. It follows that the obtained scroll contains $\langle P^{\sigma^i}, Q^{\sigma^i} \rangle$ for each i. $\square$

It has been shown ([14] for $n=3$ and [77] for general n) that the ABB-representation of a tangent subplane is a normal rational scroll. We are now ready to show when the converse is true. We will fully characterise tangent subplanes in $\mathrm{PG}(2,q^n)$, extending the result of [14] for $n=3$ to general n.

Theorem 6.4.5. *A set S of affine points of $\mathrm{PG}(2n,q)$, $n \leq q-2$, is the ABB-representation of the affine points of a tangent $\mathbb{F}_q$-subplane μ if and only if S consists of the affine points of a normal rational scroll defined by curves $\{\mathcal{C}, \mathcal{N}\}$ satisfying the following conditions for some $k|n$:*

- *(i) $\mathcal{C}$ is a normal rational curve of degree k contained in an affine k-space π, for which $\overline{\pi} \cap H_\infty$ is an element E_1 of $\mathcal{D}_k$, such that its $\mathbb{F}_{q^n}$-extension $\mathcal{C}^*$ contains all conjugate points $\{P, P^\sigma, \ldots, P^{\sigma^{k-1}}\}$ generating the spread element E_1,*
- *(ii) $\mathcal{N}$ is a normal rational curve of degree $k-1$ contained in an element E_2 of $\mathcal{D}_k$, where E_1 and E_2 are not contained in the same element of $\mathcal{D}$, such*

that its $\mathbb{F}_{q^n}$-extension $\mathcal{N}^*$ contains all conjugate points $\{Q, Q^{\sigma}, \ldots, Q^{\sigma^{k-1}}\}$ generating the spread element E_2,

(iii) the $\mathbb{F}_{q^n}$-extension of the normal rational scroll contains the lines $\langle P^{\sigma^j}, Q^{\sigma^j}\rangle$, each line contained in an indicator space Π^{σ^j} of $\mathcal{D}_k$, for all $j \in \{0, 1, \ldots, k-1\}$.

Moreover, in that case the smallest subplane containing μ and secant to l_∞ is an $\mathbb{F}_{q^k}$-subplane.

Proof. First, suppose the smallest secant subplane containing μ is an $\mathbb{F}_{q^2}$-subplane. From Theorem 6.4.1, the affine points of μ are contained in a 4-dimensional affine space intersecting H_∞ in a 3-space partitioned by lines of $\mathcal{D}_1$. In this case, we can use the characterisation of the ABB-representation in $\mathrm{PG}(4, q)$ of a tangent Baer subplane of $\mathrm{PG}(2, q^2)$ considered in [12, Theorems 3.19, 3.20 and 3.21]. This corresponds to a normal rational scroll satisfying (i), (ii), (iii) with $k = 2$, and vice versa; proving our claim. Note that the normal rational curve $\mathcal{N}$ of degree 1 is just a projective line of $\mathcal{D}_1$.

We can now consider $k > 2$. From Lemmas 6.4.2 and 6.4.3, the tangent $\mathbb{F}_q$-subplane μ is equivalent to $\pi_{\omega,0}$, where $\mathbb{F}_q(\omega) = \mathbb{F}_{q^k}$, under an element of the stabiliser G of l_∞ in $\mathrm{PGL}(3, q^n)$, say $\chi_0(\pi_{\omega,0}) = \mu$ for $\chi_0 \in G$. Note that $\pi_{\omega,0}$ consists of the following points:

$$\pi_{\omega,0} = \left\{ \left(\frac{s}{\omega+t}, \frac{1}{\omega+t}, 1 \right)_{\mathbb{F}_{q^n}} \middle| \; s, t \in \mathbb{F}_q \right\} \cup \{(t, 0, 1)_{\mathbb{F}_{q^n}} \mid t \in \mathbb{F}_q\} \; \cup \; \{(1, 0, 0)_{\mathbb{F}_{q^n}}\}.$$

In the ABB-representation, the point $(1, 0, 0)_{\mathbb{F}_{q^n}} \in \pi_{\omega,0}$ corresponds to the spread element $\{(x, 0, 0)_{\mathbb{F}_q} \mid x \in \mathbb{F}_{q^n}^*\} \in \mathcal{D}$. By definition of the ABB-map ϕ, the affine points of $\pi_{\omega,0}$ in the ABB-representation form a set W defined as follows:

$$W = \left\{ \left(\frac{s}{\omega+t}, \frac{1}{\omega+t}, 1 \right)_{\mathbb{F}_q} \middle| \; s, t \in \mathbb{F}_q \right\} \cup \left\{ (t, 0, 1)_{\mathbb{F}_q} \mid t \in \mathbb{F}_q \right\}.$$

This is a set of $q + 1$ affine lines; we can see this more clearly when we consider the set $\overline{W}$ consisting of the projective completions of all lines of W.

$$\overline{W} = \left\{ \left\langle \left(\frac{1}{\omega+t}, 0, 0 \right)_{\mathbb{F}_q}, \left(0, \frac{1}{\omega+t}, 1 \right)_{\mathbb{F}_q} \right\rangle \middle| \; t \in \mathbb{F}_q \right\} \cup \left\langle (1, 0, 0)_{\mathbb{F}_q}, (0, 0, 1)_{\mathbb{F}_q} \right\rangle$$

From the proof of Theorem 6.3.5, we know that the set

$$\mathcal{C}_\omega = \left\{ \left(0, \frac{1}{\omega + t}, 1\right)_{\mathbb{F}_q} \middle| \; t \in \mathbb{F}_q \right\} \cup \left\{ (0,0,1)_{\mathbb{F}_q} \right\}$$

is a normal rational curve satisfying the conditions in (i).

Consider the following set of points of H_∞, all contained in $\overline{W}$:

$$\begin{aligned} \mathcal{N}_\omega &= \left\{ \left(\frac{1}{\omega + t}, 0, 0\right)_{\mathbb{F}_q} \middle| \; t \in \mathbb{F}_q \right\} \cup \left\{ (1,0,0)_{\mathbb{F}_q} \right\} \\ &= \left\{ \left(\prod_{i=1}^{k-1} (\omega^{q^i} + t), 0, 0\right)_{\mathbb{F}_q} \middle| \; t \in \mathbb{F}_q \right\} \cup \left\{ (1,0,0)_{\mathbb{F}_q} \right\}. \end{aligned}$$

A similar calculation shows that $\mathcal{N}_\omega$ is a normal rational curve of degree $k-1$, contained in a projective $(k-1)$-space E_2 of $\mathcal{D}_k$, as claimed. In fact, this has been shown in e.g. [60, 61]. Note that the spread element of $\mathcal{D}$ in which E_2 is contained is not equal to the one associated to $\mathcal{C}_\omega$.

Using the map ι to embed $\mathcal{N}_\omega$ in $\mathrm{PG}(2n, q^n)$ gives the points

$$(1, \ldots, 1; 0, \ldots, 0; 0)_{\mathbb{F}_{q^n}},$$

and

$$\left(\prod_{i=1}^{k-1} (\omega^{q^i} + t), \prod_{i=1}^{k-1} (\omega^{q^{i+1}} + t), \ldots, \prod_{i=1}^{k-1} (\omega^{q^{i+n-1}} + t); 0, 0, \ldots, 0; 0 \right)_{\mathbb{F}_{q^n}},$$

for $t \in \mathbb{F}_q$. Since $\mathbb{F}_q(\omega) = \mathbb{F}_{q^k}$, the $(n+j)$-th entry is equal to the $(n+j+k)$-th entry for all $0 \leq j \leq n-1$. The extension $\mathcal{N}_\omega^*$ of $\mathcal{N}_\omega$ is the normal rational curve obtained by allowing t to range over $\mathbb{F}_{q^n}$. As before, one can see that $\mathcal{N}_\omega^*$ contains the conjugate points of the indicator set of $\mathcal{D}_k$ generating the $(k-1)$-space in which $\mathcal{N}_\omega$ lies, and hence $\mathcal{N}_\omega$ satisfies the condition of (ii).

Clearly, the points of W are precisely the affine points of the normal rational scroll $\mathcal{B}_\omega$ defined by the curves $\{\mathcal{C}_\omega, \mathcal{N}_\omega\}$ and the identity element of $\mathrm{PGL}(2, q)$, and $\mathcal{B}_\omega$ satisfies (iii).

To prove that the converse is also valid, it is again sufficient to use a counting argument.

We know from Theorem 6.3.5 that sublines l_0 external to l_∞ are in one-to-one correspondence with normal rational curves satisfying (i). The number of tangent $\mathbb{F}_q$-subplanes containing a fixed external $\mathbb{F}_q$-subline and a fixed point R of l_∞ (not on the extension of this fixed external subline) is $\frac{q^n-1}{q-1}$.

Therefore it suffices to show that for a fixed normal rational curve $\mathcal{C}$ satisfying (i) and a fixed element E of $\mathcal{D}$ (for which $\mathcal{C}$ does not lie in an n-space containing E), there are precisely $\frac{q^n-1}{q-1}$ curves $\mathcal{N}$ contained in E satisfying (ii) for which there exists a normal rational scroll $\mathcal{B}$ defined by $\{\mathcal{C}, \mathcal{N}\}$ and satisfying (iii).

Using the induced action of the stabiliser G of the line l_∞ in $\mathrm{PGL}(3, q^n)$ on $\mathrm{PG}(2n, q^n)$, we can choose w.l.o.g. $\mathcal{C} = \mathcal{C}_\omega$, where $\mathbb{F}_q(\omega) = \mathbb{F}_{q^k}$, with $k > 2$ by our assumption, and $E = \iota\phi(1, 0, 0)_{\mathbb{F}_{q^n}}$. Consider a point $A = \iota(a, 0, 0)_{\mathbb{F}_q}$ of E and the element E_2 of $\mathcal{D}_k$ containing A. We will prove that there are $q(q+1)$ choices for a point $B \in E_2$ such that A, B and the points of $\mathcal{C}_\omega$ lie on a normal rational scroll satisfying (i), (ii), (iii). We let $B = \iota(b, 0, 0)_{\mathbb{F}_q}$ for some $b \in \mathbb{F}_{q^n}$. The conjugate points generating E_2 are $\{R, R^\sigma, \ldots, R^{\sigma^{k-1}}\}$, where $R = (v)_{\mathbb{F}_{q^n}}$ with

$$\begin{aligned} v &= (1, 0, \ldots, 0, a^{q^k-1}, 0, \ldots, 0, a^{q^{n-k}-1}, 0, \ldots, 0; 0, \ldots, 0; 0) \\ &= (1, 0, \ldots, 0, b^{q^k-1}, 0, \ldots, 0, b^{q^{n-k}-1}, 0, \ldots, 0; 0, \ldots, 0; 0). \end{aligned}$$

Now there exists a unique normal rational curve $\mathcal{N}_{AB}$ satisfying (ii), i.e. such that its extension contains $\{A, B, R, R^\sigma, \ldots, R^{\sigma^{k-1}}\}$. We can see that it must be the curve defined by the following map:

$$\eta(s, t) = \left(\sum_{i=0}^{k-1} \prod_{j=0, j\neq i}^{k-1} (a^{-q^j} s - b^{-q^j} t) v^{\sigma^i} \right)_{\mathbb{F}_{q^n}}.$$

We have $R^{\sigma^i} = \eta(b^{-q^i}, a^{-q^i})$, $A = \eta(1, 0)$, $B = \eta(0, 1)$, and $\mathcal{N}_{AB} = \eta(\mathbb{F}_q \times \mathbb{F}_q)$.

From the proof of Theorem 6.3.5, the curve $\mathcal{C}_\omega^*$ can be parametrised by a map $\rho : \mathbb{F}_{q^n} \times \mathbb{F}_{q^n} \to \mathrm{PG}(2n, q)$ such that $\mathcal{C}_\omega = \rho(\mathbb{F}_q \times \mathbb{F}_q)$, and the intersection of $\mathcal{C}_\omega^*$ with the indicator sets are the points $Q^{\sigma^i} = \rho(1, -\omega^{q^i})$, $i \in \{0, \ldots, k-1\}$. By the proof of Lemma 6.4.4, there exists a normal rational scroll defined by $\{\mathcal{C}_\omega, \mathcal{N}_{AB}\}$ such that its $\mathbb{F}_{q^n}$-extension contains the lines $\langle Q^{\sigma^i}, R^{\sigma^i} \rangle$ (hence satisfies (iii)) if and only if there exists an element ψ of $\mathrm{PGL}(2, q^n)$ which fixes the canonical $\mathbb{F}_q$-subline (defined by $\mathbb{F}_q \times \mathbb{F}_q$), whence $\psi \in \mathrm{PGL}(2, q)$, and which maps $(1, -\omega)_{\mathbb{F}_{q^n}}$ to $(1, \frac{b^{-1}}{a^{-1}})_{\mathbb{F}_{q^n}}$. Since $\omega \notin \mathbb{F}_{q^2}$, there are $|\mathrm{PGL}(2, q)| = q(q^2 - 1)$ points in the orbit of $(1, -\omega)_{\mathbb{F}_{q^n}}$ under $\mathrm{PGL}(2, q)$, and hence $q(q^2 - 1)$ choices for b. As $\iota(b, 0, 0)_{\mathbb{F}_q} = \iota(\lambda b, 0, 0)_{\mathbb{F}_q}$ for all $\lambda \in \mathbb{F}_q^*$, we get that there are $q(q+1)$ allowable points $B \in E_2$.

