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Blocking sets and partial spreads in finite polar spaces

Jan De Beule

Promotor : Prof. Dr. L. Storme

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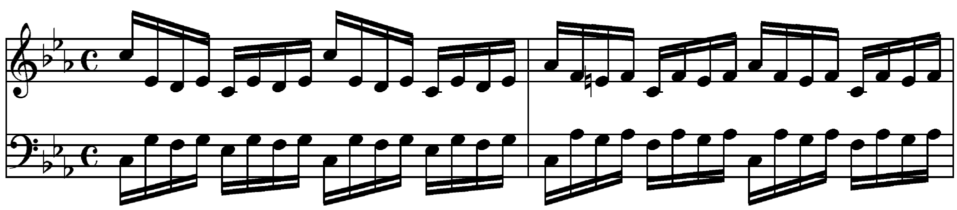

J. S. Bach, *Das Wohltemperierte Klavier, Teil I, Praeludium II.* BWV 847

Preface

The characterisation of substructures of incidence geometries, satisfying certain conditions, is a classical problem. In this thesis, we consider particular substructures of certain finite generalised quadrangles and finite classical polar spaces. Our aim is to contribute to the knowledge of these substructures.

In Chapter 1 the field of research is situated, and basic concepts are introduced. We give an overview of projective spaces, polar spaces and generalised quadrangles. For each of these geometries, we give definitions of important substructures, including those that will be considered in this thesis. Sometimes details are left for the introductory sections of each chapter. Furthermore, some historical references are given.

Chapter 2 can be seen as a stand alone part of the thesis, but the link with Chapter 3 will be made clear. In this chapter we consider an important class of finite generalised quadrangles: the so-called translation generalised quadrangles. A classical example of a translation generalised quadrangle is the non-singular parabolic quadric $\mathrm{Q}(4,q)$ in the 4-dimensional projective space $\mathrm{PG}(4,q)$.

Considering any point-line geometry, one can try to find a set of lines such that every point is covered exactly once. Such a set of lines is called a spread. Also partial spreads are interesting in many situations. Partial spreads are sets of lines covering all points at most once. Finding for instance upper bounds on the size of partial spreads, or revealing the geometrical structure of the uncovered points, is a classical problem for any partial spread of any point-line geometry.

In Chapter 2, we obtain theorems on partial spreads of translation generalised quadrangles, using characterisation theorems of quite different structures, namely minihypers in projective geometries. Minihypers have important applications in coding theory, and recent characterisation theorems are related to blocking sets of projective planes. Hence, results on partial spreads of translation generalised quadrangles become related to blocking sets of projective planes. This kind of characterisation theorems is found in e.g. certain characterisation theorems of partial spreads of projective spaces and is thus not surprising.

Trying to improve the results for the parabolic quadric $\mathrm{Q}(4, q)$, we make use of the description of this quadric as translation generalised quadrangle.

In Chapter 3, we consider a different problem. The non-singular parabolic quadric $\mathrm{Q}(6, q)$ in the six dimensional projective space $\mathrm{PG}(6, q)$ is an example of a finite classical polar space. A classical problem for all these spaces is the study of blocking sets, i.e. sets of points such that every generator contains at least one point of the set. The minimal examples are called ovoids. In the case of non-existence of ovoids, we can try to find and characterise the smallest minimal blocking sets. This is done in Chapter 3, for the finite classical polar space $\mathrm{Q}(6, q)$, $q > 16$, q even. We can obtain the characterisation using projection arguments and using technical results which are comparable to certain technical results of $T_2(\mathcal{O})$, the description of the non-singular parabolic quadric in four dimensions as translation generalised quadrangle when $\mathcal{O}$ is a conic. This fact is a clear link between Chapter 2 and Chapter 3.

Using the same techniques and recent results, we obtain a characterisation of the smallest minimal blocking sets of the finite classical polar space $\mathrm{Q}(6, q)$, q odd prime. These results are described in Chapter 4.

In Chapter 5, we lift the results of Chapter 4 to parabolic quadrics in higher dimensions. We again use projection arguments. In the last section of this chapter, we describe briefly the remaining problems.

In Chapter 6, we characterise the smallest minimal blocking sets of the Hermitian variety in even dimensional projective spaces. A lot of techniques from Chapter 4 and 5 can be used again. Furthermore, the different behaviour of the Hermitian variety becomes clear.

This thesis has been typeset with LaTeX, the music has been typeset with MusiXTeX and the figures have been created with Xfig. Cover artist: J.A.G. De Beule

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I would like to thank the people from the research group incidence geometry in our department. It is stimulating to work in the atmosphere of our group, leaded by the four geometers Prof. Dr. F. De Clerck, Prof. Dr. L. Storme, Prof Dr. J. A. Thas and Prof. Dr. H. Van Maldeghem. They are not only great teachers in geometry, but also the designers of the Ghent University GOA-project: "Incidence Geometries: Constructions, Characterizations and Applications" funding the research I did and I gratefully acknowledge the senior members of our research group for giving me a position as PhD student on the GOA-project.

It is fine to work in a dynamic research group like ours, with so many people working in geometry, and in that sense I thank Stefaan for the several pleasant and interesting mathematical chats we had. I also thank the colleagues from the department, the secretaries for their administrative help and Sara, Geert and Tom for the useful LaTeX tips.

During this period I had the opportunity to participate in conferences and to meet geometers from all over the world. These contacts were inspiring, and I thank the people who gave me all these opportunities.

In a less mathematical sense I would like to thank all my friends, some of them are also colleagues. We played a lot of card games (always during coffee breaks, but sometimes on a bad moment) and we had a lot of fun during the past years. I thank Sara for the many pleasant chats about little things, for the weekly game of badminton and for mentioning the abilities of google. It seems that we, mathematicians, sometimes first think for hours about a (computer) problem before discovering with google that somebody else in the world solved the same (computer) problem. I thank Pieter for providing the "tools" for the many secret risk evenings and I thank "Guru" Geert for the "testing of network applications".

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Jan De Beule
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Chapter 1

Introduction

This chapter introduces the basic structures and objects for this thesis. After preliminaries on finite fields, we introduce projective spaces in an axiomatic way. Although we mention the axiomatic description to emphasize the geometric way of thinking, we start as soon as possible with the introduction in an analytical way of the finite projective space. Then special objects inside projective spaces are described, such as blocking sets, spreads, quadrics, Hermitian varieties, ovals and ovoids, etc. The next two sections introduce polar spaces and generalised quadrangles, again in an axiomatic way. Basic properties are given and, in the last section, ovoids and spreads of generalised quadrangles and polar spaces are introduced.

It is beyond the scope to give the complete history of all these concepts, but, when possible, we try to describe how concepts were developed and we try to give the reference in which it was introduced.

1.1 Finite fields

Suppose that $q = p^h$, p a prime number and $h \geqslant 1$. The *finite field of order* q is always denoted by $\mathrm{GF}(q)$, it is well known that the finite field of order q is unique up to isomorphism.

If $\mathrm{GF}(q_1)$ is a subfield of $\mathrm{GF}(q_2)$, then $q_1 = p^{h_1}$ and $q_2 = p^{h_2}$, p prime, with $h_1 | h_2$.

We will sometimes use the *trace function* $\mathrm{Tr}_{q\to q_0}$ which is defined as follows. Suppose that $\mathrm{GF}(q_0)$ is a subfield of $\mathrm{GF}(q)$, then $q = q_0^d$, and

$$\mathrm{Tr}_{q\to q_0} : \mathrm{GF}(q) \to \mathrm{GF}(q_0) : x \mapsto x + x^{q_0} + x^{q_0^2} + \ldots + x^{q_0^{d-1}}.$$

1.2 Projective spaces

Projective spaces will be introduced using axioms, but only to illustrate that these structures do not need a complicated description. Quite fast we will introduce the models we work with and derive properties in the model itself, rather than doing a lot of work starting from axioms.

Starting from the axioms, one can derive a lot of geometric information on both projective spaces and planes. For projective spaces this is for instance done in [12]. Older works are for instance [101]. To read about the foundations of geometry in general and projective geometry in particular, we refer to [58]. For projective planes we refer to for instance [62] and also [66].

1.2.1 Basic definitions

A *point-line incidence structure* or *point-line geometry* is a triple $(\mathcal{P}, \mathcal{B}, \mathrm{I})$, where $\mathcal{P} \cap \mathcal{B} = \emptyset$ and where I is a symmetric incidence relation, $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$. The elements of $\mathcal{P}$ are called points and the elements of $\mathcal{B}$ are called lines.

A *non-degenerate projective space* is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ where I satisfies the following axioms.

(i) For any two distinct points p and q, there is exactly one line that is incident with p and q.[1]

(ii) Let a, b, c and d be four distinct points such that the line ab intersects the line cd. Then the line ac also intersects the line bd.

(iii) Any line is incident with at least three points.

(iv) There are at least two lines.

A *non-degenerate projective plane* is a non-degenerate projective space in which axiom (ii) is replaced by the following stronger axiom.

(ii') Any two lines have at least one point in common.

The above axioms are covering projective spaces as well as projective planes. An equivalent axiom system for projective planes is the following.

(i) For any two distinct points p and q, there is exactly one line that is incident with p and q.

[1] Since two distinct points p and q determine exactly one line, this line will often be denoted with pq.

(ii') For any two distinct lines L and M, there is exactly one point that is incident with L and M.

(iii') There exist four points of which no three are incident with the same line.

From now on we will leave the notion *non-degenerate*. We will simply talk about projective spaces and projective planes.

The structure of a projective space

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a projective space. Since two points determine exactly one line, we may identify a line with the set of points it contains. Suppose now that $A \subset \mathcal{P}$ is a subset of the point set of $\mathcal{S}$. The set A is called *linear* if every line meeting A in at least two points is completely contained in A. Define $\mathcal{B}'$ as the set of lines contained in the linear set A, $|A| \geqslant 2$, then it is clear that the incidence structure $\mathcal{S}(A) = (A, \mathcal{B}', \mathrm{I}')$ is a non-degenerate projective space or a line, with I' the incidence relation I restricted to the set A. We call $\mathcal{S}(A)$ a *linear subspace* of $\mathcal{S}$. Identifying lines with their point set, we can also identify linear subspaces with their corresponding point set. All elements of $\mathcal{B}$ are examples of linear subspaces; even the empty set, any singleton and the whole point set; hence every subset of the point set is contained in at least one linear set, and one defines the *span* of an arbitrary subset $A \subset \mathcal{P}$ as $\langle A \rangle = \bigcap\{S | A \subseteq S, S \text{ is a linear set}\}$.

A set A of points is called *linearly independent* if for any subset $A' \subset A$ and any point $p \in A \setminus A'$ we have $p \notin \langle A' \rangle$. A linearly independent set of points that spans the whole space $\mathcal{S}$ is called a *basis* of $\mathcal{S}$.

Having the concept of a basis, one can now simply define *dimension* as the number of points in a basis minus one, first having proved that every basis has the same number of points. We define the *rank* of a projective space as its dimension. The following formula is an important application. From now on, we mean "linear subspace" when we write "subspace".

Theorem 1.2.1. *(dimension formula) If U and V are two subspaces of the projective space $\mathcal{S}$, then* $\dim(\langle U, V \rangle) = \dim(U) + \dim(V) - \dim(U \cap V)$.

Lots of concepts can be developed now, but we will restrict to mentioning two important theorems, and discuss afterwards shortly their importance in classifying and characterising projective spaces.

Consider a projective space $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ of dimension at least 2. Choose different points p_1, p_2, p_3 and r_1, r_2, r_3 such that p_i and r_i are collinear with a point s, $s \neq p_i$ and $s \neq r_i$; and such that no three of the points s, p_i,

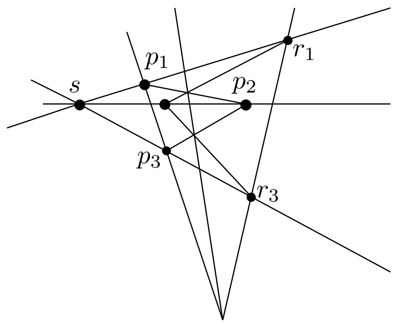

Figure 1.1: A Desargues configuration

$i = 1,2,3$ and s, r_i, $i = 1,2,3$ are collinear. The *theorem of Desargues* (or also $\mathcal{S}$ is *Desarguesian*) holds if the points $t_{ij} := p_ip_j \cap r_ir_j$, $i,j = 1,2,3$, $i \neq j$, lie on a common line.

The following theorem classifies all projective spaces of dimension at least 3. For a proof, we refer for instance to [12].

Theorem 1.2.2. *An n-dimensional projective space, $n \geqslant 3$, is Desarguesian.*

Consider again a projective space $\mathcal{S}$ of dimension at least 2. Consider two different intersecting lines L and M. Choose on both lines L and M three different points l_i and respectively m_i, $i = 1,2,3$; all six points different from $L \cap M$. The *theorem of Pappus* holds in $\mathcal{S}$ (or, $\mathcal{S}$ is *Pappian*) if the points $t_{ij} := l_im_j \cap l_jm_i$, $i,j = 1,2,3$, $i \neq j$, lie on a common line.

The link between the theorems of Desargues and Pappus is expressed in the *theorem of Hessenberg*. This theorem was already proved in [58].

Theorem 1.2.3. *If the theorem of Pappus holds in a projective space $\mathcal{S}$, then also the theorem of Desargues holds in this projective space.*

We will now introduce models of projective spaces; afterwards, the importance of the theorems of Desargues and Pappus will become clear.

The projective space $\mathrm{PG}(n, K)$

Using vector spaces over an arbitrary field (even more general: left vector spaces over skewfields), one can construct examples of projective spaces. Let K denote a field and $V(n+1, K)$ an $(n+1)$-dimensional vector space over K.

Denote by $P(V)$ the set of all i-dimensional subspaces, $0 \leqslant i \leqslant n$, of $V(n+1, K)$. Define the point set $\mathcal{P}$ as the set of 1-dimensional subspaces of

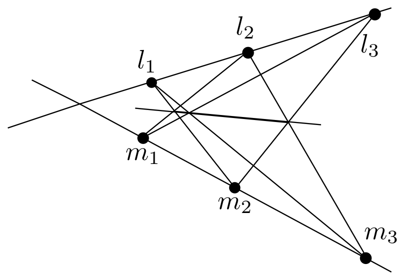

Figure 1.2: A Pappus configuration

V and the line set $\mathcal{B}$ as the set of 2-dimensional subspaces of V. If we define I as the symmetrised set theoretic containment, it is easy to check that the point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ satisfies the axioms of a projective space. We denote this geometry by $\mathrm{PG}(n, K)$.

For practical reasons, we will give an equivalent definition of $\mathrm{PG}(n, K)$. We call two vectors $\mathbf{X}$, $\mathbf{Y}$ of $V(n+1, K) \setminus \{\mathbf{0}\}$ *equivalent* if and only if $\mathbf{X} = t\mathbf{Y}$ for some $t \in K \setminus \{0\}$. The set of all equivalence classes is now the projective space $\mathrm{PG}(n, K)$, the elements of $\mathrm{PG}(n, K)$ are the points and the equivalence class of a vector $\mathbf{X}$ will be denoted by $P(\mathbf{X})$; $\mathbf{X}$ will also be called a *coordinate vector*. The points $P(\mathbf{X_1}), \ldots, P(\mathbf{X_r})$ are *linearly independent* if the set of corresponding vectors is linearly independent. For any $m \in \{-1, 0, 1, 2, \ldots, n\}$, a *subspace of geometric dimension* m or an m*-dimensional subspace* of $\mathrm{PG}(n, K)$ is just a set of points all of whose corresponding vectors form (together with the zero vector) an $(m+1)$-dimensional subspace of $V(n+1, K)$. A subspace of geometric dimension -1 is of course again the empty set. Points are 0-dimensional subspaces, lines have geometric dimension 1, planes geometric dimension 2 and hyperplanes geometric dimension $n-1$. A hyperplane is nothing else than a set of points $P(\mathbf{X})$ whose vectors satisfy a linear equation $\sum_{i=0}^{n} u_i x_i = 0$, with $(u_0, \ldots, u_n) \in V(n+1, K) \setminus \{\mathbf{0}\}$. From now on we will leave the notion *geometric* when talking about the dimension of a subspace of $\mathrm{PG}(n, K)$.

The model $P(V)$ admits to introduce immediately important concepts like linear subspaces via the elementary properties of the underlying vector space (or, more generally, the left vector space). The connection of this model with axiomatic projective spaces lies in the theorems of Desargues and Pappus.

Theorem 1.2.4. *Consider a projective space* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$.

(i) $\mathcal{S} = \mathrm{PG}(n, K)$ *for some skewfield* K *if and only if* $\mathcal{S}$ *is Desarguesian.*

(ii) $\mathcal{S} = \mathrm{PG}(n, K)$ *for some field* K *if and only if* $\mathcal{S}$ *is Pappian.*

Together with Theorem 1.2.2, we obtain the following corollary.

Corollary 1.2.5. *Any projective space* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ *of dimension at least* 3 *is a* $\mathrm{PG}(n, L)$ *for some skewfield* L.

This result is already present in [58]. A proof can also be found for instance in [12].

In the 2-dimensional case, many classes of non-Desarguesian projective planes are known. A good reference for projective planes, containing many examples, is for instance [62].

Affine spaces

Consider an n-dimensional projective space $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$. Choose a hyperplane H_∞ of $\mathcal{S}$. Define $\mathcal{P}'$ as the set of points in $\mathcal{P}$ not in H_∞ and $\mathcal{B}'$ as the set of lines in $\mathcal{B}$ not in H_∞. The incidence I' is the restriction of I to $(\mathcal{P}' \times \mathcal{B}') \cup (\mathcal{B}' \times \mathcal{P}')$. We call $\mathcal{A} = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ an *affine space of dimension* n.

Affine spaces can be introduced using axioms. We will not give axioms for affine spaces, but we mention that the above definition of an affine space is equivalent and shows that affine spaces are very closely related to projective spaces. Also the reverse construction is possible, i.e. one can consider an affine space and construct a projective space by adding the space at infinity of an affine space.

Rather than describing the possible axiomatic constructions, we just mention that models of affine spaces can also be constructed using vector spaces. Suppose $V(n, K)$ is the n-dimensional vector space over the field K. Suppose that $\mathbf{v_0} \in V(n, K)$ and that W is an i-dimensional subspace of $V(n, K)$, $0 \leqslant i \leqslant n - 1$. We define the *coset of* W *through* $\mathbf{v_0}$ as the set $\{\mathbf{v_0} + \mathbf{w} \| \mathbf{w} \in W\}$.

Denote by $A(V)$ the set of all cosets of subspaces of $V(n, K)$. Define the point set $\mathcal{P}$ as the set of the cosets of the 0-dimensional subspaces of V (the unique 0-dimensional subspace is just the zero vector) and the line set $\mathcal{B}$ as the cosets of the 1-dimensional vector spaces of V. Incidence is the symmetrised set theoretic containment. Then $\mathcal{A} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is an n-dimensional affine space, denoted by $\mathrm{AG}(n, K)$. As for the projective spaces, this construction can also be done using left vectorspaces over a skewfield L.

Duality

Duality is an important concept in projective geometry. Considering a projective plane, it is easy to see that by interchanging the role of points and lines, again a projective plane is obtained. This fact does not, however, imply that a projective plane is self-dual. Duality simply expresses that the dual of a projective plane is a (possibly different) projective plane. On the other hand, self-dual planes exist, for instance $\mathrm{PG}(2, K)$, with K an arbitrary field. Using the duality of vector spaces over a field, it is easy to see that $\mathrm{PG}(2, K)$ is self-dual. This is also true for $\mathrm{PG}(2, L)$, with L a skewfield, but a little more machinery is involved. The geometric reason for the selfduality of $\mathrm{PG}(n, L)$ is the Theorem of Desargues, which implies its own dual.

Since any projective space $\mathcal{S}$ of dimension at least 3 is a $\mathrm{PG}(n, L)$ for some skewfield L, $\mathcal{S}$ is self-dual. This duality is not expressed by interchanging points and lines, but by interchanging i-dimensional spaces with $(n-i-1)$-dimensional spaces.

We will denote the dual of a projective space $\mathcal{S}$ by $\mathcal{S}^D$.

Collineations and polarities

Although collineations can be defined for arbitrary point-line geometries, we will restrict to $\mathrm{PG}(n, K)$, K a field, for the definitions here.

If $\mathcal{S}$ and $\mathcal{S}'$ are two projective spaces $\mathrm{PG}(n, K)$ and $\mathrm{PG}(m, K)$, K a field and $n, m \geqslant 2$, a *collineation* is a bijection from the set of subspaces of $\mathcal{S}$ onto the set of subspaces of $\mathcal{S}'$ preserving incidence. In other words, if $\varphi : \mathcal{S} \to \mathcal{S}'$ is a collineation and α and β are two subspaces of $\mathrm{PG}(n, K)$, then $\alpha \subset \beta \iff \alpha^\varphi \subset \beta^\varphi$. This is a general definition for projective spaces $\mathrm{PG}(n, K)$. A classical lemma is that this definition implies $n = m$. Hence we are only left with the problem of defining collineations of projective lines, for, omitting the restriction $n \geqslant 2$ implies that arbitrary bijections between the point sets of lines are collineations. A collineation from a projective space to itself is called an *automorphism*.

When $n = 1$, consider the lines $\mathcal{S} = \mathrm{PG}(1, K)$ and $\mathcal{S}' = \mathrm{PG}(1, K')$ embedded in planes $\mathrm{PG}(2, K)$ and $\mathrm{PG}(2, K')$; then a *collineation* $\varphi : \mathcal{S} \to \mathcal{S}'$ is a map induced by a collineation of the planes. Hence if $\mathcal{S}$ and $\mathcal{S}'$ are two lines embedded in two planes $\mathcal{S}_0$ and $\mathcal{S}_0'$ respectively, and φ_0 is a collineation from $\mathcal{S}_0$ to $\mathcal{S}_0'$, mapping $\mathcal{S}$ to $\mathcal{S}'$, the map $\varphi_{|\mathcal{S}} : \mathcal{S} \to \mathcal{S}'$ is a collineation of the lines.

Consider two projective spaces $\mathcal{S} = \mathrm{PG}(n, K)$ and $\mathcal{S}' = \mathrm{PG}(n, K')$ with underlying vectorspaces $V(n+1, K)$ and $V'(n+1, K')$. It is clear that every semi-linear map $\phi : V(n+1, K) \to V'(n+1, K')$ induces a collineation $\varphi : \mathcal{S} \to \mathcal{S}'$. Also the converse is true and this theorem is called the *Fundamental theorem of projective geometry.* Hence we can describe collineations by bijective semi-linear maps from V to V' and make use of the coordinate description. Defining collineations algebraically also solves the problem of defining collineations between two projective lines.

Consider a collineation $\varphi : \mathcal{S} \to \mathcal{S}'$, then every point $p(\mathbf{X})$ is mapped onto a point $p(\mathbf{X}')$, and the relation between the two coordinate vectors of these points can be expressed by a non-singular $(n+1) \times (n+1)$ matrix A and a field isomorphism $\theta : K \to K'$: $t\mathbf{X}' = \mathbf{X}^\theta A$, where $\mathbf{X}^\theta = (x_0^\theta, \ldots, x_n^\theta)$ and $t \in K \setminus \{0\}$. When θ is the identity, φ is also called a *projectivity*.

A *coordinate frame* of $\mathrm{PG}(n, K)$ is a set of $n+2$ points such that no $n+1$ of them lie in a hyperplane. A classical result is that a projectivity is determined completely by the image of a coordinate frame. Furthermore, the group of collineations fixing a given coordinate frame pointwise is isomorphic to the automorphism group of the field K.

Consider $\mathcal{S} = \mathrm{PG}(n, K)$. A *correlation* of $\mathcal{S}$ is a collineation $\varphi : \mathcal{S} \to \mathcal{S}^D$. When φ is a projectivity from $\mathcal{S}$ to $\mathcal{S}^D$, then it is called a *reciprocity* of $\mathcal{S}$. Actually, with this definition, a correlation is a containment reversing bijection of the projective space, i.e. for any two subspaces $\alpha \subset \beta \iff \beta^\varphi \subset \alpha^\varphi$. An involutory correlation is called a *polarity*.

Suppose that φ is a polarity of $\mathrm{PG}(n, K)$. A point p is mapped onto the hyperplane p^φ, also called the *polar* of the point p. Conversely, a hyperplane π is mapped onto the point π^φ, also called the *pole* of the hyperplane π. Suppose that r, s are two points such that $r \in s^\varphi$ (and hence conversely $s \in r^\varphi$), then the points r and s are called *conjugate*, and the same definition can be applied for two hyperplanes, and, spaces of arbitrary dimension. A point p is called *self-conjugate* or *absolute* if and only if $p \in p^\varphi$.

Since a polarity is defined as a collineation, with the same arguments as for collineations, a polarity of $\mathrm{PG}(n, K)$ is determined by a non-singular $(n+1) \times (n+1)$ matrix T and a field automorphism $\theta \in Aut(K)$. The field automorphism is necessarily involutory. From the above definitions, it can be derived that a point $p(\mathbf{X})$ is self-conjugate if and only if $\mathbf{X}A(\mathbf{X}^\theta)^T = 0$. A self-conjugate subspace is sometimes called *isotropic*. The dimension of a maximal isotropic subspace of a polarity will be called the *projective index* of the polarity.

The different types of polarities of $\mathrm{PG}(n, K)$, $K = \mathrm{GF}(q)$, are given in Table 1.1. For the projective index of quadrics and Hermitian varieties we refer to Section 1.2.2. The projective index of a symplectic polarity is always $\frac{n-1}{2}$.

Finite projective spaces

From now on, we will only consider finite projective spaces. A projective space $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is finite if and only if the set $\mathcal{P}$ is finite (and hence also $\mathcal{B}$ is finite). The Theorem of Wedderburn, stating that every finite skewfield

n, q	type	A, θ	absolute points
$q = p^h$, p odd prime	orthogonal polarity	$A^T = A$, $\theta = 1$	$\mathbf{X}A\mathbf{X}^T = 0$ (a quadric)
$q = 2^h$,	pseudo polarity	$A^T = A$, not all $a_{ii} = 0$, $\theta = 1$	$\sum \sqrt{a_{ii}} x_i = 0$
n odd	symplectic polarity	$A^T = -A$, $\theta = 1$ $a_{ii} = 0$	$\mathrm{PG}(n, q)$
q square	Hermitian polarity	$(A^T)^\theta = A$, $\theta : x \mapsto x^{\sqrt{q}}$	$\mathbf{X}A(\mathbf{X}^\theta)^T = 0$ (a Hermitian variety)

Table 1.1: Polarities of $\mathrm{PG}(n, q)$

is a field, is geometricly expressed in the following theorem.

Theorem 1.2.6. *A finite Desarguesian projective space is also Pappian.*

With Theorem 1.2.2, we obtain

Corollary 1.2.7. *A finite projective space of dimension at least* 3 *is always isomorphic with* $\mathrm{PG}(n, K)$, *K a finite field* $\mathrm{GF}(q)$.

We will denote $\mathrm{PG}(n, \mathrm{GF}(q))$ by $\mathrm{PG}(n, q)$ and $\mathrm{AG}(n, \mathrm{GF}(q))$ by $\mathrm{AG}(n, q)$.

We will define some concepts and we will describe their notations. We wish to mention that a lot of these concepts can also be defined for arbitrary fields and even in the axiomatic case. Important works on finite projective geometry are certainly the series [60], [59], [61]. An older, but still interesting survey on the subject, is [39].

Combinatorics

Besides geometric information, we can now also obtain combinatorial information since the number of points, lines, etc. is finite. Since many counting arguments will be used in proofs, we will use the following information.

Consider $\mathrm{PG}(n, q)$. We will denote the set of i-dimensional subspaces by $\mathrm{PG}^{(i)}(n, q)$. Using the definition of $\mathrm{PG}(n, q)$ it is clear that $|\mathrm{PG}^{(0)}(n, q)| = \frac{q^{n+1}-1}{q-1}$, denoted by θ_n. By $\phi(n; r, q)$, we denote the cardinality of the set $\mathrm{PG}^{(r)}(n, q)$ and the number of r-dimensional subspaces through a fixed s-dimensional subspace of $\mathrm{PG}(n, q)$ is denoted by $\chi(s, r; n, q)$.

Theorem 1.2.8. **([60])**

(i) $\phi(r;n,q) = \frac{\prod_{i=n-r+1}^{n+1}(q^i-1)}{\prod_{i=1}^{r+1}(q^i-1)}$.

(ii) $\chi(s,r;n,q) = \frac{\prod_{i=r-s+1}^{n-s}(q^i-1)}{\prod_{i=1}^{n-r}(q^i-1)}$.

A finite projective space has *order* n if $n+1$ is the number of points on a line. A finite affine space has *order* n if n is the number of points on a line.

1.2.2 Quadrics and Hermitian varieties

A *quadric* in $\mathrm{PG}(n,q)$, $n \geqslant 1$, is the set of points whose coordinates satisfy an equation of the form $\sum_{\substack{i,j=0 \\ i\leqslant j}}^{n} a_{ij}X_iX_j = 0$ with not all a_{ij} equal to 0.

A *Hermitian variety* in $\mathrm{PG}(n,q^2)$, $n \geqslant 1$, is the set of points whose coordinates satisfy an equation of the form $\sum_{i,j=0}^{n} a_{ij}X_iX_j^q = 0$, not all $a_{ij} = 0$ and $a_{ij}^q = a_{ji}$ for all $i,j = 0,\ldots,n$. A quadric or Hermitian variety is called *singular* if there exists a change of the coordinate system which reduces the equation to an equation with less than $n+1$ variables.

If $n = 2$, a non-singular quadric is also called a *conic*. If $n = 2$, a non-singular Hermitian variety is also called a *Hermitian curve*.

Some definitions can be given for quadrics and Hermitian varieties in exactly the same way; we will then talk about varieties.

If a variety is singular, then it is known that the points of the variety are the points of a *cone*, i.e. all the points of the lines spanned by a point of an $(n-r)$-dimensional subspace π of $\mathrm{PG}(n,q)$ and a point of a non-singular variety $\mathcal{F}$ in an $(r-1)$-dimensional subspace π' skew to π. We will denote this cone with $\pi\mathcal{F}$. The singular points of the variety are the points of π.

Consider a variety $\mathcal{F}$. The *tangent space* in a point $p \in \mathcal{F}$ is the set of points of the lines through p intersecting $\mathcal{F}$ only in p or completely contained in $\mathcal{F}$. When p is a non-singular point of $\mathcal{F}$, the tangent space is a hyperplane and is also called the *tangent hyperplane*. When p is singular, then the tangent space is actually the whole projective space $\mathrm{PG}(n,q)$. We will denote the tangent space at the point $p \in \mathcal{F}$ by $T_p(\mathcal{F})$.

Concerning the classification of non-singular varieties, we mention the following results. In $\mathrm{PG}(2n,q)$, there is, up to collineations, only one non-singular quadric, called the *parabolic quadric*, denoted by $\mathrm{Q}(2n,q)$. There are, up to collineations, exactly two non-singular quadrics in $\mathrm{PG}(2n+1,q)$, the *hyperbolic quadric*, denoted by $\mathrm{Q}^+(2n+1,q)$, and the *elliptic quadric*, denoted by $\mathrm{Q}^-(2n+1,q)$. In $\mathrm{PG}(n,q^2)$, there is, up to collineation, exactly one non-singular Hermitian variety, denoted by $\mathrm{H}(n,q^2)$.

variety	standard equation	projective index
$Q(2n,q)$	$x_0^2 + x_1x_2 + \ldots + x_{2n-1}x_{2n}$	$n-1$
$Q^+(2n+1,q)$	$x_0x_1 + x_2x_3 + \ldots + x_{2n}x_{2n+1}$	n
$Q^-(2n+1,q)$	$f(x_0,x_1) + x_2x_3 + \ldots + x_{2n}x_{2n+1}$	$n-1$
	f is an irreducible quadratic polynomial over $GF(q)$	
$H(n,q^2)$	$x_0^{q+1} + x_1^{q+1} + \ldots + x_n^{q+1}$	$\lfloor \frac{n-1}{2} \rfloor$

Table 1.2: Standard forms and projective index of non-singular quadrics and Hermitian varieties

When q is even, every non-singular parabolic quadric $Q(2n,q)$ has a *nucleus*, i.e. a point on which every hyperplane is tangent in some point $p \in Q(2n,q)$, or, equivalently, every line on the nucleus has exactly one point in common with $Q(2n,q)$.

Consider a non-singular variety $\mathcal{F}$ in the projective space $PG(n,q)$ and consider the tangent hyperplane in a point $p \in \mathcal{F}$. It is known that $T_p(\mathcal{F}) \cap \mathcal{F} = p\mathcal{F}'$, i.e. a cone with vertex p and base a non-singular variety of the same type in a projective space $PG(n-2,q)$ not containing the vertex p.

A variety contains subspaces of the projective space. A subspace contained in the variety $\mathcal{F}$ is called *maximal* if it is not contained in an other subspace of the variety. A maximal subspace is called a *generator*. All generators have the same dimension, this is called the *projective index* and is denoted by $g(\mathcal{F})$. Table 1.2 bundles information about quadrics and Hermitian varieties.

It may seem that quadrics and Hermitian varieties are objects studied in a very analytical way. However, we mention an axiomatic description for quadrics. We will never use these axioms, but they illustrate in a simple way basic geometric properties of quadrics.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a projective space. For a set $\mathcal{A}$ of points, we define a *tangent line* as a line of $\mathcal{S}$ intersecting $\mathcal{A}$ in exactly one point, or a line contained in $\mathcal{A}$. A line intersecting $\mathcal{A}$ in exactly one point p is called *tangent in the point* p. For each point $p \in \mathcal{A}$, we define the set $\mathcal{A}_p = \{x \in \mathcal{P} \setminus \{p\} | \langle x,p\rangle$ is a tangent line of $\mathcal{A}\} \cup \{p\}$, and $\mathcal{A}_p$ is called the *tangent space of* $\mathcal{A}$ at the point p.

A *quadratic set* in $\mathcal{S}$ is a set $\mathcal{A}$ of points $\mathcal{A} \subset \mathcal{P}$ such that:

(i) Any line $L \in \mathcal{B}$ containing at least 3 points of $\mathcal{A}$ is contained in $\mathcal{A}$.

(ii) For any point $p \in \mathcal{A}$, the tangent space $\mathcal{A}_p$ is the set of points of a hyperplane or the set $\mathcal{P}$ of all points.

It is astonishing that these two axioms are sufficient to prove a lot of standard theorems on quadrics in projective spaces. A good overview can for instance be found in [12] and [61].

Looking back to Table 1.1, the absolute points of certain polarities are the points of a quadric or Hermitian variety. If $\mathcal{F}$ is a non-singular Hermitian variety with equation $\sum_{i,j=0}^{n} a_{ij}X_iX_j^q = 0$, then $A = (a_{ij})$ is the matrix of a Hermitian polarity with field automorphism $\theta : x \mapsto x^q$, and absolute points the given Hermitian variety $\mathcal{F}$. Hence there is a one-to-one correspondence between non-singular Hermitian varieties and Hermitian polarities. Suppose that $\mathcal{F}$ is a non-singular quadric with equation $\sum_{\substack{i,j=0\\ i\leqslant j}}^{n} a_{ij}X_iX_j = 0$. For q odd, define $A = (a'_{ij})$, with $a'_{ij} = a_{ij}/2$, $i < j$, $a'_{ii} = a_{ii}$ and $a'_{ij} = a_{ji}$, $j > i$. For q even and n odd, define $a'_{ij} = a'_{ji} = a_{ij}$, $i < j$, and $a_{ii} = 0$. When q is odd, A is the matrix of an orthogonal polarity with absolute points the given quadric $\mathcal{F}$. For q even and n odd, the polarity with matrix A is symplectic (and all points of $\mathrm{PG}(n,q)$ are absolute). When q and n are both even, no polarity can be associated to $\mathcal{F}$. The geometric reason is that every non-singular parabolic quadric has a nucleus.

1.2.3 Ovals, ovoids and eggs

Ovals and ovoids

An *oval* of $\mathrm{PG}(2,q)$ is a set of $q+1$ points no three collinear.

Suppose that $\mathcal{B}$ is a set of points of $\mathrm{PG}(n,q)$. A line not intersecting the set $\mathcal{B}$ will be called an *external line*, a line intersecting $\mathcal{B}$ in exactly one point will be called a *tangent line* and a line intersecting $\mathcal{B}$ in t points, $t > 1$, will be called a *t-secant line* or shortly a *secant line* or *t-secant*.

In the classification of ovals, tangent lines play an important role. Using arguments from analytic geometry and properties of finite fields, B. Segre ([83, 84]) obtained the following theorem.

Theorem 1.2.9. *In $\mathrm{PG}(2,q)$, q odd, every oval is a conic $\mathrm{Q}(2,q)$.*

This result is in sharp contrast with the q even case. When q is even, there are infinite families of ovals besides the conics, and there is no complete classification yet.

The following lemma illustrates again the importance of tangent lines.

Lemma 1.2.10. *Let $\mathcal{O}$ be an oval of $\mathrm{PG}(2,q)$, q even. The $q+1$ tangents to $\mathcal{O}$ are concurrent. The unique common point is called the* nucleus *of $\mathcal{O}$.*

A *k-arc* in $\mathrm{PG}(2, q)$ is a set of k points no three collinear. Hence an oval is a $(q + 1)$-arc of $\mathrm{PG}(2, q)$. A k-arc is called *complete* if it cannot be extended to a $(k + 1)$-arc. Let $m(2, q)$ denote the maximum size of a k-arc. Short arguments in [60] show that $m(2, q) = q + 2$ when q is even and $m(2, q) = q + 1$ when q is odd. When q is even, conics are examples of $(q + 1)$-arcs of $\mathrm{PG}(2, q)$, but Lemma 1.2.10 shows that every $(q + 1)$-arc can be extended to a $(q + 2)$-arc by adding its nucleus. Hence a $(q + 1)$-arc is never complete when q is even. A $(q + 2)$-arc of $\mathrm{PG}(2, q)$ is called a *hyperoval.* A hyperoval consisting of a conic and its nucleus is called *regular.* Since $m(2, q) = q + 1$ when q is odd, hyperovals do not exist when q is odd.

Suppose that q is even. To characterise ovals and hyperovals, often algebraic tools can be used. This is done by attaching a function $f : \mathrm{GF}(q) \to \mathrm{GF}(q)$ to any hyperoval $\mathcal{H}$. It is clear that any hyperoval $\mathcal{H}$ is projectively equivalent to the set of points $\{(1, t, f(t)) \| t \in \mathrm{GF}(q)\} \cup \{(0, 0, 1), (0, 1, 0)\}$ with $f(0) = 0$ and $f(1) = 1$. If we choose $f(t) = t^2$, then $\mathcal{H}$ is the regular hyperoval. It is possible to find conditions for a polynomial f so that it describes a hyperoval. If f is a polynomial satisfying these conditions, we sometimes use the notation $\mathcal{D}(f)$ for the corresponding hyperoval.

Hyperovals give possibilities to construct ovals of $\mathrm{PG}(2, q)$ which are not conics, even if the hyperoval is regular. Consider a regular hyperoval $\mathcal{H}$ of $\mathrm{PG}(2, q)$, i.e. $\mathcal{H} = \mathcal{O} \cup \{n\}$, n the nucleus of the conic $\mathcal{O}$. Define $\mathcal{O}_i = \mathcal{H} \setminus \{n_i\}$, $i = 1, 2$, n_i, two different points of $\mathcal{H}$. The ovals $\mathcal{O}_1$ and $\mathcal{O}_2$ are equivalent if and only if the stabiliser group of the hyperoval $\mathcal{H}$ maps n_1 on n_2. For $q = 2, 4$, the stabiliser group acts regularly on the point set of the hyperoval, while for $q > 4$, the nucleus is fixed by the stabiliser group of the hyperoval. Hence, $\mathcal{O}_1$ and $\mathcal{O}_2$ are not equivalent when $n_1 = n$ and $n_2 \neq n$. Then $\mathcal{O}_1$ is projectively equivalent to the conic with equation $x_2^2 + x_0x_1 = 0$, while $\mathcal{O}_2$ is projectively equivalent with the set $\{(1, t, t^{\frac{q}{2}}) \| t \in \mathrm{GF}(q)\} \cup \{(0, 0, 1)\}$ which is clearly not a conic. This oval is called a *pointed conic.*

The classification of all (hyper)ovals of $\mathrm{PG}(2, q)$, q even, remains open. It is considered as a hard problem. An overview of the known (hyper)ovals of $\mathrm{PG}(2, q)$, q even, is for instance [19]. We also mention "Bill Cherowitzo's hyperoval page" [33].

An *ovoid* of $\mathrm{PG}(3, q)$, is a set of $q^2 + 1$ points no three collinear. As for ovals, we can now define a *tangent plane* and a *secant plane* as a plane intersecting the ovoid $\mathcal{O}$ in exactly 1 point, respectively, at least 2 points.

Theorem 1.2.11. *Let $\mathcal{O}$ be an ovoid of $\mathrm{PG}(3, q)$, $q > 2$, then*

(i) *for any point $p \in \mathcal{O}$, the union of all tangent lines in p is a plane,*

(ii) *exactly* q^2+1 *planes of* $\mathrm{PG}(3,q)$ *meet* $\mathcal{O}$ *in a unique point and the other* q^3+q *planes meet* $\mathcal{O}$ *in an oval.*

In the classification of ovoids, the classification of ovals plays an important role. This fact is expressed in the following theorem, valid for all q. An oval contained in the ovoid $\mathcal{O}$ found by intersecting the ovoid $\mathcal{O}$ with a plane will be called a *secant plane section*. If the oval is a conic $\mathrm{Q}(2,q)$, then it will be called a *conic section*.

Theorem 1.2.12. (Barlotti [7]) *Let* $\mathcal{O}$ *be an ovoid of* $\mathrm{PG}(3,q)$, $q>2$. *If every secant plane section of* $\mathcal{O}$ *is a conic* $\mathrm{Q}(2,q)$, *then* $\mathcal{O}$ *is an elliptic quadric* $\mathrm{Q}^-(3,q)$.

Since the only ovals of $\mathrm{PG}(2,q)$, q odd, are conics $\mathrm{Q}(2,q)$, the only ovoids of $\mathrm{PG}(3,q)$, q odd, are elliptic quadrics $\mathrm{Q}^-(3,q)$.

The following theorem improves Theorem 1.2.12 considerably when q is even.

Theorem 1.2.13. (Brown [20]) *Let* $\mathcal{O}$ *be an ovoid of* $\mathrm{PG}(3,q)$, q *even. When there is at least one conic section, the ovoid* $\mathcal{O}$ *is an elliptic quadric* $\mathrm{Q}^-(3,q)$.

Besides elliptic quadrics, there is only one class of ovoids in $\mathrm{PG}(3,q)$, $q=2^{2e+1}$, $e\geqslant 1$, known; the Tits ovoids [100]. For small even q, ovoids of $\mathrm{PG}(3,q)$ are classified.

Theorem 1.2.14. (i) *When* $q=4,16$, *the only ovoids of* $\mathrm{PG}(3,q)$ *are the elliptic quadrics.*

(ii) *When* $q=8,32$, *the only ovoids of* $\mathrm{PG}(3,q)$ *are the elliptic quadrics and the Tits ovoids.*

We end this section with the definition of Tits of an ovoid.

An *ovoid* is a set $\mathcal{O}$ of points such that $|L\cap\mathcal{O}|\leqslant 2$ for any line L and for any $x\in\mathcal{O}$, the union of all lines L such that $L\cap\mathcal{O}=\{x\}$ is a hyperplane.

This definition can be used in infinite projective spaces and in non-Desarguesian projective planes.

Theorem 1.2.15. ([98], see [39]) *If* $\mathrm{PG}(n,q)$ *contains an ovoid* $\mathcal{O}$, *then* $|\mathcal{O}|=q^{n-1}+1$ *and* $n\leqslant 3$.

And hence we may state that *ovoids of Tits* unify ovals of $\mathrm{PG}(2,q)$ and ovoids of $\mathrm{PG}(3,q)$.

Eggs

Eggs can be seen as generalisations of ovals and ovoids in one concept. The most general definition can be found in [81], and we start with it.

Definition 1.2.16. Consider the projective space $\mathrm{PG}(2n+m-1,q)$, $n,m \geqslant 1$. An *egg* $\mathcal{E}$ is a set of q^m+1 pairwise disjoint $(n-1)$-dimensional subspaces $\mathrm{PG}^{(i)}(n-1,q)$, $i=0,\ldots,q^m$, every three of which generate a $(3n-1)$-dimensional subspace, and such that each element $\mathrm{PG}^{(i)}(n-1,q)$ of $\mathcal{E}$ is contained in a subspace $\mathrm{PG}^{(i)}(n+m-1,q)$ having no point in common with any $\mathrm{PG}^{(j)}(n-1,q)$ for $j \neq i$. The space $\mathrm{PG}^{(i)}(n+m-1,q)$ is called the *tangent space* of $\mathcal{E}$ at $\mathrm{PG}^{(i)}(n-1,q)$.

Eggs were not introduced as in the above definition. The first reference in which objects are defined that would later be called “eggs”, is [89]. How these objects became eggs can be read in for instance [67].

It is beyond the scope of this introduction to mention deep theorems about eggs. We only mention some basic properties. From the definition, it is possible to deduce that the elements of an egg determine the tangent spaces uniquely. Consider an egg $\mathcal{E}$, and consider an arbitrary element $\alpha \in \mathcal{E}$. Project the elements of $\mathcal{E} \setminus \{\alpha\}$ onto $\pi = \mathrm{PG}(n+m-1)$ skew to α. Then q^m mutually skew $(n-1)$-dimensional spaces are obtained and θ_{m-1} points of π do not lie in such a projected element. The tangent space at the element α intersects π in an $(m-1)$-dimensional space skew to all projected elements and necessarily contains these latter θ_{m-1} points. Hence the tangent space is determined uniquely by the elements of $\mathcal{E}$.

Finally we mention the following theorem from [81].