Since there are $\frac{q^n-1}{q-1}$ choices for A, and each $\mathcal{N}_{AB}$ contains $q(q+1)$ ordered pairs of distinct points $\{A, B\}$, we have that there are precisely $\frac{q^n-1}{q-1}q(q+1)\frac{1}{q(q+1)} = \frac{q^n-1}{q-1}$ curves satisfying (ii) which define a scroll satisfying (iii), proving the claim. $\square$

Remark. In [14], in the case $n = 3$, a curve satisfying (i) was referred to as a *special normal rational curve*, while a curve satisfying (ii) was referred to as a *special conic*.

7

Unitals with many Baer secants through a fixed point

In this chapter, we show that a unital U in $\mathrm{PG}(2, q^2)$ containing a point P, such that at least $q^2 - \epsilon$ of the secant lines through P intersect U in a Baer subline, is an ovoidal Buekenhout-Metz unital (where $\epsilon \approx 2q$ for q even and $\epsilon \approx q^{3/2}/2$ for q odd).

These results were obtained in collaboration with G. Van de Voorde [100].

7.1 Introduction

We study unitals in the Desarguesian projective plane $\mathrm{PG}(2, q^2)$, more specifically, we study *ovoidal Buekenhout-Metz unitals*. This class of unitals was first constructed by Buekenhout [27] and later extended by Metz [83]. Every known unital can be obtained by this construction, however, it remains an open problem whether all unitals arise as ovoidal Buekenhout-Metz unitals.

Combining the results of [35] (for $q > 2$ even and $q = 3$), and [95] (for $q > 3$) the following characterisation of ovoidal Buekenhout-Metz unitals was obtained.

Theorem 7.1.1. [35, 95] *If U is a unital in $\mathrm{PG}(2, q^2)$, $q > 2$, containing a point P such that all secants through P intersect U in a Baer subline, then U is an ovoidal Buekenhout-Metz unital with special point P.*

Moreover, two related characterisations were found for classical unitals.

Theorem 7.1.2. [15] *Let U be an ovoidal Buekenhout-Metz unital in $\mathrm{PG}(2, q^2)$ with special point P. If U contains a secant not through P intersecting U in a Baer subline, then U is classical.*

Theorem 7.1.3. [5] *Let U be a unital in* $\mathrm{PG}(2, p^2)$*, p prime, such that $p(p^2-2)$ secants intersect U in a Baer subline, then U is classical.*

Concerning these three results, in [41, Open problems 4], the following question was posed:

> What is the minimum required number of secants being Baer sublines, to conclude that a unital is an ovoidal Buekenhout-Metz unital?

We will improve the result of Theorem 7.1.1 by finding a new upper bound on the minimum required number of Baer secants through a fixed point of the unital. It is worth noticing that our theorem implies the result of [35] and [95] for $q \geq 16$.

Theorem 7.3.10. *Suppose q and ϵ satisfy the conditions of Table 7.1. Let U be a unital in* $\mathrm{PG}(2, q^2)$ *containing a point P such that at least $q^2 - \epsilon$ of the secants through P intersect U in a Baer subline, then U is an ovoidal Buekenhout-Metz unital with special point P.*

ϵ	Conditions
$\epsilon \leq q-3$	q even, $q \geq 16$
$\epsilon \leq 2q-7$	q even, $q \geq 128$
$\epsilon \leq \frac{\sqrt{q}q}{4} - \frac{39q}{64} - O(\sqrt{q}) + 1$	q odd, $q \geq 17$, $q = p^{2e}$, $e \geq 1$
$\epsilon \leq \frac{\sqrt{q}q}{2} - 2q$	q odd, $q \geq 17$, $q = p^{2e+1}$, $e \geq 0$
$\epsilon \leq \frac{\sqrt{q}q}{2} - \frac{67q}{16} + \frac{5\sqrt{q}}{4} - \frac{1}{12}$	q odd, $q \geq 17$, $q = p^h$, $p \geq 5$
$\epsilon \leq \frac{\sqrt{q}q}{2} - \frac{35q}{16} - O(\sqrt{q}) + 1$	q odd, $q \geq 23^2$, $q \neq 5^5, 3^6$, $q = p^h$, h even for $p = 3$

Table 7.1: Conditions for Theorem 7.3.10

7.2 Preliminaries

7.2.1 Subgeometries in the ABB-representation

In this chapter, we will use the following conventions. Consider the projective plane $\mathrm{PG}(2,q^2)$ and fix a line l_∞ at infinity. Consider the ABB-representation of $\mathrm{PG}(2,q^2)$ in $\mathrm{PG}(4,q)$ with respect to this line. The hyperplane at infinity of $\mathrm{PG}(4,q)$, corresponding to l_∞, will be denoted by H_∞ and the Desarguesian spread in H_∞ defining $\mathrm{PG}(2,q^2)$ will be denoted by $\mathcal{D}$.

We will call a Baer subline of $\mathrm{PG}(2,q^2)$ *tangent* (to l_∞) if it has one point in common with l_∞, and *external* if it has no such intersection point. Recall that in the ABB-representation, tangent sublines of $\mathrm{PG}(2,q^2)$ are in one-to-one correspondence with lines of $\mathrm{PG}(4,q)$ intersecting H_∞ in exactly one point. An external subline corresponds to a non-degenerate conic of $\mathrm{PG}(4,q)$, called a *Baer conic*, contained in a plane which meets H_∞ in a spread line of $\mathcal{D}$, external to this conic, such that the $\mathbb{F}_{q^2}$-extension of the conic has two points in common with the indicator set defining $\mathcal{D}$ (see Theorem 6.3.5). Note that, unless $q=2$, not every conic is a Baer conic. Moreover, since any two distinct Baer sublines have at most two points in common, any two distinct Baer conics share at most 2 points.

A Baer subplane will be called *secant* (to l_∞) if it meets l_∞ in $q+1$ points, and *tangent* if it meets l_∞ in one point. In the ABB-representation, secant subplanes are in one-to-one correspondence with planes of $\mathrm{PG}(4,q)$ intersecting H_∞ in a line not contained in $\mathcal{D}$. A tangent Baer subplane corresponds to the point set of $q+1$ disjoint lines, called *generator lines*, forming a ruled cubic surface, called a *Baer ruled cubic*. Such a Baer ruled cubic has a spread line $T \in \mathcal{D}$ as *line directrix*, where T corresponds to the intersection point of the tangent Baer subplane with l_∞. As a base, it has a Baer conic C in a plane disjoint from T. For each point of T, there is a unique generator line on the Baer ruled cubic through this point and a point of C. A plane through a line of $\mathcal{D} \setminus \{T\}$ intersects the Baer ruled cubic in a point or a Baer conic. For more information on the ABB-representation of sublines and subplanes of $\mathrm{PG}(2,q^2)$, we refer to [12, Section 3.4.2] or Chapter 6.

It is well known that two distinct Baer sublines spanning the plane $\mathrm{PG}(2,q^2)$, that have a common point, are contained in a unique Baer subplane. The following lemma, in terms of lines of $\mathrm{PG}(4,q)$ in the ABB-representation, can be deduced.

Lemma 7.2.1. *Two lines of* $\mathrm{PG}(4,q)$*, not contained in a plane through a line of* $\mathcal{D}$*, intersecting* H_∞ *in the same point, lie in a unique plane intersecting* H_∞ *not in a line of* $\mathcal{D}$*, i.e. they define a unique secant subplane to* l_∞*.*

Two lines of $\mathrm{PG}(4,q)$*, not in* H_∞*, through different points* P_1, P_2 *of* H_∞*, such that* P_1P_2 *is a spread line of* $\mathcal{D}$*, lie in a unique Baer ruled cubic, i.e. they define a unique tangent subplane to* l_∞*.*

7.2.2 Unitals in $\mathrm{PG}(2, q^2)$

Recall that a *unital* in $\mathrm{PG}(2,q^2)$ is a set of q^3+1 points such that every line meets U in 1 or $q+1$ points. It is easy to see that a point P of U lies on exactly one tangent line to U and on q^2 lines meeting U in $q+1$ points (including P). These last lines are called the $(q+1)$*-secants*, or short *secants*, to U. If a secant line meets a unital in a Baer subline, then we call this line a *Baer secant*.

A *classical unital* (or *Hermitian curve*) in $\mathrm{PG}(2,q^2)$ corresponds to the set of absolute points of a unitary polarity. Note that every unital in $\mathrm{PG}(2,4)$ is classical. In $\mathrm{PG}(2,q^2)$, $q>2$, there are examples of non-classical unitals.

An *ovoidal Buekenhout-Metz unital* in $\mathrm{PG}(2,q^2)$ arises from the following construction (see [27]). Consider the ABB-representation in $\mathrm{PG}(4,q)$ of $\mathrm{PG}(2,q^2)$ with respect to the line l_∞, with line spread $\mathcal{D}$ of H_∞ corresponding to the points of l_∞. Let $\mathcal{O}$ be an ovoid in a 3-space of $\mathrm{PG}(4,q)$, such that H_∞ contains exactly one point $A \in \mathcal{O}$ and such that the tangent plane of $\mathcal{O}$ at A does not contain the spread line $T \in \mathcal{D}$ incident with A. Let V be a point on T, $V \neq A$. Consider the ovoidal cone with vertex V and base $\mathcal{O}$, this point set corresponds to a unital U in $\mathrm{PG}(2,q^2)$. The line l_∞ is the tangent line to U at the point P_∞ of l_∞, where P_∞ is the point corresponding to the spread line T. We will call P_∞ the *special point* of the ovoidal Buekenhout-Metz unital U. Clearly, all secants to U at P_∞ are Baer secants.

All known unitals in $\mathrm{PG}(2,q^2)$, including the classical unital, arise as ovoidal Buekenhout-Metz unitals. Moreover, every unital in $\mathrm{PG}(2,q^2)$, with $q=2,3,4$, corresponds to an ovoidal Buekenhout-Metz unital, see [7, 93]. For $q>4$, the classification of unitals is an open problem.

7.2.3 Caps and ovoids in $\mathrm{PG}(3, q)$

We will need the following extendability results for caps in $\mathrm{PG}(3,q)$.

Theorem 7.2.2. *A cap in* $\mathrm{PG}(3,q)$ *of size at least* $q^2-\delta$*, with* δ *and* q *satisfying the conditions of Table 7.2, can be extended to an ovoid.*

δ	Conditions	Ref.
$\delta \leq \frac{q}{2} + \frac{\sqrt{q}}{2} - 1$	q even, $q > 2$	[66]
$\delta \leq q - 4$	q even, $q \geq 8$	[37]
$\delta \leq 2q - 8$	q even, $q \geq 128$	[29]
$\delta \leq \frac{\sqrt{q}q}{4} - \frac{39q}{64} - O(\sqrt{q})$	q odd, $q \geq 17$, $q = p^{2e}$, $e \geq 1$	[65]
$\delta \leq \frac{p^{e+1}q}{4} - \frac{119pq}{64} + O(p^{e+2})$	q odd, $q \geq 17$, $q = p^{2e+1}$, $e \geq 1$	[65]
$\delta \leq \frac{359q^2}{2700} + \frac{4q}{135} - \frac{94}{27}$	q odd, $q \geq 17$ prime	[65]
$\delta \leq \frac{\sqrt{q}q}{2} - \frac{67q}{16} + \frac{5\sqrt{q}}{4} - \frac{13}{12}$	q odd, $q \geq 17$, $q = p^h$, $p \geq 5$	[65]
$\delta \leq \frac{\sqrt{q}q}{2} - \frac{35q}{16} - O(\sqrt{q})$	q odd, $q \geq 23^2$, $q \neq 5^5, 3^6$, $q = p^h$ (h even for $p = 3$)	[65]

Table 7.2: Conditions for Theorem 7.2.2

Moreover, the following theorem shows that the ovoids obtained in the previous theorem are unique.

Theorem 7.2.3. [110, Theorem 2.2]

> *If K is a k-cap in* $\mathrm{PG}(n, q)$*, $n \geq 3$, q even, having size $k > (q^{n-1} + \cdots + q + 2)/2$, then K can be extended in a unique way to a complete cap.*
>
> *If K is a k-cap in* $\mathrm{PG}(n, q)$*, $n \geq 3$, q odd, of size $k > 2(q^{n-1} + \cdots + q + 2)/3$, then K can be extended in a unique way to a complete cap.*

7.3 Proof of the main theorem

We will need the following lemma which can be shown by a simple counting argument.

Lemma 7.3.1. [35, Theorem 2.1] *A Baer subplane tangent to l_∞ meets a unital in* $\mathrm{PG}(2, q^2)$ *(where the unital has l_∞ as a tangent line) in at most $2q + 2$ points.*

A Baer subplane secant to l_∞ meets a unital in $\mathrm{PG}(2, q^2)$ *(where the unital has l_∞ as a tangent line) in at most $2q + 1$ points.*

Throughout this section, we use the following notations and conventions. Let U be a unital in $\mathrm{PG}(2, q^2)$ containing a point P_∞ such that a set of at least $q^2 - \epsilon$, $\epsilon \leq q^2$, of the $(q+1)$-secants through P_∞ are Baer secants. Say l_∞ is the tangent line of U at P_∞ and consider the ABB-representation of $\mathrm{PG}(2, q^2)$ with respect to l_∞, such that the points of l_∞ correspond to the elements of the Desarguesian spread $\mathcal{D}$ in the hyperplane H_∞ of $\mathrm{PG}(4, q)$. By abuse of notation, we will use the notation U_{aff} for both the point set $U \setminus \{P_\infty\}$ in $\mathrm{PG}(2, q^2)$ and for the corresponding affine point set in $\mathrm{PG}(4, q)$.

Suppose P_∞ corresponds to the spread line T of $\mathcal{D}$. Let $\mathcal{L}$ be the set of $q^2 - \epsilon$ lines in $\mathrm{PG}(4, q)$ corresponding to Baer secants through P_∞. Every line of $\mathcal{L}$ intersects H_∞ in a point of T. Note that any plane intersecting H_∞ in T contains exactly q points of U_{aff}.

Given a unital U and its corresponding line set $\mathcal{L}$, we will consider a set $S(U)$ in the plane $\Pi = \mathrm{PG}(4, q)/T$, consisting of points with labels, induced by the lines of $\mathcal{L}$. This point set is defined as follows.

Definition 7.3.2. Consider the quotient space $\Pi = \mathrm{PG}(4, q)/T$, isomorphic to $\mathrm{PG}(2, q)$, and let $v_1, \ldots, v_{q+1}$ be the points of T. The points of $S(U)$ are the points of Π corresponding to the planes through T which contain a line of $\mathcal{L}$. We label a point R of $S(U)$ with v_j, if the line of $\mathcal{L}$ in the plane $\langle T, R\rangle$ passes through v_j.

Lemma 7.3.3. *The set $S(U)$ is a point set in* $\mathrm{AG}(2, q)$ *such that each point has exactly one label. Moreover, $S(U)$ has the property that if a point Q of $S(U)$ belongs to a line of* $\mathrm{AG}(2, q)$ *containing two points of $S(U)$ with the same label v, then Q also has label v.*

Proof. First note that the points of $S(U)$ are contained in an affine plane of $\Pi = \mathrm{PG}(4, q)/T$, since H_∞/T is a line in Π and since no plane through T in H_∞ contains a line of $\mathcal{L}$. Each point of $S(U)$ has exactly one label, as a plane through T contains at most one line of $\mathcal{L}$.

If a line m in Π contains two points of $S(U)$ with the same label, say v_k, then the 3-space $\langle T, m\rangle$ contains two lines l_1, l_2 of $\mathcal{L}$ through the point v_k. Suppose that there is a point of $S(U)$ on the line m with label v_j, $j \neq k$. This implies that there is a line of $\mathcal{L}$, say l_3, through v_j, contained in $\langle T, m\rangle$. Thus, the line l_3 meets the plane $\langle l_1, l_2\rangle$ in an affine point, which means that the secant subplane defined by l_1, l_2 contains $2q + 2$ points, a contradiction by Lemma 7.3.1. $\square$

Next, we show that the configuration of points of $S(U)$ must satisfy one of three conditions.

Lemma 7.3.4. *Suppose $q > 2$ and $k \in \mathbb{N}$, $k < \sqrt{q} - 1$. Let S be a set of $q^2 - \epsilon$, $\epsilon \leq kq$, points in $\mathrm{AG}(2, q)$, and consider a set of labels $\mathcal{V} = \{v_1, \ldots, v_{q+1}\}$, such that each point of S has exactly one label. Denote the subset of S containing all points with label v by S_v.*

Suppose that the set S has the property that if a point Q of S belongs to a line of $\mathrm{AG}(2, q)$ containing two points of S with the same label v, then Q also has label v. Then the set S satisfies one of the following conditions.

- (i) *All points of S have the same label.*
- (ii) *There are 2 distinct labels v_1 and v_2 each occurring at least $q - k$ times as labels of points of S. For $i = 1, 2$, the points of S_{v_i} lie on an affine line. These two affine lines go through a common affine point.*
- (iii) *There is a subset $\mathcal{V}^* \subseteq \mathcal{V}$ of labels, each occurring at least twice, such that for every label $v \in \mathcal{V}^*$, the points of S_v lie on an affine line. These affine lines are all parallel (i.e. their projective completions go through a common point Q_∞ at infinity). The subset $S^* \subseteq S$, consisting of points with a label in $\mathcal{V}^*$, has size at least $q^2 - \epsilon - (k^2 + k)(k^2 + k - 1) - 1$.*

Proof. First, make the following two observations.

- Suppose that there is a label v appearing $q + 2$ times or more. Take a point $P \in S$, then at least one line through P contains at least two points of S with label v. Hence, the point P also has label v, thus, all points of S have label v. We find that S has configuration (i).
- Suppose that there is a label v, such that q points of S_v lie on a line L. If S does not have configuration (i), then clearly no other point of S has label v. Moreover, if another label appears at least two times, then the line spanned by the corresponding points must be parallel to L. Hence, any label appears at most q times. There is a subset $\mathcal{V}^* \subseteq \mathcal{V}$ containing at least $q - k$ labels, such that every label appears at least twice; otherwise, there would be at most $(q - k - 1)q + (k + 2)1 = q^2 - kq - q + k + 2 < q^2 - kq$ points in S. There are at most $k + 1$ points having a label appearing only once. The subset $S^* \subseteq S$ of points having a label in $\mathcal{V}^*$ has size at least $q^2 - \epsilon - k - 1 \geq q^2 - \epsilon - (k^2 + k)(k^2 + k - 1) - 1$. Hence, S has the configuration described in (iii).

Now, there exists a label v occurring at least $q-k$ times, otherwise, there would be at most $(q+1)(q-k-1) = q^2-kq-k-1 < q^2-kq$ points in S. Suppose that there are three non-collinear points in S_v. Choose a point $P_1 \in S_v$ and consider the set Z of all lines containing P_1 and another point of S_v. Every line of Z can only contain points with label v. Consider the set $Z' \subseteq Z$ of all lines of Z that contain at most k points of S different from P_1; suppose $|Z'| = x$. Hence, the lines of Z' each contain at least $q-k-1$ affine points not in S. Since the lines of Z' contain at most all kq points not in S, we see that

$$x \leq \frac{kq}{q-k-1}.$$

However, the upper bound on the number of points of S_v, different from P_1, covered by the lines of Z' is equal to xk. We see that

$$xk \leq \frac{k^2q}{q-k-1}.$$

Moreover, when $k < \sqrt{q}-1$, we have

$$xk \leq \frac{k^2q}{q-k-1} < q-k-1.$$

As there are at least $q-k-1$ points in S_v, different from P_1, there exists a point $P_2 \in S_v$ not on a line of Z'. Hence, the line P_1P_2 contains at least $k+1$ points of S, different from P_1.

Consider a point $P_3 \in S_v$, but not on P_1P_2. There are at least $k+2$ lines through P_3 and a point of $S_v \cap P_1P_2$ containing only points of S_v. These lines cover at least $1+(k+2)(q-1)-kq = 2q-k-1 \geq q+2$ points of S, when $k < \sqrt{q}-1$ and $q > 2$. Since the label v appears at least $q+2$ times, it follows that all points of S have label v, hence, S has configuration (i).

We can now assume that if a label v appears at least $q-k$ times, then the points of S_v belong to a line. Moreover, since q points with a fixed label on a line imply configuration (i) or (iii), we can pose that $\forall v \in \mathcal{V} : |S_v| < q$. We can count that there are at least two labels v_1 and v_2 each occurring at least $q-k$ times, since otherwise there would be at most $1(q-1)+q(q-k-1) = q^2-kq-1 < q^2-kq$ points in S. Consider the lines L_1 and L_2 containing all points of S_{v_1} and S_{v_2} respectively.

If L_1 and L_2 intersect in an affine point Q, then S satisfies configuration (ii).

Now, suppose L_1 and L_2 are parallel, i.e. their projective completions intersect in a point Q_∞ at infinity. There are at least $q-k+1$ labels occurring at least twice, since otherwise there would be at most $(q-k)(q-1)+(k+1)1 = q^2-kq-q+2k+1 < q^2-kq$ points in S. A line spanned by two points with the same label (different from v_1 and v_2) must intersect both lines L_i in a point not in S. However, the line L_i, $i=1,2$, contains at most k affine points not in S. Hence, there are at most k^2 lines intersecting both lines L_i, $i=1,2$, not in Q_∞ and not in a point of S. This means that, of all the labels appearing at least twice, there are at most k^2 labels such that two points with the same label do not necessarily span a line containing Q_∞. Hence, there is a subset $\mathcal{V}^* \subseteq \mathcal{V}$ of at least $q-k^2-k+1$ labels occurring at least twice such that points with the same label do lie on a line containing Q_∞.

It follows that there are at most k^2+k-1 affine lines through Q_∞, such that the points of S on such a line do not have the same label. However, there are at most $(q+1)-(q-k^2-k+1) = k^2+k$ labels that could occur this way. Hence, at most $(k^2+k-1)(k^2+k)$ points of S have the property that a line spanned by two points with the same label does not necessarily contain Q_∞. It follows that there is a subset $S^* \subseteq S$ of at least $q^2-\epsilon-(k^2+k)(k^2+k-1) > q^2-\epsilon-(k^2+k)(k^2+k-1)-1$ points, having the property that a line spanned by two points with the same label does contain Q_∞, i.e. they have a label in $\mathcal{V}^*$. This means that S has configuration (iii). □

The following three lemmas will show that the affine point set $S(U)$, defined by the unital U, must satisfy the first configuration of Lemma 7.3.4.

Lemma 7.3.5. *Suppose $q>2$ and $k \in \mathbb{N}$, $k < \sqrt{q}-1$. Let U be a unital containing a point P_∞ such that $q^2-\epsilon$, $\epsilon \leq kq$, of the $(q+1)$-secants through P_∞ are Baer secants. The corresponding point set $S(U)$ cannot have the form (ii) of Lemma 7.3.4.*

Proof. The subset of $S(U)$ containing all points with label v_i, will be denoted by $S_{v_i}(U)$. Suppose that $S(U)$ is of the form (ii) of Lemma 7.3.4. There are two distinct labels, say v_1 and v_2, occurring at least $q-k$ times, such that for $i=1,2$, the points of $S_{v_i}(U)$ lie on an affine line L_i. The affine lines L_1 and L_2 intersect in an affine point A.

Let T be the spread line corresponding to P_∞. A line of $\mathcal{L}$ through v_1 induces a point of L_1 in the quotient space $\mathrm{PG}(4,q)/T$. Hence, all the lines of $\mathcal{L}$ containing v_1 are contained in the three-space $\Sigma_1 = \langle T, L_1\rangle$. Similarly, the lines of $\mathcal{L}$ containing v_2 are contained in the three-space $\Sigma_2 = \langle T, L_2\rangle$. Let α be the plane $\langle T, A\rangle$, then clearly α is the intersection $\Sigma_1 \cap \Sigma_2$. Moreover, as the plane α is not contained in H_∞, there are q points of U_{aff} contained in α.

There are at most $k+1$ lines, say $n_1, \ldots, n_{k+1}$, of α through v_1 which do not occur as the intersection $\langle l_i, l_j \rangle \cap \alpha$, where l_i, l_j are lines of $\mathcal{L}$ through v_1 in the three-space Σ_1. Similarly, there are at most $k+1$ lines $n'_1, \ldots, n'_{k+1}$ of α through v_2 which do not occur as the intersection $\langle l_i, l_j \rangle \cap \alpha$, where l_i, l_j are lines of $\mathcal{L}$ through v_2 in the three-space Σ_2.