Theorem 1.2.17. *Let $\mathcal{E}$ be an arbitrary egg of $\mathrm{PG}(2n+m-1,q)$, $n,m \geqslant 1$.*

(i) *$n=m$ or $n(a+1)=ma$ with a odd.*

(ii) *If q is even, then $n=m$ or $m=2n$.*

(iii) *If $n \neq m$ (respectively, $2n=m$), then each point of $\mathrm{PG}(2n+m-1,q)$ which is not contained in an element of $\mathcal{E}$ belongs to 0 or $1+q^{m-n}$ (respectively, $1+q^n$) tangent spaces of $\mathcal{E}$.*

(iv) *If $n \neq m$, then the q^m+1 tangent spaces of $\mathcal{E}$ form an egg $\mathcal{E}^*$ in the dual space of $\mathrm{PG}(2n+m-1,q)$.*

(v) *If $n \neq m$ (respectively, $2n=m$), then each hyperplane of $\mathrm{PG}(2n+m-1,q)$ which does not contain a tangent space of $\mathcal{E}$ contains 0 or $1+q^{m-n}$ (respectively, $1+q^n$) elements of $\mathcal{E}$.*

Unitals

A *unital* in $\mathrm{PG}(2, q^2)$ is a set of q^3+1 points $\mathcal{U}$, such that every line intersects $\mathcal{U}$ in either 1 or $q+1$ points. A classical example is the Hermitian curve $\mathrm{H}(2, q^2)$ in $\mathrm{PG}(2, q^2)$.

1.2.4 Blocking sets and spreads

The following paragraph is copied from [16], where some historical references can be found.

> *The name blocking set originates from game theory, where we have a set of individuals, and certain subsets called coalitions, with the property that a coalition can force a particular decision. A blocking set then is a subset that is not a coalition, but contains at least one member of each coalition, so that it can block any decision without being able to force one.*

Blocking sets

Consider the projective plane $\mathrm{PG}(2, q)$. A *blocking set* is a set of points $\mathcal{B}$ such that every line has at least one point in common with $\mathcal{B}$. A line is an example of a blocking set, but a blocking set containing a line is called a *trivial blocking set*, and a blocking set not containing a line a *non-trivial blocking set*. A blocking set $\mathcal{B}$ is *minimal* if $\mathcal{B} \setminus \{p\}$ is not a blocking set for every $p \in \mathcal{B}$. The following lemma is very useful and can for instance be found in [60]

Lemma 1.2.18. *A blocking set $\mathcal{B}$ is minimal if and only if for every point $p \in \mathcal{B}$, there is a line L such that $\mathcal{B} \cap L = \{p\}$.*

A blocking set containing k points is also called a *blocking k-set*.

Non-trivial blocking sets only exist if $q > 2$. We give some examples from [60].

A *projective triangle of side n* in $\mathrm{PG}(2, q)$ is a set $\mathcal{B}$ of $3(n-1)$ points such that

(a) on each side of the triangle $p_0p_1p_2$ there are n points of $\mathcal{B}$,

(b) the vertices p_0,p_1,p_2 are in $\mathcal{B}$,

(c) if $r_0 \in p_1p_2$ and $r_1 \in p_2p_0$ are in $\mathcal{B}$, then so is $r_2 = r_0r_1 \cap p_0p_1$.

A *projective triad of side n* is a set $\mathcal{B}$ of $3n-2$ points such that

(a) on each line of three concurrent lines L_0, L_1, L_2 there are n points of $\mathcal{B}$,

(b) the vertex $p = L_0 \cap L_1 \cap L_2 \in \mathcal{B}$,

(c) if $r_o \in L_0$ and $r_1 \in L_1$ are in $\mathcal{B}$, then so is $r = r_0 r_1 \cap L_2$.

Lemma 1.2.19. (i) *In* PG$(2, q)$*, q odd, there exists a projective triangle of side $\frac{1}{2}(q+3)$ which is a non-trivial minimal blocking set of size $\frac{3}{2}(q+1)$.*

(ii) *In* PG$(2, q)$*, q even, $q > 2$, there exists a projective triad of side $\frac{1}{2}(q+2)$ which is a non-trivial minimal blocking set of size $\frac{1}{2}(3q+2)$.*

It is clear that for a non-trivial blocking set $\mathcal{B}$ of the plane PG$(2, q)$, necessarily $q + 2 \leqslant |\mathcal{B}| \leqslant q^2 + q + 1$. The following theorem gives information about lower and upper bounds for non-trivial minimal blocking sets. Concerning the above examples, blocking k-sets in PG$(2, q)$ with $k \leqslant \frac{3(q+1)}{2}$ are called *small blocking sets.*

A *blocking set of Rédei type* is a blocking $(q + m)$-set with the property that there exists an m-secant.

The following theorem gives an upper and a lower bound for the size of a non-trivial minimal blocking set.

Theorem 1.2.20. *Let $\mathcal{B}$ be a non-trivial minimal blocking set in* PG$(2, q)$*. Then*

(i) **(Bruen [25])** $|\mathcal{B}| \geqslant q+\sqrt{q}+1$ *with equality if and only if q is a square and $\mathcal{B}$ is a Baer subplane.*

(ii) **(Bruen and Thas [29])** $|\mathcal{B}| \leqslant q\sqrt{q} + 1$, *with equality if and only if q is a square and $\mathcal{B}$ is a unital.*

Observing the above theorems, one can try to improve the bounds when q is not a square and to characterise non-trivial minimal blocking sets not containing a Baer subplane. Improvements of the bounds are for instance given by the following theorems. Let $c_p = 2^{\frac{-1}{3}}$ when $p \in \{2, 3\}$ and $c_p = 1$ when $p \geqslant 5$, p prime.

Theorem 1.2.21. *Let $\mathcal{B}$ be a non-trivial minimal blocking set of* PG$(2, q)$*, $q > 2$.*

(i) **(Blokhuis [14])** *If q is a prime, then,* $|\mathcal{B}| \geqslant \frac{3(q+1)}{2}$.

(ii) **(Blokhuis [15], Blokhuis et al. [17])** *If $q = p^{2e+1}$, p prime, $e \geqslant 1$, then* $|\mathcal{B}| \geqslant \max(q + 1 + p^{e+1}, q + 1 + c_q q^{\frac{2}{3}})$.

By Lemma 1.2.19, there exist blocking sets of size $\frac{3(q+1)}{2}$ in $\mathrm{PG}(2,q)$, q odd, hence, the above bound is sharp in the first case. Also in the second case, examples attaining the bound exist.

It is beyond the scope of this introduction to give more details about small blocking sets and Rédei type blocking sets. A good overview, on which this little overview is based, can be found in [45]. Instead of giving more details, we will introduce multiple blocking sets and blocking sets in higher dimensional spaces. Again, our overview is based on the overview in [45].

An *s-fold blocking set* in $\mathrm{PG}(2,q)$ is a set of points that intersects every line in at least s points. It is called *minimal* if no proper subset is an s-fold blocking set. A 1-fold blocking set is simply called a *blocking set*. The following theorem indicates that, to obtain an s-fold blocking set of small cardinality with $s > 1$, it is no longer interesting to include a line in the set.

Theorem 1.2.22. *Let $\mathcal{B}$ be an s-fold blocking set of* $\mathrm{PG}(2,q)$*, $s > 1$.*

(i) **(Bruen [27])** *If $\mathcal{B}$ contains a line, then $|\mathcal{B}| \geqslant sq + q - s + 2$.*

(ii) **(Ball [2])** *If $\mathcal{B}$ does not contain a line, then $|\mathcal{B}| \geqslant sq + \sqrt{sq} + 1$.*

If s is not too large, substantial improvements to this theorem have been obtained for general q. Also, for q a square and s not too large, the smallest minimal s-fold blocking sets are classified.

Theorem 1.2.23. **(Blokhuis et al. [17])** *Let $\mathcal{B}$ be an s-fold blocking set in* $\mathrm{PG}(2,q)$ *of size $s(q+1)+c$ for some $s > 1$. For a prime p, let $c_p = 2^{\frac{-1}{3}}$ for $p \in \{2,3\}$ and $c_p = 1$ for $p > 3$.*

(i) *If $q = p^{2d+1}$ and $s < \frac{q}{2} - \frac{c_p q^{\frac{2}{3}}}{2}$, then $c > c_p q^{\frac{2}{3}}$.*

(ii) *If q is a square, $s < \frac{q^{\frac{1}{4}}}{2}$ and $c < c_p q^{\frac{2}{3}}$, then $c \geqslant s\sqrt{q}$ and $\mathcal{B}$ contains the union of s pairwise disjoint Baer subplanes.*

(iii) *If $q = p^2$ and $s < \frac{q^{\frac{1}{4}}}{2}$ and $c < p\lceil \frac{1}{4} + \sqrt{\frac{p+1}{2}}\rceil$, then $c \geqslant s\sqrt{q}$ and $\mathcal{B}$ contains the union of s pairwise disjoint Baer subplanes.*

In [2], a table with the sizes of the smallest s-fold blocking sets in $\mathrm{PG}(2,q)$, $s > 1$, q small, can be found. Many examples of such blocking sets are described in [2, 3, 5].

To end this section, we introduce blocking sets in higher dimensional spaces. A *blocking set with respect to t-spaces in* $\mathrm{PG}(n,q)$ is a set $\mathcal{B}$ of points such that every t-dimensional subspace of $\mathrm{PG}(n,q)$ meets $\mathcal{B}$ in at least one point.

Theorem 1.2.24. (Bose and Burton [18]) *If B is a blocking set with respect to t-spaces in* $\mathrm{PG}(n,q)$*, then* $|B| \geqslant |\mathrm{PG}(n-t,q)|$*. Equality holds if and only if B is an $(n-t)$-dimensional subspace.*

Blocking sets with respect to t-spaces that contain an $(n-t)$-space are called *trivial*. The smallest non-trivial blocking sets with respect to t-spaces are characterised in the following theorem.

Theorem 1.2.25. (Beutelspacher [11], Heim [55]) *In* $\mathrm{PG}(n,q)$*, the smallest non-trivial blocking sets with respect to t-spaces are cones with vertex an $(n-t-2)$-space π_{n-t-2} and base a non-trivial blocking set of minimal cardinality in a plane skew to π_{n-t-2}.*

In $\mathrm{PG}(n,q)$, a blocking set with respect to hyperplanes is simply called a *blocking set*. For this case, Theorem 1.2.24 was already proved by A. A. Bruen in [26].

It is interesting to see that to block t-dimensional subspaces of a projective space, cones with base a planar blocking set can be used. Hence the important concept is still a blocking set of $\mathrm{PG}(2,q)$.

The following theorem is an improvement of Theorem 1.2.25.

Theorem 1.2.26. (Storme and Weiner [87]) *Let $\mathcal{B}$ be a blocking set in* $\mathrm{PG}(n,q)$*, $n \geqslant 3$, $q = p^h$ square, $p > 3$ prime, of cardinality smaller than or equal to the cardinality of the second smallest non-trivial blocking sets in* $\mathrm{PG}(2,q)$*. Then $\mathcal{B}$ contains a line or a planar blocking set of* $\mathrm{PG}(2,q)$*.*

For more information about the subject, we again refer to [45].

Spreads

Consider the projective space $\mathcal{P} = \mathrm{PG}(n,q)$. A *$t$-spread* of $\mathcal{P}$ is a set $\mathcal{S}$ of t-dimensional subspaces of $\mathcal{P}$ which partitions the point set of $\mathcal{P}$, i.e. every point of $\mathcal{P}$ is contained in exactly one element of $\mathcal{S}$. Since a t-spread induces a partition of the point set, $(t+1)|(n+1)$ is a necessary condition. It is shown by J. André ([1]) that this condition is also sufficient.

Let $\mathcal{S}$ be a t-spread of $\mathrm{PG}(n,q)$. Suppose that $\mathcal{L}$ is the set of $(2t+1)$-dimensional subspaces of $\mathcal{P}$ spanned by pairs of elements of $\mathcal{S}$. A t-spread is *geometric* or *normal* if for each $S \in \mathcal{S}$ and each $L \in \mathcal{L}$, either $S \subset L$ or $S \cap L = \emptyset$.

Suppose that U_1, U_2, U_3 are three pairwise disjoint t-dimensional subspaces of the projective space $\mathrm{PG}(2t+1,q)$. Consider a point $p \in U_1$. The subspace $\langle p, U_2\rangle$ intersects U_3 in a point. Hence there is a unique line through P intersecting U_2 and U_3. A line is called a *transversal to U_1,U_2 and U_3* if it intersects U_1, U_2 and U_3 in a point. Since p was arbitrary, there are θ_t transversal lines through U_1, U_2, U_3, and there are $q+1$ mutually disjoint t-spaces $U_1, \ldots, U_{q+1}$ being transversal to these lines. The set of these $q+1$ spaces is called a *t-regulus*. If $t=1$, then this set is one of the two systems of generators of a hyperbolic quadric $\mathrm{Q}^+(3,q)$. The other system is also a regulus and is called the *opposite regulus*.

Suppose now that $\mathcal{S}$ is a geometric t-spread. If for any $U_1, U_2, U_3 \in \mathcal{S}$ with $U_3 \in \langle U_1, U_2 \rangle$, the t-regulus defined by U_1, U_2, U_3 is completely contained in $\mathcal{S}$, then $\mathcal{S}$ is called *regular*. A spread containing no regulus is called *aregular*.

An important fact is the link between spreads and translation planes. Therefore we mention the following construction. Let $\mathcal{S}$ be a t-spread in $P' = \mathrm{PG}(n,q)$, $n+1 = k(t+1)$. Embed $P' = \mathrm{PG}(n,q)$ in $P = \mathrm{PG}(n+1,q)$ and define the following incidence structure $\mathcal{A} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$. The point set $\mathcal{P}$ consists of the points of $P \setminus P'$, the lines, which are the elements of $\mathcal{B}$, are the $(t+1)$-dimensional subspaces of P intersecting P' in an element of $\mathcal{S}$, and incidence is inclusion.

When $\mathcal{S}$ is a geometric t-spread, $\mathcal{A}$ is an affine space of dimension k and order q^{t+1}. The affine space $\mathcal{A}$ can be extended to a projective space Π. For $q > 2$, this projective space is Desarguesian if and only if $\mathcal{S}$ is regular ([24]). It follows that any geometric t-spread in $\mathrm{PG}(n,q)$, $n+1 = k(t+1)$, $q > 2$ and $k \geqslant 3$, is regular. The most important case is the case $n = 2t+1$, since then Π is a *translation plane*. R. H. Bruck and R. C. Bose ([23]) prove conversely that every translation plane arises from a t-spread of a projective space. For an alternative definition of translation planes, we refer for instance to [69].

We mention furthermore the following theorem about regular spreads, afterwards, we give a construction.

Theorem 1.2.27. *For any n, t, $(t+1)|(n+1)$, there is, up to collineation, a unique regular t-spread of* $\mathrm{PG}(n,q)$.

Supposing the conditions of the theorem, consider the chain of fields $\mathrm{GF}(q) \subset \mathrm{GF}(q^{t+1}) \subset \mathrm{GF}(q^{n+1})$. The field $\mathrm{GF}(q^{n+1})$ represents the projective space $\mathrm{PG}(n,q)$ as $(n+1)$-dimensional vector space over $\mathrm{GF}(q)$. Since the field $\mathrm{GF}(q^{t+1})$ is a $(t+1)$-dimensional vector space over $\mathrm{GF}(q)$, it is a t-dimensional subspace of $\mathrm{PG}(n,q)$, as well as all cosets $a \cdot \mathrm{GF}(q^{t+1})$, $a \in \mathrm{GF}(q^{n+1}) \setminus \{0\}$. Since these cosets partition the multiplicative group of $\mathrm{GF}(q^{n+1})$, they represent a t-spread of $\mathrm{PG}(n,q)$, which appears to be regular.

We now give an example of an aregular spread ([59, 17.3.3]). Let $q = p^h$, $h \geqslant 2$, and let $x^{p+1} + bx - c$ have no roots in $\mathrm{GF}(q)$. Then the set $\mathcal{S} = \{\langle (1,0,0,0), (0,1,0,0) \rangle\} \cup \{\langle (z,y,1,0), (cy^p, z^p + by^p, 0, 1) \rangle \| (y,z) \in \mathrm{GF}(q) \times \mathrm{GF}(q)\}$ is an aregular spread of $\mathrm{PG}(3,q)$.

A *partial t-spread* $\mathcal{S}$ of $\mathrm{PG}(n,q)$ is a set of mutually disjoint t-dimensional subspaces of $\mathrm{PG}(n,q)$. A partial t-spread is called *maximal* if it cannot be extended by any t-dimensional subspace of $\mathrm{PG}(n,q)$.

A *t-cover* $\mathcal{C}$ of $\mathrm{PG}(n,q)$ is a set of t-dimensional subspaces of $\mathrm{PG}(n,q)$ such that every point of $\mathrm{PG}(n,q)$ belongs to at least one element of $\mathcal{C}$. A t-cover is *minimal* when no proper subset of it is a t-cover.

t-Covers and partial t-spreads when $(t+1) \not| (n+1)$

When $(t+1) \not| (n+1)$, t-spreads do not exist. The first results in this case are combinatorial. For a partial t-spread, it is possible to compute an upper bound; for a t-cover, a lower bound.

Theorem 1.2.28. *Suppose that $n = k(t+1)+r-1$, where $1 \leqslant r \leqslant t$.*

(i) *A partial t-spread of $\mathrm{PG}(n,q)$ contains at most $q^r \frac{q^{k(t+1)}-1}{q^{t+1}-1}$ elements.*

(ii) *A t-cover of $\mathrm{PG}(n,q)$ contains at least $q^r \frac{q^{k(t+1)}-1}{q^{t+1}-1} + 1$ elements.*

Constructing examples valorises these bounds.

Theorem 1.2.29. (Beutelspacher [10])

(i) *In $\mathrm{PG}(k(t+1)+r-1,q)$, there is a partial t-spread with $q^r \frac{q^{k(t+1)}-1}{q^{t+1}-1} - q^r + 1$ elements.*

(ii) *In $\mathrm{PG}(k(t+1)+r-1,q)$, there is a t-cover with $q^r \frac{q^{k(t+1)}-1}{q^{t+1}-1} + 1$ elements.*

A more advanced theorem is the following theorem about upper bounds for partial t-spreads.

Theorem 1.2.30. *Let $\mathcal{S}$ be a partial t-spread of $\mathrm{PG}(n,q)$, $n = k(t+1)+r-1$, $1 \leqslant r \leqslant t$. Let $|\mathcal{S}| = q^r \frac{q^{k(t+1)}-1}{q^{t+1}-1} - s$. Then*

(i) **(Beutelspacher [8, 9])** $s \geqslant q-1$

(ii) **(Drake and Freeman [40])** $s > \frac{q^r-1}{2} - \frac{q^{2r-t-1}}{5}$

Furthermore, there exists an example with $s = q^r - 1$.

For more information on this topic we refer to [41].

t-Covers and partial t-spreads when $(t+1)$ divides $(n+1)$

Suppose that PG(n,q) *admits a t-spread*, i.e. $(t+1)|(n+1)$. Let $\mathcal{S}$ be a partial t-spread with cardinality $\frac{q^{n+1}-1}{q^{t+1}-1}-\delta$. Then the parameter δ is called the *deficiency* of $\mathcal{S}$. If a point p does not belong to any t-space of the t-spread $\mathcal{S}$, it is called a *hole* with respect to the spread $\mathcal{S}$.

Extendability of partial t-spreads is studied intensively. In this study, there is a clear link between blocking sets and partial t-spreads. We will give more about this in Section 2.1.4, but we mention the following theorem as illustration of this important link.

Theorem 1.2.31. *Let $\mathcal{S}$ be a maximal partial spread of* PG$(3,q)$*, of deficiency $\delta>0$. Then $\delta\geqslant\epsilon$ with $q+\epsilon$ the cardinality of the smallest non-trivial blocking sets of* PG$(2,q)$*.*

1.3 Polar spaces

Polar spaces were systematically studied in a geometric way by F. D. Veldkamp [102]. The aim of Veldkamp was to start from certain geometric axioms, to classify the structures and embed them in projective spaces. This is a quite rough description of his work. More about its history can be found in [31], but the work of F. D. Veldkamp is worth mentioning since the notion polar space is introduced there as the name for a class of structures not yet unified at that time.

J. Tits simplified and completed the theory of F. D. Veldkamp (see [99]). To start, we will recall Tits' definition of polar spaces. Afterwards, a further simplification is given, besides certain concepts and theorems.

A *polar space*[2] of rank n, $n\geqslant 2$, is a point set $\mathcal{P}$ together with a family of subsets of $\mathcal{P}$ called *subspaces*, satisfying the following axioms.

(i) A subspace, together with the subspaces it contains, is a d-dimensional projective space[3] with $-1\leqslant d\leqslant n-1$ (d is called the *dimension* of the subspace).

(ii) The intersection of two subspaces is a subspace.

(iii) Given a subspace V of dimension $n-1$ and a point $p\in\mathcal{P}\setminus V$, there is a unique subspace W such that $p\in W$ and $V\cap W$ has dimension

[2]We found this definition in [81]. In [30], we did not find the restriction $n\geqslant 2$.

[3]In [30], it is mentioned clearly that this projective space is *thick*, i.e. a line contains at least 3 points. Since we talk always about non-degenerate projective spaces, we do not mention the thickness explicitely

$n-2$; W contains all points of V that are joined to p by a line (a *line* is a subspace of dimension 1).

(iv) There exist two disjoint subspaces of dimension $n-1$.

Contrary to the definition of projective spaces and generalised quadrangles (cfr. infra), the definition of polar spaces does not use the concept of a point-line geometry. On the other hand, it is clear that the point-line structure must be hidden somewhere in the used ingredients, i.e. in the projective spaces. The subspaces of maximal dimension will also be called *generators*.

The *finite classical polar spaces* are the following structures.

(i) The non-singular quadrics in odd dimension, $\mathrm{Q}^+(2n+1,q)$ and $\mathrm{Q}^-(2n+1,q)$, together with the subspaces they contain, giving polar spaces of rank $n+1$ and n.

(ii) The non-singular parabolic quadrics in even dimension, $\mathrm{Q}(2n,q)$, together with the subspaces they contain, giving a polar space of rank n.

(iii) The points of $\mathrm{PG}(2n+1,q)$, together with the isotropic subspaces of a non-singular symplectic polarity of $\mathrm{PG}(2n+1,q)$, giving a polar space of rank n.

(iv) The non-singular Hermitian varieties in $\mathrm{PG}(2n,q^2)$, together with the subspaces they contain, $n \geqslant 2$ (respectively, $\mathrm{PG}(2n+1,q^2)$, $n \geqslant 1$), giving a polar space of rank n (respectively, rank $n+1$).

By theorems of F. D. Veldkamp ([102]) and J. Tits ([99]), all polar spaces with finite rank at least 3 are classified. In the finite case (i.e. the polar space has a finite set of points), all polar spaces are listed here above. In a certain sense, the polar spaces of rank two play a comparable role as the projective spaces of rank 2 (i.e. the projective planes). For, finite projective spaces of dimension (thus also rank) at least three are classified and are just the projective spaces over the finite field $\mathrm{GF}(q)$, while all finite polar spaces of rank at least 3 are the examples of above. For the rank two case, both non-classical projective planes as non-classical polar spaces are known. Examples of the latter ones will be given in Section 1.4.

We mention an important isomorphism between example (ii) and (iii). Suppose q is even. Then the parabolic quadric $\mathrm{Q}(2n,q)$ has a nucleus n. Projecting all points and subspaces of $\mathrm{Q}(2n,q)$ from n onto a hyperplane α of $\mathrm{PG}(2n,q)$ not containing n, we find all points of α, together with a set of subspaces of α. It is a well known result that the points of α together with the projected subspaces form an example (iii) polar space. Isomorphisms in the rank 2 case will be described in Section 1.4.

It is possible to define polar spaces as point-line geometries. This was done by F. Buekenhout and E. E. Shult ([32]). Since the most important axiom of their approach is also very useful in practical situations, we will recall this way of describing polar spaces here.

A *Shult space*[4] is a point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, with $\mathcal{B}$ a non-empty set of subsets of $\mathcal{P}$ of cardinality at least 2, such that the incidence relation I satisfies the following axiom. For each line $L \in \mathcal{B}$ and for each point $p \in \mathcal{P} \backslash L$, the point p is collinear[5] with either one or all points of the line L. A Shult space is *non-degenerate* if no point is collinear with all other points, a Shult space is *linear* if two distinct lines have at most one common point. This property implies that at most one line is incident with two points. A *subspace* X of a Shult space $\mathcal{S}$ is a non-empty set of pairwise collinear points such that any line meeting X in at least two points is contained in X. If there exists an integer n such that every chain of distinct subspaces $X_1 \subset X_2 \subset \ldots \subset X_l$ has at most n members, then $\mathcal{S}$ has finite rank n. We recall the following fundamental theorem of F. Buekenhout and E. E. Shult [32].

Theorem 1.3.1. (i) *Every non-degenerate Shult space is linear.*

(ii) *If $\mathcal{S}$ is a non-degenerate Shult space of finite rank at least* 3*, and if all lines contain at least three points, then the Shult space together with its subspaces is a polar space*[6]

1.4 Generalised quadrangles

Generalised quadrangles were introduced by J. Tits in his celebrated paper on triality [97]. In that paper, the more general class of *generalised polygons* was defined. At the same time, much research on the foundations of polar spaces was also done, and from the axioms, it is clear that finite polar spaces of rank 2 which not grids or dual grids, are just the generalised quadrangles of order (s, t), with $s > 1$ and $t > 1$. So generalised quadrangles can be studied as a certain class of polar spaces or they can be studied as a certain class of generalised polygons. Both approaches are found in the literature. Our

[4]This is the definition found in [81].

[5]Two distinct points are *collinear* if and only if there is a line incident with the two points.

[6]We found exactly this formulation in [81] and in [30], except that we added the restriction on the rank. We can now safely redefine polar spaces of rank 2.

introduction is mostly based on the standard reference for finite generalised quadrangles [81], by S. E. Payne and J. A. Thas.

A *finite generalised quadrangle*, also denoted a *GQ*, is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, for which the incidence relation satisfies the following axioms.

(i) Each point is incident with $1 + t$ lines ($t \geqslant 1$) and two distinct points are incident with at most one line[7].

(ii) Each line is incident with $1 + s$ ($s \geqslant 1$) points and two distinct lines are incident with at most one point.

(iii) If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \mathrm{I} M \mathrm{I} y \mathrm{I} L$.

The integers s and t are the *parameters* of the GQ $\mathcal{S}$ and $\mathcal{S}$ is said to have *order* (s, t). If $s = t$, then $\mathcal{S}$ is said to have *order* s. A GQ of order $(s, 1)$ is also called a *grid* and a GQ of order $(1, t)$ is also called a *dual grid*. Actually grids and dual grids are defined more generally in [81], but we will not go into detail here.

It is clear that a finite polar space of rank 2, as defined in Section 1.3 is a finite generalised quadrangle. We have to take care, because with the given definitions, not all GQs are finite polar spaces of rank 2. Therefore, we redefine a *finite polar space of rank* 2 as a finite generalised quadrangle.

There is a point-line duality for finite generalised quadrangles, since interchanging the role of points and lines, the incidence relation still satisfies the axioms. The dual of a GQ $\mathcal{S}$ (of order (s, t)) is often denoted by $\mathcal{S}^D$ and it is a GQ of order (t, s).

Suppose that $\mathcal{S}$ is a GQ. If there is a line incident with two distinct points x and y, we write $x \sim y$, and x and y are said to be *collinear*. A point is assumed collinear with itself, so $x \sim x$, and $x \not\sim y$ means that x and y are not collinear. Dually, for L and M two lines, we write $L \sim M$ or $L \not\sim M$ according as L and M are *concurrent* or *non-concurrent*, respectively.

Consider a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ of order (s, t). For $x \in \mathcal{P}$, define $x^\perp = \{y \in \mathcal{P} \| y \sim x\}$. Due to the definition of collinearity, $x \in x^\perp$. Consider a pair of distinct points (x, y), then its *trace* is defined as $x^\perp \cap y^\perp$ and is denoted by $\mathrm{tr}(x, y)$ or $\{x, y\}^\perp$. It is clear that $|\{x, y\}^\perp| = s + 1$ or $t + 1$ according as $x \sim y$ or $x \not\sim y$. For arbitrary point sets $A \subset \mathcal{P}$, we define $A^\perp = \bigcap_{x \in A} x^\perp$. For $x \neq y$, the *span* of the pair (x, y) is defined as the set $\mathrm{sp}(x, y) = \{x, y\}^{\perp\perp} = \{u \in \mathcal{P} \| u \in z^\perp, \forall z \in x^\perp \cap y^\perp\}$ and when $x \not\sim y$, it

[7]Since two points determine at most one line, we will often identify a line with the set of points it contains.

is also called the *hyperbolic line* defined by x and y. For $x \neq y$, the *closure* of the pair (x, y) is the set $\{z \in \mathcal{P} \| z^{\perp} \cap \{x,y\}^{\perp\perp} \neq \emptyset\}$ and is denoted by $\mathrm{cl}(x, y)$.

If $x \sim y$, $x \neq y$, or if $x \not\sim y$ and $|\{x,y\}^{\perp} \perp| = t + 1$, we say that the pair (x, y) is *regular*. The point x is *regular* provided (x, y) is regular for all $y \in \mathcal{P}$, $y \neq x$. The notion regularity plays an important role in a lot of characterisation and classification theorems, but it is beyond the scope of this introduction to mention these theorems. We refer to [81].

Since generalised quadrangles are point-line geometries, *collineations*, *automorphisms* and *polarities* can be defined in the usual way.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, t), and put $v = |\mathcal{P}|$ and $b = |\mathcal{B}|$. The following theorem describes important restrictions on the parameters of a GQ. The proofs can be found in [81].

Theorem 1.4.1. (i) *$v = (s + 1)(st + 1)$ and $b = (t + 1)(st + 1)$.*

(ii) *$s + t$ divides $st(s + 1)(t + 1)$.*

(iii) **(The inequality of D.G. Higman [56, 57])** *If $s > 1$ and $t > 1$, then $t \leqslant s^2$, and dually, $s \leqslant t^2$.*

(iv) *If $s \neq 1$, $t \neq 1$, $s \neq t^2$, and $t \neq s^2$, then $t \leqslant s^2 - s$ and dually $s \leqslant t^2 - t$.*

In [81], starting from the basic axioms of a GQ, a whole synthetic theory of finite generalised quadrangles is developed. Besides the development of this theory, classical examples are given, and a lot of theorems are also applied to these examples. To end this section, we will also give the classical examples and two so called non-classical examples.

The *finite classical generalised quadrangles* are the finite classical polar spaces of rank 2. They are: the quadrics $\mathrm{Q}^+(3, q)$ of order $(q, 1)$, $\mathrm{Q}(4, q)$ of order q, $\mathrm{Q}^-(5, q)$ of order (q, q^2), $\mathrm{W}(3, q)$ of order q, $\mathrm{H}(3, q^2)$ of order (q^2, q) and $\mathrm{H}(4, q^2)$ of order (q^2, q^3). The dual quadrangles of this list are called the *dual classical generalised quadrangles*. These two classes are not disjoint. The following results can be found in for instance [81].

Theorem 1.4.2. (i) *The GQ $\mathrm{Q}(4, q)$ is isomorphic to the dual of $\mathrm{W}(3, q)$.*

(ii) *The GQ $\mathrm{Q}^-(5, q)$ is isomorphic to the dual of $\mathrm{H}(3, q^2)$.*

(iii) *The GQ $\mathrm{Q}(4, q)$ (and hence $\mathrm{W}(3, q)$) is self-dual if and only if q is even.*

Hence with our definition, only the GQ $\mathrm{H}(4, q^2)^D$ is dual classical and not classical, and $\mathrm{Q}^+(3, q)$ is classical and not dual classical.

In this section, we will give two more examples of generalised quadrangles. The first example is due to J. Tits.

Consider an oval $\mathcal{O}$ in a projective plane $\pi_0 = \mathrm{PG}(2,q)$. Embed π_0 in a $\mathrm{PG}(3,q)$. Define a point set $\mathcal{P}$ as follows. Points of type (i) are the points of $\mathrm{PG}(3,q) \setminus \pi_0$. Points of type (ii) are the planes of $\mathrm{PG}(3,q)$ intersecting π_0 in a tangent line to $\mathcal{O}$, and the unique point of type (iii) is denoted by (∞). The line set contains lines of two types. Lines of type (a) are the lines of $\mathrm{PG}(3,q)$ not in π_0 intersecting π_0 in a point of $\mathcal{O}$. Lines of type (b) are the points of $\mathcal{O}$. Incidence is inherited from $\mathrm{PG}(3,q)$ for points of type (i) and lines of type (a), for points of type (ii) and lines of type (a) and for points of type (ii) and lines of type (b). Points of type (i) are never incident with lines of type (b) and (∞) is incident only with every line of type (b). It is easy to check that $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a GQ of order q. This GQ is denoted with $T_2(\mathcal{O})$, and it is well known that $T_2(\mathcal{O}) \cong \mathrm{Q}(4,q)$ if and only if $\mathcal{O}$ is a conic $\mathrm{Q}(2,q)$ (see [81] and Section 2.1.1). When $\mathcal{O}$ is not a conic, $T_2(\mathcal{O})$ is an example of a non-classical GQ.

Consider now a hyperoval $\mathcal{H}$ in $\pi_0 = \mathrm{PG}(2,q)$, q even. Embed again π_0 in $\mathrm{PG}(3,q)$. Define a point set $\mathcal{P}$ as the points of $\mathrm{PG}(3,q) \setminus \pi_0$ and the line set as the set of lines of $\mathrm{PG}(3,q)$ not in π_0 meeting $\mathcal{H}$ in a necessarily unique point. Incidence is inherited from $\mathrm{PG}(3,q)$. It is easy to check that $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a GQ of order $(q-1, q+1)$. This structure is denoted by $T_2^*(\mathcal{H})$, and it is also an example of a non-classical GQ for $q \neq 3$.

For more information about the subject, we refer again to [81].

1.5 Ovoids and spreads of polar spaces and generalised quadrangles

In this last section, we present some results on spreads and ovoids of polar spaces and GQs. From the point of view of the previous sections, it is possible to do the work for generalised quadrangles and for polar spaces separately. Since results on ovoids of particular classical GQs have important implications for ovoids of polar spaces of higher rank, we will treat the two cases at once.

An *ovoid* $\mathcal{O}$ of a polar space $\mathcal{S}$ is a set of points such that every generator meets $\mathcal{O}$ in exactly one point. A *spread* S is a set of generators partitioning the point set of $\mathcal{S}$.

Since GQs are considered as pure point-line geometries here, we define an ovoid and a spread again. An *ovoid* $\mathcal{O}$ of a GQ is a set $\mathcal{O}$ of points such that every line meets $\mathcal{O}$ in exactly one point. A *spread* S is a set of lines partitioning the point set of $\mathcal{S}$.

A lot of general theorems about ovoids and spreads can be proved and one finds a lot of information in [81]. We only mention three results which are a good illustration of the kind of theorems that can be proved in general.

Theorem 1.5.1. *If $\mathcal{O}$ (respectively, $\mathcal{R}$) is an ovoid (respectively, spread) of the GQ $\mathcal{S}$ of order (s,t), then $|\mathcal{O}| = 1 + st$ (respectively, $|\mathcal{R}| = 1 + st$).*

Theorem 1.5.2. *If the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ of order s admits a polarity, then either $s = 1$ or $2s$ is a square. Moreover the set of all absolute points (respectively, lines) of a polarity θ of $\mathcal{S}$ is an ovoid (respectively, spread) of $\mathcal{S}$.*

Theorem 1.5.3. *A GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ of order (s,t), with $s > 1$ and $t > s^2 - s$, has no ovoid.*

A very recent overview on the existence or non-existence of ovoids and spreads of finite classical polar spaces and finite classical GQs is [95]. We will mention and use some of these results in the next chapters.

Chapter 2

Maximal partial spreads of translation generalised quadrangles

Consider a GQ $\mathcal{S}$. A *partial spread* S is a set of pairwise disjoint lines of $\mathcal{S}$. A partial spread is called *maximal* if it is not contained in a larger partial spread. A point p of $\mathcal{S}$ is called a *hole with respect to* S if there is no line of S on p.

In this chapter, we investigate certain partial spreads of translation generalised quadrangles. To do this, we use the representation of translation generalised quadrangles using eggs, together with results on minihypers. For $T_2(\mathcal{O})$, we can also use combinatorial information to obtain better bounds. Furthermore, for q even and $\mathcal{O}$ a conic $\mathrm{Q}(2,q)$, we obtain a sharp result, i.e. the size of the largest example equals the theoretical upper bound. For $T_3(\mathcal{O})$, the results are less strong, and for $T_{n,m}(\mathcal{E})$, they actually illustrate a nice application of minihypers.

This chapter is based on joint work with M. R. Brown and L. Storme [21].

2.1 Preliminaries

In this section, we define the basic concepts. We start with translation generalised quadrangles, followed by more details about ovals of $\mathrm{PG}(2,q)$, minihypers and more details about spreads of $\mathrm{PG}(3,q)$.

2.1.1 Translation generalised quadrangles

We define translation generalised quadrangles following [81]. Consider a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ of order (s,t), $s \neq 1$ and $t \neq 1$. A collineation θ of $\mathcal{S}$ is a *whorl about the point* p provided θ fixes each line incident with p. If θ is the identity or θ fixes no point of $\mathcal{P} \setminus p^\perp$, then θ is called an *elation about* p. If θ fixes each point of $p^\perp$, then θ is a *symmetry about* p. The symmetries about p form a group. For each $x \sim p$, $x \neq p$, this group acts semiregularly on the set $\{L \in \mathcal{B} \| x \mathrm{~I~} L, p \not\mathrm{I}\, L\}$, and therefore its order divides t. If the order equals t, then the point p is called a *center of symmetry*. Symmetries about lines are defined dually, and a line whose symmetry group has maximal order s is called an *axis of symmetry*.

If there is a group G of elations about p acting regularly on $\mathcal{P} \setminus p^\perp$, then the GQ $\mathcal{S}$ is called an *elation generalised quadrangle* or *EGQ* with *elation group* G and *base point* p, and can be denoted by $(\mathcal{S}^{(p)}, G)$. If $(\mathcal{S}^{(p)}, G)$ is an EGQ for which G contains a full group of s symmetries about each line through p, then $\mathcal{S}$ is a *translation generalised quadrangle* or *TGQ* with *base point* p and *translation group* G.

Consider an EGQ $(\mathcal{S}^{(p)}, G)$ of order (s,t) and let y be a fixed point of $\mathcal{P} \setminus p^\perp$. Let $L_0, \ldots, L_t$ be the lines incident with p, and define z_i and M_i by $L_i \mathrm{~I~} z_i \mathrm{~I~} M_i \mathrm{~I~} y$, $0 \leqslant i \leqslant t$. Define $S_i = \{\theta \in G \| M_i^\theta = M_i\}$, $S_i^* = \{\theta \in G \| z_i^\theta = z_i\}$ and finally, $J = \{S_i \| 0 \leqslant i \leqslant t\}$. Since $(\mathcal{S}^{(p)}, G)$ is an EGQ, $|G| = s^2 t$, J is a collection of $t+1$ subgroups of order s of G, and each S_i^* has order st and contains S_i as a subgroup. Furthermore, the following two conditions are satisfied.

(K1) $S_i S_j \cap S_k = \{1\}$ for distinct i, j, k.

(K2) $S_i^* \cap S_j = \{1\}$ for distinct i, j.

The following construction is due to W. M. Kantor [64] and shows that every EGQ can be constructed as a *group coset geometry*. Consider an arbitrary group G of order $s^2 t$. Suppose $J = \{S_i \| 0 \leqslant i \leqslant t\}$ is a collection of $1+t$ subgroups of order s of G and $J^* = \{S_i^* \| 0 \leqslant i \leqslant t\}$ is a collection of $1+t$ subgroups of order st of G, where each $S_i \subset S_i^*$ and where J and J^* satisfy the conditions K1 and K2. Sometimes the couple (J, J^*) is called a *4-gonal family in* G. A 4-gonal family in G gives rise to the following incidence structure $\mathcal{S}(G, J)$, which is an EGQ with base point (∞) and elation group G.

Points of $\mathcal{S}(G, J)$ are:

(i) the elements of G,

(ii) the right cosets $S_i^* g$, $g \in G$, $0 \leqslant i \leqslant t$,

(iii) the symbol (∞).

Lines of $\mathcal{S}(G, J)$ are:

(a) right cosets $S_i g$, $g \in G$, $0 \leqslant i \leqslant t$,

(b) the symbols $[S_i]$, $0 \leqslant i \leqslant t$.

A point g of type (i) is incident with each line $S_i g$, $0 \leqslant i \leqslant t$. A point $S_i^* g$ of type (ii) is incident with $[S_i]$ and with each line $S_i h \subset S_i^* g$. The point (∞) is incident with each line $[S_i]$ of type (b). There are no further incidences. Starting from an arbitrary EGQ $(\mathcal{S}^{(p)}, G)$ and defining the groups S_i and S_i^* and the set J as above, it is noticed in [81] that $(\mathcal{S}^{(p)}, G) \cong \mathcal{S}(G, J)$. Furthermore, any $\mathcal{S}(G, J)$ is an EGQ with base point (∞) and elation group G. It is worth mentioning that the construction of $\mathcal{S}(G, J)$ is used to obtain item (i) of the following theorem. An other interesting fact is expressed in item (ii).

Theorem 2.1.1. (i) **([81, 8.2.3])** *If $(\mathcal{S}^{(p)}, G)$ is an EGQ with G abelian, then it is a TGQ.*

(ii) **([81, 8.3.2])** *The translation group of a TGQ is uniquely defined and is abelian.*

Let $(\mathcal{S}^{(p)}, G)$ be a TGQ and define S_i, S_i^* and J as above. The *kernel* K is the set of all endomorphisms α of G such that $S_i^\alpha \subset S_i$, $0 \leqslant i \leqslant t$. Since G is abelian, K is a ring. We mention that the neutral element for the addition in K is the endomorphism mapping G onto the trivial group. For the next theorem, $2 < s$ is supposed.

Theorem 2.1.2. ([81, 8.5.1]) *The kernel K of a TGQ is a field, so that $(S_i)^\alpha = S_i$, $(S_i^*)^\alpha = S_i^*$, $0 \leqslant i \leqslant t$, for all $\alpha \in K \setminus \{0\}$.*

The only cases left out by assuming $2 < s$ are the classical GQs $\mathrm{W}(2,2)$ and $\mathrm{Q}^-(5,2)$, since also $1 < t$. Before formulating the most important theorem of this section, we mention the following theorem.

Theorem 2.1.3. *The multiplicative group of the kernel of a TGQ $(\mathcal{S}^{(p)}, G)$ is isomorphic to the group of all whorls about p fixing a given $y \notin p^\perp$.*

We now will describe a model for TGQs using “ingredients from projective geometry”. This model will enable us to use tools from classical Galois

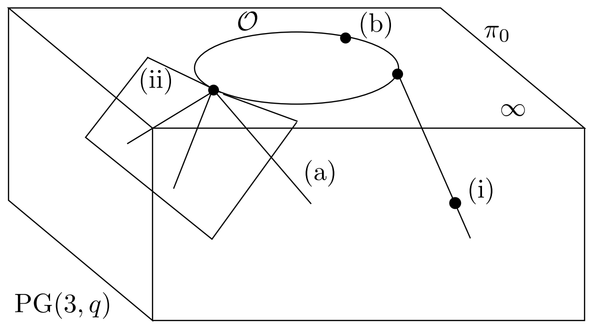

Figure 2.1: The TGQ $T_2(\mathcal{O})$

geometry to prove theorems about "abstract TGQs". The model for a TGQ is constructed using eggs.

Consider an egg $\mathcal{E}$ in $\mathrm{PG}(2n+m-1,q)$. Embed $\mathrm{PG}(2n+m-1,q)$ as a hyperplane in a $\mathrm{PG}(2n+m,q)$ and define the following point line geometry $(\mathcal{P},\mathcal{B},\mathrm{I})$, denoted by $T(n,m,q)$.

The point set $\mathcal{P}$ consists of three types of points:

(i) the points of $\mathrm{PG}(2n+m,q)\setminus\mathrm{PG}(2n+m-1,q)$,

(ii) the $(n+m)$-dimensional subspaces of $\mathrm{PG}(2n+m,q)$ which intersect $\mathrm{PG}(2n+m-1,q)$ in one of the $(n+m-1)$-dimensional tangent spaces of $\mathcal{E}$,

(iii) the symbol (∞).

The line set $\mathcal{B}$ consists of two types of lines:

(a) the n-dimensional subspaces of $\mathrm{PG}(2n+m,q)$ which intersect $\mathrm{PG}(2n+m-1,q)$ in an element of $\mathcal{E}$,

(b) the elements of $\mathcal{E}$.

A point of type (i) is only incident with lines of type (a); the incidence is inherited from $\mathrm{PG}(2n+m,q)$. A point of type (ii) is only incident with all lines of type (a) contained in it and with the unique element of $\mathcal{E}$ contained in it. The point (∞) is incident with no lines of type (a) and with all lines of type (b).

The structure $T(n,m,q)$ is a GQ of order (q^n,q^m). Furthermore, these GQs are TGQs, but also the converse is true. We mention the following theorem from [81, 8.7.1]

Theorem 2.1.4. ([81, 8.7.1]) *The point line geometry $T(n,m,q)$ is a TGQ of order (q^n,q^m), with base point (∞) and for which $\mathrm{GF}(q)$ is a subfield of the kernel. The translations of $T(n,m,q)$ induce translations of the affine space $\mathrm{AG}(2n+m,q) = \mathrm{PG}(2n+m,q) \setminus \mathrm{PG}(2n+m-1,q)$. Conversely, every TGQ for which $\mathrm{GF}(q)$ is a subfield of the kernel is isomorphic to a $T(n,m,q)$, for some egg $\mathcal{E}$ in $\mathrm{PG}(2n+m-1,q)$. It follows that the theory of TGQs is equivalent to the theory of eggs.*

We will denote the structure $T(1,1,q)$ by $T_2(\mathcal{O})$. The egg $\mathcal{E}$ is an oval $\mathcal{O}$ of the plane $\mathrm{PG}(2,q)$. The structure $T(1,2,q)$ will be denoted by $T_3(\mathcal{O})$. The egg $\mathcal{E}$ is an ovoid $\mathcal{O}$ of the projective space $\mathrm{PG}(3,q)$. In general, we will denote the structure $T(n,m,q)$ by $T_{n,m}(\mathcal{E})$.