Suppose that a point of U in α belongs to a plane $\langle l_i, l_j \rangle$, where l_i, l_j are lines of $\mathcal{L}$ through the same point of T, then the secant subplane defined by l_i, l_j contains $2q+2$ points of U, a contradiction by Lemma 7.3.1. This implies that each of the q points of U in α necessarily lies on one of the lines $n_1, \ldots, n_{k+1}$ and on one of the lines $n'_1, \ldots n'_{k+1}$. However, there are only $(k+1)^2$ such points and $q > (k+1)^2$, a contradiction. □

Consider a Baer subplane π of $\mathrm{PG}(2, q^2)$ containing the point P_∞. It is clear that π/P_∞ defines a Baer subline in the quotient space $\mathrm{PG}(2, q^2)/P_\infty$. This can be translated to the ABB-representation in the following way. Recall that a Baer subplane π, tangent to l_∞ at P_∞, corresponds to a Baer ruled cubic $\mathcal{B}$ with line directrix T. It follows that $\mathcal{B}/T$ defines a Baer conic in the quotient space $\mathrm{PG}(4, q)/T$.

Lemma 7.3.6. *Suppose $q \geq 16$ and $k \in \mathbb{N}$, $k \leq \sqrt{q}/2 - 2$. Let U be a unital containing a point P_∞ such that $q^2 - \epsilon$, $\epsilon \leq kq$, of the $(q+1)$-secants through P_∞ are Baer secants. Suppose $S(U)$ is as described in Lemma 7.3.4 case (iii), with subset $S^*(U) \subseteq S(U)$. Then there exists a Baer ruled cubic $\mathcal{B}$ in $\mathrm{PG}(4, q)$, containing two lines of $\mathcal{L} = \{l_1, \ldots, l_{q^2-\epsilon}\}$, such that the corresponding Baer conic in $\mathrm{PG}(4, q)/T$ contains at least $\lfloor \frac{q+7}{2} \rfloor$ points of $S^*(U)$.*

Proof. Consider $S(U)$ as described in Lemma 7.3.4 case (*iii*), with point Q_∞ at infinity. There is a subset $S^*(U) \subseteq S(U)$ of at least $q^2 - kq - (k^2+k)(k^2+k-1) - 1$ points of $S(U)$, such that points of $S^*(U)$ with the same label lie on an affine line containing the point Q_∞.

Choose a point $R \in S^*(U)$ having label v, this label v occurs at most q times. Hence, there are at least

$$q^2 - (k+1)q - (k^2+k)(k^2+k-1) - 1$$

points of $S^*(U)$, not with label v. We will call these points *good points*. The affine points which are not good, are called *bad points*.

Consider the line $l \in \mathcal{L}$ defined by R. We want to find a Baer ruled cubic, containing l, such that the corresponding Baer conic in $\mathrm{PG}(4, q)/T$ contains at least $\lfloor \frac{q+7}{2} \rfloor$ points of $S^*(U)$. Since such a conic always contains $R \in S^*(U)$, we want to find a conic with at least $\lfloor \frac{q+5}{2} \rfloor$ good points and at most $\lceil \frac{q-3}{2} \rceil$ bad points (one of which is R).

Consider a good point R_1 and its corresponding line $l_1 \in \mathcal{L}$. As all good points have a label different from v, the points R_1 and R have a different label. Hence, the lines l and l_1 intersect T in distinct points, so they are contained in a unique Baer ruled cubic (by Lemma 7.2.1). Consider the corresponding Baer conic C_1 in $\mathrm{PG}(4,q)/T$. If the conic C_1 contains at least $\lfloor \frac{q+5}{2} \rfloor$ good points, the result follows. Now, suppose that C_1 contains at most $\lfloor \frac{q+3}{2} \rfloor$ good points. Then there are at least $q^2-(k+1)q-(k^2+k)(k^2+k-1)-1-\frac{q+3}{2}$ good points that do not belong to C_1. Since $q \geq 4(k+1)^2$, this number is larger than zero.

Hence, we can find a good point R_2 that does not lie on C_1. The point R_2 defines a line l_2 of $\mathcal{L}$. Again, we know that the lines l and l_2 intersect T in a different point. Take the Baer ruled cubic defined by l and l_2, and consider the corresponding Baer conic C_2 in $\mathrm{PG}(4,q)/T$. Recall that two distinct Baer conics intersect in at most two points, hence C_2 meets C_1 in R and in at most one other point. If the conic C_2 contains at least $\lfloor \frac{q+5}{2} \rfloor$ good points, the result follows. So, suppose that at most $\lfloor \frac{q+3}{2} \rfloor$ points of C_2 are good points.

Since $q^2-(k+1)q-(k^2+k)(k^2+k-1)-1-2\frac{q+3}{2} > 0$, we can find a good point R_3, not contained in $C_1 \cup C_2$. Applying the same reasoning to R_3, we find a new Baer ruled cubic containing l. The corresponding Baer conic C_3 contains R and R_3, and is different from both C_1 and C_2. Thus, C_3 meets both in at most 1 point different from R.

Continuing this reasoning, suppose we have m Baer conics $C_1, \ldots, C_m$ through R, each containing at most $\lfloor \frac{q+3}{2} \rfloor$ good points. Hence, there are still at least

$$q^2-(k+1)q-(k^2+k)(k^2+k-1)-1-m\frac{q+3}{2}$$

good points not contained in one of the conics C_i, $i=1,\ldots,m$. When $m=2k^2+4$, we obtain the parabola

$$q^2-(k^2+k+3)q-(k^4+2k^3+3k^2-k+7)$$

with largest zero point equal to

$$q=q_0=\frac{(k^2+k+3)+\sqrt{(k^2+k+3)^2+4(k^4+2k^3+3k^2-k+7)}}{2}.$$

Since, by assumption,

$$\begin{aligned} q &\geq 4(k+2)^2 \\ &> q_0 = \frac{(k^2+k+3) + \sqrt{(k^2+k+3)^2 + 4(k^4+2k^3+3k^2-k+7)}}{2}, \end{aligned}$$

there is at least one good point not on $C_1 \cup \ldots \cup C_m$, say R_{m+1}. Consider the line $l_{m+1} \in \mathcal{L}$ corresponding to R_{m+1}. The Baer ruled cubic $\mathcal{B}$ defined by l and l_{m+1} induces a Baer conic C_{m+1} in $\mathrm{PG}(4,q)/T$.

There are at most $(k+1)q + (k^2+k)(k^2+k-1) + 1$ bad points contained in $\mathrm{PG}(4,q)/T$. Each conic C_i, $i = 1, \ldots, m$, contains at most $\lfloor \frac{q+3}{2} \rfloor$ good points, hence at least $\lceil \frac{q-1}{2} \rceil$ bad points, one of which is R. Since two conics have at most one bad point in common different from R, the conics $C_1, \ldots, C_m$ cover at least $1 + m\lceil \frac{q-3}{2} \rceil - \frac{m(m-1)}{2}$ bad points. The conic C_{m+1} can intersect each conic C_i, $i = 1, \ldots, m$, in at most one bad point. Hence, there are at most

$$\begin{aligned} &1 + m + \left[(k+1)q + (k^2+k)(k^2+k-1) + 1\right] - \left[1 + m\frac{q-3}{2} - \frac{m(m-1)}{2}\right] \\ &= 1 + m + (k+1)q + (k^2+k)(k^2+k-1) - m\frac{q-3}{2} + \frac{m(m-1)}{2} \end{aligned}$$

bad points contained in C_{m+1}. To check that this number is strictly smaller than $\frac{q-1}{2}$, we consider the inequality

$$(-m+2k+1)q + 2(k^2+k)(k^2+k-1) + m^2 + 4m + 3 < 0.$$

This is equivalent to

$$q > \frac{2(k^2+k)(k^2+k-1) + m^2 + 4m + 3}{m - 2k - 1},$$

which is valid when $q \geq 16$, since

$$q \geq 4(k+2)^2 > \frac{2(k^2+k)(k^2+k-1) + m^2 + 4m + 3}{m - 2k - 1}.$$

Hence, the Baer ruled cubic $\mathcal{B}$ has at most $\lceil \frac{q-3}{2} \rceil$ bad points, that is, at least $\lfloor \frac{q+5}{2} \rfloor$ good points. It follows that $\mathcal{B}$ contains at least $\lfloor \frac{q+7}{2} \rfloor$ points of $S^*(U)$ and thus satisfies the conditions of the statement. $\square$

Lemma 7.3.7. *Suppose $q \geq 16$ and $k \in \mathbb{N}$, $k \leq \sqrt{q}/2 - 2$. Let U be a unital containing a point P_∞ such that $q^2 - \epsilon$, $\epsilon \leq kq$, of the $(q+1)$-secants through P_∞*

are Baer secants. The corresponding affine point set $S(U)$ cannot have the form (iii) of Lemma 7.3.4.

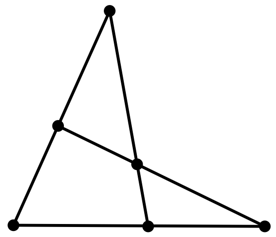
Proof. Suppose that the set $S(U)$ has the form (*iii*) of Lemma 7.3.4 with point Q_∞ at infinity. Let l_1 and l_2 be the lines of $\mathcal{L}$ defining the Baer ruled cubic $\mathcal{B}$ of Lemma 7.3.6. A tangent subplane contains (at most) $2q+2$ points of U, hence $\mathcal{B}$ contains (at most) one point of U_{aff} not on l_1 and l_2. Let μ be a plane (necessarily skew from T) containing a Baer conic C contained in $\mathcal{B}$. We can identify $\mathrm{PG}(4,q)/T$ with μ, and so the intersection points of $U \cap \mathcal{B}$ define the points R_1, R_2 in C (corresponding to l_1 and l_2 respectively) and at most one extra point R in C.

By Lemma 7.3.6, there are at least $\lfloor \frac{q+7}{2} \rfloor$ points of the Baer conic C contained in $S^*(U)$, that is, two points of $S^*(U)$ with the same label lie on a line containing Q_∞. Hence, we find at least two lines L_A and L_B through Q_∞, each intersecting C in two points with the same label. At most one of these lines, say L_B, contains the point R. Hence, L_A intersects $C \setminus \{R\}$ in two points Q_1, Q_2, having the same label v. The points Q_1 and Q_2 are each contained in a generator line of the Baer ruled cubic, say n_1 and n_2. Since Q_1 and Q_2 are different from R, for $i = 1, 2$, the line n_i either has no affine intersection point with the lines of $\mathcal{L}$ or is equal to l_1 or l_2.

Both points Q_i, $i = 1, 2$, have label v, hence, the planes $\langle T, n_i \rangle$, $i = 1, 2$, each contain a line of $\mathcal{L}$ through v, say l_{k_1} and l_{k_2} respectively. Since the line n_i is either equal to l_{k_i} or does not have an affine intersection point with l_{k_i}, both lines n_i, $i = 1, 2$, have to meet T in v. This implies that we find two generator lines of the same Baer ruled cubic having a point in common, a contradiction by the definition of a ruled cubic surface, which concludes the proof. □

As a combination of previous lemmas, we have found that $S(U)$ must satisfy configuration (*i*) of Lemma 7.3.4. We will show that in this case, the points of U on the $q^2 - \epsilon$ Baer secants are contained in a unique unital, namely an ovoidal Buekenhout-Metz unital. This leads to the conclusion that U is an ovoidal Buekenhout-Metz unital.

First, we prove that $q^2 - \epsilon$ Baer secants of an ovoidal Buekenhout-Metz unital are never contained in any other unital. We need the definition of an *O'Nan configuration*, this is a collection of four distinct lines meeting in six distinct points, as illustrated in the following picture.



It is known that an ovoidal Buekenhout-Metz unital contains no O'Nan configurations through its special point. A simple proof of this can be found in the proof of [12, Lemma 7.42].

We will call a line of $\mathrm{PG}(2, q^2)$ which is secant to a unital U', a *U'-secant.*

Lemma 7.3.8. *Consider in $\mathrm{PG}(2, q^2)$ an ovoidal Buekenhout-Metz unital U' with special point P_∞, and consider a set $\{L_1, \ldots, L_\epsilon\}$ of U'-secants through P_∞. Suppose a unital U of $\mathrm{PG}(2, q^2)$ contains P_∞ and all points of U' that do not lie on one of the ϵ secant lines L_i. If $\epsilon \leq \frac{(q-1)q}{2}$, then U and U' coincide.*

Proof. We will show that the result holds when $\epsilon = \frac{(q-1)q}{2}$, then the result easily follows for all $\epsilon \leq \frac{(q-1)q}{2}$.

Consider the set U_0 consisting of all points contained in U', but not on one of the U'-secants L_i, $i = 1, \ldots, \epsilon$. By assumption, all these points are contained in $U \cap U'$. Recall that for every unital $\widetilde{U}$, a point of $\widetilde{U}$ lies on q^2 $\widetilde{U}$-secants and a point not on $\widetilde{U}$ lies on only $q^2 - q$ $\widetilde{U}$-secants. This means, if a point Q belongs to strictly more than $q^2 - q$ lines intersecting U_0 in at least two points, then Q is contained in any unital containing all points of U_0. Hence, in that case, Q is contained in $U \cap U'$.

Consider a point $R \in U' \setminus U_0$ and say $L_1 = P_\infty R$. We will prove that there are at most $q - 2$ U'-secants M_j, containing R but different from L_1, having at most one point in common with U_0. If that is the case, then there are at least $q^2 - q + 1$ U'-secants through R containing at least two points of U_0, and hence, the point R is contained in $U \cap U'$.

Consider a U'-secant M_1, different from L_1, containing R and (at most) one point of U_0. This line intersects at least $q - 1$ U'-secants L_i, different from L_1, in a point of U', say $L_2, \ldots, L_q$.

Take a U'-secant M_2 through R, different from L_1 and M_1, containing at most 1 point of U_0. Since U' contains no O'Nan configurations through the point P_∞, there is at most one U'-secant L_i, $i \neq 1$, containing P_∞, such that the points $L_i \cap M_1$ and $L_i \cap M_2$ are both points of U'. Hence, M_2 intersects at least $q-2$ new U'-secants L_i (i.e. different from $L_1, \ldots, L_q$) in a point of U', say $L_{q+1}, \ldots, L_{2q-2}$.

Consider a third U'-secant M_3 through R, different from L_1, M_1, M_2. With the same reasoning as above, the line M_3 intersects at least $q-3$ U'-secants L_i (different from $L_1, \ldots, L_{2q-2}$) in a point of U', say $L_{2q-1}, \ldots, L_{3q-5}$.

If there are at most $q-2$ U'-secants M_j, containing R and having 0 or 1 points in common with U_0, the result follows. Otherwise, by continuing this process, the U'-secant M_{q-1} intersects at least $q-(q-1) = 1$ U'-secant L_i, different from the previously enumerated lines $L_1, \ldots, L_m$. We have found $m+1$ distinct U'-secants L_j where

$$m+1 \;=\; 1+(q-1)+(q-2)+\cdots+(q-(q-2))+1 \;=\; \frac{q(q-1)}{2}+1.$$

This is in contradiction with the restriction on the number of U'-secants L_j, since

$$\frac{q(q-1)}{2}+1 \;>\; \frac{q(q-1)}{2} \;=\; \epsilon.$$

We have proved that there are at most $q-2$ U'-secants through R containing 0 or 1 points of U_0. Hence, the point R belongs to $U \cap U'$. It follows that all points $R \in U'$ are contained in $U \cap U'$, which proves the result. $\square$

Lemma 7.3.9. *Suppose q and δ satisfy the conditions of Table 7.2. Consider a unital U containing a point P_∞ such that at least $q^2-\delta-1$ of the $(q+1)$-secants through P_∞ are Baer secants. If $S(U)$ satisfies configuration (i) of Lemma 7.3.4, then U is an ovoidal Buekenhout-Metz unital with special point P_∞.*

Proof. If the set $S(U)$ satisfies configuration (i) of Lemma 7.3.4, then all points of $S(U)$ have the same label. This implies that all $q^2-\delta-1$ lines of $\mathcal{L}$ go through a common point, say v of the line T. By Lemma 7.2.1, two lines l_i and l_j of $\mathcal{L}$ define a unique secant subplane. By Lemma 7.3.1, such a subplane has no affine intersection with any other line of $\mathcal{L}$. This means that in the 3-dimensional quotient space $\mathrm{PG}(4,q)/v$, the lines of $\mathcal{L}$ define a set K of $q^2-\delta-1$ points forming a cap. As a plane through T contains at most one line of $\mathcal{L}$, the line T defines a point in this quotient space, which extends the cap K to a cap K' of size $q^2-\delta$. By Theorems 7.2.2 and 7.2.3, the cap K can be extended to a unique ovoid $\mathcal{O}$. The cone with vertex v and base $\mathcal{O}$ defines an ovoidal Buekenhout-Metz unital U'

which has $q^2-\delta-1$ secant lines in common with U. Since $\delta+1 \leq \frac{(q-1)q}{2}$, by Lemma 7.3.8, U is an ovoidal Buekenhout-Metz unital. $\square$

Theorem 7.3.10. *Suppose that q and ϵ satisfy the conditions of Table 7.1. Let U be a unital containing a point P_∞ such that at least $q^2-\epsilon$ of the $(q+1)$-secants through P_∞ are Baer secants, then U is an ovoidal Buekenhout-Metz unital with special point P_∞.*

Proof. When q and ϵ satisfy the conditions of Table 7.1, we have $q \geq 16$ and $\epsilon \leq \min(\delta+1, \sqrt{q}q/2-2q)$, with q and δ satisfying the conditions of Table 7.2.

Consider the set $S(U)$ defined by the Baer secants to U at P_∞. By Lemma 7.3.3, this set satisfies the conditions of Lemma 7.3.4. Hence, since $q>2$ and $\epsilon<(\sqrt{q}-1)q$, the set $S(U)$ has one of the three configurations of Lemma 7.3.4. By Lemma 7.3.5 ($q>2$ and $\epsilon<(\sqrt{q}-1)q$) and Lemma 7.3.7 ($q \geq 16$ and $\epsilon \leq \sqrt{q}q/2-2q$), only the first configuration is possible. Since $\epsilon \leq \delta+1$, by Lemma 7.3.9, U is an ovoidal Buekenhout-Metz unital. $\square$

Combining Theorem 7.3.10 with Theorem 7.1.2, we obtain the following corollary.

Corollary 7.3.11. *Suppose that q and ϵ satisfy the conditions of Table 7.1. Let U be a unital in $\mathrm{PG}(2,q^2)$. If there is a point P_∞ in U that belongs to at least $q^2-\epsilon$ Baer secants, and there exists a Baer secant of U not through P_∞, then U is a classical unital.*

Summary

This appendix provides a summary of the new results obtained in this thesis. We only define the concepts necessary to understand the statements and we refer to the original text for more details.

In Chapter 1, basic concepts and definitions in finite geometry are recalled. Note that the projective space corresponding to the vector space $V(n,q)$ is denoted by $\mathrm{PG}(n-1,q)$. We use the following notation for the points of projective spaces. Consider the vector space $V \simeq \mathbb{F}_{q^{n_0}} \times \cdots \times \mathbb{F}_{q^{n_s}}$ of rank $n = \sum_{i=0}^{s} n_i$ over $\mathbb{F}_q$, for some positive integers n_i. A point P of the corresponding projective space defined by the vector $(a_0, \ldots, a_s)$, where $a_i \in \mathbb{F}_{q^{n_i}}$, will be written as $(a_0, \ldots, a_s)_{\mathbb{F}_q}$, emphasizing the fact that every $\mathbb{F}_q$-multiple of $(a_0, \ldots, a_s)$ gives rise to the point P, i.e. $(a_0, \ldots, a_s)_{\mathbb{F}_q} = \{(\lambda a_0, \ldots, \lambda a_s) \mid \lambda \in \mathbb{F}_q^*\}$.

Part I

Part I consists of two chapters, providing characterisations of elementary pseudo-caps (Chapter 2) and Desarguesian spreads (Chapter 3), both in terms of spread inducing elements.

Characterisations of elementary pseudo-caps

In Chapter 2, we study the higher dimensional equivalents of caps, arcs and ovoids.

Definition. A *pseudo-cap* is a set $\mathcal{A}$ of $(n-1)$-spaces in $\mathrm{PG}(2n+m-1,q)$ such that any three elements of $\mathcal{A}$ span a $(3n-1)$-space.

Examples of pseudo-caps in $\mathrm{PG}(kn-1,q)$ arise by applying field reduction to caps in $\mathrm{PG}(k-1,q^n)$ and if a pseudo-cap is obtained by field reduction, we call it *elementary*. Every element E_i of a pseudo-cap $\mathcal{A}$ of $\mathrm{PG}(2n+m-1,q)$ *induces* a partial spread

$$\mathcal{S}_i := \{E_1, \ldots, E_{i-1}, E_{i+1}, \ldots, E_{|\mathcal{A}|}\}/E_i$$

in the quotient space $\mathrm{PG}(n+m-1,q) \cong \mathrm{PG}(2n+m-1,q)/E_i$. Obviously, every element of an elementary pseudo-cap induces a partial spread which extends to a Desarguesian spread.

This chapter focusses on the characterisation of two types of pseudo-caps, namely pseudo-(hyper)ovals in $\mathrm{PG}(3n-1,q)$ and (weak) eggs in $\mathrm{PG}(4n-1,q)$.

Definition. A pseudo-cap in $\mathrm{PG}(3n-1,q)$ of size q^n+1, respectively q^n+2, is called a *pseudo-oval*, respectively *pseudo-hyperoval*.

Using the connection between pseudo-ovals and elation Laguerre planes, we obtain the following theorem and deduce a natural corollary.

Theorem 2.3.22. *If $\mathcal{O}$ is a pseudo-oval in $\mathrm{PG}(3n-1,q)$, $q=2^h$, $h>1$, n prime, such that the spread induced by every element of $\mathcal{O}$ is Desarguesian, then $\mathcal{O}$ is elementary.*

Corollary 2.3.23. *Let $\mathcal{H}$ be a pseudo-hyperoval in $\mathrm{PG}(3n-1,q)$, $q=2^h$, $h>1$, n prime, such that the spread induced by at least q^n+1 elements of $\mathcal{H}$ is Desarguesian, then $\mathcal{H}$ is elementary.*

Definition. A *weak egg* in $\mathrm{PG}(2n+m-1,q)$ is a pseudo-cap of size q^m+1.

A weak egg $\mathcal{E}$ in $\mathrm{PG}(2n+m-1,q)$ is called an *egg* if each element $E\in\mathcal{E}$ is contained in an $(n+m-1)$-space, T_E, which is skew from every element of $\mathcal{E}$ different from E.

An important tool in the investigation of (weak) eggs, is the following concept. A (weak) egg $\mathcal{E}$ in $\mathrm{PG}(2n+m-1,q)$, $m>n$, is *good* at an element $E\in\mathcal{E}$ if every $(3n-1)$-space containing E and at least two other elements of $\mathcal{E}$, contains exactly q^n+1 elements of $\mathcal{E}$. A (weak) egg that has at least one good element is called a *good (weak) egg*.

We provide a connection between good weak eggs and weak eggs that contain an element that induces a Desarguesian spread.

Theorem 2.4.2.

(*i*) *If a weak egg $\mathcal{E}$ in $\mathrm{PG}(2n+m-1,q)$ is good at an element E, then E induces a partial spread which extends to a Desarguesian spread.*

(*ii*) *Let $\mathcal{E}$ be a weak egg in $\mathrm{PG}(2n+m-1,q)$ for q odd or an egg in $\mathrm{PG}(2n+m-1,q)$ for q even. If an element $E\in\mathcal{E}$ induces a partial spread which extends to a Desarguesian spread, then $\mathcal{E}$ is good at E.*

We use this connection to obtain two characterisation results of weak eggs in $\mathrm{PG}(4n-1,q)$.

Theorem 2.4.5. *Suppose $n>1$, $q^n>4$, consider a weak egg $\mathcal{E}$ in $\mathrm{PG}(4n-1,q)$. Then $\mathcal{E}$ is elementary if and only if the following three properties hold:*

- *$\mathcal{E}$ is good at an element E,*

- *there exists a* $(3n-1)$*-space, disjoint from* E*, containing at least* 5 *elements* E_1*,* E_2*,* E_3*,* E_4*,* E_5 *of* $\mathcal{E}$*,*

- *all pseudo-ovals of* $\mathcal{E}$ *containing* $\{E, E_1\}$*,* $\{E, E_2\}$ *or* $\{E, E_3\}$ *are elementary.*

Theorem 2.4.9. *Consider a pseudo-cap* $\mathcal{E}$ *in* $\mathrm{PG}(4n-1, q)$*,* $q > 2$*, with* $|\mathcal{E}| > q^{n+k} + q^n - q^k + 1$*,* q *odd, and* $|\mathcal{E}| > q^{n+k} + q^n + 2$*,* q *even, where* k *is the largest divisor of* n *with* $k \neq n$*. The pseudo-cap* $\mathcal{E}$ *is elementary if and only if two of its elements induce a partial spread which extends to a Desarguesian spread.*

Corollary 2.4.10. *A weak egg in* $\mathrm{PG}(4n-1, q)$ *which is good at two distinct elements is elementary.*

A geometric characterisation of normal spreads

In Chapter 3, we obtain characterisations of spreads in terms of their normal elements.