The following theorems can be found in [81].

Theorem 2.1.5. (i) *The GQ $T_2(\mathcal{O})$ is isomorphic with the GQ $\mathrm{Q}(4,q)$ if and only if $\mathcal{O}$ is a conic $\mathrm{Q}(2,q)$ of the plane $\mathrm{PG}(2,q)$.*

(ii) *The GQ $T_3(\mathcal{O})$ is isomorphic with the GQ $\mathrm{Q}^-(5,q)$ if and only if $\mathcal{O}$ is an elliptic quadric $\mathrm{Q}^-(3,q)$ of $\mathrm{PG}(3,q)$.*

2.1.2 Ovals of $\mathrm{PG}(2,q)$ revisited and spreads of $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$

Consider a hyperoval $\mathcal{H} = \mathcal{D}(f)$ of $\mathrm{PG}(2,q)$, q even. We call $\mathcal{H}$ a *translation hyperoval* if $\mathcal{H}$ is fixed by a group of elations $(x_0,x_1,x_2) \mapsto (x_0, x_1+tx_0, x_2+f(t)x_0)$, $t \in \mathrm{GF}(q)$. This group of elations fixes the points $(0,0,1)$ and $(0,1,0)$, and hence the line $\langle(0,0,1),(0,1,0)\rangle$ and this line is called the *axis* of the hyperoval. It is clear that this elation group induces a translation group of the affine plane obtained by removing the axis from $\mathrm{PG}(2,q)$. We mention the following theorem about translation hyperovals. A proof can also be found in [60]. We will use the theorem to define translation ovals.

Theorem 2.1.6. (Payne [78]) *In $\mathrm{PG}(2,2^h)$, the set $\mathcal{D}(f)$ is a translation hyperoval if and only if $f(t) = t^{2^i}$, with $\gcd(i,h) = 1$.*

An oval $\mathcal{O}$ is a *translation oval* if and only if $\mathcal{O}$ is projectively equivalent to the set $\{(1,t,t^{2^i}) \| t \in \mathrm{GF}(q)\} \cup \{(0,0,1)\}$, with $\gcd(i,h) = 1$.

These (hyper)ovals were constructed by B. Segre in [85].

Consider $T_2(\mathcal{O})$. It is proved in [81, 12.5.2] that $T_2(\mathcal{O})$ is self-dual if and only if $\mathcal{O}$ is a translation oval. Hence for every ovoid of $T_2(\mathcal{O})$, $\mathcal{O}$ a translation oval, we find a spread.

Suppose that π is the plane of $\mathrm{PG}(3,q)$ containing the oval $\mathcal{O}$ and consider a plane π_0 intersecting π in a line external to $\mathcal{O}$. Then the q^2 points of $\pi_0 \setminus \pi$

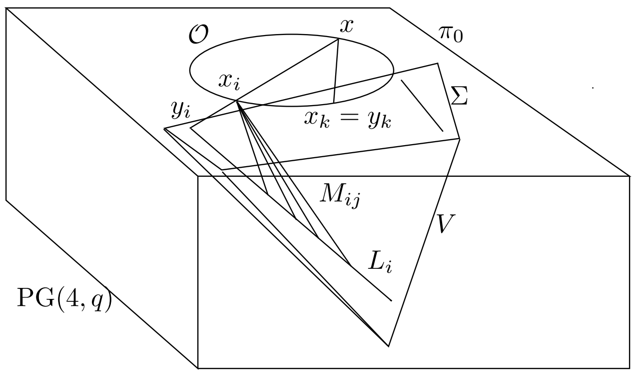

Figure 2.2: A spread of $T_3(\mathcal{O})$

together with the point (∞) constitute an ovoid of $T_2(\mathcal{O})$. This construction works for all ovals $\mathcal{O}$, but when $\mathcal{O}$ is a translation oval, we find a spread due to the self-duality of $T_2(\mathcal{O})$. We mention that an ovoid of $T_2(\mathcal{O})$ constructed in this way is also called a *planar ovoid*.

This is not the only possibility to construct spreads of $T_2(\mathcal{O})$. We recall the following theorem by S. E. Payne from [81].

Theorem 2.1.7. *If the GQ $\mathcal{S}$ of order $s \neq 1$ admits a polarity, then $2s$ is a square. Moreover the set of all absolute points (resp. lines) of a polarity θ of $\mathcal{S}$ is an ovoid (resp. a spread) of $\mathcal{S}$.*

Again from [81, 12.5.2], we know that if $q = 2^h$, h odd, and $\mathcal{O}$ is a translation oval, then $T_2(\mathcal{O})$ is self-polar. Hence, the set of all absolute lines of $T_2(\mathcal{O})$ with respect to a polarity θ is a spread of $T_2(\mathcal{O})$.

In [22], we find a good overview of this (and other) possibilities to construct spreads of $T_2(\mathcal{O})$, and relations with other subjects. For our purpose, the given constructions are sufficient.

When q is odd, the situation is less complex. Since every oval of $\mathrm{PG}(2, q)$, q odd, is a conic, $T_2(\mathcal{O}) \cong \mathrm{Q}(4, q)$ when q is odd. In for instance [81] it is proved that $\mathrm{Q}(4, q)$, q odd, has no spread.

To end this section we recall from [81] the construction of a spread of $T_3(\mathcal{O})$. This construction is valid for arbitrary q and for an arbitrary ovoid $\mathcal{O}$ of $\mathrm{PG}(3, q)$. The construction uses an arbitrary spread of $\mathrm{PG}(3, q)$.

Consider $T_3(\mathcal{O})$. Let $\mathcal{O} \in \pi_0 = \mathrm{PG}(3, q) \subset \mathrm{PG}(4, q)$. Let $x \in \mathcal{O}$ and let Σ be a plane of π_0 such that $x \notin \Sigma$. Let V be a 3-dimensional space distinct from π_0 and containing Σ. Define $L = \Sigma \cap \Sigma_x$ where Σ_x is the tangent plane to $\mathcal{O}$ in x. Suppose that S is a spread of V containing L. Define $y_i = \langle x, x_i \rangle \cap V$, $i = 1, \ldots, q^2$, for all $x_i \in \mathcal{O} \setminus \{x\}$ and denote the element of

S incident with y_i by L_i. If the lines of the plane $\Sigma_i = \langle x, x_i, L_i\rangle$, different from $\langle x, x_i\rangle$, that are incident with x_i are labelled M_{ij}, $j = 1, \ldots, q$, then it follows that $S' = \{x\} \cup \{M_{ij} \| i = 1, \ldots, q^2; j = 1, \ldots, q\}$ is a spread of $T_3(\mathcal{O})$.

2.1.3 Minihypers

Minihypers were introduced by N. Hamada and F. Tamari in [51] in the context of linear codes meeting the Griesmer bound. In two papers by P. Govaerts and L. Storme, and a paper by S. Ferret and L. Storme, new classification and characterisation results are proved, see [47], [46] and [43]. We will define minihypers in this section and recall results used in the next sections.

Definition 2.1.8. An $\{f, m; N, q\}$-*minihyper* is a pair (F, w), where F is a subset of the point set of $\mathrm{PG}(N, q)$ and w is a weight function w: $\mathrm{PG}(N, q) \to \mathbb{N}$: $x \mapsto w(x)$, satisfying

1. $w(x) > 0 \iff x \in F$,
2. $\sum_{x\in F} w(x) = f$, and
3. $\min\{\sum_{x\in H} w(x) \| H \in \mathcal{H}\} = m$, where $\mathcal{H}$ is the set of hyperplanes of $\mathrm{PG}(N, q)$.

A minihyper (F, w) is uniquely defined by its weight function. If w maps to $\{0, 1\}$, we can still use the notation (F, w), but this can also be identified with the point set F. We will also use the notation $|(F, w)| = f$. Let P be an arbitrary subset of the point set of $\mathrm{PG}(N, q)$, then $|P \cap (F, w)| = \sum_{x\in F\cap P} w(x)$. If π is an arbitrary subspace of $\mathrm{PG}(N, q)$, then $\pi \cap (F, w)$ denotes the minihyper obtained by restricting w to $\pi \cap F$.

To characterise certain minihypers, the following definition is used.

Definition 2.1.9. Denote by $\mathcal{A}$ the set of all t-dimensional subspaces of $\mathrm{PG}(N, q)$. A *sum of t-dimensional subspaces* is a weight function w: $\mathcal{A} \to \mathbb{N}$: $\pi_t \mapsto w(\pi_t)$. Such a sum induces a weight function on subspaces of smaller dimension. Let π_r be a subspace of dimension $r < t$, then $w(\pi_r) = \sum_{\pi\in\mathcal{A}, \pi\supset\pi_r} w(\pi)$. In particular, the weight of a point is the sum of the weights of the t-dimensional subspaces passing through that point. A sum of t-dimensional subspaces is said to be a *sum of n t-dimensional subspaces* if the sum of the weights of all t-dimensional subspaces of $\mathcal{A}$ is n.

With these concepts, we can immediately mention the following characterisation result from [47]. We recall that $\theta_\mu = \frac{q^{\mu+1}-1}{q-1}$.

Theorem 2.1.10. *Let $q > 2$ and $\delta < \epsilon$, where $q + \epsilon$ is the size of the smallest non-trivial blocking sets in* $\mathrm{PG}(2,q)$. *If (F,w) is a $\{\delta\theta_\mu, \delta\theta_{\mu-1}; N, q\}$-minihyper satisfying $\mu \leqslant N-1$, then w is the weight function induced on the points of* $\mathrm{PG}(N,q)$ *by a sum of δ μ-dimensional subspaces.*

The bounds mentioned in the characterisation are directly related to bounds concerning blocking sets of the plane. When q is a square, the characterisation can be improved using Baer subgeometries. We mention the following general result from [46].

Theorem 2.1.11. *A $\{\delta\theta_\mu, \delta\theta_{\mu-1}; N, q\}$-minihyper F, $q > 16$ square, $\delta < \frac{q^{\frac{5}{8}}}{\sqrt{2}} + 1$, is a unique union of pairwise disjoint μ-dimensional subspaces and subgeometries* $\mathrm{PG}(2\mu+1, \sqrt{q})$.

This theorem is proved in [46] in several steps. Although we will not need the above theorem immediately, some lemmas involving Baer subgeometries used to prove it will be useful in our application. Therefore we will mention now some more technical results.

The following lemma is a special case of a general theorem on minihypers.

Lemma 2.1.12. (Hamada and Helleseth [50]) *If (F,w) is a $\{\delta(q+1), \delta; 3, q\}$-minihyper, $\delta \leqslant \frac{(q+1)}{2}$, then any plane π intersects it in a $\{m_1(q+1)+m_0, m_1; 2, q\}$-minihyper for some integers m_0 and m_1 with $m_0 + m_1 = \delta$.*

Definition 2.1.13. Denote by $m_1(\pi)$ the integer m_1 corresponding to the plane π. If $m_1(\pi) = 0$, then π is called *poor*; if π is not poor, then it is called *rich*.

Definition 2.1.14. Suppose that q is a square. A *Baer cone* with vertex p in $\mathrm{PG}(3,q)$ is a set of points that is the union of lines on p that form a Baer subplane in the quotient space on p. The *planes* of this cone are the $q + \sqrt{q} + 1$ planes that contain $\sqrt{q} + 1$ lines of the cone.

Lemma 2.1.15. (Govaerts and Storme [46]) *Suppose that (F,w) is a $\{\delta(q+1), \delta; 3, q\}$-minihyper, $\delta \leqslant \frac{(q+1)}{2}$, q square, and suppose that every blocking set of* $\mathrm{PG}(2,q)$ *with at most $q+\delta$ points contains a line or a Baer subplane. Suppose furthermore that (F,w) contains no line. If p is a point of (F,w) with $w(p) = 1$, then the set of rich planes through p contains the set of planes of a Baer cone with vertex p.*

2.1.4 Spreads of $\mathrm{PG}(3, q)$

In this section we mention results about spreads of $\mathrm{PG}(3, q)$. We will mention extendability results and examples of maximal partial spreads of $\mathrm{PG}(3, q)$. Furthermore we will explain extendability results for spreads of $\mathrm{PG}(n, q)$, since they illustrate the link between blocking sets and spreads, and even the use of minihypers, which are defined in Section 2.1.3.

Spreads of $\mathrm{PG}(3, q)$

Theorem 1.2.31 links lower bounds for non-trivial blocking sets of $\mathrm{PG}(2, q)$ to upper bounds for maximal partial spreads of $\mathrm{PG}(3, q)$. Using this theorem and Theorems 1.2.20 and 1.2.21, we immediately conclude the following corollary.

Corollary 2.1.16. *Let S be a maximal partial spread of $\mathrm{PG}(3, q)$ of deficiency $\delta > 0$. Then*

(i) $\delta \geqslant \sqrt{q} + 1$ *when q is a square,*

(ii) $\delta \geqslant \max(1 + p^{e+1}, 1 + c_p q^{\frac{2}{3}})$, *$p$ prime, $q = p^{2e+1}$, $e \geqslant 1$,*

(iii) $\delta \geqslant \frac{q+3}{2}$ *when q is an odd prime.*

More results in this style can be found in the literature. We refer to [41] for a good overview.

The following step in investigating the problem is trying to find examples of maximal partial spreads close to the upper bound. We mention the following result.

Theorem 2.1.17. *There exists a maximal partial spread of size $q^2 - q + 2$ in $\mathrm{PG}(3, q)$.*

We will give a sketch of the proof. Consider an aregular spread S of $\mathrm{PG}(3, q)$. (For an example, see Section 1.2.4). Take an arbitrary line $L \notin S$. Denote by $\mathcal{L}$ the set of lines of S intersecting L. Then $S' = (S \setminus \mathcal{L}) \cup \{L\}$ is a partial spread of size $q^2 - q + 1$. This partial spread can only be extended with lines intersecting all $q+1$ lines of $\mathcal{L}$. If there would exist two lines L' and L'' intersecting the lines in $\mathcal{L}$, then $\mathcal{L}$ would be the complementary regulus of the regulus defined by L, L' and L''. Hence in $\mathrm{PG}(3, q)$, there exist maximal partial spreads of size $q^2 - q + 1$ or size $q^2 - q + 2$. Results of A. A. Bruen and J. A. Thas [28], J. W. Freeman [44] and D. Jungnickel [63] show that it is possible to construct maximal partial spreads of size $q^2 - q + 2$.

It was conjectured that if S is a maximal partial spread of $\mathrm{PG}(3,q)$ with positive deficiency δ, that $\delta \geqslant q-1$. This conjecture was disproved for $q=7$ by Heden [52].

Proving that the upper bound for the size of a maximal partial spread is close to the size of the largest known maximal partial spread or vice versa is the main problem. Also constructing other examples of maximal partial spreads for several values of δ receives a lot of attention. The following results give information about spectra of maximal partial spreads of $\mathrm{PG}(3,q)$.

Theorem 2.1.18. (Heden [53]) *In* $\mathrm{PG}(3,q)$*,* q *odd,* $q \geqslant 7$*, there exists a maximal partial spread of size* $q^2+1-\delta$*,* $\frac{(q^2-11)}{2} \geqslant \delta \geqslant q-1$*. In addition there also exist spreads of size* $q^2+1-\delta$ *with:*

- $\delta = \frac{(q^2+1-2n)}{2}$*,* $n = 1,2,\ldots,5$ *if* $q+1 \equiv 2$ *or* 4 *(mod* 6*),*
- $\delta = \frac{(q^2-7)}{2}$*, if* $q+1 \equiv 0$ *mod* 6 *and* $q \geqslant 17$*,*
- $\delta = \frac{(q^2+1-2n)}{2}$*,* $n = 3,5$ *if* $q = 11$*.*

The q even case is still in research, but so far there are the following results.

Theorem 2.1.19. (Heden et al. [54]) *In* $\mathrm{PG}(3,q)$*,* q *even,* $q \geqslant q_0$*, there exists a maximal partial spread of size* $q^2+1-\delta$*,* $\frac{(3q^2-q-8)}{8} \geqslant \delta \geqslant q-1$*.*

The exact value of q_0 is not yet known. It is expected that $q_0 > 16$.

An interesting link between partial spreads of projective spaces and minihypers is found in [47] and [46]. Extendability of partial t-spreads of $\mathrm{PG}(n,q)$, $(t+1)|(n+1)$, is investigated. We will briefly mention some aspects. The authors consider a partial t-spread of $\mathrm{PG}(n,q)$ with deficiency δ. The following lemma explains the structure of the set of holes with respect to the partial t-spread.

Lemma 2.1.20. *Let* S *be a partial* t*-spread in* $\mathrm{PG}(n,q)$ *of deficiency* $\delta < q$*. Let* $\mathcal{F}$ *be the set of holes with respect to* S*, then* $\mathcal{F}$ *is a set of cardinality* $\delta\theta_t$ *intersecting every hyperplane in at least* $\delta\theta_{t-1}$ *points.*

The link with minihypers is clear looking back to Definition 2.1.8 of minihypers. The following theorem is then derived in [47].

Theorem 2.1.21. (Govaerts and Storme [47]) *Let* S *be a maximal partial* t*-spread in* $\mathrm{PG}(n,q)$*,* $(t+1)|(n+1)$*, of deficiency* $\delta > 0$*. Then* $\delta \geqslant \epsilon$*, with* $q+\epsilon$ *the cardinality of the smallest non-trivial blocking sets in* $\mathrm{PG}(2,q)$*.*

The results in [47], [46] and [48] (the latter paper links partial t-spreads of polar spaces with minihypers) were inspiring for the results of this chapter. The idea now is to consider partial spreads of translation generalised quadrangles and to link results on minihypers to these partial spreads. Because translation generalised quadrangles can be built with ingredients in projective spaces, we can immediately try to fit the minihypers in the picture.

2.2 Maximal partial spreads of $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$

Our aim is to obtain upper bounds on the size of maximal partial spreads of translation generalised quadrangles. We will first consider $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$.

We will look for structure in the set of holes of a partial spread S of $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$, denoted in this section sometimes by $T_n(\mathcal{O})$. Attaching a minihyper to this set reveals such a structure immediately if the minihyper has suitable parameters. Then we try to improve the bounds by other techniques and in more particular cases.

Using minihypers to attack the problem can be motivated by other results found in the literature since minihypers are a kind of generalisation of blocking sets and links between blocking sets and spreads can be found frequently.

Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a GQ of order (s, t). Then a spread S has size $1 + st$, and a partial spread P has size $1 + st - \delta$, $\delta > 0$. We call P a partial spread of $\mathcal{S}$, with *deficiency* δ.

We first mention the following theorem.

Theorem 2.2.1. (Tallini [88]) *Consider the GQ* $\mathrm{Q}(4, q)$*, then*

(i) *if q is odd,* $\mathrm{Q}(4, q)$ *has no spreads, and if S is a partial spread, then* $|S| \leqslant q^2 - q + 1$.

(ii) *if q is even, $q \geqslant 4$, and S is a maximal partial spread of positive deficiency, then* $|S| < q^2 - q/2$.

Since $\mathrm{Q}(4, q) \cong T_2(\mathcal{O})$ when $\mathcal{O}$ is a conic $\mathrm{Q}(2, q)$, this result fits in our aim. After the examples of maximal partial spreads of $T_2(\mathcal{O})$, it will become clear why the case q even is the most interesting for $T_2(\mathcal{O})$.

Furthermore we mention that in [88], the results of Theorem 2.2.1 are obtained by embedding $\mathrm{Q}(4, q)$ in $\mathrm{Q}^+(5, q)$ and observing the fact that a spread of $\mathrm{Q}(4, q)$ is related to a blocking set with respect to the planes of $\mathrm{PG}(3, q)$.

This is not astonishing if we again observe $\mathrm{Q}(4,q) \cong T_2(\mathcal{O})$ since $T_2(\mathcal{O})$ lives in a $\mathrm{PG}(3,q)$. The interesting fact is that $T_2(\mathcal{O})$ and also $T_3(\mathcal{O})$ and the general model $T_{n,m}(\mathcal{E})$ provide the possibility to apply the same technique: attaching the set of holes to a minihyper and hence linking spreads to blocking sets.

Consider now $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$. A few times we will consider the two models at once and denote them by $T_n(\mathcal{O})$.

We recall some important notions. A spread of $T_n(\mathcal{O})$ contains q^n+1 lines. Let S denote a partial spread of $T_n(\mathcal{O})$ with deficiency δ, hence $|S| = q^n+1-\delta$.

Definition 2.2.2. Let x be a point of $\mathcal{O}$. If $x \in S$, then set $\alpha_x = q$ and if $x \notin S$ let α_x be the number of lines of S incident in $\mathrm{PG}(n+1,q)$ with x. We define the *local deficiency* δ_x of x, with respect to S, by $\delta_x = q - \alpha_x$.

Let $\pi_0 = \mathrm{PG}(n,q)$ which contains $\mathcal{O}$ and is embedded in $\mathrm{PG}(n+1,q)$ as a hyperplane. We will now define an $\{f,m;n+1,q\}$-minihyper in $\mathrm{PG}(n+1,q)$ from the set of holes of the partial spread S of $T_n(\mathcal{O})$. We remark that a partial spread contains at most one line of type (b) of the GQ, because all lines of type (b) intersect in (∞).

Definition 2.2.3. Let S be a partial spread of $T_n(\mathcal{O})$. Define w_S: $\mathrm{PG}(n+1,q) \rightarrow \mathbb{N}$ as follows:

(i) if $x \in \mathrm{PG}(n+1,q)\backslash\pi_0$ and x is a hole with respect to S, then $w_S(x) = 1$, otherwise $w_S(x) = 0$,

(ii) suppose $x \in \mathcal{O}$, define $w_S(x) = \delta_x$,

(iii) $w_S(x) = 0$, $\forall x \in \pi_0 \setminus \mathcal{O}$.

This weight function determines a set F of points of $\mathrm{PG}(n+1,q)$, i.e. $x \in F$ if and only if $w_S(x) > 0$. We will denote the defined minihyper by (F, w_S).

The point (∞) is not a hole

In this section, we suppose that the special point (∞) is covered by an element of the spread. This hypothesis leads to the following lemma.

Lemma 2.2.4. *Let S be a partial spread of $T_n(\mathcal{O})$ which covers (∞) and has deficiency $\delta < q$. Then w_S is the weight function of a $\{\delta(q+1), \delta; n+1, q\}$-minihyper (F, w_S).*

Proof. Since (∞) is covered, S contains exactly one line of type (b), which is a point of $\mathcal{O}$, denoted by p. All other $q^n - \delta$ lines of S each contain q points of type (i), so δq points of type (i) are not covered. Since the deficiency is δ, $\sum_{x \in \mathcal{O}} \delta_x = \delta$. From the definition of w_S this yields

$$|(F, w_S)| = \sum_{x \in \mathrm{PG}(n+1,q)} w_S(x) = \delta(q+1).$$

Consider an arbitrary hyperplane H of $\mathrm{PG}(n+1, q)$. If $H = \pi_0$, then $\sum_{x \in H \cap F} w_S(x) = \sum_{x \in \mathcal{O}} \delta_x = \delta$. Suppose that $H \neq \pi_0$ and that $H \cap \pi_0$ is the tangent space in a point s of $\mathcal{O}$. If $s = p$, no line of S of type (a) is on p and hence $q^n - \delta$ lines of S are intersecting H in distinct points of $\mathrm{PG}(n+1, q) \setminus \pi_0$, which are points of type (i) of $T_n(\mathcal{O})$. So H contains $q^n - (q^n - \delta) = \delta$ holes of weight one of (F, w_S). The same arguments prove, in the case $n = 2$, that a hyperplane $H \neq \pi_0$ of $\mathrm{PG}(3, q)$ skew to $\mathcal{O}$ contains δ points of (F, w_S). Suppose now that $s \neq p$. The $q^n - \delta - (q - \delta_s)$ lines of S not on s intersect H in one point of $\mathrm{PG}(n+1, q) \setminus \pi_0$. There is at most one line of S on s in H (since S is a partial spread). If no such line exists, H contains $\delta + q - \delta_s$ points of (F, w_S) of type (i), otherwise H contains $\delta - \delta_s$ points of (F, w_S) of type (i). Furthermore, the weight of s is by definition δ_s, so $\sum_{x \in H \cap F} w_S(x) = q + \delta$ or δ.

For the last case for H, we will make a distinction between $n = 2$ and $n = 3$. So let $n = 2$ and let $H \cap \pi_0 = L$ be a secant line to $\mathcal{O}$. If $L \cap \mathcal{O} = \{r, p\}$, $q^2 - \delta - (q - \delta_r)$ lines of S intersect H in a point of type (i) of $T_2(\mathcal{O})$, so adding the point r with weight δ_r, H contains at most $q + \delta$ points of (F, w_S). If there is a line on r in H, q extra points are covered. Since $q + \delta < 2q$, no two lines of S on r can lie in H. Thus $|H \cap (F, w_S)| \geqslant \delta$. If $L \cap \mathcal{O} = \{r, s\}$, $r \neq p$, $s \neq p$, we have at most $2q + \delta - \delta_r - \delta_s$ holes which are points of type (i) in H. Since S is a partial spread, no line of S on r and no line of S on s can lie in H at the same time, and since $q > \delta$, no more than two lines of S on r or s lie in H, so $|H \cap (F, w_S)| \geqslant \delta$.

Let $n = 3$ and let $\mathcal{C} = H \cap \mathcal{O}$. Using similar arguments as in the case $n = 2$, the number of points of (F, w_S) in H is at most

$$\delta + \sum_{x \in \mathcal{C}} (q - \delta_x) + \sum_{x \in \mathcal{C}} \delta_x = \delta + (q+1)q \qquad (\text{if } p \notin \mathcal{C}), \text{ or}$$

$$\delta + \sum_{x \in \mathcal{C} \setminus \{p\}} (q - \delta_x) + \sum_{x \in \mathcal{C} \setminus \{p\}} \delta_x = \delta + q^2 \qquad (\text{if } p \in \mathcal{C}).$$

For every line of S in H, q extra points of type (i) are covered, but since $\delta < q$, H contains at least δ points of (F, w_S).

So every hyperplane contains at least δ points of (F, w_S), and there exist hyperplanes which contain exactly δ points. Hence, (F, w_S) is a $\{\delta(q+1), \delta; n+1, q\}$-minihyper. ■

With this lemma, we can immediately prove the following theorem.

Theorem 2.2.5. *Let S be a partial spread with deficiency δ of $T_n(\mathcal{O})$ covering (∞). If $\delta < \epsilon$, with $q + \epsilon$ the size of the smallest non-trivial blocking sets in* $\mathrm{PG}(2, q)$*, $q > 2$, then S can be extended to a spread.*

Proof. Denote again the unique line of type (b) in S by p, a point of $\mathcal{O}$. The conditions on δ imply $\delta < q$ (Section 1.2.4) and so (F, w_S) is a $\{\delta(q+1), \delta; n+1, q\}$-minihyper. If δ satisfies the given conditions, Theorem 2.1.10 assures us that (F, w_S) is the sum of δ lines. Since the points of (F, w_S) on π_0 are points of $\mathcal{O} \setminus \{p\}$, it follows that these δ lines are lines of type (a) of $T_n(\mathcal{O})$. Those lines extend S to a spread. ■

Improvements when q is a square

From the point of view of blocking sets, there is always quite some difference when q is square or not, and this becomes clear in many theorems on bounds for non-trivial minimal blocking sets. However, not only numbers matter, also structures matter. When q is a square, Baer subgeometries arise. This is also the fact in characterisation theorems on minihypers, as we mentioned in Section 2.1.3. In this paragraph, we will use those theorems. Especially here we see the Baer subgeometries are very important. Since Lemma 2.1.15 is restricted to minihypers in 3 dimensions, we restrict to $T_2(\mathcal{O})$.

Theorem 2.2.6. *Suppose that q is a square and that S is a partial spread of $T_2(\mathcal{O})$ which covers (∞) and with deficiency $\delta \leqslant \frac{q}{4}$ such that every blocking set of* $\mathrm{PG}(2, q)$ *of size at most $q+\delta$ contains a line or a Baer subplane. Then S can be extended to a spread of $T_2(\mathcal{O})$.*

Proof. Suppose that (F, w_S) does not contain a line. Consider a hole r of type (i). Since $w_S(r) = 1$, by Lemma 2.1.15, there is a Baer cone B, with vertex r and base a Baer subplane π' in π_0, of rich planes through r. All planes of B through r are rich planes, so all lines of π' are secant or tangent lines to $\mathcal{O}$. Each rich plane contains a point x of $\mathcal{O}$ such that $\delta_x \geqslant 1$. Hence if Π is a set of rich planes through r such that no point of $\mathcal{O}$ is contained in two elements of Π we have that $\delta \geqslant |\Pi|$. Consequently, if $\mathcal{E}$ is a set of lines of π' such that no point of $\mathcal{O}$ is incident with two elements of $\mathcal{E}$, then $\delta \geqslant |\mathcal{E}|$.

Let $\mathcal{K}$ be the k-arc $\pi' \cap \mathcal{O}$. Let $\mathcal{E}$ denote the set of lines of π' external to $\mathcal{K}$, then by the previous paragraph $\delta \geqslant |\mathcal{E}|$. That is,

$$\delta \geqslant \sqrt{q} + q + 1 - k(\sqrt{q}+1) + \frac{k(k-1)}{2}, \text{ with } 0 \leqslant k \leqslant \sqrt{q}+2.$$

This latter lower bound on δ is obtained in the following way. We subtract the number of lines of π' through the k points of $\pi' \cap \mathcal{O}$. Since however every bisecant to $\pi' \cap \mathcal{O}$ is subtracted twice, we correct the lower bound $q + \sqrt{q} + 1 - k(\sqrt{q}+1)$ by adding $\frac{k(k-1)}{2}$.

For this range of values of k we have that $\delta \geqslant \frac{(q-\sqrt{q})}{2}$ which implies that $\delta > \frac{q}{4}$, a contradiction. Hence we conclude that (F, w_S) must contain a line, necessarily a line of type (a) of $T_2(\mathcal{O})$, and so S can be extended. ■

Remark. A non-trivial blocking set of size $q + \frac{q}{4} + 1$ in $\mathrm{PG}(2,q)$, q even and square, which does not contain a Baer subplane is known to exist, namely the set of points $\{(x, \mathrm{Tr}(x), 1) || x \in \mathrm{GF}(q)\} \cup \{(x, \mathrm{Tr}(x), 0) || x \in \mathrm{GF}(q) \setminus \{0\}\}$, with Tr the trace function from $\mathrm{GF}(q)$ to $\mathrm{GF}(4)$ (see [60, Chapter 13]). Thus the bounds of Theorem 2.2.6 can never exceed $\frac{q}{4}$.

The point (∞) is a hole

In the previous section, the results were basically obtained by using the characterisation of certain minihypers. In this section, we will not use the minihypers for $T_2(\mathcal{O})$. Our results for $T_2(\mathcal{O})$ will be obtained by only the use of the special properties of the model $T_2(\mathcal{O})$. This approach shows the strength of this model, particularly to make conclusions in the case $T_2(\mathcal{O}) \cong \mathrm{Q}(4,q)$.

Suppose that S is a partial spread of $T_2(\mathcal{O})$ with deficiency δ, $\delta < q$, and that the point (∞) is a hole with respect to S. The first three lemmas are short observations.

Lemma 2.2.7. (i) *There are $(\delta-1)q$ holes of type (i).*

(ii) $\sum_{x \in \mathcal{O}} \delta_x = q - 1 + \delta$.

Proof. (i) The $q^2+1-\delta$ lines of S, all of type (a), each cover q points of type (i), which gives $(\delta-1)q$ holes of type (i).
(ii) $\sum_{x\in\mathcal{O}} \delta_x = \sum_{x\in\mathcal{O}}(q-\alpha_x) = q^2+q-(q^2+1-\delta) = q+\delta-1$. ■

Lemma 2.2.8. *If S is a partial spread of $T_2(\mathcal{O})$ of deficiency $\delta < q$ (where S may or may not cover (∞)), then for each point $x \in \mathcal{O}$, the lines of* $\mathrm{PG}(3,q)$ *of S on x form an arc in the quotient geometry of x.*

Proof. It suffices to show that no plane π of $\mathrm{PG}(3,q)$ containing x and a second point x' of $\mathcal{O}$ contains more than two lines of S. Since $\delta < q$, from the proof of Lemma 2.2.7, the plane π contains at most $\delta + 2q - (\delta_x + \delta_{x'}) < 3q$ points not on lines of type (a) through x or x'. So π contains at most two lines of S on x. ■

Lemma 2.2.9. *Let S be a partial spread of $T_2(\mathcal{O})$ such that the point (∞) is not covered and such that $\delta_p = q$ for some $p \in \mathcal{O}$. Then S may be extended by adding the point p, which is a line of type (b) of $T_2(\mathcal{O})$.*

Proof. Since no point on the line p of $T_2(\mathcal{O})$ is covered by an element of S, the result follows. ■

With these lemmas, we can prove the following theorem.

Theorem 2.2.10. *Let q be even and let S be a maximal partial spread of $T_2(\mathcal{O})$ with deficiency $\delta \leqslant q-1$. Then S must cover the point (∞).*

Proof. Suppose that S does not cover (∞). Since $\sum_{x\in\mathcal{O}} \delta_x = q + \delta - 1 \leqslant 2q-2$, there must be a point $p \in \mathcal{O}$ such that $\delta_p \in \{0,1\}$. The lines of S incident with p form a $(q-1)$- or q-arc in the quotient geometry of p, which can be extended to a hyperoval $\mathcal{O}_p$ ([60]). One of the points of the quotient geometry of p extending this $(q-1)$- or q-arc corresponds to the tangent line in p to $\mathcal{O}$ in the plane π_0. Hence, π_0 is a bisecant to the hyperoval $\mathcal{O}_p$ in the quotient geometry of p. This shows that this hyperoval contains a point corresponding to a line pp' for some point $p' \in \mathcal{O} \setminus \{p\}$. Consequently, each plane of $\mathrm{PG}(3,q)$ on the line pp' may contain at most one line of S on p. Since a plane on pp' can not contain a line of S on p and a line of S on p', we have three possibilities: $\delta_p = 0$, $\delta_{p'} = q$; $\delta_p = 1$, $\delta_{p'} = q$; or $\delta_p = 1$, $\delta_{p'} = q-1$. In the first two cases, by Lemma 2.2.9 we may extend S by adding p', a contradiction, so we are left with the latter case. Now in this case $\sum_{x\in\mathcal{O}\setminus\{p,p'\}} \delta_x = \delta - 1 \leqslant q-2$, from which it follows that there exists a point $p'' \in \mathcal{O} \setminus \{p,p'\}$ such that $\delta_{p''} = 0$. By applying the arguments above to p'', it follows that S must be extendable, a contradiction. ■

Now the strength of the model $T_2(\mathcal{O})$ becomes clear. Supposing that $\mathcal{O}$ is a conic $\mathrm{Q}(2,q)$, we are just working with the classical GQ $T_2(\mathcal{O}) \cong \mathrm{Q}(4,q)$ and any point of $\mathrm{Q}(4,q)$ can play the role of the point (∞). Since the group of $\mathrm{Q}(4,q)$ acts transitively on the set of points of $\mathrm{Q}(4,q)$, we obtain the following corollary.

Corollary 2.2.11. *Consider the GQ* $\mathrm{Q}(4,q)$. *If q is even, and S is a maximal partial spread of positive deficiency, then* $|S| \leqslant q^2 - q + 1$.

We now will construct examples of maximal partial spreads of $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$.

First, we give a construction of a maximal partial spread of $T_2(\mathcal{O})$ starting from an arbitrary spread.

Theorem 2.2.12. *If $T_2(\mathcal{O})$, q even, has a spread, then $T_2(\mathcal{O})$ has a maximal partial spread of size $q^2 - q + 1$ which covers (∞).*

Proof. Suppose that S is a spread of $T_2(\mathcal{O})$ containing the line $p \in \mathcal{O}$ of type (b). Let $x \in \mathcal{O} \setminus \{p\}$. Then $S_x = (S \setminus x^\perp) \cup \{x\}$ is a partial spread of $T_2(\mathcal{O})$. Since by Lemma 2.2.8, $(S \cap x^\perp) \setminus \{p\}$ is a q-arc in the quotient geometry of x, there can be no line L of $T_2(\mathcal{O})$ such that $S \cap L^\perp = S \cap x^\perp$, and so S_x is a maximal partial spread of size $q^2 - q + 1$. ■

In Section 2.1.2, we mentioned some examples of spreads of $T_2(\mathcal{O})$, so the theorem is applicable. This theorem also shows that the bound of Corollary 2.2.11 is sharp.

Suppose that S' is a spread of $T_3(\mathcal{O})$ constructed as in Section 2.1.2. Denote the unique line of type (b) in S' by x, this is a point of $\mathcal{O}$. We will construct a partial spread P from S'. Consider a point $x_1 \in \mathcal{O} \setminus \{x\}$ and M a line of type (a) of $T_3(\mathcal{O})$ on x_1 not contained in S'. If we remove all lines of S' concurrent with M and add M, we obtain a partial spread of size $q^3 - q + 1$. There are $q + 1$ lines of S' concurrent with M. This is one line M_1 in $\langle M, \Sigma_{x_1}\rangle$, intersecting M in the point $\langle M, \Sigma_{x_1}\rangle$ of type (ii) and q lines $M_2, \ldots, M_{q+1}$ intersecting M in distinct points of type (i). We denote the points of $\mathcal{O}$ on the lines $M_1, \ldots, M_{q+1}$ by $x_1, \ldots, x_{q+1}$. Define $P = (S' \cup \{M\}) \setminus \{M_1, \ldots, M_{q+1}\}$.

Lemma 2.2.13. *If $\{x_1, \ldots, x_{q+1}\}$ is not an oval, then P is maximal.*

Proof. Suppose that some line N of $T_3(\mathcal{O})$ extends P. Necessarily, N must be a line of type (a) on one of the points $\{x_2, \ldots, x_{q+1}\}$ (by considering the covering of the points of type (ii)). Suppose that $x_2 \in N$. If we can add N, then N must intersect all lines $\{M_1, \ldots, M_{q+1}\}$ in exactly one point. So $\{M_1, \ldots, M_{q+1}\} \subset \langle M, N\rangle$ and $\{x_1, \ldots, x_{q+1}\} = \Omega \cap \langle M, N\rangle$ is an oval. The lemma follows by contraposition. ■

Lemma 2.2.14. *The GQ $T_3(\mathcal{O})$, $q > 2$, has maximal partial spreads of size $q^3 - q + 1$.*

Proof. Consider P and suppose that we can add a line N to P. We project all lines of $P = (S' \cup \{M, N\}) \setminus \{M_1, \dots, M_{q+1}\}$ from x on V[1]. Then

$$M_i \mapsto L_i \in S$$
$$M \mapsto L' \notin S$$
$$N \mapsto L'' \notin S$$

with L' incident with y_1 and L'' incident with y_2. Now L' cannot belong to S; otherwise $L' \subseteq \langle x, x_1, M_1 \rangle$. Also since N intersects the lines $M_1, \dots, M_{q+1}$, it follows that L'' meets each of the $q + 1$ lines of S meeting L'. So if L' is a line such that there is no second transversal to the $q + 1$ lines of S meeting L', then it is impossible to find a line N which satisfies all conditions. We will construct a spread of $\mathrm{PG}(3, q)$ which satisfies the necessary conditions.

Suppose that S_1 is a regular spread and $R \subseteq S_1$ is a regulus with opposite regulus R'. Replacing R by R' gives a new spread, denoted by S_2. Suppose that L' is a line of $\mathrm{PG}(3, q)$ not in S_1 nor in S_2, having exactly one point p covered by a line of R (and hence also by a line of R'). Let M_R and $M_{R'}$ be the corresponding lines of R, R' respectively, on p. Let $M_1, \dots, M_q$ be the lines of S_1 and S_2 meeting L' in the q points different from p. Since L meets M_1, M_2, M_3, it is contained in the opposite regulus generated by those three lines, and the lines of S_1 meeting L' form indeed this regulus $\{M_1, \dots, M_q, M_R\}$. Suppose that the lines $M_1, \dots, M_q, M_{R'}$ on L' have a second transversal L'' skew to L'. Since L'' meets M_1, M_2, M_3, it meets all members of the regulus including M_R. Then L'' must contain the point $p = M_r \cap M_{R'}$ and so intersects L', a contradiction. Hence there exists no such second transversal. ■

Lemma 2.2.15. *If $\mathcal{O} = \mathrm{Q}^-(3, q)$, then either P is maximal or P can be extended to a spread.*

Proof. Suppose that $\mathcal{O} = \mathrm{Q}^-(3, q)$ and that P can be extended by the line N. Then necessarily $\Omega = \langle M, N \rangle \cap \mathcal{O}$ is a conic and $M_1, \dots, M_{q+1}$ are $q + 1$ lines of a $T_2(\Omega) \cong \mathrm{Q}(4, q)$ constructed in $\langle M, N \rangle$. Because $\{M_1, \dots, M_{q+1}\} = \{M, N\}^\perp$ in $T_2(\Omega)$, they form a regulus, and $\{M, N\}$ is a subset of the opposite regulus. So if we can add one line N to P, we can add the $q - 1$ remaining lines in the opposite regulus of $\{M_1, \dots, M_{q+1}\}$. Hence P can be extended to a spread, and the result follows. ■

If $\mathcal{O}$ is not $\mathrm{Q}^-(3, q)$, then P is either maximal or may be extended by the addition of one further transversal of $\{M_1, \dots, M_{q+1}\}$. For if $\{M_1, \dots, M_{q+1}\}$

[1]See page 34 for the construction of a spread of $T_3(\mathcal{O})$ and the definition of V

has three transversals, then it follows that it is a regulus, the oval Ω is a conic and so $\mathcal{O}$ is $Q^-(3,q)$ by Theorem 1.2.13.

By the construction of S' from S, we see that the configuration of lines $\{M, M_1, \ldots, M_{q+1}\}$ is the image of respectively $\langle M, x\rangle \cap V$ and $\{\langle M_i, x\rangle \cap V : i = 1, \ldots, q+1\} \subset S$ under the projection of the spread V onto $\langle M, M_1, M_2\rangle$ from x, and conversely. Thus we may extend P if and only if we can find a spread of $\mathrm{PG}(3,q)$ that contains $q+1$ lines with two transversals and meeting some plane of $\mathrm{PG}(3,q)$ in the oval Ω.

Since, at present, only two classes of ovoids are known, we will restrict ourselves to an oval $\Omega = \{(1,t,t^\sigma) \| t \in \mathrm{GF}(q)\} \cup \{(0,0,1)\}$, $q = 2^{2e+1}$ and $\sigma^2 \equiv 2$, $e \geqslant 1$, which is found in the Tits ovoid. We use the Lüneburg spread, $S = \{\langle(1,0,0,0),(0,0,1,0)\rangle\} \cup \{\langle(s^\sigma, 1, s+t^{\sigma+1}, 0), (s+t^{\sigma+1}, 0, t^\sigma, 1)\rangle \| s,t \in \mathrm{GF}(q)\}$. If we consider the $q+1$ lines $\{\langle(1,0,0,0),(0,0,1,0)\rangle\} \cup \{\langle(t^{\sigma+2},1,0,0), (0,0,t^\sigma,1)\rangle \| t \in \mathrm{GF}(q)\}$ of S, we see that they have transversals $\langle(0,0,1,0), (0,0,0,1)\rangle$ and $\langle(1,0,0,0),(0,1,0,0)\rangle$. Furthermore, intersection with the plane $x_1 = x_2$ gives the set of $q+1$ points $\{(t^{2\sigma+2}, t^\sigma, t^\sigma, 1) \| t \in \mathrm{GF}(q)\} \cup \{(1,0,0,0)\} = \{(t^{\sigma+2}, t, t, 1) \| t \in \mathrm{GF}(q)\} \cup \{(1,0,0,0)\}$ which is equivalent to the oval Ω in the Tits ovoid. Hence

Lemma 2.2.16. *If $\mathcal{O}$ is the Tits ovoid, then $T_3(\mathcal{O})$ has a maximal partial spread of size $q^3 - q + 2$ which covers (∞).*

We already mentioned an example of a spread of $T_3(\mathcal{O})$ in Section 2.1.2. In the construction, every spread of $\mathrm{PG}(3,q)$ gives rise to a spread of $T_3(\mathcal{O})$. The following lemma generalises this construction to the construction of maximal partial spreads of $T_3(\mathcal{O})$, now using maximal partial spreads of $\mathrm{PG}(3,q)$.

Lemma 2.2.17. *Suppose that S is a maximal partial spread of* $\mathrm{PG}(3,q)$ *with deficiency δ. Then there exists a maximal partial spread of $T_3(\mathcal{O})$ with deficiency $q\delta$.*

Proof. Consider the maximal partial spread S in $\mathrm{PG}(3,q)$. We construct a partial spread S' on $T_3(\mathcal{O})$ in the same way as a spread is constructed in [81] (see also Section 2.1.2). It is straightforward to see that S' is a partial spread. We will prove that S' is maximal due to the maximality of S. By the construction of S', there are points of $\mathcal{O} \setminus \{x\}$ on which we have q lines of S' and there are points of $\mathcal{O} \setminus \{x\}$ on which there are no lines of S'. If we can extend S', then, since (∞) is already covered, we must add a line of type (a) on a point of $\mathcal{O} \setminus \{x\}$ incident with no line of S'. Suppose that $p \in \mathcal{O} \setminus \{x\}$ is such a point and that we can extend S' by M on p. By construction, if L is any line of S, then S' covers all points of $\langle x, L\rangle \setminus \pi_0$. Hence, M contains no point of $\langle x, L\rangle$ and the projection of M from x onto V is skew to L. This implies that S may be extended, a contradiction. Hence, S' is maximal and of size $q^3 + 1 - q\delta$. ■

For $T_3(\mathcal{O})$, we can bundle all information about examples of maximal partial spreads constructed in this section. Part (i) was proved by Lemma 2.2.14, part (ii) by Lemma 2.2.16, part (iii) by Lemma 2.2.17 and Theorem 2.1.18, and part (iv) by Lemma 2.2.17 and Theorem 2.1.19.