Definition. We say that an element E of an $(n-1)$-spread $\mathcal{S}$ of $\Pi = \mathrm{PG}(rn-1, q)$ is *normal* if $\mathcal{S}$ induces a spread in the $(2n-1)$-space spanned by E and any other element of $\mathcal{S}$, or equivalently, if $\mathcal{S}/E$ induces an $(n-1)$-spread in the quotient space $\Pi/E \cong \mathrm{PG}((r-1)n-1, q)$.

A spread is called *normal* if and only if all of its elements are normal. Moreover, for $r > 2$, it is well known that a spread is normal if and only if it is Desarguesian.

Definition. A *(matrix) spread set* is a family $\mathbf{M}$ of q^n $n \times n$-matrices over $\mathbb{F}_q$ such that, for every two distinct $A, B \in \mathbf{M}$, the matrix $A - B$ is also non-singular.

With every spread set $\mathbf{M}$, there corresponds an $(n-1)$-spread $\mathcal{S}(\mathbf{M})$ in $\mathrm{PG}(2n-1, q)$:

$$\mathcal{S}(\mathbf{M}) = \{E_A \mid A \in \mathbf{M}\} \cup \{E_\infty\},$$

where

$$E_A = (I, A) = \{(x, xA)_{\mathbb{F}_q} \mid x \in \mathbb{F}_q^n\}$$

and

$$E_\infty = (0, I) = \{(0, x)_{\mathbb{F}_q} \mid x \in \mathbb{F}_q^n\}.$$

The following connections are known.

$\mathcal{S}$ is a *nearfield spread*	$\Leftrightarrow$	$\mathcal{S} \cong \mathcal{S}(\mathbf{M})$ with $\mathbf{M}$ closed under multiplication;
$\mathcal{S}$ is a *semifield spread*	$\Leftrightarrow$	$\mathcal{S} \cong \mathcal{S}(\mathbf{M})$ with $\mathbf{M}$ closed under addition;
$\mathcal{S}$ is a *Desarguesian spread*	$\Leftrightarrow$	$\mathcal{S} \cong \mathcal{S}(\mathbf{M})$ with $\mathbf{M}$ closed under multiplication and under addition.

Note that, given $\mathbb{F}_q$-linear maps $a_1, \ldots, a_r$ from $\mathbb{F}_{q^n}$ to itself, the set

$$\{(a_1(x), \ldots, a_r(x))_{\mathbb{F}_q} \mid x \in \mathbb{F}_{q^n}\}$$

corresponds to an $(n-1)$-space of $\mathrm{PG}(rn-1, q)$. When choosing a basis for $\mathbb{F}_{q^n} \cong \mathbb{F}_q^n$ over $\mathbb{F}_q$, the $\mathbb{F}_q$-linear map $a_i, i = 1, \ldots, r$, is represented by an $n \times n$-matrix $A_i, i = 1, \ldots, r$, over $\mathbb{F}_q$ acting on row vectors of $\mathbb{F}_q^n$ from the right. We abuse notation and write the corresponding $(n-1)$-space of $\mathrm{PG}(rn-1, q)$ as

$$(A_1, \ldots, A_r) := \{(xA_1, \ldots, xA_r)_{\mathbb{F}_q} \mid x \in \mathbb{F}_q^n\}.$$

We obtain the following characterisations of spreads in terms of their normal elements.

Theorem 3.3.1. *An $(n-1)$-spread $\mathcal{S}$ in $\mathrm{PG}(rn-1, q), r > 2$, having r normal elements in general position is $\mathrm{P\Gamma L}$-equivalent to*

$$\mathcal{S}_r(\mathbf{M}) = \{(A_1, A_2, \ldots, A_r) \mid A_i \in \mathbf{M}\},$$

for some nearfield spread set $\mathbf{M}$.

Theorem 3.4.3. *Consider an $(n-1)$-spread $\mathcal{S}$ in $\mathrm{PG}(rn-1, q)$, $r > 2$. If $\mathcal{S}$ contains $r+1$ normal elements in general position, then $\mathcal{S}$ is a Desarguesian spread.*

Theorem 3.5.5. *Consider an $(n-1)$-spread $\mathcal{S}$ in $\mathrm{PG}(3n-1, q)$, q odd. If $\mathcal{S}$ contains 3 normal elements contained in the same $(2n-1)$-space, then $\mathcal{S}$ is $\mathrm{P\Gamma L}$-equivalent to*

$$\begin{aligned}\mathcal{T}_3(\mathbf{M}, \mathbf{M_0}) = &\{(A, B, I) \mid A, B \in \mathbf{M}\} \\ &\cup \{(I, C, 0) \mid C \in \mathbf{M_0}\} \cup \{(0, I, 0)\},\end{aligned}$$

for some spread set $\mathbf{M_0}$ and a semifield spread set $\mathbf{M}$.

Part II

In Part II, we consider linear representations (Chapter 4) and their graphs (Chapter 5).

Definition. Let $\mathcal{K}$ be a point set in a hyperplane $H_\infty \cong \mathrm{PG}(n,q)$ of $\mathrm{PG}(n+1,q)$. The *linear representation* $T_n^*(\mathcal{K})$ of $\mathcal{K}$ is a point-line incidence structure $(\mathcal{P},\mathcal{L})$ with natural incidence, point set $\mathcal{P}$ and line set $\mathcal{L}$ as follows:

$\mathcal{P}$: the affine points of $\mathrm{PG}(n+1,q)$, i.e. not contained in H_∞,

$\mathcal{L}$: the lines of $\mathrm{PG}(n+1,q)$ intersecting H_∞ exactly in a point of $\mathcal{K}$.

The isomorphism problem for linear representations

In Chapter 4, we study the isomorphism problem for linear representations. For this we need the definition of the closure of a point set.

Definition. If a point set S contains a frame of $\mathrm{PG}(n,q)$, then its *closure* $\widehat{S}$ consists of the points of the smallest n-dimensional subgeometry of $\mathrm{PG}(n,q)$ containing all points of S.

We make the distinction between the case where the closure of $\mathcal{K}$ is H_∞ or a non-trivial subgeometry of H_∞. The following result is obtained for $\widehat{\mathcal{K}} = H_\infty$.

Theorem 4.2.10. *Let $q > 2$. Let $\mathcal{K}$ and $\mathcal{K}'$ denote point sets in $H_\infty = \mathrm{PG}(n,q)$ such that*

- *there is no plane of H_∞ intersecting $\mathcal{K}$ in two intersecting lines, or in two intersecting lines minus their intersection point;*
- *the closure $\widehat{\mathcal{K}}$ is equal to H_∞.*

If α is an isomorphism of incidence structures between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$, then α is induced by an element of $\mathrm{P\Gamma L}(n+2,q)_{H_\infty}$ mapping $\mathcal{K}$ to $\mathcal{K}'$.

When $\widehat{\mathcal{K}}$ is equal to a subgeometry of H_∞, we need to view the linear representation in an other setting, that is, we need to consider the generalised linear representation isomorphic to $T_n^*(\mathcal{K})$.

Definition. Let $\mathcal{K}$ be a set of disjoint $(t-1)$-dimensional subspaces in $\Pi_\infty \cong \mathrm{PG}(m,q)$, q a prime power. Embed Π_∞ as a hyperplane in $\mathrm{PG}(m+1,q)$. The *generalised linear representation* $T_{m,t-1}^*(\mathcal{K})$ of $\mathcal{K}$ is the incidence structure $(\mathcal{P}',\mathcal{L}')$ with natural incidence for which:

$\mathcal{P}'$: the *affine* points of $\mathrm{PG}(m+1,q)$, i.e. not contained in Π_∞,

$\mathcal{L}'$: the t-spaces of $\mathrm{PG}(m+1,q)$ containing a $(t-1)$-space of $\mathcal{K}$, but not lying in Π_∞.

Consider a point set $\mathcal{K}$ in $H_\infty \cong \mathrm{PG}(n,q^t)$ such that its closure $\widehat{\mathcal{K}}$ is an n-dimensional $\mathbb{F}_q$-subgeometry $\mathcal{S}$ of H_∞. Consider the linear representation $T_n^*(\mathcal{K})$ embedded in $\mathrm{PG}(n+1,q^t)$. The points of the hyperplane $H_\infty \cong \mathrm{PG}(n,q^t)$ in $\mathrm{PG}(n+1,q^t)$ can be represented as $(t-1)$-dimensional spaces of a Desarguesian spread $\mathcal{D}_\infty$ in $J_\infty \cong \mathrm{PG}(t(n+1)-1,q)$ under a field reduction map $\mathcal{F}$. The affine points of $\mathrm{PG}(n+1,q^t)$ with respect to H_∞ can be identified with the affine points of the space $\mathrm{PG}(t(n+1),q) \setminus J_\infty$. The lines of $\mathrm{PG}(n+1,q^t)$ intersecting H_∞ in a point of $\mathcal{K}$ correspond to the t-dimensional spaces of $\mathrm{PG}(t(n+1),q)$ meeting J_∞ in an element of $\mathcal{F}(\mathcal{K}) \subset \mathcal{D}_\infty$. In this way we obtain the generalised linear representation $T^*_{t(n+1)-1,t-1}(\mathcal{F}(\mathcal{K}))$, which is thus clearly isomorphic to $T_n^*(\mathcal{K})$.

Theorem 4.5.1. *Let $\mathcal{K}$ and $\mathcal{K}'$ denote point sets in $H_\infty \cong \mathrm{PG}(n,q^t)$, $t>1$, such that the closures $\widehat{\mathcal{K}}$ and $\widehat{\mathcal{K}'}$ are non-trivial n-dimensional subgeometries of H_∞. Suppose $\widehat{\mathcal{K}} \cong \mathrm{PG}(n,q)$ and let α be an isomorphism between $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$. Then α is induced by an element of $\mathrm{P\Gamma L}(t(n+1)+1,q)_{J_\infty}$ mapping $\mathcal{F}(\mathcal{K})$ onto $\mathcal{F}(\mathcal{K}')$.*

Using these results, we obtain that, under a mild condition, isomorphic linear representations $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$ lead to isomorphic point sets $\mathcal{K}$ and $\mathcal{K}'$.

Theorem 4.5.3. *Let $\mathcal{K}$ and $\mathcal{K}'$ be two point sets of $H_\infty = \mathrm{PG}(n,q)$, $q>2$, $n>1$, each containing a frame, such that $\langle\mathcal{K}\rangle = \langle\mathcal{K}'\rangle = H_\infty$. If $\widehat{\mathcal{K}} = H_\infty$, suppose furthermore that there is no plane of H_∞ intersecting $\mathcal{K}$ in two intersecting lines, or in two intersecting lines minus their intersection point. The linear representations $T_n^*(\mathcal{K})$ and $T_n^*(\mathcal{K}')$ are isomorphic if and only if the point sets $\mathcal{K}$ and $\mathcal{K}'$ are $\mathrm{P\Gamma L}$-equivalent.*

Infinite families of semisymmetric graphs

In Chapter 5 we give a general construction leading to non-isomorphic families $\Gamma_{n,q}(\mathcal{K})$ of connected q-regular semisymmetric graphs of order $2q^{n+1}$ embedded in $\mathrm{PG}(n+1,q)$, by using the linear representation $T_n^*(\mathcal{K})$ of a particular point set $\mathcal{K}$ of size q.

Definition. We say that a graph is *vertex-transitive* if its automorphism group acts transitively on the vertices. Similarly, a graph is *edge-transitive* if its automorphism group acts transitively on the edges. A graph is *semisymmetric* if it is regular and edge-transitive but not vertex-transitive.

We introduce the concept of the incidence graph of a linear representation.

Definition. The *point-line incidence graph* $\Gamma_{n,q}(\mathcal{K})$ of the linear representation $T_n^*(\mathcal{K})$ is the bipartite graph with as classes the point set $\mathcal{P}$ and line set $\mathcal{L}$ of $T_n^*(\mathcal{K})$ and adjacency corresponding to the natural incidence of the structure $T_n^*(\mathcal{K})$.

Theorem 5.5.1. *Let $\mathcal{K}$ be a point set of $H_\infty = \mathrm{PG}(n,q)$ of size q spanning H_∞ such that every point of $H_\infty \backslash \mathcal{K}$ belongs to at least one tangent line to $\mathcal{K}$, and such that $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ acts transitively on the points of $\mathcal{K}$. Then the graph $\Gamma_{n,q}(\mathcal{K})$ is a connected semisymmetric graph.*

In this summary, we only give a brief overview of all constructions of Chapter 5. These are represented in Table 5.1. Note that we use the abbreviation NRC for a normal rational curve. If $q = p$ is prime with $|\mathcal{K}| = q$ and $\mathcal{K}$ contains a frame, then every automorphism of $T_n^*(\mathcal{K})$ is geometric, that is, every automorphism is induced by a collineation of the ambient space. When $q = p^h$ is not prime and the closure $\widehat{\mathcal{K}}$ is isomorphic to a subgeometry $\mathrm{PG}(n, q_0)$, where q_0 statisfies $q = q_0^k$ for some k, then one can obtain an automorphism group which is $\mathrm{ng}(q_0, k) := \frac{1}{k} q_0^{\frac{k(k-1)}{2}} \prod_{i=1}^{k-1}(q_0^i - 1)$ times larger than the geometric one. When $\mathcal{K}$ contains a frame, this is the full automorphism group.

$\mathcal{K}$	Condition	$\lvert\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K}))\rvert$	Ref.
basis	$q = n+1$	$> hq^{n+1}(q-1)q!\mathrm{ng}(p,h)$	§5.4.1
frame	$q = n+2$	$hq^{n+1}(q-1)^n q!\mathrm{ng}(p,h)$	§5.4.1
$\subset$ NRC	$q \geq n+3$	$hq^{n+2}(q-1)^2$	§5.4.2
$\subset$ q-arc	$q > 4$ even	$hq^5(q-1)^2$	§5.4.3
$\subset$ Glynn-arc	$q = 9$	$9^6 8^2$	§5.4.4
$\subset Q^-(3,q)$	$q > 4$ square	$2hq^5(q-1)^2\mathrm{ng}(\sqrt{q},2)$	§5.5.1
$\subset$ Tits ovoid	$q = 2^{2(2e+1)}$	$hq^5(q-1)(\sqrt{q}-1)\mathrm{ng}(\sqrt{q},2)$	§5.5.2
$\subset Q^+(3,q)$	$q > 4$ square	$2hq^5(q-1)(\sqrt{q}-1)^2\mathrm{ng}(\sqrt{q},2)$	§5.5.3
$\subset$ cone $V\mathcal{O}$	$q = q_0^k$	$kq^{2n+1}(q-1)^2\lvert\mathrm{P\Gamma L}(n,q_0)_{\mathcal{O}}\rvert\mathrm{ng}(q_0,k)$	§5.5.4

Table 5.1: Overview of all constructions of Chapter 5

Part III

In Part III, we consider substructures in the *André/Bruck-Bose representation* or *ABB-representation* of $\mathrm{PG}(2,q^n)$ in $\mathrm{PG}(2n,q)$. This representation is obtained as follows. Let $\mathcal{D}$ be a Desarguesian $(n-1)$-spread in $\mathrm{PG}(2n-1,q)$. Embed $\mathrm{PG}(2n-1,q)$ as a hyperplane H_∞ in $\mathrm{PG}(2n,q)$. Consider the following incidence structure $A(\mathcal{D}) = (\mathcal{P},\mathcal{L})$, where incidence is natural:

$\mathcal{P}$: the *affine points*, i.e. the points of $\mathrm{PG}(2n,q)\backslash H_\infty$,

$\mathcal{L}$: the n-spaces of $\mathrm{PG}(2n,q)$ intersecting H_∞ in an element of $\mathcal{D}$.

The incidence structure $A(\mathcal{D})$ is a Desarguesian affine plane $\mathrm{AG}(2,q^n)$. Its projective completion $\overline{A(\mathcal{D})} \cong \mathrm{PG}(2,q^n)$ can be found by adding H_∞ as the line l_∞ at infinity where the elements of $\mathcal{D}$ correspond to the points of l_∞.

Subgeometries in the ABB-representation

In Chapter 6, we investigate the ABB-representation of $\mathbb{F}_{q^k}$-sublines and $\mathbb{F}_{q^k}$-subplanes of $\mathrm{PG}(2,q^n)$. We will settle the characterisation of the following cases:

- $\mathbb{F}_{q^k}$-sublines tangent to or contained in the line at infinity:
  for general k and n (Theorem 6.3.3 and Theorem 6.3.8),
- $\mathbb{F}_q$-sublines external to the line at infinity:
  for general n (Theorem 6.3.5),
- $\mathbb{F}_{q^k}$-subplanes secant to the line at infinity:
  for general k and n (Theorem 6.4.1),
- $\mathbb{F}_q$-subplanes tangent to the line at infinity:
  for general n (Theorem 6.4.5).

These characterisations involve the unique *indicator set* of a Desarguesian spread, which is obtained as follows. Embed $\Lambda = \mathrm{PG}(rn-1,q)$ as a subgeometry of $\Lambda^* = \mathrm{PG}(rn-1,q^n)$. The subgroup of $\mathrm{P\Gamma L}(rn,q^n)$ fixing Λ pointwise is isomorphic to $\mathrm{Aut}(\mathbb{F}_{q^n}/\mathbb{F}_q)$; take a generator σ of this group. Consider a Desarguesian $(n-1)$-spread $\mathcal{D}$ in Λ. There exists an $(r-1)$-space ν in Λ^* skew to the subgeometry Λ, such that for every point $P \in \nu$, the $(n-1)$-space $\langle P, P^\sigma, \ldots, P^{\sigma^{n-1}}\rangle$ intersects Λ in an element of $\mathcal{D}$. The set $\{\nu, \nu^\sigma, \ldots, \nu^{\sigma^{n-1}}\}$ is called the *indicator set* of $\mathcal{D}$ and is unique.

We prove that a Desarguesian $(n-1)$-spread $\mathcal{D}$ of $\mathrm{PG}(rn-1, q)$ has a unique Desarguesian $(k-1)$-*subspread* $\mathcal{D}_k$ for each $k|n$, that is, a Desarguesian $(k-1)$-spread of $\mathrm{PG}(rn-1, q)$ that induces a $(k-1)$-spread in each element of $\mathcal{D}$.

We start with the characterisations of sublines.

Theorem 6.3.3.

(i) *The affine points of an $\mathbb{F}_{q^k}$-subline in $\mathrm{PG}(2, q^n)$ tangent to l_∞ correspond to the points of a k-dimensional affine space π in the ABB-representation, such that $\overline{\pi} \cap H_\infty$ is an element of $\mathcal{D}_k$.*

(ii) *Conversely, let π be a k-dimensional affine space of Σ such that $\overline{\pi}$ intersects H_∞ in a spread element of $\mathcal{D}_k$. Then the points of π correspond to the affine points of an $\mathbb{F}_{q^k}$-subline tangent to l_∞.*

Recall that a *normal rational curve* in $\mathrm{PG}(k, q)$, $2 \le k \le q-2$, is a $(q+1)$-arc PGL-equivalent to the $(q+1)$-arc

$$\{(0, \ldots, 0, 1)_{\mathbb{F}_q}\} \cup \{(1, t, t^2, t^3, \ldots, t^k)_{\mathbb{F}_q} \mid t \in \mathbb{F}_q\}.$$

We say that a point set $\mathcal{C}$ in $\mathrm{PG}(N, q)$ is a normal rational curve of *degree* (or *order*) l if and only if it is a normal rational curve in a l-dimensional subspace of $\mathrm{PG}(N, q)$. Consider a normal rational curve $\mathcal{C}$ of $\mathrm{PG}(k, q)$, $2 \le k \le q-2$, and the embedding of $\mathrm{PG}(k, q)$ as a subgeometry of $\mathrm{PG}(k, q^n)$. There is a unique normal rational curve $\mathcal{C}^*$ in $\mathrm{PG}(k, q^n)$ containing the points of $\mathcal{C}$ and we call this the $\mathbb{F}_{q^n}$-*extension* $\mathcal{C}^*$ of $\mathcal{C}$.

Theorem 6.3.5. *A set of points $\mathcal{C}$ in $\mathrm{PG}(2n, q)$, $n \le q-2$, is the ABB-representation of an $\mathbb{F}_q$-subline m of $\mathrm{PG}(2, q^n)$ external to l_∞ if and only if*

(i) *$\mathcal{C}$ is a normal rational curve of degree k contained in a k-space intersecting H_∞ in an element of $\mathcal{D}_k$,*

(ii) *its extension $\mathcal{C}^*$ to $\mathrm{PG}(2n, q^n)$ intersects the indicator set $\{\Pi, \Pi^\sigma, \ldots, \Pi^{\sigma^{k-1}}\}$ of $\mathcal{D}_k$ in k conjugate points.*

Moreover, the smallest subline containing m and tangent to l_∞ is an $\mathbb{F}_{q^k}$-subline.

Theorem 6.3.8. *Let $\mathcal{S}$ be a set of q^k+1 elements of the Desarguesian spread $\mathcal{D}$ of $H_\infty = \mathrm{PG}(2n-1, q)$, $q > 2$. Then the following statements are equivalent:*

(i) *$\mathcal{S}$ is the ABB-representation of an $\mathbb{F}_{q^k}$-subline of l_∞,* (*ii*) *for any three elements of $\mathcal{S}$, the unique regulus through them is contained in $\mathcal{S}$,*

(*iii*) *there exists a $(2k-1)$-dimensional subspace of H_∞ intersecting each element of $\mathcal{S}$ in a $(k-1)$-dimensional space,*

(*iv*) *there exists a $(2k-1)$-dimensional subspace of H_∞ intersecting each element of $\mathcal{S}$ in a $(k-1)$-dimensional space of $\mathcal{D}_k$.*

Secondly, we consider the characterisations of subplanes.

Theorem 6.4.1. *A set Π of affine points in $\mathrm{PG}(2n,q)$ is the ABB-representation of the affine points of an $\mathbb{F}_{q^k}$-subplane in $\mathrm{PG}(2,q^n)$ secant to l_∞ if and only if*

(*i*) *Π is a $2k$-dimensional affine space,*

(*ii*) *its projective completion $\overline{\Pi}$ intersects H_∞ in a $(2k-1)$-space which intersects q^k+1 elements of $\mathcal{D}$ in exactly a $(k-1)$-space.*

Moreover, this $(2k-1)$-space intersects each of the q^k+1 spread elements of $\mathcal{D}$ in a $(k-1)$-space of $\mathcal{D}_k$.

Consider two normal rational curves $\mathcal{C}_1$ and $\mathcal{C}_2$ of degree k and l respectively. Embed both curves in $\mathrm{PG}(N,q)$, $N \geq k+l+1$, such that the subspaces they span, of dimension k and l respectively, are disjoint.

Let ρ_1, ρ_2 be maps from $\mathrm{PG}(1,q) \to \mathrm{PG}(N,q)$ defined by

$$\rho_1 : (s,t)_{\mathbb{F}_q} \mapsto \left(\sum_{i=0}^{k} s^{k-i}t^i e_i\right)_{\mathbb{F}_q},$$

$$\rho_2 : (s,t)_{\mathbb{F}_q} \mapsto \left(\sum_{i=0}^{l} s^{l-i}t^i f_i\right)_{\mathbb{F}_q},$$

for defining vectors e_i, f_i such that $\mathcal{C}_1 = \mathrm{Im}(\rho_1)$ and $\mathcal{C}_2 = \mathrm{Im}(\rho_2)$.

A *normal rational scroll of bidegree* $\{k,l\}$ defined by $\{\mathcal{C}_1, \mathcal{C}_2\}$ consists of the set of lines of $\mathrm{PG}(N,q)$ defined as follows

$$\{\langle \rho_1(P), \rho_2(\psi(P))\rangle \mid P \in \mathrm{PG}(1,q)\},$$

where ψ is an element of $\mathrm{PGL}(2,q)$.

Theorem 6.4.5. *A set S of affine points of* $\mathrm{PG}(2n,q)$*,* $n \leq q-2$*, corresponds to the affine points of a tangent* $\mathbb{F}_q$*-subplane* μ *if and only if* S *consists of the affine points of a normal rational scroll defined by curves* $\{\mathcal{C},\mathcal{N}\}$ *satisfying for some* $k|n$*:*

(*i*) $\mathcal{C}$ *is a normal rational curve of degree* k *contained in an affine* k*-space* π*, for which* $\overline{\pi} \cap H_\infty$ *is an element* E_1 *of* $\mathcal{D}_k$*, such that its* $\mathbb{F}_{q^n}$*-extension* $\mathcal{C}^*$ *contains all conjugate points* $\{P, P^\sigma, \ldots, P^{\sigma^{k-1}}\}$ *generating the spread element* E_1*,*

(*ii*) $\mathcal{N}$ *is a normal rational curve of degree* $k-1$ *contained in an element* E_2 *of* $\mathcal{D}_k$*, where* E_1 *and* E_2 *are not contained in the same element of* $\mathcal{D}$*, such that its* $\mathbb{F}_{q^n}$*-extension* $\mathcal{N}^*$ *contains all conjugate points* $\{Q, Q^\sigma, \ldots, Q^{\sigma^{k-1}}\}$ *generating the spread element* E_2*,*

(*iii*) *the* $\mathbb{F}_{q^n}$*-extension of the normal rational scroll contains the lines* $\langle P^{\sigma^j}, Q^{\sigma^j}\rangle$*, each line contained in an indicator space* Π^{σ^j} *of* $\mathcal{D}_k$*, for all* $j \in \{0,1,\ldots,k-1\}$*.*

Moreover, in that case the smallest subplane containing μ *and secant to* l_∞ *is an* $\mathbb{F}_{q^k}$*-subplane.*

Unitals with many Baer secants through a fixed point

In Chapter 7 we obtain a characterisation of ovoidal Buekenhout-Metz unitals in $\mathrm{PG}(2,q^2)$. A *unital* U in $\mathrm{PG}(2,q^2)$ is a set of q^3+1 points such that every line meets U in 1 or $q+1$ points. All known unitals in $\mathrm{PG}(2,q^2)$ arise as ovoidal Buekenhout-Metz unitals.