Theorem 2.2.18. (i) *$T_3(\mathcal{O})$ has a maximal partial spread of size q^3-q+1 which covers (∞).*

(ii) *If $\mathcal{O}$ is the Tits ovoid, then $T_3(\mathcal{O})$ has a maximal partial spread of size $q^3 - q + 2$ which covers (∞).*

(iii) *Let q be odd, $q \geqslant 7$, then $T_3(\mathcal{O})$ has maximal partial spreads of size $q^3 + 1 - \delta$, with $\delta = nq$, $\frac{q^2-11}{2} \geqslant n \geqslant q - 1$. For certain values of q, other values for δ are possible:*

- $\delta = \frac{q^3-(2n-1)q}{2}$, *$n = 1, 2, \ldots, 5$ if $q + 1 \equiv 2$ or 4 (mod 6),*
- $\delta = \frac{q^3-7q}{2}$, *if $q + 1 \equiv 0$ mod 6 and $q \geqslant 17$,*
- $\delta = \frac{q^3-(2n-1)q}{2}$, *$n = 3, 5$ if $q = 11$.*

(iv) *Let q be even, $q \geqslant q_0$* (cf. Theorem 2.1.19), *then $T_3(\mathcal{O})$ has maximal partial spreads of size $q^3 + 1 - \delta$, with $\delta = nq$, $\frac{3q^2-q-8}{8} \geqslant n \geqslant q - 1$.*

2.3 Maximal partial spreads of $T_{n,m}(\mathcal{E})$

Consider now a GQ $T_{n,m}(\mathcal{E})$ different from $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$ as defined in Section 2.1.1. The GQ $\mathcal{S} = T_{n,m}(\mathcal{E})$ has order (q^n, q^m) and hence if S is a spread of $\mathcal{S}$, then $|S| = 1 + q^{n+m}$ and the point (∞) is covered by a unique line of type (b), i.e. an $(n-1)$-dimensional space $\alpha \in \mathcal{E}$. As in Section 2.2 we denote by π_0 the $(2n+m-1)$-dimensional space containing $\mathcal{E}$ embedded as a hyperplane in $\mathrm{PG}(2n+m, q)$.

Suppose that S is a partial spread of size $q^{n+m} + 1 - \delta$ covering the point (∞). Denote by α the unique line of type (b) contained in S.

We start with an adapted version of Definition 2.2.2.

Definition 2.3.1. Let α be an element of $\mathcal{E}$. If $\alpha \in S$, then set $A_\alpha = q^n$ and if $\alpha \notin S$, then let A_α denote the number of lines of type (a) incident in $\mathrm{PG}(2n+m, q)$ with α. We define the *local deficiency* Δ_α of α with respect to S, by $\Delta_\alpha = q^n - A_\alpha$.

It is clear that for any partial spread S with deficiency δ and covering (∞), $\sum_{\alpha\in\mathcal{E}} \Delta_\alpha = \delta$. We can now define the *local deficiency* δ_x of a point $x \in \alpha$, with $\alpha \in \mathcal{E}$, as $\delta_x = \Delta_\alpha$. We now have $\sum_{\substack{\alpha\in\mathcal{E}\\ x\in\alpha}} \delta_x = \delta\theta_{n-1}$.

We now can generalise Definition 2.2.3. For any point $x \in \mathrm{PG}(2n+m-1, q)$, we say that $x \in \mathcal{E}$ if and only if $x \in \alpha$, with α an element of $\mathcal{E}$.

Definition 2.3.2. Let S be a partial spread of $T_{n,m}(\mathcal{E})$, $n \geqslant 2$. Define $w_S : \mathrm{PG}(2n+m, q) \to \mathbb{N}$ as follows:

(i) if $x \in \mathrm{PG}(2n+m, q) \setminus \pi_0$ and x is a hole with respect to S, then $w_S(x) = 1$, otherwise $w_S(x) = 0$.

(ii) if $x \in \mathcal{E}$, define $w_S(x) = \delta_x$.

(iii) $w_S(x) = 0$, $\forall x \in \pi_0 \setminus \mathcal{E}$.

This weight function determines a minihyper (F, w_S) of $\mathrm{PG}(2n+m, q)$. The following lemma generalises Lemma 2.2.4.

Lemma 2.3.3. *Let S be a partial spread of $T_{n,m}(\mathcal{E})$, $n \geqslant 2$, which covers (∞) and has deficiency $\delta < q$. Then w_S is the weight function of a $\{\delta\theta_n, \delta\theta_{n-1}; 2n+m, q\}$-minihyper (F, w_S).*

Proof. Since (∞) is covered, S contains exactly one line of type (b), which is an element of $\mathcal{E}$, denoted by α. All other $q^{n+m} - \delta$ lines of S each contain q^n points of type (i), so δq^n points of type (i) are not covered. Since the deficiency is δ, $\sum_{x\in\mathcal{E}} \delta_x = \delta\theta_{n-1}$. From the definition of w_S, this yields

$$|(F, w_S)| = \sum_{x\in\mathrm{PG}(2n+m,q)} w_S(x) = \delta(q^n + \theta_{n-1}) = \delta\theta_n.$$

Consider now an arbitrary hyperplane H of $\mathrm{PG}(2n+m, q)$. If $H = \pi_0$, then $\sum_{x\in H\cap F} w_S(x) = \sum_{x\in\mathcal{E}} \delta_x = \delta\theta_{n-1}$.

Suppose that $H \neq \pi_0$. Any element of $\mathcal{E}$ has an $(n-2)$-dimensional intersection with H or is completely contained in H. Let K be the set of elements of $\mathcal{E}$ intersecting H in an $(n-2)$-dimensional subspace and let L be the set of elements of $\mathcal{E}$ completely contained in H. Put $k = |K|$ and $l = |L|$. Clearly $k + l = q^m + 1$.

For any $\beta \in K$, $\beta \neq \alpha$, $q^n - \Delta_\beta$ lines of type (a) intersect H in an $(n-1)$-dimensional subspace, covering q^{n-1} points of type (i) of H (i.e. points of $H \setminus \pi_0$). Suppose that $\gamma \in L$, $\gamma \neq \alpha$. Then every line of type (a) on γ intersects H in an $(n-1)$-dimensional subspace, necessarily γ, or is completely contained in H. We define m_γ^H as the number of lines of S of type (a) on γ covering q^n further points of type (i) in H, so $m_\gamma^H \leqslant q^n$. Furthermore there are no lines of type (a) of the partial spread S on α.

We have to distinguish two cases. Suppose that $\alpha \in K$. Then $\sum_{\beta\in K}(q^n - \Delta_\beta)q^{n-1}$ points of H of type (i) are covered by the elements of S through the elements of K, and there are at most $q^{2n+m-1} - (k-1)q^n q^{n-1} + \sum_{\beta\in K}\Delta_\beta q^{n-1}$ holes of type (i) in H.

Furthermore, $|H \cap (F, w_S) \cap \pi_0| = \sum_{\beta\in K}\Delta_\beta\theta_{n-2} + \sum_{\gamma\in L}\Delta_\gamma\theta_{n-1}$. Since $\sum_{\beta\in K}\Delta_\beta(\theta_{n-2}+q^{n-1}) = \sum_{\beta\in K}\Delta_\beta\theta_{n-1}$ and $\sum_{\beta\in K}\Delta_\beta\theta_{n-1} + \sum_{\gamma\in L}\Delta_\gamma\theta_{n-1} = \delta\theta_{n-1}$, we have

$$\begin{aligned}|H \cap (F, w_S)| &\leqslant q^{2n+m-1} - (k-1)q^n q^{n-1} + \textstyle\sum_{\beta\in K}\Delta_\beta q^{n-1} \\ &\quad + \textstyle\sum_{\beta\in K}\Delta_\beta\theta_{n-2} + \sum_{\gamma\in L}\Delta_\gamma\theta_{n-1} \\ &= q^{2n+m-1} - (k-1)q^{2n-1} + \delta\theta_{n-1}.\end{aligned} \tag{2.1}$$

Taking into account these values, we find the equation: $|H \cap (F, w_S)| = q^{2n+m-1} - (q^m - l)q^{2n-1} - q^n\sum_{\gamma\in L} m_\gamma^H + \delta\theta_{n-1}$, or,

$$|H \cap (F, w_S)| = q^n(lq^{n-1} - \sum_{\gamma\in L} m_\gamma^H) + \delta\theta_{n-1}. \tag{2.2}$$

If $A = lq^{n-1} - \sum_{\gamma\in L} m_\gamma^H < 0$, then still $Aq^n + \delta\theta_{n-1} \geqslant 0$, a contradiction since $\delta < q$. Hence $A \geqslant 0$ and $|H \cap (F, w_S)| \geqslant \delta\theta_{n-1}$.

Suppose now that $\alpha \in L$. There is little difference with the previous case. It is clear that $m_\alpha^H = 0$. Hence, Inequality (2.1) becomes

$$|H \cap (F, w_S)| \leqslant q^{2n+m-1} - kq^n q^{n-1} + \delta\theta_{n-1}. \tag{2.3}$$

and Equation (2.2) becomes

$$|H \cap (F, w_S)| = q^n((l-1)q^{n-1} - \sum_{\gamma\in L} m_\gamma^H) + \delta\theta_{n-1}. \tag{2.4}$$

Again, if $A = (l-1)q^{n-1} - \sum_{\gamma\in L} m_\gamma^H < 0$, then still $Aq^n + \delta\theta_{n-1} \geqslant 0$, a contradiction since $\delta < q$.

So for an arbitrary hyperplane H, $|H \cap (F, w_S)| \geqslant \delta\theta_{n-1}$ and $|\pi_0 \cap (F, w_S)| = \delta\theta_{n-1}$, hence $\min\{\sum_{x\in H} w_S(x) \| H$ is a hyperplane of $\mathrm{PG}(2n+m, q)\} = \delta\theta_{n-1}$.

We conclude that (F, w_S) is a $\{\delta\theta_n, \delta\theta_{n-1}; 2n+m, q\}$-minihyper. ■

We now can formulate a version of Theorem 2.2.5 for $T_{n,m}(\mathcal{E})$.

Theorem 2.3.4. *Let S be a partial spread of $\mathcal{S} = T_{n,m}(\mathcal{E})$, $n > 1$, which covers (∞) and has deficiency $\delta < \epsilon$, with $q + \epsilon$ the size of the smallest non-trivial blocking sets in* $\mathrm{PG}(2, q)$*, $q > 2$, then S can be extended to a spread.*

Proof. The condition on δ implies that $\delta < q$ and so (F, w_S) is a $\{\delta\theta_n, \delta\theta_{n-1}; 2n + m, q\}$-minihyper. By Theorem 2.1.10, (F, w_S) is the sum of δ n-dimensional spaces. Since the points of $(F, w_S) \cap \pi_0$ are the points of some elements of $\mathcal{E}$, counted with multiplicity, the n-dimensional spaces of (F, w_S) necessarily intersect π_0 in an element of $\mathcal{E}$, and hence are lines of type (a) of $T_{n,m}(\mathcal{E})$. Those lines extend S to a spread. ■

We now will describe an example of a TGQ on which the above theorem can be applied. Therefore we need some extra definitions. The overview we give here is based on parts of [96] and [68].

We start with the construction of a GQ via "q-clans".

Define $G = \{(\alpha, c, \beta) \| \alpha, \beta \in \mathrm{GF}(q)^2, c \in \mathrm{GF}(q)\}$. Define a binary operation $\cdot$ on G by

$$(\alpha, c, \beta) \cdot (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta\alpha'^T, \beta + \beta')$$

It is clear that $G, \cdot$ is a group with center $C = \{(\mathbf{0}, c, \mathbf{0}) \| c \in \mathrm{GF}(q)\}$, with $\mathbf{0} = (0, 0)$. Let $\mathcal{C} = \{A_u \| u \in \mathrm{GF}(q)\}$ be a set of q distinct upper triangular 2×2-matrices over $\mathrm{GF}(q)$. Then $\mathcal{C}$ is called a *q-clan* provided $A_u - A_r$ is anisotropic whenever $u \neq r$. This condition expresses that $\alpha(A_u - A_r)\alpha^T = 0$ only has the trivial solution $\alpha = (0, 0)$. Let

$$A_u = \begin{pmatrix} x_u & y_u \\ 0 & z_u \end{pmatrix}$$

with $x_u, y_u, z_u, u \in \mathrm{GF}(q)$. When q is odd, put $K_u = A_u + A_u^T$, then $\mathcal{C}$ is a q-clan if and only if

$$-\det(K_u - K_r) = (y_u - y_r)^2 - 4(x_u - x_r)(z_u - z_r) \tag{2.5}$$

is a non-square of $\mathrm{GF}(q)$ whenever $r, u \in \mathrm{GF}(q)$, $r \neq u$. When q is even, $\mathcal{C}$ is a q-clan if and only if

$$y_u \neq y_r \text{ and } \mathrm{Tr}_{q\to 2}((x_u + x_r)(z_u + z_r)(y_u + y_r)^{-2}) = 1 \tag{2.6}$$

whenever $r, u \in \mathrm{GF}(q)$, $r \neq u$.

Define a family of subgroups of G by $A(u) = \{(\alpha, \alpha A_u \alpha^T, \alpha K_u) \in G \| \alpha \in \mathrm{GF}(q)\}$, $u \in \mathrm{GF}(q)$ and $A(\infty) = \{(0, 0, \beta) \in G \| \beta \in \mathrm{GF}(q)^2\}$. Put $J = \{A(u) \| u \in \mathrm{GF}(q) \cup \{\infty\}\}$. Then $J^* = \{A^*(u) = A(u)C, u \in \mathrm{GF}(q) \cup \{\infty\}\}$.

Theorem 2.3.5. *The pair (J, J^*) is a 4-gonal family if and only if $\mathcal{C}$ is a q-clan. Hence if $\mathcal{C}$ is a q-clan, then it defines a GQ of order (q^2, q).*

Now we introduce flocks of quadratic cones and we describe the connection with q-clans.

A *flock* of the quadratic cone K with vertex v of $\mathrm{PG}(3, q)$ is a partition of the points of $K \setminus \{v\}$ into q pairwise disjoint irreducible conics. A flock is determined by q planes not on v. If these q planes share a common line, then the flock is called *linear*.

J. A. Thas shows in [93] that the conditions (2.5) and (2.6) are precisely the conditions for the equations $x_uX_0 + z_uX_1 + y_uX_2 + X_3 = 0$, $x_u, y_u, z_u, u \in \mathrm{GF}(q)$, to constitute a flock of the quadratic cone with equation $X_0X_1 = X_2^2$ and vertex $(0, 0, 0, 1)$ in $\mathrm{PG}(3, q)$. We conclude:

Theorem 2.3.6. *To any flock $\mathcal{F}$ of the quadratic cone of* $\mathrm{PG}(3, q)$ *corresponds a GQ of order (q^2, q).*

If $\mathcal{F}$ is a flock, then we denote the corresponding GQ by $\mathcal{S}(\mathcal{F})$. Let us now consider an example of a flock GQ. Let q be odd. Consider q planes π_t with equation $tX_0 - mt^\sigma X_1 + X_3 = 0$, $t \in \mathrm{GF}(q)$, m a fixed non-square in $\mathrm{GF}(q)$ and $\sigma \in \mathrm{Aut}(\mathrm{GF}(q))$. These q planes form a flock $\mathcal{F}_K$ of the cone $X_0X_1 = X_2^2$ in $\mathrm{PG}(3, q)$ and the flock is linear if and only if $\sigma = 1$. The corresponding GQ $\mathcal{S}(\mathcal{F}_K)$ is called the *Kantor semifield (flock) GQ*. This GQ is a TGQ for some base line; hence the point line dual is a TGQ for some base point and hence $\mathcal{S}(\mathcal{F})^D = T_{n,m}(\mathcal{E})$ for some egg $\mathcal{E}$ of $\mathrm{PG}(2n + m - 1, q')$.

We now are interested in the parameters n, m of $\mathcal{E}$. This GQ is studied in the literature and it is known from [79, 82] that the kernel of $\mathcal{S}(\mathcal{F}_K)^D$ is the fixed field of σ. Hence, if $\sigma \neq 1$, then $\mathcal{S}(\mathcal{F}_K)^D$ is a TGQ with kernel $\mathrm{GF}(q')$, the fixed field of σ, and necessarily $q'^n = q$. Hence $\mathcal{S}(\mathcal{F})^D$ is a TGQ of order (q'^n, q'^{2n}), or $\mathcal{S}(\mathcal{F}_K)^D = T_{n,m}(\mathcal{E})$ with necessarily $n > 1$, making $\mathcal{S}(\mathcal{F}_K)$ different from $T_3(\mathcal{O})$.

Theorem 2.3.7. (Payne [80]) *The TGQ $\mathcal{S}(\mathcal{F}_K)^D = T_{n,m}(\mathcal{E})$ is isomorphic to its translation dual.*

The TGQ $\mathcal{S}(\mathcal{F}_K)^D$ has a spread. This is not merely an observation; quite a lot of definitions and lemmas are needed to prove this. We will not go into detail about the existence of a spread, but we mention that the basic result can be found in [94]. For more information, we refer to [96].

Hence, by Theorem 2.3.4,

Theorem 2.3.8. *Let S be a partial spread of $\mathcal{S}(\mathcal{F}_K)^D$, $\mathcal{S}(\mathcal{F}_K)$ the Kantor semifield flock GQ, which covers (∞) and has deficiency $\delta < \epsilon$, with $q + \epsilon$*

the size of the smallest non-trivial minimal blocking sets in $\mathrm{PG}(2, q)$, $q > 2$, *then* S *can be extended to a spread.*

Chapter 3

The smallest minimal blocking sets of $\mathrm{Q}(6,q)$, q even

It is known that $\mathrm{Q}(6,q)$, q even, has no ovoids ([92]). The natural question is how the smallest sets of points blocking every generator look like. In this chapter we investigate this problem. We also develop ideas useful for the forthcoming chapters.

We start with an introductory section. Basic ideas and results are explained. Furthermore, technical results for $\mathrm{Q}(4,q)$ are mentioned. The second section contains the result itself, while the third section contains the proofs of the technical results mentioned in the introduction.

The present chapter is based on joint work with L. Storme [37].

3.1 Introduction

From now on, we will frequently use the notation π_n for an n-dimensional subspace of a projective space.

Consider a polar space $\mathcal{S}$. A *blocking set* $\mathcal{K}$ is a set of points of $\mathcal{S}$ such that every generator meets $\mathcal{K}$ in at least one point. A blocking set $\mathcal{K}$ is called *minimal* if and only if $\mathcal{K} \setminus \{p\}$ is not a blocking set for every point $p \in \mathcal{K}$. This definition is equivalent with the following property. If $\mathcal{K}$ is a minimal blocking set, then for every $p \in \mathcal{K}$, there exists a generator π such that $\pi \cap \mathcal{K} = \{p\}$.

An ovoid is an example of a minimal blocking set. Furthermore no set smaller than an ovoid can block all generators. It will become clear that also ovoids of polar spaces play an important role in characterising blocking sets of polar spaces different from ovoids. This is the main goal of this chapter, and we will characterise the smallest minimal blocking sets of $Q(6,q)$, q even, $q > 16$.

An important idea is found in [92] and recalled in [76]. Although the main result of the latter paper is not used in this chapter, it recalls the following easy to prove lemma, which is an important fact.

Lemma 3.1.1. *If the quadric* $Q(2n,q)$*,* $n \geqslant 2$*, admits an ovoid, then each quadric* $Q(2m,q)$*,* $n \geqslant m \geqslant 2$*, admits an ovoid.*

Consider an ovoid $\mathcal{O}$ of $Q(2n,q)$, $n \geqslant 3$, and consider all lines of $Q(2n,q)$ on a point $p \in Q(2n,q) \setminus \mathcal{O}$. In the base of the tangent cone $T_p(Q(2n,q)) \cap Q(2n,q)$, the lines $\langle p,r\rangle$, $r \in \mathcal{O} \cap T_p(Q(2n,q))$, give rise to an ovoid $\mathcal{O}'$ of $Q(2n-2,q)$. This observation leads immediately to the above lemma.

More important for us is the generalisation to blocking sets; something that will be one of the first lemmas of the next section.

Suppose now that $\mathcal{K}$ is a minimal blocking set of $Q(6,q)$, q even, of size at most q^3+q. It will become clear that $\mathcal{K}$ gives rise to minimal blocking sets of $Q(4,q)$, q even. Concerning these objects, we have the following theorem.

Theorem 3.1.2. (Eisfeld et al. [42]) *Let* $\mathcal{B}$ *be a blocking set of* $Q(4,q)$*, q even,* $q > 16$*, of size* q^2+1+r*, with* $0 < r \leqslant \sqrt{q}$*. Then* $\mathcal{B}$ *consists of an ovoid and r extra points. Hence, a minimal blocking set of* $Q(4,q)$*, q even,* $q > 16$*, has size* q^2+1+r*,* $r > \sqrt{q}$*.*

Theorem 3.1.2 and a generalisation of Lemma 3.1.1 will give combinatorial information about blocking sets of $Q(6,q)$.

Combinatorial and geometric information on minimal blocking sets of $Q(4,q)$ will be the key elements in solving the problem. Since $Q(4,q) \cong W(3,q)^D$, and $Q(4,q)$ is self-dual if and only if q is even, also results on covers can be useful. Before mentioning some of the used results, we give some more definitions.

Consider a GQ $\mathcal{S}$ of order (s,t). A *blocking set* of $\mathcal{S}$ is a set $\mathcal{B}$ of points such that every line of $\mathcal{S}$ meets $\mathcal{B}$ in at least one point. Recall that this definition is consistent with the definition of a blocking set of a polar space. A concept like minimality is defined in the usual way. A *cover* $\mathcal{C}$ is a set of lines of $\mathcal{S}$ such that every point is contained in at least one line of $\mathcal{C}$. It is clear that blocking sets and covers are dual concepts. A cover $\mathcal{C}$ is called *minimal* if no proper subset of $\mathcal{C}$ is still a cover of $\mathcal{S}$. A point p is called a *multiple point* or an *excess point* of $\mathcal{C}$ if it is contained in at least two lines of $\mathcal{C}$. The *excess* of p is the number of lines of $\mathcal{C}$ on p minus one. The *weight* of a line with respect to a given cover is the minimum of the excesses of the points belonging to this line.

We now can mention results from [42]. For the definition of a "sum of lines", we refer to Definition 2.1.9, Section 2.1.3. The next theorem was a basic argument in [42] to prove Theorem 3.1.2.

Theorem 3.1.3. *Let $\mathcal{C}$ be a minimal cover of* $\mathrm{Q}(4,q)$. *Let* $|\mathcal{C}| = q^2+1+r$, *with* $q+r$ *smaller than the cardinality of the smallest non-trivial blocking sets in* $\mathrm{PG}(2,q)$. *Then the multiple points form a sum of lines, contained in* $\mathrm{Q}(4,q)$, *where the weight of a line in this sum is equal to the weight of this line with respect to the cover, and with the sum of the weights of the lines equal to* r.

We will use an adapted version of Theorem 3.1.3, admitting a larger excess. The proof will be given in Section 3.3, Lemma 3.3.1.

Lemma 3.1.4. *Let $\mathcal{C}$ be a minimal cover of* $\mathrm{Q}(4,q)$, $|\mathcal{C}| = q^2+1+r$, $0 < r \leqslant q-1$. *If each multiple point has excess at least* $\sqrt{q}$, *then the set E of multiple points is a sum of lines, with the sum of the weights of the lines equal to* r.

The following lemma is also the adaption of a theorem from [42] for q even. For the proof, we refer to Section 3.3, Lemma 3.3.6.

Lemma 3.1.5. *A minimal cover of* $\mathrm{Q}(4,q)$, q *even, of size* q^2+1+r, *having only points of positive excess at least* $\sqrt{q}$ *satisfies* $r \geqslant \frac{q+4}{6}$.

Finally, we will prove a quite technical result on covers of $\mathrm{Q}(4,q)$. The proof can be found in Section 3.3, Lemma 3.3.7 and Lemma 3.3.8.

Lemma 3.1.6. *Suppose that $\mathcal{C}$ is a minimal cover of* $\mathrm{Q}(4,q)$ *of size* q^2+1+r, $0 < r < q$, *for which there is a line L not in $\mathcal{C}$ such that every point of L lies on* $r+1$ *lines of $\mathcal{C}$, but all other points of* $\mathrm{Q}(4,q)$ *lie on one line of $\mathcal{C}$, then* $(r+2)|q$ *or* $r = q-1$. *Furthermore,* $r \leqslant \frac{q}{2}-2$ *is impossible.*

Suppose now that $\mathcal{K}$ is a minimal blocking set of $\mathrm{Q}(6,q)$. A *multiple line* with respect to $\mathcal{K}$ is a line of $\mathrm{Q}(6,q)$ meeting $\mathcal{K}$ in at least two points. Using a projection argument, we will show that the multiple lines must contain *many* points. It will become clear that the lemmas imply that the number of points on a multiple line is at least $q-1$. A final argument shows the following theorem.

Theorem 3.1.7. *Let $\mathcal{K}$ be a minimal blocking set of* $\mathrm{Q}(6,q)$, q *even,* $|\mathcal{K}| \leqslant q^3+q$, $q \geqslant 32$. *Then there is a point* $p \in \mathrm{Q}(6,q) \setminus \mathcal{K}$ *with the following property:* $T_p(\mathrm{Q}(6,q)) \cap \mathrm{Q}(6,q) = p\mathrm{Q}(4,q)$ *and $\mathcal{K}$ consists of all the points of the lines L on p meeting* $\mathrm{Q}(4,q)$ *in an ovoid $\mathcal{O}$, minus the point p itself, and* $|\mathcal{K}| = q^3+q$.

3.2 Proof of the theorem

When possible, proofs are given for general q. Suppose for this section that $\mathcal{K}$ is a minimal blocking set of $Q(6,q)$, q even, $|\mathcal{K}| = q^3 + 1 + \delta$, $0 < \delta < q$. The first observation is very short, but will be very useful.

Lemma 3.2.1. *If p is a point of $Q(6,q)$, $p \in \mathcal{K}$, then $|T_p(Q(6,q)) \cap \mathcal{K}| \leqslant 1+\delta$.*

Proof. Since $\mathcal{K}$ is minimal, there exists a generator π on p such that $\pi \cap \mathcal{K} = \{p\}$. Consider π. On the q^2 lines of π not on p, there are q planes of $Q(6,q)$ different from π. Every one of these q^3 planes needs to be blocked by a point of $\mathcal{K} \setminus T_p(Q(6,q))$. None of these points of $\mathcal{K} \setminus T_p(Q(6,q))$ is double counted. Hence there are at most $\delta + 1$ points in $T_p(Q(6,q)) \cap \mathcal{K}$. ■

Corollary 3.2.2. *Every plane of $Q(6,q)$ contains at most $1+\delta$ points of $\mathcal{K}$.*

Proof. Suppose that $\pi \cap \mathcal{K} = \{p_1, \ldots, p_n\}$. Since $|T_{p_1}(Q(6,q)) \cap \mathcal{K}| \leqslant 1+\delta$, $|\pi \cap \mathcal{K}| \leqslant 1+\delta$. ■

Lemma 3.2.1 also provides the possibility to generalise the idea of Lemma 3.1.1.

Lemma 3.2.3. *If $p \in Q(6,q) \setminus \mathcal{K}$, then p projects the points of $T_p(Q(6,q)) \cap \mathcal{K}$ onto a minimal blocking set of $Q(4,q)$, with $Q(4,q)$ the base of the cone $T_p(Q(6,q))$.*

Proof. Consider a point $p \in Q(6,q) \setminus \mathcal{K}$. Then $T_p(Q(6,q))$ intersects $Q(6,q)$ in a singular quadric with vertex p and base a non-singular quadric $Q(4,q)$ in a hyperplane of $T_p(Q(6,q))$. It is clear that the lines of $Q(6,q)$ on p and a point of $\mathcal{K}$ meet $Q(4,q)$ in a blocking set $\mathcal{B}$. Suppose that $\mathcal{B}$ is not minimal. Then there is a point $s \in \mathcal{B}$ such that every line of $Q(4,q)$ on s contains another point of $\mathcal{B}$. But $\mathcal{B}$ is the projection of $T_p(Q(6,q)) \cap \mathcal{K}$. If p projects $s' \in \mathcal{K}$ on s, then there are $q+1$ planes on $\langle p, s' \rangle$ containing at least one other point of $\mathcal{K}$, hence $|T_{s'}(Q(6,q)) \cap \mathcal{K}| > 1+\delta$, a contradiction. We conclude that $\mathcal{B}$ is minimal. ■

After these lemmas for general q, we suppose for the remainder of this section that q is even and $q \geqslant 32$. We now will prove that multiple lines contain a lot of points. We rely on Theorem 3.1.2.

Lemma 3.2.4. *If L is a line of $Q(6,q)$, then $|L \cap \mathcal{K}| = 0, 1$ or $|L \cap \mathcal{K}| \geqslant 1+\sqrt{q}$.*

Proof. Suppose that $2 \leqslant |L \cap \mathcal{K}| < 1 + \sqrt{q}$. There exists a generator π of $\mathrm{Q}(6, q)$ through L such that $\pi \cap \mathcal{K} = L \cap \mathcal{K}$. For, if every generator of $\mathrm{Q}(6, q)$ on L would contain more points of $\mathcal{K}$ than $L \cap \mathcal{K}$, then every point of $L \cap \mathcal{K}$ would contain more than $1+q$ points of $\mathcal{K}$ in its tangent cone, a contradiction. We count the pairs (u, v) with $u \in \pi \setminus L$ and $v \in \mathcal{K} \setminus \pi$ and $u \in T_v(\mathrm{Q}(6, q))$. Since $u \in \pi \setminus L$ and $|L \cap \mathcal{K}| \geqslant 2$, u cannot project $T_u(\mathrm{Q}(6, q)) \cap \mathcal{K}$ on an ovoid of $\mathrm{Q}(4, q)$, so $|T_u(Q(6, q)) \cap \mathcal{K}| > q^2+1+\sqrt{q} > q^2+|L \cap \mathcal{K}|$ if $|L \cap \mathcal{K}| < 1+\sqrt{q}$. So we find a lower bound of $q^2(q^2 + 1)$ for the number of pairs.

If v is a point of $\mathcal{K} \setminus \pi$, then $T_v(\mathrm{Q}(6, q))$ intersects π in a line, so with v correspond q or 0 points of $\pi \setminus L$, hence $(q^3 + 1 + \delta - |L \cap \mathcal{K}|)q$ is an upper bound for the number of pairs, and since $|L \cap \mathcal{K}| \geqslant 2$, we can increase the upper bound to $(q^3 + \delta - 1)q$. So necessarily $(q^3 + \delta - 1)q \geqslant (q^2 + 1)q^2$ or $q^4 + q^2 - q \geqslant q^4 + q^2$, which is not possible. ■

The next step is to prove that if a generator of $\mathrm{Q}(6, q)$ contains more than one point of $\mathcal{K}$, it contains a lot of points of $\mathcal{K}$, and they are collinear.

Lemma 3.2.5. *If π is a generator of $\mathrm{Q}(6, q)$, then $|\pi \cap \mathcal{K}| = 1$ or $|\pi \cap \mathcal{K}| \geqslant 1 + \sqrt{q}$, and all points of $\pi \cap \mathcal{K}$ lie on a line of $\mathrm{Q}(6, q)$.*

Proof. If $|\pi \cap \mathcal{K}| \geqslant 2$, then by the previous lemma, π contains at least $\sqrt{q} + 1$ points of $\mathcal{K}$ on a line L. Suppose that π contains another point $p \in \mathcal{K}$ not on L. There are at least $\sqrt{q} + 1$ lines on p containing at least $\sqrt{q} + 1$ points of $\mathcal{K}$; the point p included. Hence π would contain at least $\sqrt{q}(\sqrt{q} + 1) + 1 > 1 + q$ points of $\mathcal{K}$, a contradiction. ■

The following two lemmas give information on how the points of $\mathcal{K}$ can be concentrated in a tangent cone.

Lemma 3.2.6. *Suppose that $p \notin \mathcal{K}$. If there is a generator on p containing exactly one point of $\mathcal{K}$, then $|T_p(\mathrm{Q}(6, q)) \cap \mathcal{K}| \leqslant q^2+q$, else $|T_p(\mathrm{Q}(6, q)) \cap \mathcal{K}| \geqslant (\sqrt{q} + 1)(q^2 + 1)$.*

Proof. Suppose that π is a generator of $\mathrm{Q}(6, q)$ on p containing exactly one point $s \in \mathcal{K}$. Consider the $q^2 - q$ lines of π not through s or p. Each line lies in q generators of $\mathrm{Q}(6, q)$ different from π. The $q(q^2 - q)$ generators of $\mathrm{Q}(6, q)$ on these lines of π are blocked by at least one point of $\mathcal{K}$, so at most $q^2 + q$ points remain for $T_p(\mathrm{Q}(6, q)) \cap \mathcal{K}$. If all generators on p contain at least $\sqrt{q} + 1$ points of $\mathcal{K}$, then $|T_p(\mathrm{Q}(6, q)) \cap \mathcal{K}| \geqslant (\sqrt{q} + 1)(q^2 + 1)$. ■

Lemma 3.2.7. *Suppose that π is a generator of $\mathrm{Q}(6, q)$ containing at least $\sqrt{q}+1$ points of $\mathcal{K}$ on the line L. Let $p \in \pi \setminus L$, then $|T_p(\mathrm{Q}(6, q)) \cap \mathcal{K}| \leqslant q^2+q$.*

Proof. Let $\pi \cap \mathcal{K} = \{s_0, \dots, s_n\}$ $(n \geqslant \sqrt{q})$. If every generator on p contains more than one point of $\mathcal{K}$, then also the $q+1$ generators on $\langle s_0, p\rangle$ contain more than one point of $\mathcal{K}$, while the only point of $\mathcal{K}$ they share is the point s_0. Hence $|T_{s_0}(\mathrm{Q}(6,q)) \cap \mathcal{K}| > 1+q$, a contradiction. By the previous lemma, $|T_p(\mathrm{Q}(6,q)) \cap \mathcal{K}| \leqslant q^2 + q$. ■

Consider now a generator π which contains at least $\sqrt{q}+1$ points of $\mathcal{K}$ on a line L. If $p \in \pi \setminus L$, the preceding lemma ensures us that $|T_p(\mathrm{Q}(6,q)) \cap \mathcal{K}| \leqslant q^2 + q$, while p cannot project the points of $T_p(\mathrm{Q}(6,q)) \cap \mathcal{K}$ on an ovoid of $\mathrm{Q}(4,q)$, since the generator π is projected on a line of $\mathrm{Q}(4,q)$ containing at least $\sqrt{q}+1$ points of the projected blocking set $\mathcal{B}$. Hence $\mathcal{B}$ is the dual of a cover $\mathcal{C}$, $|\mathcal{C}| = q^2+1+r$ and $\mathcal{C}$ only has multiple points of excess at least $\sqrt{q}$. Moreover $\mathcal{B}$ is minimal by Lemma 3.2.3. Lemma 3.1.4 shows that the multiple points of $\mathcal{C}$ form a sum of lines with the sum of the weights of the lines equal to r and from Lemma 3.1.5, it follows that $r \geqslant (q+4)/6$. For the next lemma we use the fact that the multiple points of $\mathcal{C}$ form a sum of lines. For the blocking set $\mathcal{B}$ this means the following. There exist some points (possibly one), such that all lines of $\mathrm{Q}(4,q)$ on these points are multiple lines. We call this *a sum of pencils.* Since $\mathcal{B}$ is minimal, such a point does not belong to $\mathcal{B}$. Call all the lines on such a point a pencil. The *weight* of the pencil is then the minimum number of points of $\mathcal{B}$ a line of the pencil contains minus one. The sum of the weights of the pencils is equal to r.

We now present a lemma which already starts indicating that $\mathcal{K}$ must look like a cone over an ovoid of $\mathrm{Q}(4,q)$.

Lemma 3.2.8. *There exists a point $p' \in \mathrm{Q}(6,q) \setminus \mathcal{K}$ only lying on q^2+1 lines each having at least $\sqrt{q}+1$ points of $\mathcal{K}$, and these lines meet $\mathrm{Q}(4,q)$ in an ovoid, where $\mathrm{Q}(4,q)$ is the base of the cone $T_{p'}(\mathrm{Q}(6,q))$.*

Proof. Consider a generator π, $\pi \cap \mathcal{K} = \{s_0, \dots, s_n\}$, $n > 0$. Necessarily $n \geqslant \sqrt{q}$ and all points of $\pi \cap \mathcal{K}$ lie on a line L. Consider $s \in \pi \setminus L$. By Lemma 3.2.7, $|T_s(\mathrm{Q}(6,q)) \cap \mathcal{K}| \leqslant q^2+q$, and s does not project the points of $T_s(\mathrm{Q}(6,q)) \cap \mathcal{K}$ on an ovoid. As explained before, s projects these points on a blocking set $\mathcal{B}$ of $\mathrm{Q}(4,q)$ such that the multiple lines form a sum of pencils. Consider such a pencil of $\mathrm{Q}(4,q)$ represented by the point p. The $q+1$ lines of the pencil are the projections of $q+1$ lines $M_1, \dots, M_{q+1}$ of $\mathrm{Q}(6,q)$, all containing at least $\sqrt{q}+1$ points of $\mathcal{K}$ and intersecting the line $\langle s, p\rangle$. Consider the point $p' = \langle s, p\rangle \cap M_1$. Suppose that $|T_{p'}(\mathrm{Q}(6,q)) \cap \mathcal{K}| \leqslant q^2+q$. Then p' lies on at most $(q-1)/\sqrt{q} \leqslant \sqrt{q}$ lines of $\{M_1, \dots, M_{q+1}\}$. If we now project from p', then there are at most $q^2+q-\sqrt{q}$ and at least q^2+1 projected points since M_1 is projected on one point of $\mathrm{Q}(4,q)$. Some lines M_i are not projected on one point. Again we find a blocking set $\mathcal{B}$ with the multiple lines forming a sum of pencils; there are at least $q+1-\sqrt{q}$ projected lines M_i having at least $\sqrt{q}+1$ projected points. So since the sum of pencils consists of at most $\sqrt{q}$ distinct pencils, all lines through the intersection point of the projected lines M_i have at least $\sqrt{q}+1$ points; but the projection of M_1 consists of one point, a contradiction. Hence $|T_{p'}(\mathrm{Q}(6,q)) \cap \mathcal{K}| \geqslant (\sqrt{q}+1)(q^2+1)$. Lemma 3.2.6 implies that all generators on p' contain at least $\sqrt{q}+1$ collinear points of $\mathcal{K}$. In such a generator on p', the points of $\mathcal{K}$ lie on one line. If one of these lines does not pass through p', then by Lemma 3.2.7, $|T_{p'}(\mathrm{Q}(6,q)) \cap \mathcal{K}| \leqslant q^2+q$, a contradiction, so all these lines pass through p' and contain at least $\sqrt{q}+1$ points of $\mathcal{K}$. Hence every generator on p' contains exactly one line on p' which contains at least $\sqrt{q}+1$ points of $\mathcal{K}$, which implies that p' projects these lines on an ovoid of $\mathrm{Q}(4,q)$. ∎

By Lemma 3.1.5, it is possible to increase the number of points of $\mathcal{K}$ on a multiple line.

Lemma 3.2.9. *If a line L of $\mathrm{Q}(6,q)$ contains more than one point of $\mathcal{K}$, then it contains at least $(q+10)/6$ points of $\mathcal{K}$.*

Proof. Consider a generator π on L. As in the proof of Lemma 3.2.4, we count the number of pairs (u,v), $u \in \pi \setminus L$ and $v \in \mathcal{K} \setminus \pi$ and $u \in T_v(\mathrm{Q}(6,q))$. Suppose now that $|L \cap \mathcal{K}| < (q+10)/6$. A point u projects the points of $T_u(\mathrm{Q}(6,q)) \cap \mathcal{K}$ on a blocking set $\mathcal{B}$ of size q^2+1+r satisfying the conditions of Lemma 3.1.5, so $r \geqslant (q+4)/6$, so $q^2+1+(q+4)/6 \leqslant |T_u(\mathrm{Q}(6,q)) \cap \mathcal{K}| \leqslant q^2+q$. As in the proof of Lemma 3.2.4, the hypothesis $|L \cap \mathcal{K}| < (q+10)/6$ leads to a contradiction. ∎

From the proof of Lemma 3.2.8, on every multiple line L, there is a point p' lying on q^2+1 multiple lines. We stress this result in the following corollary.

Corollary 3.2.10. *Every multiple line L contains at least $(q+10)/6$ points of $\mathcal{K}$ and every multiple line L has a point not in $\mathcal{K}$ lying on q^2+1 multiple lines to $\mathcal{K}$.*

Finally, we can prove the main theorem of this section.

Theorem 3.2.11. *Let $\mathcal{K}$ be a minimal blocking set of $\mathrm{Q}(6,q)$, $|\mathcal{K}| \leqslant q^3+q$, q even, $q \geqslant 32$. Then there is a point $p \in \mathrm{Q}(6,q) \setminus \mathcal{K}$ with the following property: $T_p(\mathrm{Q}(6,q)) \cap \mathrm{Q}(6,q) = p\mathrm{Q}(4,q)$ and $\mathcal{K}$ consists of all the points of the lines L on p meeting $\mathrm{Q}(4,q)$ in an ovoid $\mathcal{O}$, minus the point p itself. Moreover, $|\mathcal{K}| = q^3+q$.*

Proof. By Lemma 3.2.8 and Corollary 3.2.10, there is a point $p \notin \mathcal{K}$ such that there are q^2+1 lines on p each containing at least $(q+10)/6$ points of $\mathcal{K}$. Consider p and a multiple line L on p such that $|L \cap \mathcal{K}|$ is minimal. Denote $|L \cap \mathcal{K}| = s$ (then $s \geqslant (q+10)/6$). Consider a generator π on L. By Lemma 3.2.7, $|T_u(\mathrm{Q}(6,q)) \cap \mathcal{K}| \leqslant q^2+q$ for each $u \in \pi \setminus L$, and u projects the points of $T_u(\mathrm{Q}(6,q)) \cap \mathcal{K}$ on a minimal blocking set $\mathcal{B}$ of Q(4, q). As explained after Lemma 3.2.7, $|\mathcal{B}| \geqslant q^2+1+(q+4)/6$, the excess of a multiple line is at least $(q+4)/6$ and the multiple lines form a sum of pencils with the sum of the weights of the pencils equal to r; with $|\mathcal{B}| = q^2+1+r$.

Now count as in Lemma 3.2.4 the pairs (u, v) with $u \in \pi \setminus L$ and $v \in \mathcal{K} \setminus \pi$ and $u \in T_v(\mathrm{Q}(6,q))$. If we suppose that $|T_u(\mathrm{Q}(6,q)) \cap \mathcal{K}| > q^2+s$ for all points $u \in \pi \setminus L$, we obtain the inequality $q^4+q^2-q \geqslant q^2(q^2+1)$, a contradiction. So we find a point $u \in \pi \setminus L$ such that $|T_u(\mathrm{Q}(6,q)) \cap \mathcal{K}| \leqslant q^2+s$. Equality is needed since the multiple lines of the projection form a sum of pencils and the sum of the weights of the pencils is equal to r, with $|\mathcal{B}| = q^2+1+r$, and $\mathcal{B}$ the projected blocking set. Since $s-1$ is the minimal weight of the secants to $\mathcal{K}$, there is at least one pencil with weight $s-1$, the sum of the weights in the sum of the pencils is at least $s-1$, but $|T_u(\mathrm{Q}(6,q)) \cap \mathcal{K}| \leqslant q^2+s$, so $|\mathcal{B}| \leqslant q^2+1+(s-1)$, and there is exactly one pencil of weight $s-1$, all multiple lines have excess $s-1$ and $|\mathcal{B}| = q^2+1+(s-1)$. By the dual of Lemma 3.1.6, $s-1 = q-2$ or $s-1 = q-1$.

Hence the minimal number of points of $\mathcal{K}$ on a secant line on p is $q-1$ or q. Suppose that $s-1 = q-2$. So far, we have q^2+1 lines $L_0, \ldots, L_{q^2}$ on p containing at least $q-1$ points, and $|T_p(\mathrm{Q}(6,q)) \cap \mathcal{K}| \geqslant (q^2+1)(q-1)$. Consider a point $p' \neq p$, $p' \in L_0 \setminus \mathcal{K}$, $|L_0 \cap \mathcal{K}| = q-1$. There are at least q^2 points of $\mathcal{K}$ necessary to block the $q^2(q+1)$ generators on p' not on p, hence $|T_p(\mathrm{Q}(6,q)) \cap \mathcal{K}| \leqslant q^3+q-q^2 = (q^2+1)(q-1)+1$ and all multiple lines on p, except maybe one, contain exactly $q-1$ points of $\mathcal{K}$. At least q^2 points of the $q^2(+1)$ points of $\mathcal{K} \setminus T_p(\mathrm{Q}(6,q))$ lie in each $T_u(\mathrm{Q}(6,q))$, $u \neq p$, $u \notin \mathcal{K}$ and u lying on a secant to $\mathcal{K}$ on p. No two of these $q^2(+1)$ points are collinear on Q(6, q), otherwise they are on a secant not on p containing at least $q-1$ points of $\mathcal{K}$, this secant lies in a tangent cone containing at least $(q^2+1)(q-1)$ points of $\mathcal{K}$, which is not possible since $|T_p(\mathrm{Q}(6,q)) \cap \mathcal{K}| \geqslant (q^2+1)(q-1)$. By Lemma 3.2.5, a generator cannot contain two secants to $\mathcal{K}$, hence p projects the points of $T_p(\mathrm{Q}(6,q)) \cap \mathcal{K}$ on an ovoid of Q(4, q). This also implies that two distinct points u and u', $u \neq p \neq u'$, each lying on a secant to $\mathcal{K}$ on p, cannot be collinear on Q(6, q). Select $q+2$ points $u_1, \ldots, u_{q+2}$, each lying on a secant to $\mathcal{K}$ on p, $u_i \notin \mathcal{K}$, $u_i \neq p$. These points define at least a 3-dimensional space. In the intersection $\bigcap_{i=1}^{q+2} T_{u_i}(\mathrm{Q}(6,q))$ lie at least q^2-q-1 points of $\mathcal{K} \setminus T_p(\mathrm{Q}(6,q))$

since each intersection $T_{u_i}(\mathrm{Q}(6,q)) \cap \mathcal{K}$ contains at least q^2 of the $q^2(+1)$ points of $\mathcal{K} \setminus T_p(\mathrm{Q}(6,q))$. Hence these q^2-q-1 points of $\mathcal{K} \setminus T_p(\mathrm{Q}(6,q))$ lie in the intersection of tangent hyperplanes on $q+2$ non-collinear points of $\mathrm{Q}(6,q)$ and are two by two non-collinear. The smallest space which can contain q^2-q-1 such points is a 3-space. Since they lie in the intersection of $q+2$ tangent hyperplanes on non-collinear points of $\mathrm{Q}(6,q)$, they lie in a 3-space necessarily intersecting $\mathrm{Q}(6,q)$ in an elliptic quadric. Since the 3-space containing this quadric is the intersection of tangent hyperplanes, this 3-space contains the nucleus of $\mathrm{Q}(6,q)$; a contradiction since an elliptic quadric does not have a nucleus.

The hypothesis $s-1=q-2$ leads us to a contradiction and $s-1=q-1$. Hence there are q^2+1 lines on p containing q points of $\mathcal{K}$, $|\mathcal{K}|=q^3+q$ and p projects the points of $T_p(\mathrm{Q}(6,q)) \cap \mathcal{K}$ on an ovoid of $\mathrm{Q}(4,q)$. ■

Since all ovoids of $\mathrm{PG}(3,32)$ have been classified by C. M. O'Keefe, T. Penttila and G. Royle [75], we have a complete classification of the smallest blocking sets of $\mathrm{Q}(6,32)$. Namely, the only ovoids of $\mathrm{PG}(3,32)$ are the elliptic quadric and the Tits-ovoid. Since $\mathrm{Q}(4,q)$, q even, is isomorphic to the GQ $\mathrm{W}(3,q)$, q even, and since, by results of B. Segre ([86]) and J.A. Thas ([90]), a set $\mathcal{K}$ is an ovoid of $\mathrm{PG}(3,q)$, q even, if and only if it is an ovoid of $\mathrm{W}(3,q)$, q even, the classification of the ovoids of $\mathrm{Q}(4,32)$ is obtained. We conclude:

Theorem 3.2.12. *Let $q=32$. Let $\mathcal{K}$ be a minimal blocking set of $\mathrm{Q}(6,q)$, $|\mathcal{K}| \leqslant q^3+q$. Then there is a point $p \in \mathrm{Q}(6,q) \setminus \mathcal{K}$ with the following property: $T_p(\mathrm{Q}(6,q)) \cap \mathrm{Q}(6,q) = p\mathrm{Q}(4,q)$ and $\mathcal{K}$ consists of all the points of the lines L on p meeting $\mathrm{Q}(4,q)$ in an elliptic quadric or a Tits-ovoid $\mathcal{O}$, minus the point p itself. Moreover $|\mathcal{K}|=q^3+q$.*

3.3 Proof of the technical results

In this section, we will give a proof of Lemmas 3.1.4, 3.1.5 and 3.1.6. We can start immediately with Lemma 3.1.4.

Lemma 3.3.1. *Let $\mathcal{C}$ be a minimal cover of $\mathrm{Q}(4,q)$, $|\mathcal{C}|=q^2+1+r$, $0<r\leqslant q-1$. If each multiple point has excess at least $\sqrt{q}$, then the set E of multiple points is a sum of lines, with the sum of the weights of the lines equal to r.*

Proof. Denote by $e(p)$ the excess of a point p. Counting the points according to their excess, we find $|E|=r(q+1)$. Every hyperplane of $\mathrm{PG}(4,q)$ intersects E in $r \mod q$ points, since $|\mathcal{C}| \equiv 1+r \mod q$, every hyperplane intersects $\mathrm{Q}(4,q)$ in $1 \mod q$ points and every hyperplane intersects a line of $\mathcal{C}$ in $1 \mod q$ points.

Suppose that π is a plane of $\mathrm{PG}(4,q)$, $\pi \cap E = \emptyset$, and suppose that $\pi_1, \ldots, \pi_{q+1}$ are the $q+1$ hyperplanes on π. Since $|\pi_i \cap E| \equiv r \mod q$, put $|\pi_i \cap E| = r + l_i q$. Hence $r(q+1) = \sum_{i=1}^{q+1} |\pi_i \cap E| = r(q+1) + q\sum_{i=1}^{q+1} l_i$, from which $\sum_{i=1}^{q+1} l_i = 0$. So if α is a hyperplane on π, $\pi \cap E = \emptyset$, then $|\alpha \cap E| = \sum_{p \in \alpha \cap E} e(p) = r$.

Suppose that $p \in E$, where $e(p) = k$ is the minimal excess of a point of E. Necessarily $k \geqslant \sqrt{q}$. Since the total excess is $r(q+1)$, the number of distinct points of E, not counted according to their excess, is at most $\sqrt{q}(q+1)$. We can find a plane π on p only sharing p with E. Suppose again that $\pi_1, \ldots, \pi_{q+1}$ are the $q+1$ hyperplanes on π. Put again $|\pi_i \cap E| = r + l_i q$, then $|E| = |E \cap \pi| + |E \setminus \pi| = k + \sum_{p \in E \setminus \pi} e(p) = k + \sum_{i=1}^{q+1}(r + l_i q - k) = -qk + r(q+1) + q\sum_{i=1}^{q+1} l_i$, from which $\sum_{i=1}^{q+1} l_i = k$. Hence there is a hyperplane π_i such that $r + q \leqslant |\pi_i \cap E| \leqslant r + kq$ or, since for $x \in E$, $e(x) = k \geqslant \sqrt{q}$, we have $|\{x \in E \cap \pi_i\}| \leqslant q + r/k \leqslant q + r/\sqrt{q} \leqslant q + \sqrt{q}$. If $\pi' \subset \pi_i$, and π' is a plane such that $\pi' \cap E = \emptyset$, then $|\pi_i \cap E| = r$, hence every plane of π_i contains at least one point of E. This shows that $\pi_i \cap E$ blocks every plane of π_i. Since $q + 1 \leqslant |\{x \in E \cap \pi_i\}| \leqslant q + \sqrt{q}$, $\pi_i \cap E$ contains a line (recall the definitions in Section 1.2.4 and Theorem 1.2.20, or [26], also mentioned in Section 1.2.4), necessarily on p, since π only shares p with E.

We prove by induction on the excess of the multiple points of E that there is a line in E on every point of E. As induction hypothesis, we suppose that on all points of E with excess at most k lies a line completely in E. Suppose then that p is a point with minimum excess $k' > k$. As above, there is a plane π such that $E \cap \pi = \{p\}$; the same counting as above proves the existence of a hyperplane $\pi_i \supset \pi$ such that $q + r \leqslant |\pi_i \cap E| \leqslant r + k'q$. Again, $\pi_i \cap E$ must block all planes in π_i. If a line of E lies in π_i, then it must be a line on p; hence we can suppose that the lines contained in E having points of weight at most k in π_i intersect π_i in one point. If N is the number of distinct points of π_i of excess less than k', then N points of excess at least $\sqrt{q}$ are lying on a line in E, intersecting π_i in exactly one point. Hence $|\{x \in E \cap \pi_i\}| - N \leqslant (k'q + r - N\sqrt{q})/k' \leqslant q + (q - 1 - N\sqrt{q})/(\sqrt{q}+1)$. To be allowed to do the substitution $(r - N\sqrt{q})/k' \leqslant (r - N\sqrt{q})/(\sqrt{q}+1)$, we must have $r \geqslant N\sqrt{q}$, which we will prove in the next paragraph. Working further with the above inequality, we find $|\{x \in E \cap \pi_i\}| \leqslant q + \sqrt{q} - 1 + N/(\sqrt{q}+1)$. In the next paragraph we also prove that the number of distinct lines contained in E is at most $(q-1)/\sqrt{q} \leqslant \sqrt{q}$; so $|\{x \in E \cap \pi_i\}| \leqslant q + \sqrt{q}$. As in the preceding paragraph, $\pi_i \cap E$ blocks every plane of π_i, hence $\pi_i \cap E$ must contain a line, and since $\pi \subset \pi_i$ only shares p with E, this line must pass through p.

We now will prove that the condition $r \geqslant N\sqrt{q}$ holds. Suppose that L and M are two distinct lines contained in E, where k is the minimal excess of the points of L, and where k' is the minimal excess of the points of M. Suppose that $L \cap M = \{t\}$. Suppose that $e(t) = k + k' - s$, $s > 0$. We can find a plane π only sharing t with E. We know from the second paragraph that $\sum_{i=1}^{q+1} l_i = k + k' - s$, if π_i $(i = 1, \ldots, q+1)$ are the $q+1$ hyperplanes on π. Now $\langle L, \pi\rangle$ contains $q+1$ points with excess at least k. Hence $|\langle L, \pi\rangle \cap E| \geqslant kq + r$ and $|\langle M, \pi\rangle \cap E| \geqslant k'q + r$, which implies $\sum_{i=1}^{q+1} l_i \geqslant k+k'$, a contradiction. (If $\langle L, \pi\rangle = \langle M, \pi\rangle$, then the same counting arguments hold, $|\langle L, \pi\rangle \cap E| \geqslant (k+k')q$.) We conclude $e(t) \geqslant k + k'$. Hence every line contained in E gives rise to at least $\sqrt{q}(q+1)$ of the total excess $r(q+1) \leqslant q^2 - 1$, which implies that at most $(q-1)/\sqrt{q} < \sqrt{q}$ lines are contained in E and $N\sqrt{q} \leqslant r$.

In the final paragraph, we show by induction that E is a sum of lines. Suppose that E consists of one line L, and suppose that the minimal excess of a point on L is k. If we remove this line k times, then a set E' of excess $(r-k)(q+1)$ remains. But if a point has excess $k' > k$, repeating the previous arguments, we find a hyperplane π_i with at least $q+1$ distinct points of E', since this hyperplane intersects E' in a blocking set in π_i; but the only points of E' are the points of L; so every point of L still has positive excess. So every point of L has excess larger than k. We conclude that the weight of L is r. Suppose as the induction hypothesis that the result is true if E would consist of s distinct lines and suppose now that E consists of $s+1$ distinct lines. As above, look for the minimal excess k of a point of E. Consider a line L through this point lying in E and subtract k from the excess of every point on L. If a point p on L has positive excess larger than k, then it lies on a second line in E. Namely, from the standard arguments, for the remaining weighted set E', we can find a hyperplane π_i through a plane π with $\pi \cap E' = \{p\}$, sharing at least $q+1$ points with E'; this means that the remaining points must form a blocking set in π_i. Since we can describe E' as a union of at most $\sqrt{q}$ lines, a line L' of E must lie in π_i, this line L' must pass through p; so p lies on a second line L'; and then $e(p) \geqslant k + k'$ with k' the minimal excess of the points of L'. So if we subtract $k(q+1)$ from the total excess of E by subtracting k from the excess of every point of L, the remaining set E' has size $(r-k)(q+1)$ (counting the points according to their excess), and every point of E' still has excess at least $\sqrt{q}$, so by the induction hypothesis E' is a sum of lines. Then E is a sum of lines with the sum of the weights of the lines equal to r. ■

To prove Lemma 3.1.5, we need two extra definitions, a result of Bichara and Korchmáros and a lemma on covers of $Q(4, q)$.

Definition 3.3.2. Let Ω be a $(q+2)$-set in a projective plane. A point $p \in \Omega$ is an *internal nucleus* of Ω if every line on p contains exactly two points of Ω.

Theorem 3.3.3. (Bichara and Korchmáros [13]) *If S is a $(q+2)$-set in $\mathrm{PG}(2,q)$, q even, and if S has r internal nuclei, $r > q/2$, then every point of Ω is an internal nucleus.*

Definition 3.3.4. Suppose that $\mathcal{C}$ is a cover of $\mathrm{Q}(4,q)$. A line L of $\mathrm{Q}(4,q)$ is called a *good* line if $L \notin \mathcal{C}$ and L contains no multiple points of $\mathcal{C}$.

Lemma 3.3.5. *A cover $\mathcal{C}$ of $\mathrm{Q}(4,q)$ of size q^2+1+r, $0 \leqslant r \leqslant q$, always has a good line.*

Proof. There are q^2+1+r lines in $\mathcal{C}$ and at most $r(q+1)$ distinct excess points; all lying on at most $q-1$ lines not in $\mathcal{C}$. So there are at most $r(q+1)(q-1)+q^2+1+r$ lines of $\mathrm{Q}(4,q)$ which are not good. For $0 \leqslant r \leqslant q$, this number is smaller than the total number of lines of $\mathrm{Q}(4,q)$, so for $0 \leqslant r \leqslant q$, the cover $\mathcal{C}$ has at least one good line. ■

We now can prove Lemma 3.1.5.

Lemma 3.3.6. *A minimal cover of $\mathrm{Q}(4,q)$, q even, of size q^2+1+r, only having points of positive excess at least $\sqrt{q}$ satisfies $r \geqslant (q+4)/6$.*

Proof. Let L be a good line of $\mathcal{C}$ and let $M_1,\ldots,M_{q+1}$ be the lines of $\mathcal{C}$ intersecting L. Define $L^\perp$ as the plane $\langle L,n\rangle$, with n the nucleus of $\mathrm{Q}(4,q)$. Since $L^\perp$ is tangent to $\mathrm{Q}(4,q)$, $L^\perp \cap \mathrm{Q}(4,q) = L$ and $M_j \not\subset \langle L, M_i\rangle$ if $i \neq j$. Hence the planes $L^\perp, \langle L, M_i\rangle$, $i=1,\ldots,q+1$, define a $(q+2)$-set S in the quotient geometry Δ of L. There are q^2+q+1 3-spaces on L, $q+1$ of them are tangent hyperplanes, so there are q^2 3-spaces α on L such that $\alpha \cap \mathrm{Q}(4,q)$ is a hyperbolic quadric $\mathrm{Q}^+(3,q)$ in α. Consider a tangent hyperplane α on L. It contains exactly one line M_i. So every line in Δ on the point $L^\perp$ contains exactly the two points $L^\perp$ and $\langle L, M_i\rangle$ of S. Consider now the q^2 3-spaces α on L for which $\alpha \cap \mathrm{Q}(4,q) = \mathrm{Q}^+(3,q)$. Count the number of pairs (M,Q) for which $M \in \mathcal{C}$ and Q is a hyperbolic quadric on L and M. If $M \cap L = \{p\}$, then there are q 3-spaces α on $\langle L, M\rangle$ such that $\alpha \cap \mathrm{Q}(4,q)$ is a $\mathrm{Q}^+(3,q)$ containing M. If $M \cap L = \emptyset$, then there is exactly one 3-space $\alpha = \langle L, M\rangle$ such that $\alpha \cap \mathrm{Q}(4,q) = \mathrm{Q}^+(3,q)$. We find $q(q+1)+q^2+r-q = 2q^2+r$ pairs. Since there are q^2 $\mathrm{Q}^+(3,q)$ on L, each one of them contains exactly 2 lines of $\mathcal{C}$, except for at most r, which contain at least 3 lines of $\mathcal{C}$. So at least $q+2-3r$ elements of S are internal nuclei of S since at most $3r$ elements of S lie on a line in Δ defining a $\mathrm{Q}^+(3,q)$ through L with at least three lines

M_i. Suppose that $q+2-3r > q/2$, then by Theorem 3.3.3, every point of S is an internal nucleus of S, meaning that S only has 0- and 2-secants. So if L is a good line and $r < (q+4)/6$, then the hyperbolic quadrics of $\mathrm{Q}(4,q)$ on L always contain 0 or 2 lines of $\mathcal{C}$ intersecting L.

Since $r \leqslant q-1$, the multiple points of $\mathcal{C}$ form a sum of lines, with the sum of the weights of the lines equal to r (Lemma 3.3.1). Every line has weight at least $\sqrt{q}$. Let N be a line of this sum. Then $N \notin \mathcal{C}$ since $\mathcal{C}$ is minimal. If N has weight k, then N is intersected by at least $(k+1)(q+1)$ lines of $\mathcal{C}$ with $k \geqslant \sqrt{q}$. Interpreting $N^\perp$ and the planes $\langle N, M\rangle$, with M a line of $\mathcal{C}$ intersecting N, in the quotient geometry Δ' of N, we define a set S', now of size at least $(k+1)(q+1)+1$. Consider a line of Δ' not containing $N^\perp$, but containing at least 3 points of S'. This line defines a hyperbolic quadric of $\mathrm{Q}(4,q)$ through N and containing at least 3 lines N_1, N_2, N_3 of $\mathcal{C}$ intersecting N. The regulus of this hyperbolic quadric containing N cannot contain a good line, since otherwise the hyperbolic quadric cannot contain 3 lines of $\mathcal{C}$ intersecting N; so every line of that regulus must either belong to $\mathcal{C}$ or must contain multiple points. Since $N_1 \in \mathcal{C}$, every line of the regulus of N belonging to $\mathcal{C}$ has a multiple point. Since the multiple points form a sum of lines $\mathcal{L}$, with the sum of the weights of the lines equal to $r < (q+4)/6$, and where every line of $\mathcal{L}$ has at least weight $\sqrt{q}$, there are at most $\sqrt{q}/6$ distinct lines in that sum. Hence the previous situation is only possible if a line N' of this sum intersects N and lies on this hyperbolic quadric. So the solids of hyperbolic quadrics of $\mathrm{Q}(4,q)$ through N containing at least 3 lines of $\mathcal{C}$ intersecting N must pass through one of at most $\sqrt{q}/6$ planes through N.

Define S'' as the set of points in Δ' which correspond with the planes $\langle N, N'\rangle$ and $N^\perp$, $N' \in \mathcal{L}$, $|\mathcal{L}| \leqslant \sqrt{q}/6$; then $|S''| \leqslant \sqrt{q}/6+1$. A line of Δ' not on $N^\perp$ containing at least 3 elements of S' must pass through one of the points of $S'' \setminus \{N^\perp\}$ and since there are lines of Δ' on $N^\perp$ containing more than 2 points of S', all lines in Δ' containing at least 3 points of S' pass through a point of S''.

There are at least $(q+1)(\sqrt{q}+1)$ lines intersecting N defining at least $(q+1)(\sqrt{q}+1)$ points of S'. Consider a point p of S' different from $N^\perp$. Suppose that n lines on p contain one other point p_i of S'. Consider all the points $p_1, \ldots, p_n, p$, and the at least $(q+1)(\sqrt{q}+1)-(n+1)$ points of $S' \setminus \{p_1, \ldots, p_n, p\}$. Those points must lie on at most $\sqrt{q}/6+1$ lines through the points of S'', a contradiction. We conclude that $r \geqslant (q+4)/6$. ■

For the proof of Lemma 3.1.6, we will use the model $T_2(\mathcal{O})$ for $\mathrm{Q}(4,q)$. We will prove Lemma 3.1.6 in two smaller lemmas. Considering $T_2(\mathcal{O})$, a line of type (b) will be denoted by $[p]$, $p \in \mathcal{O}$. For the definition and basic results about $T_2(\mathcal{O})$, we refer to Section 2.1.1 and in particular Theorem 2.1.5.

Lemma 3.3.7. *Suppose that $\mathcal{C}$ is a minimal cover of $Q(4, q) \cong T_2(\mathcal{O})$ of size $q^2 + 1 + r$, $0 < r < q$, for which there is a line L not in $\mathcal{C}$ such that every point of L lies on $r + 1$ lines of $\mathcal{C}$, but all other points of $T_2(\mathcal{O})$ lie on one line of $\mathcal{C}$, then $(r + 2)|q$ or $r = q - 1$.*

Proof. Suppose that $r < q-2$. We may assume that L is a line of type (b), denoted by $[p_0]$. Since (∞) has excess r, $\mathcal{C}$ contains $r + 1$ lines of type (b), $[p_1], \ldots, [p_{r+1}]$, and every tangent plane to $\mathcal{O}$ in p_0 contains $r+1$ lines of type (a). There remain $q^2+1+r-(r+1)-q(r+1) = q(q-r-1)$ lines of type (a) of $\mathcal{C}$. On each point $p_{r+2}, \ldots, p_q$ of $\mathcal{O}$, there are q lines of $\mathcal{C}$. Consider a plane $\alpha \neq \pi_0$, π_0 the plane containing $\mathcal{O}$, on the line p_0p_q. Since $r < q-2$, there are other points p_i than p_0 and p_q such that $[p_i] \notin \mathcal{C}$. There are $q + q(r+1)$ lines of type (a) of $\mathcal{C}$ on p_q and p_0, so $q^2+1+r-(r+1)-q-q(r+1) = q^2-q(r+2)$ lines of type (a) of $\mathcal{C}$ are intersecting α in exactly one point of type (i). Hence $q(r + 2)$ points of α must be covered by lines on p_0 or p_q in α, so each plane on p_qp_0 different from π_0 contains exactly $r + 2$ lines on either p_q or p_0 since the only multiple points are points of type (ii) and (∞). So it is possible to partition the q lines of $\mathcal{C}$ through p_q into sets of size $r + 2$; so $(r + 2)|q$ and the lemma follows. ■

Lemma 3.3.8. *Under the same conditions as in the previous lemma, $r \leqslant q/2 - 2$ is impossible.*

Proof. We now assume that L is a line of type (a), so (∞) has excess 0 and $\mathcal{C}$ contains exactly one line $[p_0]$ of type (b). Denote by L the line of type (a) such that every point on it lies on $r + 1$ lines $L_1, \ldots, L_{r+1}$ of $\mathcal{C}$; $L \notin \mathcal{C}$. By the transitivity of the stabilizer group of $Q(4, q)$ on the point set, we can assume that the line of $\mathcal{C}$ passing through (∞) does not intersect L, then $p_0 \notin L$.

Suppose that $p_1 \in L \cap \mathcal{O}$. Then one tangent plane on p_1 lies on $r + 1$ lines of $\mathcal{C}$, the other $q - 1$ tangent planes lie on one line of $\mathcal{C}$, so there are $(q-1)+r+1 = q+r$ lines of type (a) of $\mathcal{C}$ on p_1. The other points $p_2, \ldots, p_q$ of $\mathcal{O}$ lie on q lines of type (a) of $\mathcal{C}$. Consider the point p_2 and all planes different from π_0, $\mathcal{O} \subseteq \pi_0$, on it. A tangent plane to $\mathcal{O}$ on p_2 contains exactly one line of type (a). A plane α, $\alpha \neq \pi_0$, on $\langle p_2, p_0\rangle$ is intersected in a point of type (i) by $q^2 + r - q$ lines of type (a) of $\mathcal{C}$, so there are at least $q - r > 0$ points of type (i) not covered by those lines; at most $r + 1$ of those lines intersect α in the point $\alpha \cap L$; so at most q points of type (i) of α are not covered by those lines and hence α must contain exactly one line of type (a) of $\mathcal{C}$. A plane α, $\alpha \neq \pi_0$, on $\langle p_2, p_i \rangle$, $i = 3, \ldots, q$, is intersected in a point of type (i) by $q^2 + r - 2q$ lines of type (a) of $\mathcal{C}$. As in the previous case, there are now between $2q$ and $2q - r$ points of type (i) not covered by those lines; so two lines of $\mathcal{C}$ lie in the plane α. It is possible that these 2 lines both lie on p_i or on p_2. If one line lies on p_i and the other on p_2, they intersect in $\alpha \cap L$.

A plane α, $\alpha \neq \pi_0$, on $\langle p_1, p_2 \rangle$ different from $\langle p_2, L \rangle$ is intersected by exactly $q^2 + r - q - (q + r) = q^2 - 2q$ lines of type (i) of $\mathcal{C}$. Since $L \cap \alpha = \{p_1\}$, these lines intersect α in different points, hence exactly two lines of $\mathcal{C}$ either on p_1 or p_2 lie in α. Finally we consider the plane $\langle p_2, L \rangle$. There are $2q + r$ lines of type (a) of $\mathcal{C}$ on p_1 and p_2. In the $q - 1$ planes on $\langle p_1, p_2 \rangle$ different from π_0 and $\langle p_2, L \rangle$ lie exactly 2 lines; so $r + 2$ lines remain, in $\langle p_2, L \rangle$. Since the intersection points must lie on L, these lines pass either through p_1 or p_2.

Suppose that the $r + 2$ lines pass through p_2. Consider the quotient geometry Δ of p_2, with relation to $\mathrm{PG}(3, q)$. The q lines of type (a) of $\mathcal{C}$ on p_2 together with the tangent line to $\mathcal{O}$ in the point p_2 and the line $\langle p_2, p_0 \rangle$ define a set S of $q+2$ points in Δ. All lines of Δ intersect S in $0, 1$ or 2 points, except for one line M (the line corresponding with $\langle p_2, L \rangle$); M intersects S in $r+2$ points $s_1, \ldots, s_{r+2}$. Define $S' = (S \setminus M) \cup \{s_1\}$, then S' is a $(q-r+1)$-arc which can be extended in $r + 1$ ways to a $(q - r + 2)$-arc. If $r \leqslant q/2 - 2$, then $q - r + 1 \geqslant q - (q/2 - 2) + 1 \geqslant q/2 + 3$. A $(q/2 + 2)$-arc in $\mathrm{PG}(2, q)$, q even, can be extended in a unique way to a complete arc [60, Cor. 10.3]. Hence S must be a $(q + 2)$-arc and the plane $\langle p_2, L \rangle$ contains $r + 2$ lines of $\mathcal{C}$ on the point p_1, since all lines in Δ must intersect S in 0 or 2 points. Since we can repeat the arguments for all the points $p_2, \ldots, p_q$, we conclude that all the planes $\langle p_2, L \rangle, \ldots, \langle p_q, L \rangle$ contain $r + 2$ lines of type (a) of $\mathcal{C}$ on p_1. So $(q - 1)(r + 2) \leq q + r$ which is impossible. ■

These last two lemmas lead to Lemma 3.1.6.

3.4 Final remarks

We mention that Theorem 3.1.7 follows from a more general result, which was independently proved by K. Metsch.

Theorem 3.4.1. (Metsch [70]) *Every set of points of* $\mathrm{W}(2n + 1, q)$ *that meets all generators has at least* $q^{n+1} + q^{n-1}$ *points. Equality can only occur for even* q *and then the set consists of the points outside the vertex of a cone with a vertex of dimension* $n - 2$ *over an ovoid in a* $\mathrm{W}(3, q)$.

Since for q even, $\mathrm{Q}(6, q) \cong \mathrm{W}(5, q)$, Theorem 3.1.7 can be derived from this theorem. However, it is interesting to compare the two different proofs.

Our proof focusses on the projection of the blocking set. It also relies very heavily on the results on blocking sets of Q$(4, q)$, q even.

The proof of K. Metsch starts immediately with purely combinatorial arguments and focusses on the general case W$(2n+1, q)$, $n > 1$. Bounds on the smallest minimal blocking sets different from an ovoid of W$(3, q)$ are not needed.

It is interesting to see that also in the proof of K. Metsch, a lot of arguments are proved for q even and q odd. Of course, the arguments can not be translated completely since W$(3, q)$, q odd, has no ovoid, [81]. The lower bound achieved in [70] is improved in [45], using an inductive argument and a result from [42]. An example of a minimal blocking set of W$(2n+1, q)$ is also constructed in [45]. The question whether the lower bound is sharp remains open however.

Chapter 4

The smallest minimal blocking sets of $Q(6, q)$, q odd prime

The existence or non-existence of ovoids of $Q(6, q)$, q odd, is not completely solved. It is known that $Q(6, q)$ has ovoids for $q = 3^r$, $r \geqslant 1$. Furthermore, a recent result of S. Ball [4], together with a result of S. Ball, P. Govaerts and L. Storme [6], implies that the only ovoids of $Q(4, q)$, q an odd prime, are elliptic quadrics $Q^-(3, q)$. Then a result of J. A. Thas and C. M. O'Keefe [76], implies that $Q(6, q)$ has no ovoids when $q > 3$ is an odd prime.

Again we can try to find the smallest sets of points meeting every generator of $Q(6, q)$, when it has no ovoids. We can also try to find the smallest minimal blocking sets of $Q(6, q)$, q odd, different from an ovoid of $Q(6, q)$, when $Q(6, q)$ has ovoids.

The present chapter is based on joint work with L. Storme [36] and joint work with K. Metsch [34].

4.1 Introduction

For this chapter we use the notations introduced in Section 3.1. Since we are dealing with $Q(6, q)$, q odd, we can use the associated polarity, and we will denote it by $\perp$. Hence, if p is a point of $Q(6, q)$, the notation $T_p(Q(6, q))$ will be replaced with $p^\perp$.

The situation is more complex compared with $Q(6, q)$, q even. First, we mention the central result of [76].

Theorem 4.1.1. *If for some q, $q > 3$, every ovoid of $Q(4, q)$ is an elliptic quadric, then $Q(6, q)$ has no ovoid.*

Recently, the following result was shown ([4] and [6]).

Theorem 4.1.2. *If q is prime, then every ovoid of* $\mathrm{Q}(4, q)$ *is an elliptic quadric.*

Hence, we can conclude that $\mathrm{Q}(6, q)$, $q > 3$ prime, has no ovoids. Concerning the non-existence of ovoids of $\mathrm{Q}(6, q)$, q odd, this is the only known result.

When $q = 3^h$, some examples of ovoids of $\mathrm{Q}(6, q)$ are known. We refer to [65], [91] and [92].

Two questions now arise. Is it possible to classify the smallest minimal blocking sets of $\mathrm{Q}(6, q)$, q odd, when it has no ovoids and, can we classify the smallest minimal blocking sets different of ovoids of $\mathrm{Q}(6, q)$, q odd, when $\mathrm{Q}(6, q)$ has ovoids?

Concerning the general question, it seems reasonable to think about the techniques of Chapter 3, since the proofs are independent of the existence of ovoids of $\mathrm{Q}(6, q)$. If we want to use the same arguments, it is clear that a replacement for Theorem 3.1.2 is needed. Finding a "q-odd version" of that theorem seems very difficult. Therefore we can only present results for $q = 3, 5, 7$. For these small values, it was possible to obtain some results on minimal blocking sets of $\mathrm{Q}(4, q)$ which replaced Theorem 3.1.2, although we needed a computer search. We will mention all obtained results, therefore we need a few more concepts.

Recall the definition of a blocking set and a cover of a GQ from Section 3.1. The concepts *multiple point, excess of a point* and *weight of a line with respect to a cover* were also given in Section 3.1 and we will dualise these concepts now. Suppose that $\mathcal{B}$ is a blocking set of the GQ $\mathcal{S}$. A *multiple line* is a line of $\mathcal{S}$ meeting $\mathcal{B}$ in more than one point. The *excess* of a line L is the number of points of $\mathcal{B}$ it contains minus one. The *weight* of a point with respect to a given blocking set is the minimum of the excesses of the lines on this point.

Again in Section 3.1, we mentioned a special case (Theorem 3.1.3) of the following theorem.

Theorem 4.1.3. (Eisfeld et al. [42]) *Let $\mathcal{C}$ be a cover of a classical generalised quadrangle $\mathcal{S}$ of order (q, t) embedded in* $\mathrm{PG}(d, q)$. *Let $|\mathcal{C}| = qt + 1 + r$, with $q + r$ smaller than the cardinality of the smallest non-trivial blocking sets in* $\mathrm{PG}(2, q)$. *Then the multiple points of $\mathcal{C}$ form a sum of lines of* $\mathrm{PG}(d, q)$, *where the weight of a line in this sum is equal to the weight of this line with respect to the cover, and with the sum of the weights of the lines equal to r.*[1]

[1] This is exactly the formulation found in [42], except for the notation $\mathcal{S}$ for the GQ, and where we replaced "contained in Q" by "of $\mathrm{PG}(d, q)$"

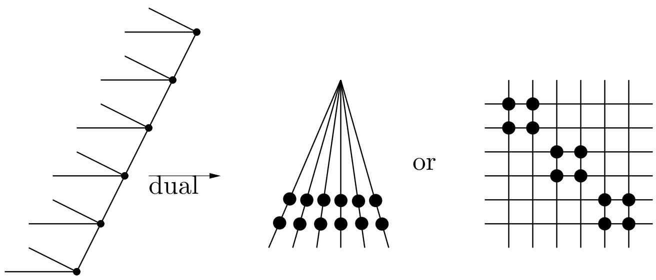

Figure 4.1: A line of $\mathrm{PG}(3,q)$ dualises to a pencil or a grid

The concept *sum of lines* was defined in Section 2.1.3, Definition 2.1.9. Since all classical generalised quadrangles are embedded in some projective space $\mathrm{PG}(n,q)$, the above theorem describes the set of multiple points in terms of projective points and projective lines of the ambient projective space. However, one has to be extremely careful in the case of $\mathrm{W}(3,q)$ concerning the interpretation of this theorem. It is clear that all the multiple points are points of the GQ $\mathcal{S}$. This does not, however, imply in general that the lines of the sum are lines of the GQ $\mathcal{S}$. If $\mathcal{S}$ is a quadric, for instance $\mathrm{Q}(4,q)$, then all lines of the sum of lines are indeed lines of the quadric, since otherwise projective points not belonging to the quadric would become multiple points of the given cover of the quadric. If $\mathcal{S}$ is $\mathrm{W}(3,q)$, which is the situation we need in this chapter, then every point of $\mathrm{PG}(3,q)$ is a point of $\mathrm{W}(3,q)$, hence it is possible that projective lines which are not lines of the GQ $\mathcal{S}$, are lines of the sum of lines describing the set of multiple points. Hence the interpretation is that the sum of lines is a sum of lines of $\mathrm{PG}(3,q)$.

Consider a cover $\mathcal{C}$ of the GQ $\mathcal{S} = \mathrm{W}(3,q)$, satisfying the conditions of Theorem 4.1.3. This cover $\mathcal{C}$ dualises to a blocking set $\mathcal{B}$ of the GQ $\mathcal{S}' = \mathrm{Q}(4,q)$. The sum of multiple lines can now be described by *pencils*, i.e. $q+1$ lines of $\mathcal{S}'$ on a point, the dual of a line of $\mathrm{PG}(3,q)$ which is also a line of $\mathcal{S}$, and *reguli*, lying in 3-dimensional spaces intersecting $\mathcal{S}' = \mathrm{Q}(4,q)$ in a hyperbolic quadric $\mathrm{Q}^+(3,q)$, corresponding to the $q+1$ points on a line of $\mathrm{PG}(3,q) \setminus \mathcal{S}$. We will be interested in the situation where only pencils occur in this sum.

The following lemma is based on similar results from [48].

Lemma 4.1.4. *Suppose that $\mathcal{C}$ is a cover of $\mathcal{S} = \mathrm{W}(3,q)$, of size q^2+1+r, with $q+r$ smaller than the cardinality of the smallest non-trivial blocking set in $\mathrm{PG}(2,q)$, such that the multiple points of $\mathcal{C}$ are a sum $\mathcal{A}$ of lines of $\mathrm{PG}(3,q)$. If L is a line of $\mathcal{A}$, L not a line of $\mathrm{W}(3,q)$, then $L^\perp \in \mathcal{A}$, with $\perp$ the symplectic polarity corresponding to $\mathrm{W}(3,q)$.*

Proof. Suppose that L is a line of $\mathcal{A}$, but not a line of $\mathcal{S}$. Since $L \notin \mathcal{S}$, $L \notin \mathcal{C}$, so L is intersected by at least $2q+2$ lines of $\mathcal{C}$. Then also $L^\perp$ is intersected by these at least $2q+2$ lines of $\mathcal{C}$.

If $L^\perp \notin \mathcal{A}$, then $L^\perp$ intersects at most r lines of $\mathcal{A}$, so the sum of the excesses of the points of $L^\perp$ is at most r. But it is at least $2q+2-(q+1) = q+1$, so also $L^\perp \in \mathcal{A}$. ■

Consider now a blocking set $\mathcal{B}$ of $\mathrm{Q}(4,q)$, q odd, of size $|\mathcal{B}| = q^2+1+r$. This corresponds to a cover $\mathcal{C}$ of $\mathrm{W}(3,q)$. If $r=1$, then Theorem 4.1.3 implies that the multiple points of $\mathcal{C}$ lie on a unique line L of $\mathrm{PG}(3,q)$, with weight 1. This implies that $L^\perp = L$, since otherwise the sum of lines would contain two lines with weight 1, a contradiction since the sum of the weights of the lines equals $r=1$. If $r=2$, then either the multiple points of $\mathcal{C}$ lie on a unique line with weight 2, or on two lines with weight one. If we suppose that all lines of the sum have weight 2, then, by the same arguments as for $r=1$, the sum of lines consists of a unique line of weight 2 belonging to $\mathrm{W}(3,q)$. We can formulate the following corollary.

Corollary 4.1.5. *Let $\mathcal{B}$ be a minimal blocking set of $\mathrm{Q}(4,q)$, q odd, $|\mathcal{B}| = q^2+1+r$, $q+r$ smaller than the cardinality of the smallest non-trivial blocking sets in $\mathrm{PG}(2,q)$. If $r=1$, then the multiple lines pass through a common point $p \in \mathrm{Q}(4,q) \setminus \mathcal{B}$. If $r=2$, and all multiple lines have excess 2, then all multiple lines pass through a common point $p \in \mathrm{Q}(4,q) \setminus \mathcal{B}$.*

This corollary is not a complete replacement of Theorem 3.1.2, since it only gives information on how the multiple lines are structured. Still it enables to prove for $q=3$, and for $q=5,7$ with the aid of a computer, the following results.

Lemma 4.1.6. ([36]) *If $\mathcal{B}$ is a minimal blocking set of $\mathrm{Q}(4,3)$ different from an ovoid, then $|\mathcal{B}| > 11$.*

Proof. Suppose that $\mathcal{B}$ is a minimal blocking set of $\mathrm{Q}(4,3)$, $|\mathcal{B}| = 11$. Note that an ovoid $\mathcal{O}$ of $\mathrm{Q}(4,3)$ has size 10. Then there exists a point $p \notin \mathcal{B}$, such that all 4 lines of $\mathrm{Q}(4,3)$ on p contain 2 points of $\mathcal{B}$ (Corollary 4.1.5), i.e. $p^\perp \cap \mathcal{B} = \{p_1, \dots, p_8\}$. There remain 3 points r_1,r_2,r_3 in $\mathcal{B}$. Those points cannot be collinear with the points $p_1, \dots, p_8$, since the only 2-secants to $\mathcal{B}$ pass through p. Hence, $(\cap_{i=1}^3 r_i^\perp) \cap p^\perp \cap \mathrm{Q}(4,3) = p^\perp \setminus (\mathcal{B} \cup \{p\}) = \{s_1, \dots, s_4\}$. But the polar space of $\langle s_1, s_2, s_3, s_4 \rangle$ is at most a line, so this would be a line containing r_1, r_2, r_3, a contradiction. ■

Lemma 4.1.7. ([38]) *If $\mathcal{B}$ is a minimal blocking set of $\mathrm{Q}(4,q)$, $q=5,7$, different from an ovoid of $\mathrm{Q}(4,q)$, then $|\mathcal{B}| > q^2+2$.*

Lemma 4.1.8. **([38])** *There is no minimal blocking set $\mathcal{B}$ of size* 52 *on* $\mathrm{Q}(4,7)$ *such that there is one point of* $\mathrm{Q}(4,7)$ *with* $q+1$ *lines on it being blocked by exactly three points of* $\mathcal{B}$.

These lemmas were as replacement for Theorem 3.1.2 sufficient to classify the smallest minimal blocking sets of $\mathrm{Q}(6,q)$, $q=3,5,7$, different from an ovoid.

The above mentioned approach only gives results for $q=3,5,7$. Therefore one can wonder if the recent Theorem 4.1.2 can be useful to prove the classification for all q prime. It will become clear that in the proofs of the characterisation, we often find ovoids of $\mathrm{Q}(4,q)$. When those ovoids all are elliptic quadrics, it is easier to handle intersections of ovoids. Therefore we can try to find the classification using basically the result of Theorem 4.1.2.

The goal of this chapter is to prove the following theorem.

Theorem 4.1.9. *Let $\mathcal{K}$ be a minimal blocking set of* $\mathrm{Q}(6,q)$*, q an odd prime, different from an ovoid of* $\mathrm{Q}(6,q)$ *and with* $|\mathcal{K}| \leqslant q^3+q$*. Then there is a point $p \in \mathrm{Q}(6,q) \setminus \mathcal{K}$ with the following property: $p^\perp \cap \mathrm{Q}(6,q) = p\mathrm{Q}(4,q)$ and $\mathcal{K}$ consists of all the points of the lines L on p meeting $\mathrm{Q}(4,q)$ in an elliptic quadric $\mathrm{Q}^-(3,q)$, minus the point p itself, and $|\mathcal{K}| = q^3+q$.*

4.2 The smallest minimal blocking sets of $\mathrm{Q}(6,q)$, $q=3,5,7$

In this section, we will discuss Theorem 4.1.9 for $q=3,5,7$. We will give a complete proof only for $q=3$, since the cases $q=5,7$ will be handled in a different way in the next section.

For this section, suppose that $\mathcal{K}$ is a minimal blocking set of $\mathrm{Q}(6,q)$, different from an ovoid, and with $|\mathcal{K}| = q^3+1+\delta$, $0<\delta<q$. At first, we recall Lemma 3.2.1, Corollary 3.2.2 and Lemma 3.2.3 from Section 3.2. The proofs were given in Section 3.2 for general q.

Lemma 4.2.1. *If p is a point of* $\mathrm{Q}(6,q)$*, $p \in \mathcal{K}$, then $|p^\perp \cap \mathcal{K}| \leqslant 1+\delta$.*

Corollary 4.2.2. *Every generator of* $\mathrm{Q}(6,q)$ *contains at most $1+\delta$ points of $\mathcal{K}$.*

Lemma 4.2.3. *If $p \in \mathrm{Q}(6,q) \setminus \mathcal{K}$, then p projects the points of $p^\perp \cap \mathcal{K}$ onto a minimal blocking set of* $\mathrm{Q}(4,q)$.

We now suppose that $q=3,5,7$. One of the key lemmas in Section 3.2 was Lemma 3.2.4. We can now prove a replacement for $q=3,5,7$ using Lemma 4.1.6, Lemma 4.1.7 and Lemma 4.1.8.

Lemma 4.2.4. *Suppose that $\mathcal{K}$ is a minimal blocking set of $Q(6,q)$, $q = 3,5,7$, not an ovoid, and with $|\mathcal{K}| \leqslant q^3+q$. If L is a line of $Q(6,q)$, then $|L \cap \mathcal{K}| = 0,1$ or $|L \cap \mathcal{K}| \geqslant 3$.*

Proof. Suppose that $|L \cap \mathcal{K}| = 2$. Consider a generator $\pi \subseteq Q(6,q)$ such that $L \subset \pi$ and $L \cap \mathcal{K} = \pi \cap \mathcal{K}$. Suppose that every generator on L contains a point of $\mathcal{K}$ not on $L \cap \mathcal{K}$. This would imply $|p^\perp \cap \mathcal{K}| > 1+q$, for every $p \in L \cap \mathcal{K}$, a contradiction with Lemma 4.2.1. Hence, such a generator π exists. Count the pairs (u,v), $u \in \pi \setminus L$, $v \in \mathcal{K} \setminus \pi$, $u \in v^\perp$. By Lemma 4.1.6, Lemma 4.1.7 and Lemma 4.2.3, $|u^\perp \cap \mathcal{K}| > q^2+2$. We find a lower bound of $q^2(q^2+2)$. If $v \in \mathcal{K} \setminus \pi$, then $v^\perp$ intersects π in a line, hence to v correspond q or 0 points of $\pi \setminus L$, which gives $(q^3+q-|L\cap\mathcal{K}|)q$ as upper bound. Necessarily $(q^3+q-2)q \geqslant (q^2+1)q^2$, a contradiction. Hence $|L \cap \mathcal{K}| < 2$, or $|L\cap\mathcal{K}| > 2$. ■

In [36], this latter lemma is, together with further steps, sufficient to prove Theorem 4.1.9 for $q = 5,7$.

Suppose now that $q = 3$. Lemma 4.2.1 and Lemma 4.2.4 lead immediately to the following corollary.

Corollary 4.2.5. *If π is a generator of $Q(6,3)$, then $|\pi \cap \mathcal{K}| = 1$ or $|\pi \cap \mathcal{K}| = 3$, and all points of $\pi \cap \mathcal{K}$ are collinear.*

This corollary leads to

Corollary 4.2.6. *Consider a line L of $Q(6,3)$ with the property that $|L \cap \mathcal{K}| = 3$, and let $L \setminus \mathcal{K} = \{p\}$. Then $\mathcal{K} \subseteq p^\perp \cap Q(6,3)$.*

Proof. If $r \in \mathcal{K} \setminus p^\perp$, then $r \in r'^\perp$, for some $r' \in L \cap \mathcal{K}$. But then $|r'^\perp \cap \mathcal{K}| \geqslant |L \cap \mathcal{K}| + 1 = 4$. This contradicts Lemma 4.2.1. ■

We now can prove the following theorem.

Theorem 4.2.7. *Let $\mathcal{K}$ be a minimal blocking set of $Q(6, q = 3)$, different from an ovoid, with $|\mathcal{K}| \leqslant q^3+q$. Then there is a point $p \in Q(6,3) \setminus \mathcal{K}$ with the following property: $p^\perp \cap Q(6,3) = pQ(4,3)$ and $\mathcal{K}$ consists of all the points of the lines L on p meeting $Q(4,3)$ in an elliptic quadric $Q^-(3,3)$, minus the point p itself, and $|\mathcal{K}| = q^3+q$.*

Proof. We know from the previous corollary that there exists a point $p \in Q(6,3) \setminus \mathcal{K}$, such that $\mathcal{K} \subseteq p^\perp \cap Q(6,3)$. Suppose that some line L of $Q(6,q)$ through p intersecting $\mathcal{K}$ in at least one point contains a point r, $r \neq p$, $r \notin \mathcal{K}$.