An *ovoidal Buekenhout-Metz unital* in $\mathrm{PG}(2,q^2)$ can be constructed as follows. Consider the ABB-representation in $\mathrm{PG}(4,q)$ of $\mathrm{PG}(2,q^2)$ with respect to the line l_∞, with line spread $\mathcal{D}$ of H_∞ corresponding to the points of l_∞. Let $\mathcal{O}$ be an ovoid in a 3-space of $\mathrm{PG}(4,q)$, such that H_∞ contains exactly one point $A \in \mathcal{O}$ and such that the tangent plane of $\mathcal{O}$ at A does not contain the spread line $T \in \mathcal{D}$ incident with A. Let V be a point on T, $V \neq A$. Consider the ovoidal cone with vertex V and base $\mathcal{O}$, this point set corresponds to a unital U in $\mathrm{PG}(2,q^2)$. The point P_∞ of l_∞, corresponding to the spread line $T \in \mathcal{D}$, is called the *special point* of the ovoidal Buekenhout-Metz unital U.

First, we consider a lemma on the intersection of ovoidal Buekenhout-Metz unitals with other unitals.

Lemma 7.3.8. *Consider in* $\mathrm{PG}(2,q^2)$ *an ovoidal Buekenhout-Metz unital* U' *with special point* P_∞*, and consider a set* $\{L_1,\ldots,L_\epsilon\}$ *of* U'*-secants through* P_∞*. Sup-*

pose a unital U of $\mathrm{PG}(2,q^2)$ *contains* P_∞ *and all points of* U' *that do not lie on one of the* ϵ *secant lines* L_i. *If* $\epsilon \leq \frac{(q-1)q}{2}$, *then* U *and* U' *coincide.*

A secant of a unital U is a *Baer secant* to U if it intersects U in a Baer subline of $\mathrm{PG}(2,q^2)$. The main result of Chapter 7 is the following characterisation of ovoidal Buekenhout-Metz unitals in terms of Baer secants. This result relies on extension results saying that large caps in $\mathrm{PG}(3,q)$ are contained in unique ovoids.

Theorem 7.3.10. *Suppose* q *and* ϵ *satisfy the conditions of Table 7.1. Let* U *be a unital in* $\mathrm{PG}(2,q^2)$ *containing a point* P *lying on at least* $q^2-\epsilon$ *Baer secants to* U, *then* U *is an ovoidal Buekenhout-Metz unital with special point* P.

As a corollary, we obtain a characterisation for the classical unital.

Corollary 7.3.11. *Suppose that* q *and* ϵ *satisfy the conditions of Table 7.1. Let* U *be a unital in* $\mathrm{PG}(2,q^2)$. *If there is a point* P_∞ *in* U *that belongs to at least* $q^2-\epsilon$ *Baer secants, and there exists a Baer secant of* U *not through* P_∞, *then* U *is a classical unital.*

ϵ	Conditions
$\epsilon \leq q-3$	q even, $q \geq 16$
$\epsilon \leq 2q-7$	q even, $q \geq 128$
$\epsilon \leq \frac{\sqrt{q}q}{4} - \frac{39q}{64} - O(\sqrt{q}) + 1$	q odd, $q \geq 17$, $q = p^{2e}$, $e \geq 1$
$\epsilon \leq \frac{\sqrt{q}q}{2} - 2q$	q odd, $q \geq 17$, $q = p^{2e+1}$, $e \geq 0$
$\epsilon \leq \frac{\sqrt{q}q}{2} - \frac{67q}{16} + \frac{5\sqrt{q}}{4} - \frac{1}{12}$	q odd, $q \geq 17$, $q = p^h$, $p \geq 5$
$\epsilon \leq \frac{\sqrt{q}q}{2} - \frac{35q}{16} - O(\sqrt{q}) + 1$	q odd, $q \geq 23^2$, $q \neq 5^5, 3^6$, $q = p^h$, h even for $p = 3$

Table 7.1: Conditions for Theorem 7.3.10 and Corollary 7.3.11

Samenvatting

Deze appendix geeft een beknopte samenvatting van de resultaten bekomen in deze thesis. Voor meer details verwijzen we naar de originele tekst.

Hoofdstuk 1 introduceert de algemene notaties en definities in eindige meetkunde. De projectieve ruimte die overeenkomt met de vectorruimte $V(n,q)$ wordt genoteerd als $\mathrm{PG}(n-1,q)$.

Deel I bestaat uit twee hoofdstukken. We beschouwen twee karakterisaties, enerzijds van elementaire pseudo-caps in Hoofdstuk 2, anderzijds van Desarguesiaanse spreads in Hoofdstuk 3. Beide karakterisaties werden bekomen door te kijken naar spreads in deelruimten, namelijk in de quotiëntruimten van (sommige) elementen.

In **Hoofdstuk 2** bestuderen we het equivalent van caps, bogen en ovoïden in hogere dimensie. Een *pseudo-cap* is een verzameling $\mathcal{A}$ van $(n-1)$-ruimten in $\mathrm{PG}(m-1,q)$ waarvoor geldt dat elke drie elementen uit $\mathcal{A}$ een $(3n-1)$-ruimte opspannen. *Elementaire* voorbeelden van pseudo-caps in $\mathrm{PG}(kn-1,q)$ ontstaan door veldreductie toe te passen op caps in $\mathrm{PG}(k-1,q^n)$. Elk element E van een pseudo-cap in $\mathrm{PG}(2n+m-1,q)$ *induceert* een partiële spread $\mathcal{S}$ in de quotiëntruimte $\mathrm{PG}(n+m-1,q) \cong \mathrm{PG}(2n+m-1,q)/E$. Door te veronderstellen dat sommige elementen *Desarguesiaanse spreads* induceren, bekomen we karakterisaties van elementaire *pseudo-hyperovalen* in $\mathrm{PG}(3n-1,q)$ (Sectie 2.3) en van elementaire *eieren* in $\mathrm{PG}(4n-1,q)$ (Sectie 2.4).

In **Hoofdstuk 3** verkrijgen we karakterisaties van $(n-1)$-spreads in $\mathrm{PG}(rn-1,q)$ in termen van hun normale elementen. Een element E van een $(n-1)$-spread $\mathcal{S}$ in $\mathrm{PG}(rn-1,q)$ is *normaal* als $\mathcal{S}$ een spread induceert in elke $(2n-1)$-ruimte opgespannen door E en een ander element van $\mathcal{S}$. Het is welbekend dat, als $r>2$, een spread *Desarguesiaans* is als en slechts als al zijn elementen normaal zijn. We tonen aan dat dit ook waar is onder een zwakkere veronderstelling, namelijk, een $(n-1)$-spread $\mathcal{S}$ in $\mathrm{PG}(rn-1,q)$, $r>2$, is Desarguesiaans als en slechts als hij ten minste $r+1$ normale elementen in algemene positie heeft.

In **Deel II** beschouwen we lineaire representaties (Hoofdstuk 4) en hun corresponderende grafen (Hoofdstuk 5). Een *lineaire representatie* $T_n^*(\mathcal{K})$ is een punt-rechte incidentiestructuur ingebed in een projectieve ruimte $\mathrm{PG}(n+1,q)$, volledig bepaald door een puntenverzameling $\mathcal{K}$ bevat in een hypervlak H_∞.

Hoofdstuk 4 bestudeert het *isomorfisme probleem* voor lineaire representaties. Onder de voorwaarde dat $\mathcal{K}$ een geraamte bevat en dat geen enkel vlak $\mathcal{K}$ snijdt in enkel twee snijdende rechten, bewijzen we dat twee lineaire representaties $T_n^*(\mathcal{K})$ en $T_n^*(\mathcal{K}')$ isomorf zijn als en slechts als de puntenverzamelingen $\mathcal{K}$ en $\mathcal{K}'$ isomorf zijn. Onder dezelfde voorwaarden, bepalen we de volledige automorfismegroep van $T_n^*(\mathcal{K})$. Als de sluiting van $\mathcal{K}$ gelijk is aan het hypervlak H_∞, dan correspondeert elk automorfisme met een collineatie van de omringende ruimte. Als de sluiting van $\mathcal{K}$ gelijk is aan een niet-triviale deelmeetkunde van H_∞, dan bestaan er steeds automorfismen die niet overeenkomen met collineaties.

In **Hoofdstuk 5** beschouwen we de incidentiegraaf van een lineaire representatie $T_n^*(\mathcal{K})$. Hierdoor bekomen we een algemene constructie die leidt tot niet-isomorfe families van samenhangende q-reguliere semisymmetrische grafen van orde $2q^{n+1}$. Gebruikmakende van de resultaten van het vorige hoofdstuk, kunnen we in de meeste gevallen de volledige automorfismegroep van deze grafen bepalen.

In **Deel III** bestuderen we deelstructuren in de *André/Bruck-Bose representatie*, kortweg *ABB-representatie*, van $\mathrm{PG}(2, q^n)$ in $\mathrm{PG}(2n, q)$.

Hoofdstuk 6 onderzoekt de ABB-representatie van $\mathbb{F}_{q^k}$-deelrechten en $\mathbb{F}_{q^k}$-deelvlakken van $\mathrm{PG}(2, q^n)$. We bekomen een karakterisatie in de volgende gevallen: $\mathbb{F}_{q^k}$-deelrechten die raken aan of bevat zijn in de rechte op oneindig; $\mathbb{F}_q$-deelrechten disjunct aan de rechte op oneindig; $\mathbb{F}_{q^k}$-deelvlakken die een deelrechte gemeenschappelijk hebben met de rechte op oneindig; en $\mathbb{F}_q$-deelvlakken die een punt gemeenschappelijk hebben met de rechte op oneindig.

In **Hoofdstuk 7** bekomen we een karakterisatie van Buekenhout-Metz unitalen in $\mathrm{PG}(2, q^2)$. Een *unitaal* U in $\mathrm{PG}(2, q^2)$ is een verzameling van $q^3 + 1$ punten zodat elke rechte U snijdt in 1 of $q + 1$ punten. Elke gekende unitaal correspondeert met een *Buekenhout-Metz unitaal*, deze unitalen komen overeen met ovoïdale kegels in de ABB-representatie van $\mathrm{PG}(2, q^2)$ in $\mathrm{PG}(4, q)$. We tonen aan dat een unitaal U in $\mathrm{PG}(2, q^2)$ een Buekenhout-Metz unitaal is als en slechts als U een punt P bezit waarvoor er minstens $q^2 - \epsilon$ secanten door P de unitaal U snijden in een deelrechte, waarbij ϵ orde $2q$ heeft voor q even en orde $\frac{q^{3/2}}{2}$ voor q oneven.

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Index

Publications

All results presented in this document have appeared previously (partially, identically or modified) in the following publications, ordered chronologically.

- S. Rottey and L. Storme. Maximal partial line spreads of non-singular quadrics. *Des. Codes Cryptogr.* **72** (2014), 33–51.
- P. Cara, S. Rottey and G. Van de Voorde. The isomorphism problem for linear representations and their graphs. *Adv. Geom.* **14** (2014), 353–367.
- P. Cara, S. Rottey and G. Van de Voorde. A construction for infinite families of semisymmetric graphs revealing their full automorphism group. *J. Algebr. Comb.* **39** (2014), 967–988.
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- S. Rottey, J. Sheekey and G. Van de Voorde. Subgeometries in the André/Bruck-Bose representation. *Finite Fields Appl.* **35** (2015), 115–138.
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- S. Gravier, A. Parreau, S. Rottey, L. Storme and E. Vandomme. Identifying codes in vertex-transitive graphs and strongly regular graphs. *Electron. J. Combin.* **22** (2015), Paper 4.6, 26 pp.
- S. Rottey and G. Van de Voorde. Unitals with many Baer secants through a fixed point. *Submitted. arXiv:1503.05341.*
- S. Rottey and J. Sheekey. A geometric characterisation of Desarguesian spreads. *Submitted. arXiv:1504.01915.*

Everything will be okay in the end.
If it's not okay, it's not the end.
– Fernando Sabino