Since $r \in \mathrm{Q}(6,3) \setminus \mathcal{K}$, we have that $|r^\perp \cap \mathcal{K}| \geqslant 10$, and since $\mathcal{K} \subseteq p^\perp$, $|L^\perp \cap \mathcal{K}| \geqslant 10$. Suppose that $|L \cap \mathcal{K}| = 1$, since $|L \cap \mathcal{K}| = 2$ is excluded by Lemma 4.2.4; then at least 9 points of $L^\perp \cap \mathcal{K}$ lie in the 4 planes of $\mathrm{Q}(6,q)$ on L, but not on L. Hence, at least one of those planes contains at least 4 points of $\mathcal{K}$, a contradiction with Corollary 4.2.5. So the only possibility is that p lies on q^2+1 distinct 3-secants to $\mathcal{K}$ which project from p onto an ovoid of $\mathrm{Q}(4,3)$, which is necessarily an elliptic quadric $\mathrm{Q}^-(3,3)$. ■

4.3 The smallest minimal blocking sets of $\mathrm{Q}(6,q)$, $q > 3$, q prime

In this section, we will give a complete proof for Theorem 4.1.9 for all $q>3$ prime. The proof is heavily relying on Theorem 4.1.2. Furthermore, also general results on blocking sets of quadrics play a role. At first, we generalise slightly the concept of a blocking set of a polar space.

Suppose that $\mathcal{S}$ is a polar space of rank n, hence having generators of dimension $n-1$. A *blocking set with respect to s-dimensional spaces* is a set $\mathcal{K}$ of points of $\mathcal{S}$ such that every s-dimensional subspace of $\mathcal{S}$ meets $\mathcal{K}$ in at least one point. When $s=n-1$, we call $\mathcal{K}$ simply a *blocking set* of $\mathcal{S}$, so the new definition coincides with the old one. All concepts concerning blocking sets like minimality can be generalised straightforwardly.

The first result we state is a theorem concerning blocking sets of elliptic quadrics. The version we mention here is a special case of a more general theorem on blocking sets with respect to c-dimensional subspaces of the elliptic quadric $\mathrm{Q}^-(2n+1,q)$, $0 \leqslant c \leqslant n-1$, which can be found in [72]. The special case we state can also be found in [71].

Theorem 4.3.1. *Suppose that $\mathcal{K}$ is a minimal blocking set of the elliptic quadric $\mathrm{Q}^-(5,q)$, $|\mathcal{K}| \leqslant q^3+q$, then there exists a point $p \in \mathrm{Q}^-(5,q) \setminus \mathcal{K}$ such that $\mathcal{K} = (p^\perp \cap \mathrm{Q}^-(5,q)) \setminus \{p\}$.*

The next result is a theorem concerning blocking sets with respect to lines of $\mathrm{Q}^+(2n+1,q)$. We state the particular case $n=2$. The general theorem can be found in [73].

Theorem 4.3.2. *Let $\mathcal{K}$ be a blocking set with respect to lines of $\mathrm{Q}^+(5,q)$, $|\mathcal{K}| \leqslant q|\mathrm{Q}^+(3,q)| = q^3+2q^2+q$. Then $\mathcal{K}$ contains a blocking set with respect to lines that is contained in a hyperplane of $\mathrm{PG}(5,q)$. In particular, $|\mathcal{K}| \geqslant |\mathrm{Q}(4,q)| = q^3+q^2+q+1$.*

Suppose now that $\mathcal{K}$ is a minimal blocking set of $Q(6, q)$, $q > 3$, q prime, $|\mathcal{K}| = q^3 + 1 + \delta$, $1 \leqslant \delta < q$. A point $p \in Q(6, q) \setminus \mathcal{K}$ will be called a *small point* when $|p^\perp \cap \mathcal{K}| = q^2 + 1$.

We will try to find as much as possible 3-dimensional elliptic quadrics in the set $\mathcal{K}$. The first lemma immediately gives cones over elliptic quadrics.

Lemma 4.3.3. *If $p \in Q(6, q) \setminus \mathcal{K}$, then $|p^\perp \cap \mathcal{K}| \geqslant q^2 + 1$. If equality holds, then there exists a 4-dimensional space α_p on p that meets $Q(6, q)$ in a cone over an elliptic quadric $Q^-(3, q)$ with vertex p and such that each one of the $q^2 + 1$ lines of $Q(6, q)$ in α_p on p meets $\mathcal{K}$ in a unique point.*

Proof. Recall Lemma 4.2.3, which implies that $|p^\perp \cap \mathcal{K}| \geqslant q^2 + 1$ and if we have equality, projecting $p^\perp \cap \mathcal{K}$ from p we find an ovoid $\mathcal{O}$ of the base $Q(4, q)$ of the cone $p^\perp \cap Q(6, q)$. This ovoid is now necessarily an elliptic quadric $Q^-(3, q)$. Define $\alpha_p := \langle Q^-(3, q), p\rangle$. Then α_p is a 4-dimensional subspace satisfying the conditions of the lemma. ■

Lemma 4.3.4. *If π is a generator of $Q(6, q)$ meeting $\mathcal{K}$ in a unique point p, then π contains a small point. Also, every line of π that contains a small point but not p, contains a second small point.*

Proof. Count pairs (r, s) with $r \in \pi \setminus \{p\}$ and $s \in \mathcal{K} \setminus \{p\}$ such that $r \in s^\perp$. This gives

$$\sum_{r \in \pi \setminus \{p\}} (|r^\perp \cap \mathcal{K}| - 1) \leqslant (|\mathcal{K}| - 1)(q + 1),$$

since every point of $\mathcal{K} \setminus \{p\}$ is perpendicular to $q+1$ points of π. As the right hand side is at most $(q^3 + q - 1)(q + 1) < (q^2 + q)(q^2 + 1)$, it follows that $|r^\perp \cap \mathcal{K}| - 1 < q^2 + 1$ for at least one point $r \in \pi \setminus \{p\}$. Hence, π contains a small point r.

Now consider a line L of π containing a small point r but not p. As r is a small point, every generator of $Q(6, q)$ on r meets $\mathcal{K}$ in a unique point. Hence, $|L^\perp \cap \mathcal{K}|$ is equal to the number $q + 1$ of generators of $Q(6, q)$ on L. Count the number of pairs (t, s) with $t \in L$ and $s \in \mathcal{K}$ such that $t \in s^\perp$. Then the $q + 1$ points of $L^\perp \cap \mathcal{K}$ occur in $q + 1$ such pairs and every other point of $\mathcal{K}$ occurs in exactly one such pair. This implies that

$$\sum_{t \in L} |t^\perp \cap \mathcal{K}| \leqslant (q + 1)^2 + (|\mathcal{K}| - q - 1) = |\mathcal{K}| + q^2 + q.$$

The left hand side is at most $q^3 + q^2 + 2q = (q + 1)(q^2 + 1) + (q - 1)$, so at most $q - 1$ points $t \in L$ can have $|t^\perp \cap \mathcal{K}| \geqslant q^2 + 2$. ■

Lemma 4.3.5. *Suppose that L is a line of $\mathrm{Q}(6,q)$, $L\cap\mathcal{K}=\emptyset$, and containing at least two small points. Then there exists a plane β meeting $\mathrm{Q}(6,q)$ in a conic $\mathrm{Q}(2,q)$ and such that $L^\perp\cap\mathcal{K}$ consists of the $q+1$ points of this conic.*

Proof. Let p and r be small points on the line L. From Lemma 4.3.3, we find two 4-dimensional subspaces α_p and α_r. Since $L\cap\mathcal{K}=\emptyset$, it is clear that $L\cap\alpha_p=\{p\}$ and $L\cap\alpha_r=\{r\}$, hence $\alpha_p\cap\alpha_r\subset L^\perp$ and $\alpha_p\cap\alpha_r\cap L=\emptyset$. Since the subspaces α_p, α_r and $L^\perp$ have dimension 4, the subspace $\alpha_p\cap\alpha_r$ has dimension 2. Since $L^\perp\cap\mathrm{Q}(6,q)$ is a cone $L\mathrm{Q}(2,q)$, the plane $\alpha_p\cap\alpha_r$ meets $\mathrm{Q}(6,q)$ in a conic $\mathrm{Q}(2,q)$. Lemma 4.3.3 also implies that $\alpha_p\cap\mathcal{K}=p^\perp\cap\mathcal{K}$ and $\alpha_r\cap\mathcal{K}=r^\perp\cap\mathcal{K}$. Hence, $\alpha_p\cap\alpha_r\cap\mathrm{Q}(6,q)\cap\mathcal{K}=p^\perp\cap r^\perp\cap\mathcal{K}=L^\perp\cap\mathcal{K}$. So the conic $\alpha_p\cap\alpha_r\cap\mathrm{Q}(6,q)$ belongs to $\mathcal{K}$. ■

Lemma 4.3.6. *Suppose that p is a small point and suppose that r is a point of $\alpha_p\cap\mathcal{K}$. Then the set $\alpha_p\cap\mathcal{K}$ can be written as the union of q plane conics sharing two by two only the point r. There are at least $\frac{1}{2}(q+1)$ different ways to do this.*

Proof. Suppose that π is a generator of $\mathrm{Q}(6,q)$ on the line $\langle p,r\rangle$. Let $L_1,L_2,\ldots,L_q$ be the lines of π on p different from $\langle p,r\rangle$. By Lemma 4.3.3 and Lemma 4.3.4, each line L_i contains a small point r_i different from p. Lemma 4.3.5 shows that $C_i:=\langle p,r_i\rangle^\perp\cap\mathcal{K}$ is a conic that is contained in $\alpha_p\cap\mathcal{K}$. Each conic C_i contains r. Two different conics C_i and C_j only share r, because $C_i\cap C_j\subset L_i^\perp\cap L_j^\perp=\pi^\perp$ and $\pi^\perp$ meets $\mathrm{Q}(6,q)$ only in π. Thus, the q conics C_i describe $\alpha_p\cap\mathcal{K}$ as required.

The same can be done for each one of the $q+1$ generators on the line $\langle p,r\rangle$. We show that each conic arises from at most two generators. It is sufficient to show this for C_1. It is clear that $C_1^\perp$ is a 3-dimensional subspace meeting $\mathrm{Q}(6,q)$ in an elliptic quadric $\mathrm{Q}^-(3,q)$ or in a hyperbolic quadric $\mathrm{Q}^+(3,q)$. But, in every generator on $\langle p,r\rangle$ that gives rise to C_1, we have a line L_1 on p such that $C_1\subset L_1^\perp$. This implies that $C_1^\perp$ meets $\mathrm{Q}(6,q)$ in a hyperbolic quadric $\mathrm{Q}^+(3,q)$. Since a point of a hyperbolic quadric $\mathrm{Q}^+(3,q)$ lies on two lines of $\mathrm{Q}^+(3,q)$, this shows that C_1 can arise only from at most two planes. ■

The fact that we are dealing only with elliptic quadrics as ovoids of $\mathrm{Q}(4,q)$ is a very strong condition. This becomes clear with the next lemma, where we really find elliptic quadrics in the 4-dimensional spaces α_p associated to small points p.

Lemma 4.3.7. *For every small point p, the set $p^\perp\cap\mathcal{K}$ is an elliptic quadric $\mathrm{Q}^-(3,q)$.*

Proof. Suppose that p is a small point and that r is a point of $p^\perp \cap \mathcal{K} = \alpha_p \cap \mathcal{K}$. Since we supposed that $q > 3$, we find from Lemma 4.3.6 three different families of conics $\{C_i \| 1 \leqslant i \leqslant q\}$, $\{D_i \| 1 \leqslant i \leqslant q\}$ and $\{E_i \| 1 \leqslant i \leqslant q\}$, two conics of the same family only share the point r and the union of the q conics of one family is the set $\alpha_p \cap \mathcal{K}$.

A conic D_i that is not in the family $\{C_i\}$ shares with each conic C_i the point r and at most one further point. Thus such a conic D_i meets each conic C_j in r and a second point. This shows that two conics from different families are distinct and share two points. Then $C_1 \cap D_1$ consists of two points r and s, and the supporting planes span a 3-space β.

Every conic E_i that does not contain s, contains r and one more point from C_1 and D_1, which implies that E_i is contained in β. Hence $q-1$ of the conics E_i lie in β. The same argument now shows that β contains $q-1$ conics of all of the three families. Now it is easy to see that $\alpha_p \cap \mathcal{K}$ is contained in β, which proves the lemma. ■

Lemma 4.3.8. *If there exists a* 5*-dimensional subspace containing at least* q^3+1 *points of* $\mathcal{K}$*, then* $\mathcal{K}$ *is as stated in Theorem 4.1.9.*

Proof. Denote the 5-dimensional subspace containing at least q^3+1 points of $\mathcal{K}$ by α.

We will consider the different possibilities for the structure of $\alpha \cap \mathrm{Q}(6, q)$ but we start with a remark that applies for all three possibilities for this structure. If there is a line L of $\alpha \cap \mathrm{Q}(6, q)$ that does not meet $\mathcal{K}$, then every generator of $\mathrm{Q}(6, q)$ on L that is not contained in α meets $\mathcal{K}$ in a point outside α. As $|\alpha \cap \mathcal{K}| \geqslant q^3+1$, there can be at most $|\mathcal{K}| - (q^3+1) \leqslant q-1$ such generators on such a line.

If $\alpha \cap \mathrm{Q}(6, q) = \mathrm{Q}^-(5, q)$, then any generator (which is a line) of this $\mathrm{Q}^-(5, q)$ lies on $q+1$ generators of $\mathrm{Q}(6, q)$ that do not lie in α, so the previous remark shows that all generators of $Q^-(5, q)$ meet $\mathcal{K}$. Then Theorem 4.3.1 gives the structure of $\mathcal{K}$.

Now consider the case that α is tangent to $\mathrm{Q}(6, q)$ at a point p, that is, $\alpha \cap \mathrm{Q}(6, q)$ is a cone pQ with vertex p over a base $Q = \mathrm{Q}(4, q)$. Then every line of the cone that does not pass through p lies in q generators of $\mathrm{Q}(6, q)$ that are not contained in α, so the remark above implies that all lines of the cone pQ that do not pass through p meet $\mathcal{K}$. Thus, each $\mathrm{Q}(4, q)$ of pQ contains at least q^2+1 points of $\mathcal{K}$ with equality if and only if $\mathrm{Q}(4, q) \cap \mathcal{K}$ is an ovoid of $\mathrm{Q}(4, q)$. Count the number of pairs (Q_4, r), $Q_4 \subset pQ$ a quadric $\mathrm{Q}(4, q)$, $r \in \mathcal{K} \cap Q_4$. We find $q^5(q^2+1) \leqslant |\mathcal{K} \cap \alpha| q^4$, which implies that $|\mathcal{K} \cap \alpha| = q^3 + q$, each such $\mathrm{Q}(4, q)$ meets $\mathcal{K}$ in an ovoid and that $\mathcal{K}$ is contained in pQ (but $p \notin \mathcal{K}$). By the hypothesis on q and Theorem 4.1.2, for each $\mathrm{Q}(4,q)$ of $p\mathrm{Q}(4,q)$, the ovoid $\mathrm{Q}(4,q)\cap\mathcal{K}$ is an elliptic quadric $\mathrm{Q}^-(3,q)$. Consider one elliptic quadric $Q_1 := \mathrm{Q}^-(3,q)$ of $p\mathrm{Q}(4,q)$ that is contained in $\mathcal{K}$. Then the cone pQ_1 contains p and q^3+q further points. We show that $\mathcal{K}$ consists of these latter q^3+q points, proving the desired structure of $\mathcal{K}$. Suppose the contrary, then there would be a point r in $p\mathrm{Q}(4,q)\cap\mathcal{K}$ outside pQ_1. Then $\langle Q_1, r\rangle$ is a 4-dimensional space intersecting $p\mathrm{Q}(4,q)$ in a parabolic quadric $\mathrm{Q}_r(4,q)$. This latter $\mathrm{Q}_r(4,q)$ shares Q_1 and also r with $\mathcal{K}$. This is false since it must share an elliptic quadric with $\mathcal{K}$.

Consider finally the case $\alpha\cap\mathrm{Q}(6,q)=\mathrm{Q}^+(5,q)$. To block all lines of $\mathrm{Q}^+(5,q)$, at least $|\mathrm{Q}(4,q)| = q^3+q^2+q+1$ points are needed (Theorem 4.3.2). Hence, there exists a line L of $\mathrm{Q}^+(5,q)$ not meeting $\mathcal{K}$. In $\mathrm{Q}(6,q)$, the line L lies in $q+1$ generators and two of these lie in $\mathrm{Q}^+(5,q)$. The remaining $q-1$ planes of $\mathrm{Q}(6,q)$ on L meet $\mathcal{K}$ in points outside α. Thus $|\alpha\cap\mathcal{K}| \leqslant |\mathcal{K}|-(q-1)\leqslant q^3+1$. Hence, $|\alpha\cap\mathcal{K}| = q^3+1$ and exactly $q-1$ points of $\mathcal{K}$ do not lie in α. Also, the $q-1$ points of $\mathcal{K}\setminus\alpha$ are perpendicular to L and similarly to every line of $\alpha\cap\mathrm{Q}(6,q)$ that is skew to $\mathcal{K}$. Since the $q-1$ points span different planes with L, they are two by two non-perpendicular. Then three of these points span a plane π that meets $\mathrm{Q}(6,q)$ in a conic C. This shows that all lines of $\mathrm{Q}^+(5,q)$ that do not meet $\mathcal{K}$ lie in $C^\perp$, which intersects $\mathrm{Q}(6,q)$ in a $\mathrm{Q}^\pm(3,q)$. Since L lies in $C^\perp\cap\mathrm{Q}(6,q)$, $C^\perp\cap\mathrm{Q}(6,q)=\mathrm{Q}^+(1,q)$. Since $q+1$ points are enough to block all lines of a $\mathrm{Q}^+(3,q)$, this implies that we can adjoin $q+1$ points to $\alpha\cap\mathcal{K}$ in order to obtain a set $\mathcal{B}$ blocking all lines of $\mathrm{Q}^+(5,q)$. Then $|\mathcal{B}|\leqslant|\alpha\cap\mathcal{K}|+q+1 = q^3+q+2$, a contradiction since $\mathcal{B}$ must contain at least q^3+q^2+q+1 points (Theorem 4.3.2). ■

Lemma 4.3.9. *Suppose that $p\in\mathcal{K}$, then there exists a 4-dimensional subspace α on p such that $\alpha\cap\mathcal{K}$ contains at least $q+1$ elliptic quadrics $\mathrm{Q}^-(3,q)$ all containing p.*

Proof. Consider a generator π of $\mathrm{Q}(6,q)$ meeting $\mathcal{K}$ in the unique point $p\in\mathcal{K}$. Such a generator exists since $\mathcal{K}$ is a minimal blocking set. Lemma 4.3.4 implies that π contains at least $q+1$ small points, denoted here by r_i, $i=1,\ldots,q+1$. Lemma 4.3.7 implies that the set $Q_{r_i} := r_i^\perp\cap\mathcal{K}$ is a 3-dimensional elliptic quadric $Q_{r_i} := \mathrm{Q}^-(3,q)$ for every small point r_i. If r_1 and r_2 are small points of the generator π such that the line r_1r_2 does not contain p, then Lemma 4.3.5 implies that $Q_{r_1}\cap Q_{r_2}$ is a conic.

First we consider the case that we find three non-collinear small points r_1,r_2,r_3 in π that generate with p different lines. Then the quadrics Q_{r_1}, Q_{r_2} and Q_{r_3} meet two by two in a conic, but they do not share a conic (since this conic would be perpendicular to the generator $\pi=\langle r_1,r_2,r_3\rangle$ and $\pi^\perp\cap\mathrm{Q}(6,q)$ is a 3-space only sharing π with $\mathrm{Q}(6,q)$). Thus Q_{r_1}, Q_{r_2} and Q_{r_3}

span together a 4-dimensional subspace β. From Lemma 4.3.4 we know that every line of π on r_1 that does not contain any of the points r_2, r_3, p contains a second small point r'. Then r' either does not lie on pr_2 or not on pr_3. We may assume that it does not lie on pr_2. Then r_1, r_2, r' are non-collinear and on different lines with p, so as before Q_{r_1}, Q_{r_2} and $Q_{r'}$ span together a 4-dimensional subspace. Of course, this 4-dimensional subspace is β, so $Q_{r'} \subseteq \beta$. Since there are q choices for r', r_2 and r_3 included, this case is handled.

Now consider the case that all small points of π lie on two lines of π on p. Let these lines be $L = \{p, p_1, \ldots, p_q\}$ and $M = \{p, r_1, \ldots, r_q\}$. Then Lemma 4.3.4 implies that all points p_i and r_i are small points. For different i, the elliptic quadrics Q_{p_i} only share p; this is because the points p_i are small, so every generator on the line L meets $\mathcal{K}$ only in one point which is p. Thus different i yield different 3-dimensional subspaces $\alpha_i := \langle Q_{p_i} \rangle$ and similarly different 3-dimensional subspaces $\beta_i := \langle Q_{r_i} \rangle$. From Lemma 4.3.5, we know that each subspace $\alpha_i \cap \beta_j$ is a plane that meets $\mathrm{Q}(6, q)$ in a conic, which is $Q_{p_i} \cap Q_{r_j}$. Different pairs (i, j) give different planes, because different quadrics Q_{p_i} and also different quadrics Q_{r_j} only share the point p. If two of the α_i, say α_1 and α_2, meet in a plane, then $\gamma := \langle \alpha_1, \alpha_2 \rangle$ is a 4-dimensional subspace; in this case, every β_i meets α_1 and α_2 in different planes and thus is contained in γ, which proves the claim of the lemma. Hence, assume that different α_i share at most a line, and similarly for the β_i. Each β_i meets α_1 and α_2 in planes, so $\gamma := \langle \alpha_1, \alpha_2 \rangle$ is a 5-dimensional space, and $N := \alpha_1 \cap \alpha_2$ is a line on p. These two planes $\beta_i \cap \alpha_1$ and $\beta_i \cap \alpha_2$ share a line since they lie in the 3-dimensional subspace β_i, so they share the line N. Hence, $N \subseteq \beta_i \subseteq \gamma$ for all i. By the same argument, N is contained in all subspaces α_i, since $\alpha_i \cap \beta_1$ and $\alpha_i \cap \beta_2$ are planes in α_i; so they intersect in a line; this line is contained in $\beta_1 \cap \beta_2 = N$. Then N lies in the perp of all points p_i and all points r_i, which implies that N is a tangent line on p meeting $\mathrm{Q}(6, q)$ only in p. Then the q elliptic quadrics Q_{r_i} cover $q^3 + 1$ points of $\gamma \cap \mathrm{Q}(6, q)$ and these points lie in $\mathcal{K}$. Thus we can apply Lemma 4.3.8, so $\mathcal{K}$ lies in a 4-dimensional subspace. But this is a contradiction, as the quadrics Q_{r_i} are contained in $\mathcal{K}$ and span a 5-dimensional subspace. ■

Lemma 4.3.10. *The set $\mathcal{K}$ is as described in Theorem 4.1.9.*

Proof. Consider a 4-space γ as in Lemma 4.3.9. Then $\gamma \cap \mathcal{K}$ contains q quadrics $\mathrm{Q}_i^-(3, q)$. Since two such quadrics share at most $q + 1$ points, it follows that

$$|\gamma \cap \mathcal{K}| \geqslant q(q^2 + 1) - \binom{q}{2}(q + 1) = \frac{q(q^2 + 3)}{2} \ .$$

Since $\gamma \cap \mathrm{Q}(6,q)$ contains quadrics $\mathrm{Q}_i^-(3,q)$, then $\gamma \cap \mathrm{Q}(6,q)$ is a $\mathrm{Q}(4,q)$ or a cone with point vertex over a $\mathrm{Q}^-(3,q)$. Assume that $\gamma \cap \mathrm{Q}(6,q) = \mathrm{Q}(4,q)$. Consider two $\mathrm{Q}_i^-(3,q)$ contained in $\gamma \cap \mathcal{K}$, and choose a point p that lies in exactly one of them. Then $p^\perp$ meets the other $\mathrm{Q}_i^-(3,q)$ in a $\mathrm{Q}(2,q)$, since $p^\perp \cap \mathrm{Q}(4,q)$ is a cone $p\mathrm{Q}(2,q)$. The $q+1$ lines of this cone all intersect the other $\mathrm{Q}_i^-(3,q)$ in one point. Thus $p^\perp$ contains $1+(q+1)$ points of $\mathcal{K}$, a contradiction with Lemma 4.2.1.

Hence, $\gamma \cap \mathrm{Q}(6,q)$ is a cone with point vertex s over a $\mathrm{Q}^-(3,q)$. Since $|\gamma \cap \mathcal{K}| > q$, we have $s \notin \mathcal{K}$ by Lemma 4.2.1. Put $b := |\gamma \cap \mathcal{K}|$, and denote by $\mathcal{M}$ the set of points of $\gamma \cap \mathrm{Q}(6,q)$ that do not lie in $\mathcal{K}$ and that are different from s. Then $b + |\mathcal{M}| = q(q^2+1)$. Put $\mathcal{K}' := \mathcal{K} \setminus \gamma$. Then $b + |\mathcal{K}'| = |\mathcal{K}| \leqslant q^3 + q$, so $|\mathcal{K}'| \leqslant |\mathcal{M}|$.

Let $r \in \mathcal{M}$. Then $|r^\perp \cap \mathcal{K}| \geqslant q^2+1$ by Lemma 4.3.3, but the argument of the proof of Lemma 4.3.3 even shows that at least q^2+1 lines of $\mathrm{Q}(6,q)$ on r meet $\mathcal{K}$. Since r lies on a unique line of $\gamma \cap \mathrm{Q}(6,q) = s\mathrm{Q}^-(3,q)$, it follows that r lies on at least q^2 lines of $\mathrm{Q}(6,q)$ that meet $\mathcal{K}$ in a point but, do not lie in γ. Hence, $|r^\perp \cap \mathcal{K}'| \geqslant q^2$ for $r \in \mathcal{M}$.

Let $r' \in \mathcal{K}'$. We first show that r' is not perpendicular to s. Assume the contrary. Then sr' is a line of $\mathrm{Q}(6,q)$ and lies in $q+1$ generators of $\mathrm{Q}(6,q)$. Each one of these generators lies in $s^\perp$ and thus meets γ in a line on s. Since $\gamma \cap \mathcal{K}$ contains elliptic quadrics $\mathrm{Q}_i^-(3,q)$, each such line meets $\mathcal{K}$. It follows that each one of the $q+1$ generators on sr' meets $\gamma \cap \mathcal{K}$. But then $r'^\perp$ contains $q+1$ points of $\mathcal{K}$, a contradiction with Lemma 4.2.1. Hence, r' is not perpendicular to s. Then $\beta := r'^\perp \cap \gamma$ is a 3-dimensional subspace that meets $\mathrm{Q}(6,q)$ in a $\mathrm{Q}^-(3,q)$. We know that $\gamma \cap \mathcal{K}$ contains at least $q+1$ elliptic quadrics $\mathrm{Q}_i^-(3,q)$. Such an elliptic quadric $\mathrm{Q}_i^-(3,q)$ does not intersect β in a conic, or else $|r'^\perp \cap \mathcal{K}| \geqslant 1+(1+q)$. This contradicts Lemma 4.2.1. If they all meet β in only one point, that is, each one of the $q+1$ 3-dimensional subspaces spanned by these elliptic quadrics meets β in a tangent plane of $\beta \cap \mathrm{Q}(6,q) = \mathrm{Q}^-(3,q)$, then not all these tangent planes can be equal since only q hyperplanes of γ through a plane of β intersect $\mathrm{Q}(6,q)$ in an elliptic quadric. This implies that the $q+1$ elliptic quadrics cover at least two points of $\beta \cap \mathcal{K}$, so $|\beta \cap \mathcal{K}| \geqslant 2$, which implies that $|\beta \cap \mathcal{M}| = q^2+1-|\beta \cap \mathcal{K}| \leqslant q^2-1$.

Count the number of pairs $(r,r') \in \mathcal{M} \times \mathcal{K}'$, with $r \in r'^\perp$, to obtain

$$|\mathcal{M}|q^2 \leqslant \sum_{r\in\mathcal{M}} |r^\perp \cap \mathcal{K}'| = \sum_{r'\in\mathcal{K}'} |r'^\perp \cap \mathcal{M}| \leqslant |\mathcal{K}'|(q^2-1).$$

Since $|\mathcal{K}'| \leqslant |\mathcal{M}|$, this implies that $\mathcal{M} = \emptyset$. Hence, all points of the cone $s\mathrm{Q}^-(3,q)$, except the vertex s, lie in $\mathcal{K}$. This proves the lemma. ■

We conclude Sections 4.2 and 4.3 with the obtained theorem

Theorem 4.3.11. *Let $\mathcal{K}$ be a minimal blocking set of $Q(6, q)$, q an odd prime, different from an ovoid of $Q(6, q)$, and with $|\mathcal{K}| \leqslant q^3 + q$. Then there is a point $p \in Q(6, q) \setminus \mathcal{K}$ with the following property: $p^\perp \cap Q(6, q) = pQ(4, q)$ and $\mathcal{K}$ consists of all the points of the lines L on p meeting $Q(4, q)$ in an elliptic quadric $Q^-(3, q)$, minus the point p itself, and $|\mathcal{K}| = q^3 + q$.*

In the next chapter, we will obtain a similar result on the smallest blocking sets of $Q(2n, q)$, $n \geqslant 3$, q odd prime.

Chapter 5

The smallest minimal blocking sets of $Q(2n, q)$, q odd prime

In this chapter, we "lift" the results of the previous chapter to higher dimensions. At first, we describe the known results concerning the existence or non-existence of ovoids of the quadrics $Q(2n, q)$, $n \geqslant 3$, q odd. Although the situation for $Q(6, q)$, q odd, is not completely solved, it is known that $Q(2n, q)$, q odd, $n \geqslant 4$, has no ovoids [49]. However, it will become clear that existence or non-existence of ovoids of $Q(6, q)$ will have important implications.

The present chapter is based on joint work with L. Storme [36].

5.1 Introduction

In the previous chapter, we already mentioned the known results on the existence and non-existence of ovoids of $Q(6, q)$, q odd. Consider now the parabolic quadric $Q(8, q)$, then the following result is known.

Theorem 5.1.1. (Gunawardena and Moorhouse [49]) *The polar space $Q(8, q)$, q odd, has no ovoids.*

Hence, using Lemma 3.1.1, we immediately find that the polar spaces $Q(2n, q)$, q odd, $n \geqslant 4$, have no ovoids.

We can find examples of minimal blocking sets of these polar spaces, and to describe these examples, we first introduce the following concept.

Suppose that $\alpha\mathcal{O}$ is a cone with vertex the k-dimensional subspace α and base some set $\mathcal{O}$ of points lying in some subspace π, $\pi \cap \alpha = \emptyset$. Then the *truncated cone* $\alpha^*\mathcal{O}$ is defined as $\alpha\mathcal{O} \setminus \alpha$, hence as the set of points of the cone

$\alpha\mathcal{O}$ where the points of the vertex α are removed from. If α is the empty subspace, then $\alpha^*\mathcal{O} = \mathcal{O}$. In this chapter, we will often use this concept.

Consider now the polar space $\mathrm{Q}(2n,q)$, for some q odd, such that $\mathrm{Q}(6,q)$ has no ovoids. All $\mathrm{Q}(2n,q)$, q odd prime, $q > 3$, satisfy this condition. It is clear that the truncated cone $\pi^*_{n-3}\mathcal{O}$, π_{n-3} an $(n-3)$-dimensional subspace contained in $\mathrm{Q}(2n,q)$ and $\mathcal{O}$ an ovoid of $\mathrm{Q}(4,q)$, the base of the cone $\pi^{\perp}_{n-3} \cap \mathrm{Q}(2n,q)$ (with $\perp$ the polarity of $\mathrm{Q}(2n,q)$), constitutes a minimal blocking set of $\mathrm{Q}(2n,q)$ of size $q^n + q^{n-2}$. This is the known example for $\mathrm{Q}(6,q)$ of the previous chapter.

Consider now $\mathrm{Q}(2n,q)$ for some q odd such that $\mathrm{Q}(6,q)$ has ovoids, for instance when $q = 3^r$, $r \geqslant 1$. Then we can construct smaller minimal blocking sets of $\mathrm{Q}(2n,q)$, $n \geqslant 4$, now using ovoids of $\mathrm{Q}(6,q)$. It is clear that the truncated cone $\pi^*_{n-4}\mathcal{O}$, π_{n-4} an $(n-4)$-dimensional subspace contained in $\mathrm{Q}(2n,q)$ and $\mathcal{O}$ an ovoid of $\mathrm{Q}(6,q)$, the base of the cone $\pi^{\perp}_{n-4} \cap \mathrm{Q}(2n,q)$ (with $\perp$ the polarity of $\mathrm{Q}(2n,q)$), is a minimal blocking set of $\mathrm{Q}(2n,q)$ of size $q^n + q^{n-3}$.

In this chapter, we characterise the smallest minimal blocking sets of $\mathrm{Q}(2n,q)$, $n \geqslant 4$, q an odd prime. To obtain the classification, it will become clear that results concerning the smallest minimal blocking sets of $\mathrm{Q}(6,q)$ (different from an ovoid) are crucial. Since the results of the previous chapter are restricted to q an odd prime, we cannot omit this restriction in this chapter. Furthermore, when $q > 3$, prime, we can use the fact that all ovoids of $\mathrm{Q}(4,q)$ are elliptic quadrics and that $\mathrm{Q}(6,q)$ has no ovoids.

In Section 5.2, we will prove the following theorem.

Theorem 5.1.2. *The smallest minimal blocking sets of* $\mathrm{Q}(2n,q)$*,* $q > 3$ *prime,* $n \geqslant 4$*, are truncated cones* $\pi^*_{n-3}\mathrm{Q}^-(3,q)$*,* $\pi_{n-3} \subset \mathrm{Q}(2n,q)$*,* $\mathrm{Q}^-(3,q) \subset \pi^{\perp}_{n-3} \cap \mathrm{Q}(2n,q)$*, and have size* $q^n + q^{n-2}$*.*

In Section 5.5, we will prove the following theorem

Theorem 5.1.3. *The smallest minimal blocking sets of* $\mathrm{Q}(2n,q=3)$*,* $n \geqslant 4$ *are truncated cones* $\pi^*_{n-4}\mathcal{O}$*,* $\pi_{n-4} \subset \mathrm{Q}(2n,q)$*,* $\mathcal{O}$ *an ovoid of* $\mathrm{Q}(6,q) \subset \pi^{\perp}_{n-4} \cap \mathrm{Q}(2n,q)$*, and have size* $q^n + q^{n-3}$*.*

We will use an induction hypothesis to prove the two theorems. To end this section, we give a generalisation of Lemma 3.2.1 and Lemma 3.2.3. These lemmas will be used in both Sections 5.2 and 5.5.

Lemma 5.1.4. *Suppose that* $\mathcal{K}$ *is a minimal blocking set of* $\mathrm{Q}(2n,q)$*, different from an ovoid, with* $|\mathcal{K}| = q^n + \delta$*,* $1 < \delta \leqslant q^{n-2}$*. Suppose that* $p \in \mathcal{K}$*, then* $|p^{\perp} \cap \mathcal{K}| \leqslant \delta$*.*

Proof. By the minimality of $\mathcal{K}$, there exists a generator π_{n-1} of $\mathrm{Q}(2n, q)$ such that $\pi_{n-1} \cap \mathcal{K} = \{p\}$. There are q^{n-1} hyperplanes in π_{n-1} not passing through the point p, and there are q generators different from π_{n-1} on each hyperplane of π_{n-1}, which must contain at least one point of $\mathcal{K}$. Hence, at least $q^{n-1} \cdot q$ points of $\mathcal{K}$ lie outside $p^\perp$, and so $|p^\perp \cap \mathcal{K}| \leqslant \delta$. ■

Lemma 5.1.5. *Suppose that $\mathcal{K}$ is a minimal blocking set of $\mathrm{Q}(2n, q)$, different from an ovoid, with $|\mathcal{K}| = q^n + \delta$, $1 < \delta \leqslant q^{n-2}$. If p is a point of $\mathrm{Q}(2n, q) \setminus \mathcal{K}$, $n \geqslant 4$, then the points of $p^\perp \cap \mathcal{K}$ are projected from p onto $\mathcal{K}_p$, a minimal blocking set of $Q = \mathrm{Q}(2n-2, q)$, the base of the cone $p^\perp \cap \mathrm{Q}(2n, q)$.*

Proof. Choose Q as fixed base of the cone $p^\perp \cap \mathrm{Q}(2n, q)$. Denote by $\mathcal{K}_p$ the projection of the set $p^\perp \cap \mathcal{K}$ from p. Suppose now that $\mathcal{K}_p$ is not minimal, then there exists a point $p' \in \mathcal{K}_p$ such that every generator π_{n-2} of Q through p' contains at least one other point of $\mathcal{K}_p$. There are $(q+1)(q^2+1)\ldots(q^{n-2}+1)$ generators of Q on p', and every point of $\mathcal{K}_p \setminus \{p'\}$ that lies in $p'^\perp \cap Q$ can block $(q+1)(q^2+1)\ldots(q^{n-3}+1)$ of these generators. So if $\mathcal{K}_p$ is not minimal, then at least $q^{n-2}+1$ points of $\mathcal{K}_p$ different from p' are needed to block all generators on p'. Hence, for some point $r \in \mathcal{K}$ on the line pp', $|r^\perp \cap \mathcal{K}| > q^{n-2}$, a contradiction with the previous lemma. We conclude $\mathcal{K}_p$ to be minimal. ■

5.2 Part 1: Q$(6, q)$ has no ovoids

For this section, we suppose that $q > 3$ is an odd prime. This implies that every ovoid of $\mathrm{Q}(4, q)$ is an elliptic quadric $\mathrm{Q}^-(3, q)$ and that $\mathrm{Q}(6, q)$ has no ovoids.

Suppose that $\mathcal{K}$ is a minimal blocking set of $\mathrm{Q}(2n, q)$, $|\mathcal{K}| = q^n + \delta$, $1 < \delta \leqslant q^{n-2}$, $n \geqslant 4$. As induction hypothesis, we suppose that Theorem 5.1.2 is proved for $\mathrm{Q}(2n-2, q)$. This hypothesis is satisfied for $n = 4$, i.e. the theorem is true for $\mathrm{Q}(6, q)$.

Before we start, we define $b_n := q^{n-2}(q^2+1) = |\pi^*_{n-3}\mathrm{Q}^-(3, q)|$, $n \geqslant 2$.

Lemma 5.2.1. *Let r be a point of $\mathrm{Q}(2n, q) \setminus \mathcal{K}$, then $|r^\perp \cap \mathcal{K}| \geqslant b_{n-1}$. If equality holds, then there exists an $(n+1)$-dimensional subspace $\overline{\alpha}_r$, such that $r \in \overline{\alpha}_r \subset r^\perp$ and $\overline{\alpha}_r \cap \mathrm{Q}(2n, q) = \pi_{n-3}\mathrm{Q}^-(3, q)$. In other words, the points of $\mathcal{K} \cap r^\perp$ are projected from r onto a truncated cone $\pi^*_{n-4}\mathrm{Q}^-(3, q) \subset \overline{\alpha}_r$, $r \notin \pi_{n-4}$. Moreover, a line L on r contained in $\mathrm{Q}(2n, q)$ meets $\mathcal{K}$ if and only if $L \subset \overline{\alpha}_r \setminus \pi_{n-3}$.*

Proof. Applying the induction hypothesis to the base of the cone $r^\perp \cap \mathrm{Q}(2n, q) = r\mathrm{Q}(2n-2, q)$, we find that $|r^\perp \cap \mathcal{K}| \geqslant b_{n-1}$. If equality holds, then necessarily r projects the points of $r^\perp \cap \mathcal{K}$ onto a truncated cone $\pi_{n-4}^* Q^-(3,q)$. The only lines of $Q(2n,q)$ on r that meet $\mathcal{K}$ are the lines of $Q(2n,q) \cap \overline{\alpha}_r$, passing through r, but not lying in π_{n-3}. The $(n+1)$-dimensional space $\overline{\alpha}_r$ is the space $\langle r, \pi_{n-4} Q^-(3,q)\rangle$. ■

The next lemma proves that equality occurs for some points $r \in Q(2n,q) \setminus \mathcal{K}$. If there is equality, the previous lemma assures that the projected structure is a truncated cone. It will be crucial to prove that not only the projection of $r^\perp \cap \mathcal{K}$ is a truncated cone, but also the set $r^\perp \cap \mathcal{K}$ itself is a truncated cone.

Lemma 5.2.2. *There exists a point $r \in Q(2n,q)\setminus\mathcal{K}$ such that $|r^\perp \cap \mathcal{K}| = b_{n-1}$.*

Proof. Count the elements of the set $S = \{(p,r) \| p \in \mathcal{K}, r \in Q(2n,q)\setminus\mathcal{K}, p \in r^\perp\}$. For every point $p \in \mathcal{K}$, at most $q\theta_{2n-3}$ points of $p^\perp \cap Q(2n,q)$ are points of $Q(2n,q) \setminus \mathcal{K}$. With $|\mathcal{K}| \leqslant b_n$, we find $b_n q \theta_{2n-3}$ as upper bound U for $|S|$. Suppose now that for every $r \in Q(2n,q)\setminus\mathcal{K}$, there are at least $b_{n-1}+1$ points $p \in \mathcal{K}$, satisfying $p \in r^\perp$. Then we find as lower bound for $|S|$, again using $|\mathcal{K}| \leqslant b_n$, the number $L = (b_{n-1}+1)(\theta_{2n-1} - b_n)$. We find for $n \geqslant 7$

$$\begin{aligned} U - L &= -q^{2n-2} + q^{2n-3} - q^{2n-4} - q^{2n-6} - q^{2n-7} - \ldots - q^{n+1} \\ &\quad -q^n - 2q^{n-1} - q^{n-2} - 2q^{n-3} - q^{n-4} - \ldots - q - 1. \end{aligned}$$

For $n = 4, 5$ and 6, we find $-q^6 + q^5 - q^4 - q^3 - q^2 - 2q - 1$, $-q^8 + q^7 - q^6 - 2q^4 - q^3 - 2q^2 - q - 1$ and $-q^{10} + q^9 - q^8 - q^6 - 2q^5 - q^4 - 2q^3 - q^2 - q - 1$ respectively. Since $U - L < 0$ if we suppose that $|r^\perp \cap \mathcal{K}| > b_{n-1}$ for all points $r \in Q(2n,q) \setminus \mathcal{K}$, we find that there must exist a point $r \in Q(2n,q) \setminus \mathcal{K}$ with $|r^\perp \cap \mathcal{K}| = b_{n-1}$. ■

Lemma 5.2.3. *Suppose that $L \subset Q(2n,q)$ is a line for which $L \cap \mathcal{K} = \emptyset$. If $|L^\perp \cap \mathcal{K}| = b_{n-2}$, then $|\mathcal{K}| = b_n$ and $|r^\perp \cap \mathcal{K}| = b_{n-1}$ for all points $r \in L$.*

Proof. By Lemma 5.2.1, $|r_i^\perp \cap \mathcal{K}| \geqslant b_{n-1}$ for all points $r_i \in L$. The sets $r_i^\perp \cap \mathcal{K}$ have exactly b_{n-2} points in common, which implies that $|\mathcal{K}| = |\bigcup_{i=0}^{q}(r_i^\perp \cap \mathcal{K})| \geqslant (q+1)(b_{n-1} - b_{n-2}) + b_{n-2} = b_n \geqslant |\mathcal{K}|$. Hence, $|r_i^\perp \cap \mathcal{K}| = b_{n-1}$ for all points $r_i \in L$ and $|\mathcal{K}| = b_n$. ■

At this point, we know that there exist points $r \in Q(2n,q) \setminus \mathcal{K}$ with $|r^\perp \cap \mathcal{K}| = b_{n-1}$. We will now prove that, for such points r, the set $r^\perp \cap \mathcal{K}$ is a truncated cone $\pi_{n-4}^* Q^-(3,q)$. The following lemma plays a crucial role.

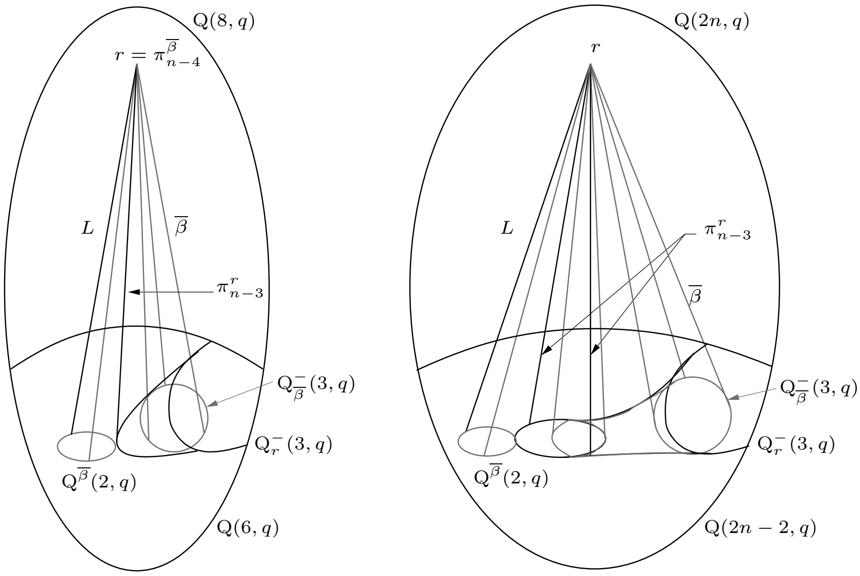

Figure 5.1: The situation in Lemma 5.2.4

Lemma 5.2.4. *Suppose that* $r \in \mathrm{Q}(2n, q) \setminus \mathcal{K}$ *such that* $|r^{\perp} \cap \mathcal{K}| = b_{n-1}$. *If* $\overline{\beta}$ *is a hyperplane of* $\overline{\alpha}_r$ *on* r *not containing the vertex* π_{n-3} *of the cone* $\overline{\alpha}_r \cap \mathrm{Q}(2n, q)$, *then the points of* $\overline{\beta} \cap \mathcal{K}$ *lie in an* $(n-1)$*-dimensional subspace* β *of* $\overline{\beta}$, $r \notin \beta$, *and* $\beta \cap \mathrm{Q}(2n, q) = \pi^{\beta}_{n-5}\mathrm{Q}^{-}(3, q)$.

Proof. Since $\overline{\beta}$ is a hyperplane of $\overline{\alpha}_r$ on r not containing the vertex π^r_{n-3} of the cone $\overline{\alpha}_r \cap \mathrm{Q}(2n, q) = \pi^r_{n-3}\mathrm{Q}^-_r(3, q)$, $\overline{\beta} \cap \mathrm{Q}(2n, q)$ is a cone with base $\mathrm{Q}^-_{\overline{\beta}}(3, q)$ and vertex $\pi^{\overline{\beta}}_{n-4}$, an $(n-4)$-dimensional subspace on r. When $n = 4$, this subspace is the point r itself. The properties of the polarity associated to $\mathrm{Q}(2n, q)$ imply that $\overline{\beta}^{\perp} \cap \mathrm{Q}(2n, q) = \pi^{\overline{\beta}}_{n-4}\mathrm{Q}^{\overline{\beta}}(2, q)$, and this cone meets the cone $\overline{\alpha}_r$ in the space π^r_{n-3}. Thus there must exist a line L of $\mathrm{Q}(2n, q)$ contained in $\overline{\beta}^{\perp}$ such that $L \cap \overline{\alpha}_r = \{r\}$, $L \not\subseteq \overline{\alpha}_r^{\perp}$. Since $L \subset \overline{\beta}^{\perp}$, we find $\overline{\beta} = L^{\perp} \cap \overline{\alpha}_r$. By Lemma 5.2.1, L does not meet $\mathcal{K}$.

Since $L^{\perp} \cap \mathcal{K} \subset r^{\perp} \cap \mathcal{K} \subseteq \overline{\alpha}_r$, it is clear that $L^{\perp} \cap \mathcal{K} = \overline{\beta} \cap \mathcal{K}$. Lemma 5.2.1 implies that $|L^{\perp} \cap \mathcal{K}| = b_{n-2}$. Suppose that p is a point of $L \setminus \{r\}$, then Lemma 5.2.3 implies that $|p^{\perp} \cap \mathcal{K}| = b_{n-1}$. By Lemma 5.2.1, there exists an $(n+1)$-dimensional subspace $\overline{\alpha}_p$ that meets $\mathrm{Q}(2n, q)$ in the cone $\pi^p_{n-3}\mathrm{Q}^-_p(3, q)$ and $p^{\perp} \cap \mathcal{K} \subset \overline{\alpha}_p$. Furthermore, $\overline{\alpha}_p$ contains b_{n-1} points of $\mathcal{K}$, while $L^{\perp}$ contains b_{n-2} points of $\mathcal{K}$, hence $L^{\perp}$ intersects $\overline{\alpha}_p$ in a hyperplane $\overline{\beta}'$ of $\overline{\alpha}_p$, with $p \in \overline{\beta}'$. We conclude that $L^{\perp} \cap \mathcal{K}$ is a subset of $\overline{\beta}$ and $\overline{\beta}'$. The spaces $\overline{\beta}$ and $\overline{\beta}'$ are different since $\overline{\beta}$ does not contain the line L and so $p \notin \overline{\beta}$.

Hence, $L^\perp \cap \mathcal{K}$ lies in the $(n-1)$-dimensional subspace $\overline{\beta} \cap \overline{\beta}'$; it cannot lie in a subspace of lower dimension by Lemma 5.2.1. It is impossible that $r \in \beta = \overline{\beta} \cap \overline{\beta}'$; or else r projects the points of $\beta \cap \mathcal{K}$ onto an $(n-2)$-dimensional subspace, but the projected points form a truncated cone $\pi^*_{n-5}\mathrm{Q}^-(3,q)$ which lies in a space of dimension $n-1$. The subspace $\beta = \overline{\beta} \cap \overline{\beta}'$ intersects $\mathrm{Q}(2n,q)$ in a cone $\pi^\beta_{n-5}\mathrm{Q}^-(3,q)$, since $\langle \beta, r\rangle = \overline{\beta} \subseteq r^\perp$ and $\overline{\beta}$ intersects $\mathrm{Q}(2n,q)$ in $\pi^{\overline{\beta}}_{n-4}\mathrm{Q}^-_{\overline{\beta}}(3,q)$. ■

Lemma 5.2.5. *Suppose that r is a point of $\mathrm{Q}(2n,q) \setminus \mathcal{K}$ such that $|r^\perp \cap \mathcal{K}| = b_{n-1}$. Then there exists an n-dimensional subspace α_r, $r \notin \alpha_r$, such that $\alpha_r \cap \mathrm{Q}(2n,q) = \pi_{n-4}\mathrm{Q}^-(3,q)$ and such that the truncated cone $\pi^*_{n-4}\mathrm{Q}^-(3,q)$ is equal to the set $r^\perp \cap \mathcal{K}$.*

Proof. Consider the $(n+1)$-dimensional space $\overline{\alpha}_r$ with $\overline{\alpha}_r \cap \mathrm{Q}(2n,q) = \pi_{n-3}\mathrm{Q}^-(3,q)$. Suppose that $\overline{\beta}_1$ is a hyperplane of $\overline{\alpha}_r$, not containing π_{n-3} and containing the point r. By Lemma 5.2.4, $\overline{\beta}_1$ contains an $(n-1)$-dimensional subspace β_1, $r \notin \beta_1$, such that $\beta_1 \cap \mathrm{Q}(2n,q) = \pi^{\beta_1}_{n-5}\mathrm{Q}^-_{\beta_1}(3,q)$ and $\overline{\beta}_1 \cap \mathcal{K} = \beta_1 \cap \mathcal{K} = \pi^{\beta_1 *}_{n-5}\mathrm{Q}^-_{\beta_1}(3,q)$. Choose a conic $\mathrm{Q}(2,q) \subset \mathrm{Q}^-_{\beta_1}(3,q)$. We can find a hyperplane $\overline{\beta}_2$ of $\overline{\alpha}_r$, $\overline{\beta}_2 \neq \overline{\beta}_1$, $r \in \overline{\beta}_2$, $\pi_{n-3} \nsubseteq \overline{\beta}_2$, $\beta_1 \nsubseteq \overline{\beta_2}$, but $\pi^{\beta_1}_{n-5}\mathrm{Q}(2,q) \subseteq \overline{\beta}_2$. Again, by Lemma 5.2.4, we find an $(n-1)$-dimensional subspace β_2, $r \notin \beta_2$, $\beta_2 \cap \mathrm{Q}(2n,q) = \pi^{\beta_2}_{n-5}\mathrm{Q}^-_{\beta_2}(3,q)$, and $\overline{\beta}_2 \cap \mathcal{K} = \beta_2 \cap \mathcal{K} = \pi^{\beta_2 *}_{n-5}\mathrm{Q}^-_{\beta_2}(3,q)$. Necessarily, $\pi^{\beta_1}_{n-5} = \pi^{\beta_2}_{n-5}$, and $\mathrm{Q}(2,q) \subset \mathrm{Q}^-_{\beta_2}(3,q) \neq \mathrm{Q}^-_{\beta_1}(3,q)$.

Define $\pi_1 := \langle \mathrm{Q}^-_{\beta_1}(3,q)\rangle$ and $\pi_2 := \langle \mathrm{Q}^-_{\beta_2}(3,q)\rangle$. Consider the n-dimensional space $\gamma = \langle \pi^{\beta_1}_{n-5}, \pi_1, \pi_2\rangle$. The two solids π_1 and π_2 are skew to π_{n-3}, hence, $\pi_{n-3} \nsubseteq \gamma$. Furthermore, $r \notin \gamma$, since then γ would be an n-dimensional subspace on r, not containing π_{n-3}, spanned by points of $r^\perp \cap \mathcal{K}$, a contradiction with Lemma 5.2.4. We conclude that $\gamma \cap \mathrm{Q}(2n,q) = \pi^\gamma_{n-4}\mathrm{Q}^-_\gamma(3,q)$.

Choose an arbitrary conic $\mathrm{Q}'(2,q) \subset \mathrm{Q}^-_{\beta_1}(3,q)$, $\mathrm{Q}'(2,q) \neq \mathrm{Q}(2,q)$, and $|\mathrm{Q}'(2,q) \cap \mathrm{Q}(2,q)| = 2$. Consider the $q+1$ $(n-1)$-dimensional subspaces δ_i of γ through the $(n-2)$-dimensional subspace $\langle \mathrm{Q}'(2,q), \pi^{\beta_1}_{n-5}\rangle$. One δ_i, say δ_1, is the space $\langle \pi^\gamma_{n-4}, \mathrm{Q}'(2,q)\rangle$. Consider now a space δ_i, $i \neq 1$. It is clear that δ_i is spanned by points of $\mathcal{K}$, since the quadrics $\mathrm{Q}^-_{\beta_j}(3,q) \subset \mathcal{K}$, $j = 1,2$, and δ_i intersects the spaces π_j in distinct conics of $\mathrm{Q}^-_{\beta_j}(3,q)$, $j = 1,2$, or contains π_1. If p is a point of $\mathrm{Q}(2n,q)$, $p \in \delta_i \setminus (\beta_1 \cup \beta_2 \cup \pi^\gamma_{n-4})$, such that $p \notin \mathcal{K}$, then by Lemma 5.2.1, the line $\langle r, p\rangle$ meets $\mathcal{K}$ in exactly one point t. But then the space $\langle t, \delta_i\rangle \subseteq \overline{\alpha}_r$ is an n-dimensional subspace on r, not containing π_{n-3} and spanned by points of $r^\perp \cap \mathcal{K}$, a contradiction with Lemma 5.2.4. We conclude that every point $p \in (\gamma \cap \mathrm{Q}(2n,q)) \setminus \langle \mathrm{Q}'(2,q), \pi^\gamma_{n-4}\rangle$ lies in $\mathcal{K}$.

Letting vary the conic $\mathrm{Q}'(2,q)$, we can reach every point $p \in (\mathrm{Q}(2n,q)) \cap \gamma) \setminus \pi^\gamma_{n-4}$, since the intersection of all conics of $\mathrm{Q}^-_{\beta_1}(3,q)$, sharing two points with $Q(2,q)$ is empty. Hence, $\gamma \cap \mathcal{K} = \pi^*_{n-4}Q^-_\gamma(3,q)$, the space γ is the space α_r. ■

Lemma 5.2.6. *The set $\mathcal{K}$ is a truncated cone $\pi^*_{n-3}Q^-(3,q)$.*

Proof. From Lemma 5.2.2, we find a point $r \in Q(2n,q) \setminus \mathcal{K}$ satisfying $|r^\perp \cap \mathcal{K}| = b_{n-1}$. The n-dimensional subspace α_r from Lemma 5.2.5 meets $Q(2n,q)$ in a cone $\pi^r_{n-4}Q^-_r(3,q)$. Choose $Q = Q(2n-2,q)$ as the base of the cone $r^\perp \cap Q(2n,q)$ in such a way that $\langle Q \rangle$ contains the cone $\pi^r_{n-4}Q^-_r(3,q)$. Let L be a line of $Q(2n,q)$ on r such that $L \not\subseteq \pi^{r\perp}_{n-4} \cap Q(2n,q)$, which implies that $L^\perp$ does not contain the vertex π^r_{n-4} of α_r. Thus $L^\perp$ meets α_r in a hyperplane of α_r, and this hyperplane meets $Q(2n,q)$ in a cone $\pi_{n-5}Q^-(3,q)$. Note that $n \geqslant 4$. If $n = 4$, then this hyperplane meets $Q(2n,q)$ in an elliptic quadric $Q^-(3,q)$.

As $L^\perp \cap \mathcal{K}$ is contained in $r^\perp \cap \mathcal{K} \subseteq \alpha_r \cap \mathcal{K}$, it follows that $L^\perp \cap \mathcal{K}$ is a truncated cone $\pi^{L*}_{n-5}Q^-_L(3,q)$. Hence, $|L^\perp \cap \mathcal{K}| = b_{n-2}$. By Lemma 5.2.3, $|t^\perp \cap \mathcal{K}| = b_{n-1}$ for all points $t \in L$. Every point $t \in L$ gives rise to a truncated cone $t^\perp \cap \mathcal{K} = \pi^{t*}_{n-4}Q^-_t(3,q)$, and all these truncated cones share the truncated cone $L^\perp \cap \mathcal{K} = \pi^{L*}_{n-5}Q^-_L(3,q)$. Denote the subspace spanned by $L^\perp \cap \mathcal{K}$ by β_L.

Every point of $\mathcal{K}$ is collinear with a point of L, which implies that $\mathcal{K}$ is the union of these $q+1$ cones. It follows that $|\mathcal{K}| = b_n$, and that $\mathcal{K}$ is contained in the union of the $q+1$ n-dimensional subspaces α_s, $s \in L$, that share the $(n-1)$-dimensional subspace β_L.

Consider now a second line L' of $Q(2n,q)$ on r such that $L' \not\subseteq \pi^{r\perp}_{n-4} \cap Q(2n,q)$ and choose it in such a way that $\beta_L \not\subseteq L'^\perp$. This is possible since $\langle \beta_L, r\rangle^\perp = \langle r, \pi^L_{n-5}Q(2,q)\rangle$ has only dimension $n-1$. Then, as for L, the subspace $\beta_{L'} := \langle L'^\perp \cap \mathcal{K}\rangle$ has dimension $n-1$, and is contained in α_t for all $t \in L'$. We have $\beta_{L'} \neq \beta_L$. Let p be a point of L' with $p \neq r$. Then α_p has dimension n and meets α_r in $\beta_{L'}$. Furthermore, $\beta_{L'} \cap \mathcal{K} = \pi^{L'*}_{n-5}Q^-_{L'}(3,q)$ and $|Q^-_{L'}(3,q) \cap Q^-_L(3,q)| \geqslant 1$.

Varying the point $t \in L'$, the tangent hyperplanes $t^\perp$ vary over the hyperplanes through $L'^\perp$, hence, every point of the $(n-4)$-dimensional spaces π^s_{n-4}, $s \in L$, lies in some $t^\perp$, $t \in L'$. Every point of π^s_{n-4}, $s \in L$, lies on lines with q points of $\mathcal{K}$, to the points of $Q^-_L(3,q) \cap Q^-_{L'}(3,q)$, and hence belongs to one of the vertices π^t_{n-4}, $t \in L'$.

Consider a fixed point $s \in L \setminus \{r\}$, fixed points $p_1 \in \pi^r_{n-4}$, $p_2 \in \pi^s_{n-4}$, $p_1, p_2 \notin \pi^r_{n-4} \cap \pi^s_{n-4} = \pi^L_{n-5}$. Consider a fixed point $u \in \pi^{r*}_{n-4}Q^-_r(3,q)$, then it is possible to select a line L'', satisfying the conditions of L', for which $u \in L''^\perp$. Then the preceding arguments show that the set $\langle u, p_2\rangle \setminus \{p_2\}$ is contained in $\mathcal{K}$.

Consider an arbitrary line M of $\pi_{n-4}^{r*}\mathrm{Q}_r^-(3,q)$ passing through p_1 and containing q points of $\mathcal{K}$. The q^2 points of $\langle M, p_2\rangle \setminus \langle p_1, p_2\rangle$ all lie in $\mathcal{K}$; this implies that the truncated cone $\langle \pi_{n-4}^r, \pi_{n-4}^s\rangle^*\mathrm{Q}_r^-(3,q)$ lies in $\mathcal{K}$. Since $|\mathcal{K}| = |\langle \pi_{n-4}^r, \pi_{n-4}^s\rangle^*\mathrm{Q}_r^-(3,q)| = b_n$, this truncated cone must be equal to $\mathcal{K}$. ■

We may conclude Theorem 5.1.2.

Theorem 5.2.7. *The smallest minimal blocking sets of* $\mathrm{Q}(2n,q)$, $q > 3$ *prime,* $n \geqslant 4$, *are truncated cones* $\pi_{n-3}^*\mathrm{Q}^-(3,q)$, $\pi_{n-3} \subset \mathrm{Q}(2n,q)$, $\mathrm{Q}^-(3,q) \subset \pi_{n-3}^\perp \cap \mathrm{Q}(2n,q)$, *and have size* $q^n + q^{n-2}$.

5.3 Interlude 1: $\mathrm{Q}(8,q)$, $q = 3$

In this section, we suppose that $\mathrm{Q}(6,q)$ has ovoids. This is true for all $q = 3^r$, $r \geqslant 1$. For some values of q, different non-isomorphic classes of ovoids are known ([65], [91] and [92]). Suppose now that $\mathcal{O}$ is an ovoid of $\mathrm{Q}(6,q)$. A short observation learns that $\langle\mathcal{O}\rangle \cap \mathrm{Q}(6,q) = \mathrm{Q}(6,q)$. For, let $\Omega = \langle\mathcal{O}\rangle$. It is impossible that $\Omega \cap \mathrm{Q}(6,q)$ is any singular quadric. For, assume that $\langle\mathcal{O}\rangle \cap \mathrm{Q}(6,q) = \pi_s Q$, a cone with vertex π_s, an s-dimensional subspace, $s \geqslant 0$, and base Q, a non-singular quadric of dimension at most 4. Then π_s projects $\mathcal{O}$ onto an ovoid of Q. However, no non-singular quadric of dimension at most four has ovoids of size $q^3 + 1$. If $\langle\mathcal{O}\rangle \cap \mathrm{Q}(6,q) = \mathrm{Q}(4,q)$, then $\mathcal{O}$ must necessarily be an ovoid of $\mathrm{Q}(4,q)$; impossible since $|\mathcal{O}| > q^2 + 1$. If $\langle\mathcal{O}\rangle \cap \mathrm{Q}(6,q) = \mathrm{Q}^+(5,q)$, then $\mathcal{O}$ must be an ovoid of $\mathrm{Q}^+(5,q)$; impossible since $|\mathcal{O}| > q^2 + 1$. Finally, $\langle\mathcal{O}\rangle \cap \mathrm{Q}(6,q) = \mathrm{Q}^-(5,q)$ is impossible, since $\mathrm{Q}^-(5,q)$ has no ovoids.

In this section, we will prove Theorem 5.1.3, without the restriction $q = 3$, for $n = 4$, and supposing that $\mathrm{Q}(6,q)$ has an ovoid and that the smallest minimal blocking set of $\mathrm{Q}(6,q)$ different from an ovoid, is a truncated cone $p^*\mathcal{O}$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(4,q)$. These results are valid for $q = 3$ (Theorem 4.2.7). Suppose for this section that $\mathcal{K}$ is a minimal blocking set of $\mathrm{Q}(8,q)$, q odd, $|\mathcal{K}| = q^4 + \delta$, $\delta \leqslant q$. The way the result is proved, looks very similar to the proofs of the results of Chapter 3 and Chapter 4.

Lemma 5.3.1. *If L is a line of* $\mathrm{Q}(8,q)$, *then* $|L \cap \mathcal{K}| = 0, 1$ *or* $|L \cap \mathcal{K}| = q$.

Proof. Suppose that $q - 1 \geqslant |L \cap \mathcal{K}| \geqslant 2$. Consider a generator π on L such that $L \cap \mathcal{K} = \pi \cap \mathcal{K}$. By Lemma 5.1.4, such a generator exists. Count the pairs (u, v), $u \in \pi \setminus L$ and $v \in \mathcal{K} \setminus \pi$, $u \in v^\perp$. Since $u \in \pi \setminus L$ and $|L \cap \mathcal{K}| \geqslant 2$, u cannot project $u^\perp \cap \mathcal{K}$ on an ovoid of $\mathrm{Q}(6,q)$, so $|u^\perp \cap \mathcal{K}| \geqslant q^3 + q \geqslant q^3 + 1 +$

$|L \cap \mathcal{K}|$ since the projection is a minimal blocking set of Q$(6, q)$ (Lemma 5.1.5). We find a lower bound of $(q^3 + q^2)(q^3 + q - |L \cap \mathcal{K}|) \geqslant (q^3 + q^2)(q^3 + 2)$. The first factor comes from the number of points in $\pi \setminus L$. If $v \in \mathcal{K} \setminus \pi$, then $v^\perp$ intersects π in a plane, hence to v correspond $q^2 + q$ or q^2 points of $\pi \setminus L$. So we find $(q^4 + \delta - |L \cap \mathcal{K}|)(q^2 + q)$ as upper bound for the number of pairs (u, v), and since $2 \leqslant |L \cap \mathcal{K}|$, we can change this upper bound to $(q^4 + \delta - 2)(q^2 + q)$. So necessarily $(q^4 + \delta - 2)(q^2 + q) \geqslant (q^3 + q^2)(q^3 + 2)$ or, since $\delta \leqslant q$, $(q^4 + q - 2)(q^2 + q) \geqslant (q^3 + q^2)(q^3 + 2)$, a contradiction. ■

Corollary 5.3.2. *If π is a generator of* Q$(8, q)$*, then $|\pi \cap \mathcal{K}| = 1$ or $|\pi \cap \mathcal{K}| = q$, and all points of $\pi \cap \mathcal{K}$ lie on a line.*

Proof. If $|\pi \cap \mathcal{K}| \geqslant 2$, then any line L spanned by two points of $\pi \cap \mathcal{K}$ contains already q points of $\mathcal{K}$. Lemma 5.1.4 admits no further points of $\mathcal{K}$ in π. ■

Lemma 5.3.3. *Suppose that $p \notin \mathcal{K}$. If there is a generator π on p containing exactly* 1 *point of $\mathcal{K}$, then $|p^\perp \cap \mathcal{K}| \leqslant q^3 + q$, else $|p^\perp \cap \mathcal{K}| = q(q^3 + 1)$.*

Proof. Suppose that $p \in \pi$, $\pi \cap \mathcal{K} = \{s\}$. There are $q^3 - q^2$ planes in π not through s or p. All generators of Q$(8, q)$, different from π on these planes only share points with $\mathcal{K} \setminus \pi$. For, a point r of $\mathcal{K} \setminus \pi$ has a tangent hyperplane not containing π; so π intersects this tangent hyperplane in a plane Ω; this plane Ω and r define a unique generator. Hence $q(q^3 - q^2)$ points of $\mathcal{K}$ are needed to block them; so at most $q^4 + q - (q^4 - q^3) = q^3 + q$ points of $\mathcal{K}$ lie in $p^\perp \cap \mathcal{K}$. If every generator on p contains q points of $\mathcal{K}$, then $|p^\perp \cap \mathcal{K}| = q^4 + q$. ■

Lemma 5.3.4. *Suppose that π is a generator of* Q$(8, q)$*, $|\pi \cap \mathcal{K}| = q$, $\langle \pi \cap \mathcal{K} \rangle = L$. If $p \in \pi \setminus L$, then $|p^\perp \cap \mathcal{K}| = q^3 + q$.*

Proof. If all generators on p contain q points of $\mathcal{K}$, then in particular also all generators on $\langle s, p \rangle$, $s \in \pi \cap \mathcal{K}$, hence $|s^\perp \cap \mathcal{K}| > q$, contradicting Lemma 5.1.4. So, $|p^\perp \cap \mathcal{K}| \leqslant q^3 + q$. Since p must project the points of $p^\perp \cap \mathcal{K}$ on $\mathcal{K}_p$, a minimal blocking set of Q$(6, q)$ different from an ovoid (Lemma 5.1.5), $|\mathcal{K}_p| \geqslant q^3 + q$. Hence $|p^\perp \cap \mathcal{K}| = q^3 + q$. ■

We end with the following theorem.

Theorem 5.3.5. *If* Q$(6, q)$*, q odd, has an ovoid and the smallest minimal blocking set of* Q$(6, q)$*, different from an ovoid, is a truncated cone $p^*\mathcal{O}$, with $\mathcal{O}$ an ovoid of* Q$(4, q)$*, then the smallest minimal blocking set of* Q$(8, q)$*, q odd, is a truncated cone $p^*\mathcal{O}'$, with $\mathcal{O}'$ an ovoid of* Q$(6, q)$*.*

Proof. Consider a generator π of Q$(8, q)$ such that $|\pi \cap \mathcal{K}| = q$, $\langle \pi \cap \mathcal{K} \rangle = L$. Consider $s \in \pi \setminus L$. By the previous lemma, $|s^\perp \cap \mathcal{K}| = q^3 + q$, and s projects the points of $T_s(\mathrm{Q}(8, q)) \cap \mathcal{K}$ onto a minimal blocking set $\mathcal{K}_s$ of Q$(6, q)$, being a truncated cone $p^*\mathcal{O}$, $\mathcal{O}$ an ovoid of Q$(4, q)$. The $q^2 + 1$ lines containing q points of $\mathcal{K}_s$ are projections of $q^2 + 1$ lines M_i of Q$(8, q)$, $|M_i \cap \mathcal{K}| = q$. Suppose that $M_i \cap \langle s, p \rangle = p'_i$. Suppose that $|p'^\perp_i \cap \mathcal{K}| \leqslant q^3 + q$. For some i, for instance $i = 1$, $p' = p'_i$ lies on a line M_i. The point p' projects $p'^\perp \cap \mathcal{K}$ necessarily on a minimal blocking set of Q$(6, q)$ which is an ovoid, hence $|p'^\perp \cap \mathcal{K}| = q^3 + q$. Consider now all generators on the line M_1. There are $(q^2 + 1)(q + 1)$ such generators and they are all blocked by the points of $M_1 \cap \mathcal{K}$. Since there are $q^3(q^2 + 1)(q + 1)$ generators left in $p'^\perp \cap \mathrm{Q}(8, q)$, and every point of $p'^\perp \cap \mathcal{K}$ blocks $(q^2 + 1)(q + 1)$ of them, every generator on p' not on M_1 contains exactly 1 point of $\mathcal{K}$. This is a contradiction since every generator on a plane $\langle p', M_j \rangle$ is a generator on p' containing q points of $\mathcal{K}$. We conclude that $|p'^\perp \cap \mathcal{K}| = q(q^3 + 1) = q^4 + q$, so $|\mathcal{K}| = q^4 + q$, every generator through p' contains q collinear points of $\mathcal{K}$, and these q collinear points of $\mathcal{K}$ in a generator through p' lie on a line through p' (Lemma 5.3.4 Furthermore, p' projects all points of $\mathcal{K}$ on an ovoid of Q$(6, q)$. ■

Using the results on Q$(6, 3)$ from Chapter 4, we obtain the following theorem.

Theorem 5.3.6. *The smallest minimal blocking sets of* Q$(8, q = 3)$ *are truncated cones $p^*\mathcal{O}$, $p \in \mathrm{Q}(8, q)$, $\mathcal{O}$ an ovoid of* Q$(6, q) \subset p^\perp \cap \mathrm{Q}(8, q)$, *and have size $q^4 + q$.*

5.4 Interlude 2: geometric behaviour of ovoids of Q$(6, q)$

In this section we explore some geometric properties of ovoids of Q$(6, q)$.

The first lemma is an observation we made in the previous section.

Lemma 5.4.1. *The points of $\mathcal{O}$ span the 6-dimensional projective space* PG$(6, q)$.

The second lemma is an observation which was implicitly made in [76].

Lemma 5.4.2. *The ovoid $\mathcal{O}$ does not contain an elliptic quadric* Q$^-(3, q)$.

Proof. Suppose the contrary, i.e., some Q$^-(3, q) \subseteq \mathcal{O}$. Since $\mathcal{O}$ spans the 6-dimensional space, there is a point $p \in \mathcal{O} \setminus \mathrm{Q}^-(3, q)$. The space $\langle p, \mathrm{Q}^-(3, q) \rangle$ intersects Q$(6, q)$ in a parabolic quadric Q$(4, q)$, containing at least $q^2 + 2$

points of $\mathcal{O}$, a contradiction, since any $\mathrm{Q}(4,q)$ can intersect $\mathcal{O}$ in at most q^2+1 points, the number of points of an ovoid of $\mathrm{Q}(4,q)$. ■

In Chapter 4, we already mentioned the important result of [6]. The main result of [4] is the following theorem, which is used in [6] to prove Theorem 4.1.2.

Theorem 5.4.3. (Ball [4]) *Let $\mathcal{O}$ be an ovoid of $\mathrm{Q}(4,q)$, $q=p^h$, p prime. Every elliptic quadric $\mathrm{Q}^-(3,q)$ on $\mathrm{Q}(4,q)$ intersects $\mathcal{O}$ in $1 \mod p$ points.*

This theorem leads, also in [6], to the following interesting property of ovoids of $\mathrm{Q}(6,q)$.

Theorem 5.4.4. (Ball et al. [6]) *An ovoid $\mathcal{O}$ of $\mathrm{Q}(6,q)$, $q=p^h$, p prime, intersects every elliptic quadric $\mathrm{Q}^-(5,q)$ on $\mathrm{Q}(6,q)$ in $1 \mod p$ points.*

We use Theorem 5.4.3 to prove the following lemma. It shows also that "$t \mod p$" results have important applications.

Lemma 5.4.5. *The ovoid $\mathcal{O}$ does not contain any ovoid $\mathcal{O}'$ of $\mathrm{Q}(4,q) \subseteq \mathrm{Q}(6,q)$.*

Proof. Suppose the contrary, i.e., suppose that there is some ovoid $\mathcal{O}'$ of $\mathrm{Q}(4,q) \subseteq \mathrm{Q}(6,q)$, with $\mathcal{O}' \subseteq \mathcal{O}$. By the previous lemma, we may suppose that $\mathcal{O}'$ is not an elliptic quadric and hence, $\langle\mathcal{O}'\rangle$ is a 4-dimensional projective space α, such that $\alpha \cap \mathrm{Q}(6,q) = \mathrm{Q}(4,q)$. Since $\mathcal{O}$ spans the 6-dimensional space, we can choose a point $p \in \mathcal{O} \setminus \alpha$. Since α contains an ovoid of $\mathrm{Q}(4,q)$, $p \notin \alpha^\perp$, hence $p^\perp \cap \mathrm{Q}(4,q) = \mathrm{Q}^\pm(3,q)$, or $p^\perp \cap \mathrm{Q}(4,q) = r\mathrm{Q}(2,q)$ which is a tangent cone to $\mathrm{Q}(4,q)$. All these 3-dimensional quadrics intersect $\mathcal{O}'$ in $1 \mod p$ points, hence, at least one point $r \in \mathcal{O}'$ belongs to $p^\perp$, a contradiction. ■

We call a hyperplane α of $\mathrm{PG}(6,q)$ hyperbolic, elliptic respectively, if $\alpha \cap \mathrm{Q}(6,q) = \mathrm{Q}^+(5,q)$, $\alpha \cap \mathrm{Q}(6,q) = \mathrm{Q}^-(5,q)$ respectively.

Corollary 5.4.6. *Any hyperbolic hyperplane α has the property that $\langle \alpha \cap \mathcal{O}\rangle = \alpha$.*

Proof. Suppose that α is a 5-dimensional subspace such that $\alpha \cap \mathrm{Q}(6,q) = \mathrm{Q}^+(5,q)$. Then necessarily α intersects $\mathcal{O}$ in an ovoid $\mathcal{O}'$ of $\mathrm{Q}^+(5,q)$. Since any ovoid of $\mathrm{Q}(4,q)$ is not contained in $\mathcal{O}$, the ovoid $\mathcal{O}'$ spans the 5-dimensional space α. ■

With the aid of the computer, we also found the following result for $q=3$.

Lemma 5.4.7. *Any elliptic hyperplane α of $\mathrm{Q}(6,3)$ has the property that $\langle \alpha \cap \mathcal{O}\rangle = \alpha$.*

5.5 Part 2: $\mathrm{Q}(6,q)$ has ovoids

For this section, we suppose that $\mathrm{Q}(6,q)$ has ovoids and that the smallest minimal blocking sets of $\mathrm{Q}(6,q)$ different from ovoids are truncated cones $p^*\mathcal{O}$, $p \in \mathrm{Q}(6,q)$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(4,q) \subseteq p^\perp\mathrm{Q}(6,q)$. This hypothesis is true for $q=3$ (Theorem 4.2.7). We will prove in this section the following theorem.

Theorem 5.5.1. *Suppose that $\mathrm{Q}(6,q)$ has ovoids and that the smallest minimal blocking sets of $\mathrm{Q}(6,q)$, different from ovoids, are truncated cones $p^*\mathcal{O}$, $p \in \mathrm{Q}(6,q)$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(4,q)$. Then the smallest minimal blocking sets of $\mathrm{Q}(2n,q)$, $n \geqslant 5$, are truncated cones $\pi^*_{n-4}\mathcal{O}$, $\pi_{n-4} \subset \mathrm{Q}(2n,q)$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(6,q) \subset \pi^\perp_{n-4} \cap \mathrm{Q}(2n,q)$, and have size $q^n + q^{n-3}$.*

Suppose that $\mathcal{K}$ is a minimal blocking set of $\mathrm{Q}(2n,q)$, $|\mathcal{K}| = q^n + \delta$, $1 < \delta \leqslant q^{n-3}$, $n \geqslant 5$. As induction hypothesis, we suppose that Theorem 5.5.1 is proved for $\mathrm{Q}(2n-2,q)$. This hypothesis is satisfied for $n=5$, i.e. the theorem is true for $\mathrm{Q}(8,q)$ (Theorem 5.3.5).

Before we start, we define $b_n := q^{n-3}(q^3+1) = |\pi^*_{n-4}\mathcal{O}|$, $n \geqslant 4$.

It will become clear that we can repeat the proof chain from Section 5.2.

Lemma 5.5.2. *Let r be a point of $\mathrm{Q}(2n,q) \setminus \mathcal{K}$, then $|r^\perp \cap \mathcal{K}| \geqslant b_{n-1}$. If equality holds, then there exists an $(n+3)$-dimensional subspace $\overline{\alpha}_r$, such that $r \in \overline{\alpha}_r \subset r^\perp$ and $\overline{\alpha}_r \cap \mathrm{Q}(2n,q) = \pi_{n-4}\mathrm{Q}(6,q)$. Moreover, every line L on r contained in $\mathrm{Q}(2n,q)$ meets $\mathcal{K}$ if and only if L meets a truncated cone $\pi^*_{n-5}\mathcal{O}$ contained in $\overline{\alpha}_r \cap \mathrm{Q}(2n,q)$, with $\mathcal{O}$ an ovoid of a quadric $\mathrm{Q}(6,q)$, and such that the points of $\mathcal{K} \cap r^\perp$ are projected from r onto a truncated cone $\pi^*_{n-5}\mathcal{O} \subset \overline{\alpha}_r$, $r \notin \pi_{n-5}$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(6,q)$.*

Proof. Applying the induction hypothesis to the base of the cone $r^\perp \cap \mathrm{Q}(2n,q) = r\mathrm{Q}(2n-2,q)$, we find that $|r^\perp \cap \mathcal{K}| \geqslant b_{n-1}$. If equality holds, then necessarily r projects the points of $r^\perp \cap \mathcal{K}$ onto a truncated cone $\pi^*_{n-5}\mathcal{O}$, hence only lines $L \subset \mathrm{Q}(2n,q)$ on r meeting $\pi^*_{n-5}\mathcal{O}$ meet $\mathcal{K}$. The $(n+3)$-dimensional space $\overline{\alpha}_r$ is the space $\langle r, \pi_{n-5}\mathcal{O}\rangle$, since $\mathcal{O}$ spans a 6-dimensional space. ∎

As in Section 5.2, also here we prove that equality occurs for some points $r \in \mathrm{Q}(2n,q) \setminus \mathcal{K}$. If there is equality, the previous lemma assures us that the projected structure is a truncated cone we look for. It will again be crucial to prove that not only the projection of $r^\perp \cap \mathcal{K}$ is a truncated cone, but also that the set $r^\perp \cap \mathcal{K}$ is a truncated cone.

Lemma 5.5.3. *There exists a point $r \in \mathrm{Q}(2n,q)\setminus\mathcal{K}$ such that $|r^\perp \cap \mathcal{K}| = b_{n-1}$.*

Proof. Count the number of elements of the set $S = \{(p, r) \| p \in \mathcal{K}, r \in \mathrm{Q}(2n, q) \setminus \mathcal{K}, p \in r^\perp\}$. For every point $p \in \mathcal{K}$, at most $q\theta_{2n-3}$ points of $p^\perp \cap \mathrm{Q}(2n, q)$ are points of $\mathrm{Q}(2n, q) \setminus \mathcal{K}$. With $|\mathcal{K}| \leqslant b_n$, we find $b_n q\theta_{2n-3}$ as upper bound U for $|S|$. Suppose now that for every $r \in \mathrm{Q}(2n, q) \setminus \mathcal{K}$, there are at least $b_{n-1} + 1$ points $p \in \mathcal{K}$, $p \in r^\perp$. Then we find as lower bound for $|S|$, again using $|\mathcal{K}| \leqslant b_n$, the number $L = (b_{n-1} + 1)(\theta_{2n-1} - b_n)$. We find for $n \geqslant 8$,

$$\begin{aligned} U - L &= -q^{2n-2} - q^{2n-3} + q^{2n-4} - q^{2n-5} - q^{2n-6} - q^{2n-8} - \ldots - q^n \\ &\quad -2q^{n-1} - q^{n-2} - q^{n-3} - 2q^{n-4} - q^{n-5} - \ldots - q - 1 \end{aligned}$$

For $n = 5, 6$ and 7, we find $-q^8 - q^7 + q^6 - q^5 - 2q^4 - q^2 - 2q - 1$, $-q^{10} - q^9 + q^8 - q^7 - q^6 - q^5 - q^4 - q^3 - 2q^2 - q - 1$ and $-q^{12} - q^{11} + q^{10} - q^9 - q^8 - 2q^6 - q^5 - q^4 - 2q^3 - q^2 - q - 1$ respectively. Since this number $U - L < 0$ if we suppose that $|r^\perp \cap \mathcal{K}| > b_{n-1}$ for all points $r \in \mathrm{Q}(2n, q) \setminus \mathcal{K}$, we find that there must exist a point $r \in \mathrm{Q}(2n, q) \setminus \mathcal{K}$ with $|r^\perp \cap \mathcal{K}| = b_{n-1}$. ■

Lemma 5.5.4. *Suppose that $L \subset \mathrm{Q}(2n, q)$ is a line, $L \cap \mathcal{K} = \emptyset$. If $|L^\perp \cap \mathcal{K}| = b_{n-2}$, then $|\mathcal{K}| = b_n$ and $|r^\perp \cap \mathcal{K}| = b_{n-1}$ for all points $r \in L$.*

Proof. By Lemma 5.5.2, $|r_i^\perp \cap \mathcal{K}| \geqslant b_{n-1}$ for all points $r_i \in L$. The sets $r_i^\perp \cap \mathcal{K}$ have exactly b_{n-2} points in common, which implies that $|\mathcal{K}| = |\bigcup_{i=0}^{q}(r_i^\perp \cap \mathcal{K})| \geqslant (q+1)(b_{n-1} - b_{n-2}) + b_{n-2} = b_n \geqslant |\mathcal{K}|$. Hence, $|r_i^\perp \cap \mathcal{K}| = b_{n-1}$ for all points $r_i \in L$ and $|\mathcal{K}| = b_n$. ■

Lemma 5.5.5. *Suppose that $r \in \mathrm{Q}(2n, q) \setminus \mathcal{K}$ such that $|r^\perp \cap \mathcal{K}| = b_{n-1}$. If $\overline{\beta}$ is a hyperplane of $\overline{\alpha}_r$ on r not containing the vertex π_{n-4} of the cone $\overline{\alpha}_r \cap \mathrm{Q}(2n, q)$, then the points of $\overline{\beta} \cap \mathcal{K}$ lie in an $(n+1)$-dimensional subspace β of $\overline{\beta}$, $r \notin \beta$.*

Proof. Since $\overline{\beta}$ is a hyperplane of $\overline{\alpha}_r$ on r not containing the vertex π^r_{n-4} of the cone $\overline{\alpha}_r \cap \mathrm{Q}(2n, q) = \pi^r_{n-4}\mathrm{Q}(6, q)$, $\overline{\beta} \cap \mathrm{Q}(2n, q)$ is a cone with base $\mathrm{Q}^{\overline{\beta}}(6, q)$ and vertex $\pi^{\overline{\beta}}_{n-5}$, an $(n-5)$-dimensional subspace on r. When $n = 5$, this subspace is the point r itself. The properties of the polarity associated to $\mathrm{Q}(2n, q)$ imply that $\overline{\beta}^\perp \cap \mathrm{Q}(2n, q) = \pi^{\overline{\beta}}_{n-5}\mathrm{Q}^+_{\overline{\beta}}(1, q)$, and this cone meets the cone $\overline{\alpha}_r$ in the space π^r_{n-4}. Thus there must exist a line L of $\mathrm{Q}(2n, q)$ contained in $\overline{\beta}^\perp$ such that $L \cap \overline{\alpha}_r = \{r\}$. Since $L \subset \overline{\beta}^\perp$, we find $\overline{\beta} = L^\perp \cap \overline{\alpha}_r$. By Lemma 5.5.2, L does not meet $\mathcal{K}$.

Since $L^\perp \cap \mathcal{K} \subseteq r^\perp \cap \mathcal{K} \subseteq \overline{\alpha}_r$, it is clear that $L^\perp \cap \mathcal{K} = \overline{\beta} \cap \mathcal{K}$. Lemma 5.5.2 implies that $|L^\perp \cap \mathcal{K}| = b_{n-2}$. Suppose that p is a point of $L \setminus \{r\}$. Lemma 5.5.4

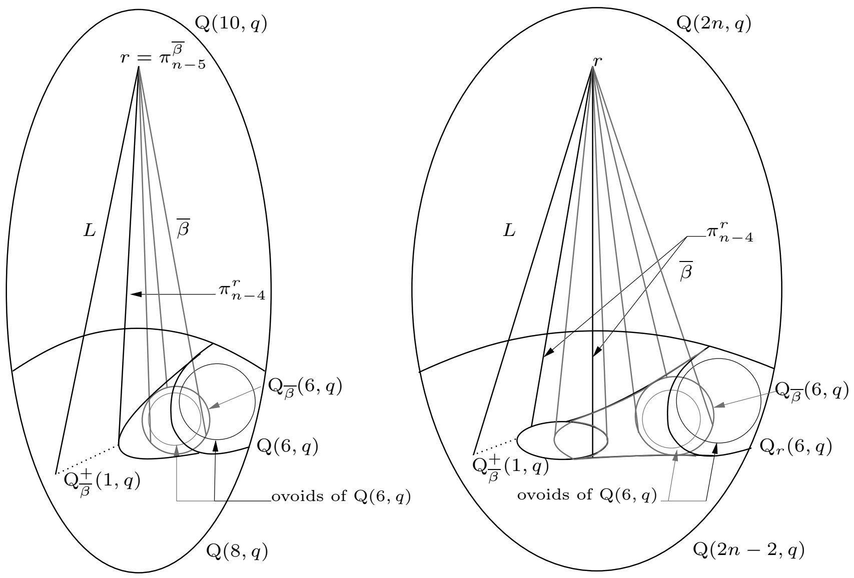

Figure 5.2: The situation in Lemma 5.5.5

implies that $|p^\perp \cap \mathcal{K}| = b_{n-1}$. By Lemma 5.5.2, there exists an $(n+3)$-dimensional subspace $\overline{\alpha}_p$ that meets $\mathrm{Q}(2n,q)$ in the cone $\pi^p_{n-4}\mathrm{Q}_p(6,q)$ and $p^\perp \cap \mathcal{K} \subset \overline{\alpha}_p$. Furthermore, $\overline{\alpha}_p$ contains b_{n-1} points of $\mathcal{K}$, while $L^\perp$ contains b_{n-2} points of $\mathcal{K}$, hence $L^\perp$ intersects $\overline{\alpha}_p$ in a hyperplane $\overline{\beta}'$ of $\overline{\alpha}_p$, with $p \in \overline{\beta}'$. We conclude that $L^\perp \cap \mathcal{K}$ is a subset of $\overline{\beta}$ and $\overline{\beta}'$. The spaces $\overline{\beta}$ and $\overline{\beta}'$ are different since $\overline{\beta}$ does not contain the line L, and so $p \notin \overline{\beta}$. Hence, $L^\perp \cap \mathcal{K}$ lies in the $(n+1)$-dimensional subspace $\overline{\beta} \cap \overline{\beta}'$; it cannot lie in a subspace of lower dimension by Lemma 5.5.2. It is impossible that $r \in \beta = \overline{\beta} \cap \overline{\beta}'$; or else r projects the points of $\beta \cap \mathcal{K}$ onto an n-dimensional subspace, but the projected points form a truncated cone $\pi^*_{n-6}\mathcal{O}$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(6,q)$, which lies in a space of dimension $n+1$. The subspace $\beta = \overline{\beta} \cap \overline{\beta}'$ intersects $\mathrm{Q}(2n,q)$ in a cone $\pi^\beta_{n-6}\mathrm{Q}(6,q)$, since $\langle \beta, r\rangle = \overline{\beta} \subseteq r^\perp$ and $\overline{\beta}$ intersects $\mathrm{Q}(2n,q)$ in $\pi^{\overline{\beta}}_{n-5}\mathrm{Q}^{\overline{\beta}}(6,q)$. ■

Lemma 5.5.6. *Suppose that r is a point of $\mathrm{Q}(2n,q) \setminus \mathcal{K}$ such that $|r^\perp \cap \mathcal{K}| = b_{n-1}$. Then there exists an $(n+2)$-dimensional subspace α_r, $r \notin \alpha_r$, such that $\alpha_r \cap \mathrm{Q}(2n,q) = \pi_{n-5}\mathrm{Q}^r(6,q)$, and such that the truncated cone $\pi^*_{n-5}\mathcal{O}$, $\mathcal{O}$ an ovoid of $\mathrm{Q}^r(6,q)$, is equal to the set $r^\perp \cap \mathcal{K}$.*

Proof. Consider the $(n+3)$-dimensional space $\overline{\alpha}_r$ with $\overline{\alpha}_r \cap \mathrm{Q}(2n,q) = \pi_{n-4}\mathrm{Q}(6,q)$. Suppose that $\overline{\beta}_1$ is a hyperplane of $\overline{\alpha}_r$, not containing π_{n-4} and containing the point r. By Lemma 5.5.5, $\overline{\beta}_1$ contains an $(n+1)$-dimensional subspace β_1, $r \notin \beta_1$, such that $\beta_1 \cap \mathrm{Q}(2n, q) = \pi_{n-6}^{\beta_1}\mathrm{Q}^{\beta_1}(6, q)$ and $\overline{\beta}_1 \cap \mathcal{K} = \beta_1 \cap \mathcal{K} = \pi_{n-6}^{\beta_1 *}\mathcal{O}^{\beta_1}$, $\mathcal{O}^{\beta_1}$ an ovoid of $\mathrm{Q}^{\beta_1}(6, q)$. Define $\pi_1 := \langle \mathcal{O}^{\beta_1} \rangle$. Choose a hyperbolic hyperplane $\alpha \subseteq \pi_1$, $\alpha \cap \mathrm{Q}^{\beta_1}(6, q) = \mathrm{Q}_\alpha^+(5, q)$. We can find a hyperplane $\overline{\beta}_2$ of $\overline{\alpha}_r$, $\overline{\beta}_2 \neq \overline{\beta}_1$, $r \in \overline{\beta}_2$, $\pi_{n-4} \nsubseteq \overline{\beta}_2$, but $\pi_{n-6}^{\beta_1}\mathrm{Q}_\alpha^+(5, q) \subseteq \overline{\beta}_2$. Again, by Lemma 5.5.5, we find an $(n+1)$-dimensional subspace β_2, $r \notin \beta_2$, $\beta_2 \cap \mathrm{Q}(2n, q) = \pi_{n-6}^{\beta_2}\mathrm{Q}^{\beta_2}(6, q)$, $\overline{\beta_2} \cap \mathcal{K} = \beta_2 \cap \mathcal{K} = \pi_{n-6}^{\beta_2 *}\mathcal{O}^{\beta_2}$, $\mathcal{O}^{\beta_2}$ an ovoid of $\mathrm{Q}^{\beta_2}(6, q)$. Necessarily, $\pi_{n-6}^{\beta_1} = \pi_{n-6}^{\beta_2}$, and $\mathrm{Q}_\alpha^+(5, q) \subset \mathrm{Q}^{\beta_2}(6, q) \neq \mathrm{Q}^{\beta_1}(6, q)$. Define now $\pi_2 := \langle \mathcal{O}^{\beta_2} \rangle$.

Consider the $(n+2)$-dimensional space $\gamma = \langle \pi_{n-6}^{\beta_1}, \pi_1, \pi_2 \rangle$. The two 6-dimensional spaces π_1 and π_2 are skew to π_{n-4}, hence, $\pi_{n-4} \nsubseteq \gamma$. Furthermore, $r \notin \gamma$, since then γ would be an $(n+2)$-dimensional subspace on r, not containing π_{n-4}, spanned by points of $r^\perp \cap \mathcal{K}$, a contradiction with Lemma 5.5.5. We conclude that $\gamma \cap \mathrm{Q}(2n, q) = \pi_{n-5}^{\gamma}\mathrm{Q}^{\gamma}(6, q)$.

Choose now an arbitrary hyperplane α', $\alpha' \neq \alpha$, of π_1, such that $\langle \alpha' \cap \mathcal{O}^{\beta_1} \rangle = \alpha'$. This is possible, since all hyperbolic hyperplanes have this property (Corollary 5.4.6), and, for $q = 3$, all elliptic hyperplanes have this property (Lemma 5.4.7). Consider the $q+1$ $(n+1)$-dimensional spaces $\delta_i \subset \gamma$ through the n-dimensional space $\langle \alpha', \pi_{n-6}^{\beta_1} \rangle$. One of them, say δ_1, is the space $\langle \alpha', \pi_{n-5}^{\gamma} \rangle$. Consider now a space δ_i, $i \neq 1$. This space δ_i intersects π_2 in a 5-dimensional space through the 4-dimensional space $\epsilon := \alpha \cap \alpha'$. At most two 5-dimensional spaces through ϵ are tangent hyperplanes to $\mathrm{Q}^{\beta_2}(6, q)$, hence, at least $q-2$ elliptic and hyperbolic hyperplanes of $\mathrm{Q}^{\beta_2}(6, q)$ on ϵ remain, hence, at least $q-2$ spaces δ_i are possibly spanned by points of $\mathcal{K}$. For $q \geqslant 5$, at least one of them is a hyperbolic hyperplane and forr $q = 3$, we can use both the elliptic and hyperbolic hyperplanes, so at least one such δ_i is spanned by points of $\mathcal{K}$. Consider such a δ_i, spanned by points of $\mathcal{K}$. If p is a point, $p \in (\delta_i \cap \mathrm{Q}(2n, q)) \setminus (\beta_1 \cup \beta_2 \cup \pi_{n-5}^{\gamma})$, such that $p \notin \mathcal{K}$, then by Lemma 5.5.2, the line $\langle r, p \rangle$ meets $\mathcal{K}$ in exactly one point t. But then the space $\langle t, \delta_i \rangle \subseteq \overline{\alpha}_r$ is an $(n+2)$-dimensional subspace on r, not containing π_{n-4} and spanned by points of $\mathcal{K}$, a contradiction with Lemma 5.5.5. We conclude that every point $p \in (\gamma \cap \mathrm{Q}(2n, q)) \setminus \langle \alpha', \pi_{n-5}^{\gamma} \rangle$ lies in $\mathcal{K}$, provided p lies in some subspace δ_i (which depends on the choice of α'), spanned by points of $\mathcal{K}$.

Letting vary the 5-dimensional space α', we can reach every point $p \in (\gamma \cap \mathrm{Q}(2n, q) \setminus \pi_{n-5}^{\gamma}$, since the intersection of all hyperbolic hyperplanes of π_1 is empty. We complete the proof by showing that every point $p \in (\gamma \cap \mathrm{Q}(2n, q)) \setminus \pi_{n-5}^{\gamma}$ lies in an $(n+1)$-dimensional space not on r, spanned by points of $\mathcal{K}$, and not containing π_{n-5}^{γ}.

Consider $p \in (\gamma \cap \mathrm{Q}(2n, q)) \setminus (\beta_1 \cup \beta_2 \cup \pi_{n-5}^{\gamma})$. The $(n-4)$-dimensional subspace $\langle \pi^{\gamma}_{n-5}, p\rangle \subseteq \gamma$ intersects the $(n+1)$-dimensional space β_2 in an $(n-5)$-dimensional space ζ. (If $n = 5$, then this is a point u belonging to π_2). If $n > 5$, then ζ intersects π_2 in exactly one point u.

Choose a point $x \in (\beta_2 \cap \mathcal{K}) \setminus \zeta$, $x \notin \beta_1$. This is possible since we excluded at most one point of $\mathcal{O}^{\beta_2}$, namely the point $u \in \zeta \cap \pi_2$. It is impossible that $\mathcal{O}^{\beta_2} = \{u\} \cup (\mathcal{O}^{\beta_1} \cap \mathcal{O}^{\beta_2})$ since $\langle \mathcal{O}^{\beta_1} \cap \mathcal{O}^{\beta_2}\rangle$ intersects $\mathrm{Q}^{\beta_1}(6, q)$ in a hyperbolic quadric, and an ovoid of a hyperbolic quadric contains $q^2 + 1$ points. Hence, $x \in (\beta_2 \cap \mathcal{K}) \setminus \zeta$, $x \notin \beta_1$ exists.

The line $\langle p, x\rangle$ intersects β_1 in exactly one point $y \notin \pi^{\beta_1}_{n-6}$, else $\langle p, y\rangle \subseteq \zeta$, but $x \notin \zeta$.

The space $\langle y, \pi^{\beta_1}_{n-6}\rangle$ intersects π_1 in exactly one point z. If $z \in \alpha$ and $z = y$, then $\langle x, y\rangle = \langle x, z\rangle \subseteq \pi_2$, so $p \in \beta_2$, which is false. If $z \in \alpha$ and $z \neq y$, then $y \in \beta_2$ and hence, $p \in \beta_2$. We conclude that $z \notin \alpha$. Choose one 5-dimensional space $\alpha' \subseteq \pi_1$, $\alpha \neq \alpha'$, through z such that $\langle \alpha' \cap \mathcal{O}^{\beta_1}\rangle = \alpha'$. Then $\langle \pi^{\beta_1}_{n-6}, z, \alpha', x\rangle = \langle \pi^{\beta_1}_{n-6}, \alpha', x\rangle$ is an $(n+1)$-dimensional subspace of γ not containing π^{γ}_{n-5}. For, suppose that $\pi^{\gamma}_{n-5} \subseteq \Omega := \langle \pi^{\beta_1}_{n-6}, \alpha', x\rangle$, then since $z \in \alpha'$, $z \in \Omega$ and $\pi^{\beta_1}_{n-6} \subseteq \Omega$, also $y \in \Omega$. Furthermore, $x \in \Omega$ and $y \in \Omega$, which implies $p \in \Omega$. Finally, $\pi^{\gamma}_{n-5} \subseteq \Omega$, $p \in \Omega$, which implies $u \in \Omega$. Hence, selecting α' in such a way that $u \notin \langle x, \alpha'\rangle$ will imply that $\pi^{\gamma}_{n-5} \not\subseteq \langle \pi^{\beta_1}_{n-6}, \alpha', x\rangle$. This is possible. For, $\langle \pi_1, \pi_2\rangle$ is a 7-dimensional space, while $\langle x, \alpha'\rangle$ is a 6-dimensional space intersecting π_2 in a hyperplane. All hyperbolic 5-spaces of π_1 on z intersect only in z, hence, all spaces $\langle x, \alpha'\rangle$ only intersect in the line $\langle x, z\rangle$, So we can find an α' through z, such that $\langle x, \alpha'\rangle$ does not contain the point u. ■

Lemma 5.5.7. *The set $\mathcal{K}$ is a truncated cone $\pi^*_{n-4}\mathcal{O}$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(6, q)$.*

Proof. From Lemma 5.5.3, we find a point $r \in \mathrm{Q}(2n, q) \setminus \mathcal{K}$ satisfying $|r^\perp \cap \mathcal{K}| = b_{n-1}$. The $(n+2)$-dimensional subspace α_r from Lemma 5.5.6 meets $\mathrm{Q}(2n, q)$ in a cone $\pi^r_{n-5}\mathrm{Q}^r(6, q)$. Choose $Q = \mathrm{Q}(2n-2, q)$ as the base of the cone $r^\perp \cap \mathrm{Q}(2n, q)$ in such a way that $\langle Q\rangle$ contains the cone $\pi^r_{n-5}\mathrm{Q}^r(6, q)$. Let L be a line of $\mathrm{Q}(2n, q)$ on r such that $L \not\subseteq \pi^{r\perp}_{n-5} \cap Q$, which implies that $L^\perp$ does not contain the vertex π^r_{n-5} of α_r. Thus $L^\perp$ meets α_r in a hyperplane of α_r, and this hyperplane meets $\mathrm{Q}(2n, q)$ in a cone $\pi^L_{n-6}\mathrm{Q}^L(6, q)$. Note that $n \geqslant 5$. If $n = 5$, then this hyperplane meets $\mathrm{Q}(2n, q)$ in a quadric $\mathrm{Q}^L(6, q)$.

As $L^\perp \cap \mathcal{K}$ is contained in $r^\perp \cap \mathcal{K} = \alpha_r \cap \mathcal{K}$, it follows that $L^\perp \cap \mathcal{K}$ is a truncated cone $\pi^{L*}_{n-6}\mathcal{O}^L$, $\mathcal{O}^L$ an ovoid of $\mathrm{Q}^L(6, q)$. Hence, $|L^\perp \cap \mathcal{K}| = b_{n-2}$. By Lemma 5.5.4, $|s^\perp \cap \mathcal{K}| = b_{n-1}$ for all points $s \in L$. Every point s gives rise to a truncated cone $s^\perp \cap \mathcal{K} = \pi^{s*}_{n-5}\mathcal{O}^s$, $\mathcal{O}^s$ an ovoid of $\mathrm{Q}^s(6, q)$, and all these truncated cones share the truncated cone $L^\perp \cap \mathcal{K} = \pi^{L*}_{n-6}\mathcal{O}^L$. Denote the subspace spanned by $L^\perp \cap \mathcal{K}$ by β_L.

Every point of $\mathcal{K}$ is collinear with a point of L, which implies that $\mathcal{K}$ is the union of these $q+1$ cones. It follows that $|\mathcal{K}| = b_n$, and that $\mathcal{K}$ is contained in the union of the $q+1$ $(n+2)$-dimensional subspaces α_s, $s \in L$, that share the $(n+1)$-dimensional subspace β_L.

Consider now a second line L' of $\mathrm{Q}(2n, q)$ on r such that $L' \not\subseteq \pi_{n-5}^{r\perp} \cap \mathrm{Q}(2n, q)$ and choose it in such a way that $\beta_L \not\subseteq L'^{\perp}$. This is possible since $\langle \beta_L, r\rangle^{\perp} = \langle r, \pi_{n-6}^L \mathrm{Q}^+(1, q)\rangle$ has only dimension $n-3$. Then, as for L, the subspace $\beta_{L'} := \langle L'^{\perp} \cap \mathcal{K}\rangle$ has dimension $n+1$ and is contained in α_s for all $s \in L'$. We have $\beta_L \neq \beta_{L'}$. Let p be a point of L' with $p \neq r$. Then α_p has dimension $n+2$ and meets α_r in $\beta_{L'}$. Furthermore, $\beta_{L'} \cap \mathrm{Q}(2n, q) = \pi_{n-6}^{L'*} \mathrm{Q}^{L'}(6, q)$, $\beta_{L'} \cap \mathcal{K} = \pi_{n-6}^{L'*} \mathcal{O}^{L'}$, $\mathcal{O}^{L'}$ an ovoid of $\mathrm{Q}^{L'}(6, q)$ and $|\mathcal{O}_{L'} \cap \mathcal{O}_L| \geqslant 1$, since $\mathcal{O}_L$ intersects every hyperplane of $\langle \mathcal{O}_L\rangle$ by Lemma 5.4.4

Varying the point $t \in L'$, the tangent hyperplanes $t^{\perp}$ vary over the hyperplanes through $L'^{\perp}$, hence, every point of the $(n-5)$-dimensional spaces π_{n-5}^s, $s \in L$, lies in some $t^{\perp}$, $t \in L'$. Every point of π_{n-5}^s, $s \in L$, lies on lines with q points of $\mathcal{K}$, to the points of $\mathcal{O}^L \cap \mathcal{O}^{L'}$, and hence belongs to one of the vertices π_{n-5}^t, $t \in L'$.

Consider a fixed point $s \in L \setminus \{r\}$, fixed points $p_1 \in \pi_{n-5}^r$, $p_2 \in \pi_{n-5}^s$, $p_1, p_2 \notin \pi_{n-5}^r \cap \pi_{n-5}^s = \pi_{n-6}^L$. Consider a fixed point $u \in \pi_{n-5}^{r*} \mathcal{O}^r$, then it is possible to select a line L'', satisfying the conditions of L', for which $u \in L''^{\perp}$. Then the preceding arguments show that the set $\langle u, p_2\rangle \setminus \{p_2\}$ is contained in $\mathcal{K}$.

Consider an arbitrary line M of $\pi_{n-5}^{r*}\mathcal{O}^r$ passing through p_1 and containing q points of $\mathcal{K}$. The q^2 points of $\langle M, p_2\rangle \setminus \langle p_1, p_2\rangle$ all lie in $\mathcal{K}$; this implies that the truncated cone $\langle \pi_{n-5}^r, \pi_{n-5}^s\rangle^* \mathcal{O}^r$ lies in $\mathcal{K}$. Since $|\mathcal{K}| = |\langle \pi_{n-5}^r, \pi_{n-5}^s\rangle^* \mathcal{O}^r| = b_n$, this truncated cone must be equal to $\mathcal{K}$. ■

We may conclude Theorem 5.5.1.

Theorem 5.5.8. *Suppose that* $\mathrm{Q}(6, q)$ *has ovoids and that the smallest minimal blocking sets of* $\mathrm{Q}(6, q)$ *different from ovoids are truncated cones* $p^*\mathcal{O}$, $p \in \mathrm{Q}(6, q)$, $\mathcal{O}$ *an ovoid of* $\mathrm{Q}(4, q)$. *Then the smallest minimal blocking sets of* $\mathrm{Q}(2n, q)$, $n \geqslant 4$, *are truncated cones* $\pi_{n-4}^*\mathcal{O}$, $\pi_{n-4} \subset \mathrm{Q}(2n, q)$, $\mathcal{O}$ *an ovoid of* $\mathrm{Q}(6, q) \subset \pi_{n-4}^{\perp} \cap \mathrm{Q}(2n, q)$, *and have size* $q^n + q^{n-3}$.

Together with Theorem 4.2.7, we find finally Theorem 5.1.3.

Theorem 5.5.9. *The smallest minimal blocking sets of* $\mathrm{Q}(2n, q = 3)$, $n \geqslant 4$, *are truncated cones* $\pi_{n-4}^*\mathcal{O}$, $\pi_{n-4} \subset \mathrm{Q}(2n, q)$, $\mathcal{O}$ *an ovoid of* $\mathrm{Q}(6, q) \subset \pi_{n-4}^{\perp} \cap \mathrm{Q}(2n, q)$, *and have size* $q^n + q^{n-3}$.

5.6 Final remarks

Two main problems are left to characterise the smallest minimal blocking sets of the polar space Q$(2n, q)$, $n \geqslant 3$, q odd, in general.

Firstly, the existence or non-existence of ovoids of Q$(6, q)$, q odd, q not prime and $q \neq 3^h$, $h \geqslant 1$, is not solved, although it is conjectured in [76] that Q$(6, q)$ has ovoids if and only if $q = 3^h$, $h \geqslant 1$. As we have seen in the previous sections, it is possible to formulate characterisation theorems only supposing that the smallest minimal blocking sets of Q$(6, q)$, different from ovoids of Q$(6, q)$, are known. The existence of ovoids of Q$(6, q)$ changes of course the characterisation of minimal blocking sets of Q$(2n, q)$, $n \geqslant 4$, but to prove the theorems in the higher dimensional case, the characterisation for Q$(6, q)$ is the most important part.

In the previous sections, characterisation theorems were obtained for the $q > 3$ prime case using the classification of ovoids of Q$(4, q)$, q prime. It is for instance known that the ovoids of Q$(4, 9)$ are classified ([103]), but it seems that using this classification to characterise the smallest minimal blocking sets of Q$(6, 9)$, different from an ovoid, is much more complicated than doing the characterisation of the smallest minimal blocking sets of Q$(6, q)$, $q > 3$, q prime, using the fact that all ovoids of Q$(4, q)$, q prime, are elliptic quadrics. Since presently it is not known that, for general q odd, q not a prime, the classification of ovoids of Q$(4, q)$ will be obtained, and, since a possible characterisation can give many different cases, it seems more interesting to use the techniques of Section 4.2, where the classification of the smallest minimal blocking sets of Q$(6, q)$, $q = 3, 5, 7$, is obtained using a lower bound on the size of the smallest minimal blocking sets of Q$(4, q)$, $q = 3, 5, 7$, different from an ovoid of Q$(4, q)$, $q = 3, 5, 7$.

This leads us to the second main problem, and, in the point of view of the previous section, the most important: finding a lower bound on the size of the smallest minimal blocking sets of Q$(4, q)$, q odd, different from an ovoid. In general, even minimal blocking sets of Q$(4, q)$, q odd, of size $q^2 + 2$ are not yet excluded.

Chapter 6

The smallest minimal blocking sets of $H(2n, q^2)$

It is known that the Hermitian variety $H(2n, q^2)$, $n \geqslant 2$, has no ovoids, [92]. As for the other polar spaces having no ovoids, we can look how the smallest sets of points blocking every generator, look like.

It will become clear that the Hermitian variety $H(2n, q^2)$ behaves very nicely. Although the characterisation result is comparable to the result for $Q(2n, q)$, $q > 3$ prime, some important lemmas are easier to prove. Especially the low dimensional case $H(4, q^2)$ can be handled in a more straightforward way than $Q(6, q)$, $q > 3$ prime, while the extension of the result to $H(2n, q^2)$, $n > 2$, is very similar to the proof for $Q(2n, q)$, $q > 3$, $n > 3$, q prime.

The present chapter is based on joint work with K. Metsch [35].

6.1 Introduction

In a paper of J.A. Thas [92], the non-existence for ovoids of $H(2n, q^2)$ is proved. The proof is based on a counting argument, and goes on with the use of geometric properties of $H(2n, q^2)$, proving the non-existence of an ovoid.

In one of his papers, K. Metsch states that whenever the non-existence of ovoids of some polar space $\mathcal{P}$ can be proved "easily", then there is hope to determine the smallest sets of points blocking every generator. With "easily" is meant a proof based on a short counting argument. It is interesting to see that it is indeed possible to prove the non-existence of ovoids of $H(4, q^2)$ with a short argument. The proof is taken from [74].

Lemma 6.1.1. (Thas [92]) *The polar space $H(4, q^2)$ has no ovoids.*

Proof. Suppose that $\mathrm{H}(4, q^2)$ has an ovoid $\mathcal{O}$. Consider an arbitrary point $p \in \mathcal{O}$. Then p lies on q^6 lines intersecting $\mathrm{H}(4, q^2)$ in $1 + q$ points. Since $|\mathcal{O}| = q^5 + 1$, we find a line L on p intersecting $\mathrm{H}(4, q^2)$ in $1 + q$ points and intersecting $\mathcal{O}$ in exactly the point p. Every point of $\mathcal{O} \setminus \{p\}$ lies in $x^\perp$, for a unique point $x \in L \setminus \{p\}$. If $x \in \mathrm{H}(4, q^2)$, then x lies on $q^3 + 1$ lines of $\mathrm{H}(4, q^2)$ that all meet $\mathcal{O}$ in exactly one point and hence, $|x^\perp \cap \mathcal{O}| = q^3 + 1$. If $x \notin \mathrm{H}(4, q^2)$, then $x^\perp \cap \mathrm{H}(4, q^2) = \mathrm{H}(3, q^2)$ and $x^\perp \cap \mathcal{O}$ is an ovoid of this $\mathrm{H}(3, q^2)$. Hence again, $|x^\perp \cap \mathcal{O}| = q^3 + 1$. It follows that $|\mathcal{O}| - 1 = q^2(q^3 + 1)$, a contradiction. ■

An argument very similar to this counting will show that a minimal blocking set of $\mathrm{H}(4, q^2)$ necessarily contains "a lot" more points than $q^5 + 1$. This behaviour can also be found in the polar spaces $\mathrm{Q}^-(5, q)$ and $\mathrm{W}(2n + 1, q)$, while for other polar spaces more arguments are needed to prove the non-existence of ovoids, when possible.

After the use of counting arguments, we will proceed with the proof for $\mathrm{H}(4, q^2)$ in a way quite similar to the proof for $\mathrm{Q}(6, q)$, q prime.

6.2 The smallest minimal blocking sets of $\mathrm{H}(4, q^2)$

In this section, we will prove the following theorem.

Theorem 6.2.1. *Let $\mathcal{K}$ be a minimal blocking set of $\mathrm{H}(4, q^2)$, $|\mathcal{K}| \leqslant q^5 + q^2$. Then there exists a point $p \in \mathrm{H}(4, q^2)$ such that $\mathcal{K} = p^*\mathrm{H}(2, q^2) \subseteq p^\perp \cap \mathrm{H}(4, q^2)$, and $|\mathcal{K}| = q^5 + q^2$.*

For this section, we suppose that $\mathcal{K}$ is a minimal blocking set of $\mathrm{H}(4, q^2)$, $|\mathcal{K}| = q^5 + \delta$, $1 \leqslant \delta \leqslant q^2$. We start with the following traditional lemma.

Lemma 6.2.2. *If p is a point of $\mathrm{H}(4, q^2)$, $p \in \mathcal{K}$, then $|p^\perp \cap \mathcal{K}| \leqslant \delta$.*

Proof. Since $\mathcal{K}$ is minimal, there exists a generator L of $\mathrm{H}(4, q^2)$ such that $L \cap \mathcal{K} = \{p\}$. Each point of $L \setminus \{p\}$ lies on q^3 generators of $\mathrm{H}(4, q^2)$ which meet $\mathcal{K}$ in at least one point different from p. Considering the $q^2 \cdot q^3$ such generators which meet $\mathcal{K}$ in a point of $\mathcal{K} \setminus p^\perp$, we find $|p^\perp \cap \mathcal{K}| \leqslant |\mathcal{K}| - q^5 = \delta$. ■

Lemma 6.2.3. *For all points $r \in \mathrm{PG}(4, q^2) \setminus \mathcal{K}$ holds $|r^\perp \cap \mathcal{K}| \geqslant q^3 + 1$.*

Proof. Suppose that $r \in \mathrm{H}(4,q^2) \setminus \mathcal{K}$. Then each one of the q^3+1 generators of $\mathrm{H}(4,q^2)$ through r meets $\mathcal{K}$ in at least one point of $\mathcal{K}$. Hence, $|r^\perp \cap \mathcal{K}| \geqslant q^3+1$. If $r \in \mathrm{PG}(4,q^2) \setminus \mathrm{H}(4,q^2)$, then $r^\perp \cap \mathrm{H}(4,q^2) = \mathrm{H}(3,q^2)$, and all generators of $\mathrm{H}(3,q^2)$ are generators of $\mathrm{H}(4,q^2)$ and must meet $\mathcal{K}$ in at least one point, hence again $|r^\perp \cap \mathcal{K}| \geqslant q^3+1$. ■

Lemma 6.2.4. *If $p \in \mathcal{K}$, then $|p^\perp \cap \mathcal{K}| \geqslant q^2-q+1$.*

Proof. There are q^6 lines of $\mathrm{PG}(4,q^2)$ on p not in the tangent cone $p^\perp$. We call these lines *secants* on p. At most $|\mathcal{K}|-1$ of those secant lines meet $\mathcal{K}$ in a second point, hence we find that at least $q^6-q^5+1-\delta$ secant lines L on p meet $\mathcal{K}$ only in p. We prove that $L^\perp \cap \mathcal{K} \neq \emptyset$ for each such secant line L on p. Therefore we count the set of pairs $\{(r,s) \| r \in \mathcal{K}, s \in L, r \in s^\perp\}$. Each point $r \in L^\perp \cap \mathcal{K}$ occurs q^2+1 times since $r \in s^\perp$ for all $s \in L$. Every other point $r \in \mathcal{K}$ occurs just once. By the previous lemma, every point $s \in L \setminus \{p\}$ occurs in at least q^3+1 pairs. Hence,

$$|\mathcal{K}| + |L^\perp \cap \mathcal{K}|q^2 \geqslant |p^\perp \cap \mathcal{K}| + q^2(q^3+1).$$

As $|\mathcal{K}| \leqslant q^5+q^2$, we find $L^\perp \cap \mathcal{K} \neq \emptyset$. So far, we have $q^6-q^5+1-\delta$ secant lines L on p satisfying $L^\perp \cap \mathcal{K} \neq \emptyset$. It is clear that $L^\perp \cap \mathcal{K} \subset p^\perp \cap \mathcal{K}$. The subspaces $L^\perp \cap \mathcal{K}$ are planes of the 3-dimensional space $p^\perp$ not passing through p. As every point of $(p^\perp \cap \mathcal{K}) \setminus \{p\}$ lies in q^4 such planes of $p^\perp$, it follows that $|(p^\perp \setminus \{p\}) \cap \mathcal{K}|q^4 \geqslant q^6-q^5+1-\delta$. Hence, $|p^\perp \cap \mathcal{K}| \geqslant q^2-q+1$. ■

The power of using the polarity together with the geometric structure of the Hermitian variety becomes clear with Lemma 6.2.4. The first two lemmas can be repeated for e.g. the parabolic quadric $\mathrm{Q}(6,q)$, q odd. But with the same arguments, we can never reach a conclusion like in Lemma 6.2.4. The reason is that all $(2n-1)$-dimensional non-singular subvarieties of $\mathrm{H}(2n,q^2)$ have generators of the same dimension as $\mathrm{H}(2n,q^2)$. This is not the fact for e.g. $\mathrm{Q}(6,q)$. It contains hyperbolic quadrics in 5 dimensions, also containing planes as generators, but it also contains elliptic quadrics in 5 dimensions, which only contain lines as generators.

Lemmas 6.2.2, 6.2.3 and 6.2.4 also lead to the following corollary.

Corollary 6.2.5. $\mathrm{H}(4,q^2)$ *has no ovoids.*

Proof. An ovoid satisfies the hypothesis on $\mathcal{K}$ for this section, with $\delta=1$. Lemma 6.2.2 and Lemma 6.2.4 imply $\delta \geqslant q^2-q+1$, a contradiction. ■

The lower bound on δ in Lemma 6.2.4 is strong enough to work immediately towards the desired structure for $\mathcal{K}$. We first introduce the following notation. If p is a point, $p \in \mathrm{H}(4, q^2)$, then $w_p + 1$ is the smallest number of points of $\mathcal{K}$ that lie on a generator of $\mathrm{H}(4, q^2)$ on p. Since $\mathcal{K}$ is minimal, $w_p = 0$ for all $p \in \mathcal{K}$. For a line L, we define $w_L = \sum_{p \in L} w_p$.

The following lemma is comparable to Lemma 3.2.6.

Lemma 6.2.6. *If r is a point, $r \in \mathrm{H}(4, q^2) \setminus \mathcal{K}$, $w_r = 0$, then $|r^\perp \cap \mathcal{K}| \leqslant q^3 - q^2 + q + \delta$.*

Proof. Let L be a generator on r meeting $\mathcal{K}$ in exactly one point p. Each one of the $q^2 - 1$ points of $L \setminus \{p, r\}$ lies on q^3 further generators meeting $\mathcal{K}$. Hence,

$$(|p^\perp \cap \mathcal{K}| - 1) + (|r^\perp \cap \mathcal{K}| - 1) \leqslant |\mathcal{K}| - 1 - (q^2 - 1)q^3.$$

From Lemma 6.2.4, we have $|p^\perp \cap \mathcal{K}| \geqslant q^2 - q + 1$, and $|\mathcal{K}| = q^5 + \delta$. Hence, $|r^\perp \cap \mathcal{K}| \leqslant q^3 - q^2 + q + \delta$. ■

The next lemma implies already a certain structure for $\mathcal{K}$. Finding a line of $\mathrm{H}(4, q^2)$ intersecting $\mathcal{K}$ in more than one point implies that there exists a point on that generator such that all generators on that point intersect $\mathcal{K}$ in more than one point.

Lemma 6.2.7. *If L is a generator of $\mathrm{H}(4, q^2)$ meeting $\mathcal{K}$ in more than one point, then L contains a point $s \in \mathrm{H}(4, q^2) \setminus \mathcal{K}$, with $w_s > 0$.*

Proof. Suppose that L meets $\mathcal{K}$ in more than one point. Suppose that $w_s = 0$ for all $s \in L \setminus \mathcal{K}$. Define $k + 1 = |L \cap \mathcal{K}|$. We again count the number of pairs in $\{(r, s) \| r \in \mathcal{K}, s \in L, r \in s^\perp\}$. Each one of the $k + 1$ points of $L \cap \mathcal{K}$ occurs in $q^2 + 1$ pairs, every other point of $\mathcal{K}$ occurs in exactly one pair. We can apply Lemma 6.2.2 to the points of $L \cap \mathcal{K}$ and Lemma 6.2.6 to the points of $L \setminus \mathcal{K}$. Hence,

$$|\mathcal{K}| + (1 + k)q^2 \leqslant (1 + k)\delta + (q^2 - k)(q^3 + \delta - q^2 + q + 1).$$

Since $|\mathcal{K}| = q^5 + \delta$, we obtain $k(q^3 + q + 1) \leqslant q^2(\delta - q^2 + q)$. Since $k \geqslant 1$ and $\delta \leqslant q^2$, this is a contradiction. ■

We can now prove that a generator contains exactly one point of $\mathcal{K}$ or contains "a lot" of points of $\mathcal{K}$. We mention the extra condition $q > 2$ for the next lemma. The case $q = 2$ will be handled separately.

Lemma 6.2.8. *Suppose that $q > 2$. If $p \in \mathrm{H}(4, q^2) \setminus \mathcal{K}$ and $w_p > 0$, then $w_p \geqslant q^2 - q$.*

Proof. Define $b = w_p$. Let L be a generator on p, $|L \cap \mathcal{K}| = w_p + 1 = b + 1$. Count again the number of pairs in the set $\{(r,s) \| r \in \mathcal{K}, s \in L, r \in s^\perp\}$. We find

$$|\mathcal{K}| + (1+b)q^2 = \sum_{s \in L} |s^\perp \cap \mathcal{K}|.$$

If $s \in L \cap \mathcal{K}$, then, by Lemma 6.2.4, $|s^\perp \cap \mathcal{K}| \geqslant q^2 - q + 1$. If $s \in L \setminus (\mathcal{K} \cup \{p\})$, s lies on q^3+1 generators meeting $\mathcal{K}$, hence $|s^\perp \cap \mathcal{K}| \geqslant q^3 + b + 1$. Furthermore, all q^3+1 generators on p meet $\mathcal{K}$ in at least $1 + w_p = 1 + b$ points. Hence,

$$|\mathcal{K}| + (1+b)q^2 \geqslant (1+b)(q^2 - q + 1) + (q^2 - b - 1)(q^3 + b + 1) + (q^3 + 1)(1 + b),$$

giving $\delta \geqslant (b+1)(q^2 - q - b + 1)$. As $\delta \leqslant q^2$ and supposing that $q > 2$, we find $b < 1$ or $b > q^2 - q - 1$, hence, since we supposed that $w_p = b > 0$, $w_p = b \geqslant q^2 - q$. ■

A final step is now sufficient to prove the result for $\mathrm{H}(4,q^2)$, $q > 2$.

Lemma 6.2.9. *Suppose that* $q > 2$*, then there exists a point* $r \in \mathrm{H}(4,q^2) \setminus \mathcal{K}$ *such that* $\mathcal{K}$ *is the truncated cone* $r^*\mathrm{H}(2,q^2) \subseteq r^\perp \cap \mathcal{K}$*.*

Proof. Since $\mathcal{K}$ is not an ovoid, there exists a generator L meeting $\mathcal{K}$ in more than 1 point. By Lemma 6.2.7 and Lemma 6.2.8, L contains a point r with $w_r \geqslant q^2 - q$. Hence, $r^\perp$ contains at least $(q^3+1)(w_r+1) = (q^3+1)(q^2-q+1)$ points of $\mathcal{K}$.

There cannot be a second point $r' \in \mathrm{H}(4,q^2)$ with this property, because otherwise we could conclude that

$$|\mathcal{K}| \geqslant 2(q^3+1)(q^2-q+1) - |r^\perp \cap r'^\perp \cap \mathcal{K}|$$

As $|\mathcal{K}| \leqslant (q^3+1)q^2$ and

$$|r^\perp \cap r'^\perp \cap \mathcal{K}| \leqslant |r^\perp \cap r'^\perp \cap \mathrm{H}(4,q^2)| \leqslant q^3 + 1,$$

this is a contradiction.

Hence, every generator that meets $\mathcal{K}$ in more than 1 point, must pass through r. However, Lemma 6.2.4 implies that every point of $\mathcal{K}$ lies on a generator meeting $\mathcal{K}$ in more than 1 point. Hence, all points of $\mathcal{K}$ lie in $r^\perp$. As $r \notin \mathcal{K}$ and as $\mathcal{K}$ meets all generators, it is clear that $\mathcal{K}$ must consist of all points of $r^\perp \cap \mathrm{H}(4,q^2)$ different from r, since each point $x \in r^\perp \cap \mathrm{H}(4,q^2)$, with $x \neq r$, lies on a generator M with $M \cap r^\perp = \{x\}$. ■

In the final lemma, we handle the case $q = 2$.

Lemma 6.2.10. *Suppose that $q = 2$, then there exists a point $r \in \mathrm{H}(4, q^2) \setminus \mathcal{K}$ such that $\mathcal{K}$ is the truncated cone $r^*\mathrm{H}(2, q^2) \subseteq r^\perp \cap \mathcal{K}$.*

Proof. Lemma 6.2.2 implies that every generator meets $\mathcal{K}$ in at most $\delta \leqslant q^2 = 4$ points. Suppose that there exists a generator L meeting $\mathcal{K}$ in exactly 4 points, and let r be the point of L not in $\mathcal{K}$. Then Lemma 6.2.2 implies that $\delta = q^2 = 4$ and $p^\perp \cap \mathcal{K} = L \cap \mathcal{K}$ for all points $p \in L \cap \mathcal{K}$. As every point of $\mathcal{K}$ is collinear to at least one point of L, it follows that $\mathcal{K} \subseteq r^\perp$, which finishes the proof. Hence, the lemma is proved if we can prove that there exists a generator meeting $\mathcal{K}$ in 4 points.

Assume the contrary, hence every generator meets $\mathcal{K}$ in 1, 2 or 3 points. Let s_1, s_2 and s_3 be the number of generators of $\mathrm{H}(4, q^2)$ meeting $\mathcal{K}$ in 1, 2 and 3 points, respectively. Then $s_1 + s_2 + s_3 = (q^3 + 1)(q^5 + 1)$. Also $s_1 + 2s_2 + 3s_3 = |\mathcal{K}|(q^3 + 1)$, since every point of $\mathcal{K}$ lies on $q^3 + 1$ generators of $\mathrm{H}(4, q^2)$. Lemma 6.2.4 implies that $|p^\perp \cap \mathcal{K}| \geqslant 3$ for $p \in \mathcal{K}$, so we have $2s_2 + 6s_3 \geqslant 2|\mathcal{K}|$. The two equations and the inequality together imply that $s_3 > 0$. Hence, there exists a generator that meets $\mathcal{K}$ in 3 points.

Consider a point $r \in \mathrm{H}(4, q^2) \setminus \mathcal{K}$, put $b := w_r$, and let L be a generator on r that meets $\mathcal{K}$ in exactly $1 + b$ points. Since no generator contains 4 points of $\mathcal{K}$, $b \leqslant 2$. Lemma 6.2.4 implies that $|s^\perp \cap \mathcal{K}| \geqslant 3$, for every point $s \in L \cap \mathcal{K}$, so $s^\perp \cap \mathcal{K}$ contains at least $2 - b$ points of $\mathcal{K}$ not lying on L. Each one of the $q^2 - 1 - b$ points s of L not lying in $\mathcal{K} \cup \{r\}$ lies on $q^3 + 1$ generators, so q^3 of these meet $\mathcal{K}$ in points outside L. This implies that there are at least $(1 + b)(2 - b) + (q^2 - 1 - b)q^3$ points in $\mathcal{K}$ not lying on L and not collinear with r. Hence, $|r^\perp \cap \mathcal{K}| \leqslant |\mathcal{K}| - (1 + b)(2 - b) - (q^2 - 1 - b)q^3$. Hence, $|r^\perp \cap \mathcal{K}| \leqslant 10 + b(b + 7)$. If $b = 0$, this gives $|r^\perp \cap \mathcal{K}| \leqslant 10$; as r lies on $q^3 + 1 = 9$ generators, this implies that all generators on r meet $\mathcal{K}$ in 1 point, except for maybe one generator that contains two points of $\mathcal{K}$. If $b = 1$, then $|r^\perp \cap \mathcal{K}| \leqslant 18$. Since $b = w_r = 1$, $w_r + 1 = 2$ is the smallest number of points of $\mathcal{K}$ on every generator on r. Hence, all $q^3 + 1$ generators on r meet $\mathcal{K}$ in exactly 2 points and $|r^\perp \cap \mathcal{K}| = 18$.

As we have seen above, there exists a generator that meets $\mathcal{K}$ in 3 points. Then $w_r = 0$ and $w_r = 1$ is impossible for the 2 points $r \in L \setminus \mathcal{K}$. Hence, $w_r \geqslant 2$ for the 2 points $r \in L \setminus \mathcal{K}$. There are 17 generators containing one of these 2 points, and each generator meets $\mathcal{K}$ in at least 3 points. This implies $|\mathcal{K}| \geqslant 17 \cdot 3$, a contradiction. ∎

Lemmas 6.2.9 and 6.2.10 together imply Theorem 6.2.1.

6.3 The smallest minimal blocking sets of $\mathrm{H}(2n, q^2)$, $n > 2$

In this section, we prove the following theorem.

Theorem 6.3.1. *Let $\mathcal{K}$ be a minimal blocking set of $\mathrm{H}(2n, q^2)$, $n > 2$, $|\mathcal{K}| \leqslant q^{2n-2}(q^3+1)$. Then there exists an $(n-2)$-dimensional subspace $\pi_{n-2} \subset \mathrm{H}(2n, q^2)$ such that $\mathcal{K}$ is the truncated cone $\pi^*_{n-2}\mathrm{H}(2, q^2) \subseteq \pi^\perp_{n-2} \cap \mathrm{H}(2n, q^2)$, and $|\mathcal{K}| = q^{2n-2}(q^3+1)$.*

We suppose for this section that $\mathcal{K}$ is a minimal blocking set of $\mathrm{H}(2n, q^2)$, $n > 2$, $|\mathcal{K}| \leqslant b_n$, with b_n defined as

$$b_n := q^{2n-2}(q^3+1) = |\pi^*_{n-2}\mathrm{H}(2, q^2)|.$$

To prove the theorem, we use induction on n. The case $n = 2$ is handled in Section 6.2.

Lemma 6.3.2. *Let r be a point of $\mathrm{H}(2n, q^2) \setminus \mathcal{K}$. Then $|r^\perp \cap \mathcal{K}| \geqslant b_{n-1}$. If equality holds, then there exists an $(n+1)$-dimensional subspace $\overline{\alpha}_r$, such that $r \in \overline{\alpha}_r \subset r^\perp$ and $\overline{\alpha}_r \cap \mathrm{H}(2n, q^2) = \pi_{n-2}\mathrm{H}(2, q^2)$. Moreover, every line L on r contained in $\mathrm{H}(2n, q^2)$ meets $\mathcal{K}$ if and only if $L \subset \overline{\alpha}_r \setminus \pi_{n-2}$. In other words, the points of $\mathcal{K} \cap r^\perp$ are projected from r onto a truncated cone $\pi^*_{n-3}\mathrm{H}(2, q^2) \subset \overline{\alpha}_r$, $r \notin \pi_{n-3}$.*

Proof. Applying the induction hypothesis to the base of the cone $r^\perp \cap \mathrm{H}(2n, q^2) = r\mathrm{H}(2n-2, q^2)$, we find that $|r^\perp \cap \mathcal{K}| \geqslant b_{n-1}$. If equality holds, then necessarily r projects the points of $r^\perp \cap \mathcal{K}$ onto a truncated cone $\pi^*_{n-3}\mathrm{H}(2, q^2)$. The $(n+1)$-dimensional space $\overline{\alpha}_r$ is the space $\langle r, \pi_{n-3}\mathrm{H}(2, q^2)\rangle$. ■

The next lemma is a generalisation of Lemma 5.2.3 and also Lemma 5.5.4. The importance of the generalisation will become clear afterwards.

Lemma 6.3.3. *Suppose that $\beta \subset \mathrm{H}(2n, q^2)$ is an s-dimensional subspace with $\beta \cap \mathcal{K} = \emptyset$. Then $|\beta^\perp \cap \mathcal{K}| \geqslant b_{n-s-1}$. Equality implies that $|\mathcal{K}| = b_n$ and $|r^\perp \cap \mathcal{K}| = b_{n-1}$ for all $r \in \beta$.*

Proof. Count the number of pairs in the set $\{(r, t) \| r \in \mathcal{K}, t \in \beta, r \in t^\perp\}$. We obtain

$$|\beta^\perp \cap \mathcal{K}|\frac{q^{2s+2}-1}{q^2-1} + (|\mathcal{K}| - |\beta^\perp \cap \mathcal{K}|)\frac{q^{2s}-1}{q^2-1} = \sum_{t \in \beta} |t^\perp \cap \mathcal{K}|.$$

Using $|\mathcal{K}| \leqslant b_n$, $|r^\perp \cap \mathcal{K}| \geqslant b_{n-1}$ for all $r \in \mathrm{H}(2n, q^2) \setminus \mathcal{K}$ and $b_n = q^2 b_{n-1}$, we find that $|\beta^\perp \cap \mathcal{K}| \geqslant b_{n-s-1}$.

Suppose that $|\beta^\perp \cap \mathcal{K}| = b_{n-s-1}$. Using again that $|r^\perp \cap \mathcal{K}| \geqslant b_{n-1}$ for all $r \in \mathrm{H}(2n, q^2) \setminus \mathcal{K}$, we now find that $|\mathcal{K}| \geqslant b_n$, which implies that $|\mathcal{K}| = b_n$ and henceforth also $|r^\perp \cap \mathcal{K}| = b_{n-1}$ for all $r \in \beta$. ∎

It is not possible to prove the lemma for hyperplanes of generators of quadrics. With the next lemma, it turns out that exactly the hyperplanes of generators case is very useful to prove that equality occurs in Lemma 6.3.2.

Lemma 6.3.4. *There exists a point $r \in \mathrm{H}(2n, q^2) \setminus \mathcal{K}$ such that $|r^\perp \cap \mathcal{K}| = b_{n-1}$.*

Proof. Consider a point $p \in \mathcal{K}$ and a generator π_n meeting $\mathcal{K}$ only in p. Such a generator exists, since we assumed $\mathcal{K}$ to be minimal. The generator π_n contains q^{2n-2} hyperplanes π_{n-1} not containing p. Suppose that $|\pi_{n-1}^\perp \cap \mathcal{K}| \geqslant q^3+2$ for all the subspaces π_{n-1}. Since all points $p \in \mathcal{K} \setminus \pi_n$ belong to exactly one $\pi_{n-1}^\perp$, we find that $|\mathcal{K}| \geqslant q^{2n-2}(q^3+1)+1$, where we counted the point p separtely, contradicting with $|\mathcal{K}| \leqslant b_n$. Applying Lemma 6.3.3 to any subspace π_{n-1}, with $|\pi_{n-1} \cap \mathcal{K}| \leqslant q^3+1$, gives the lemma. ∎

Also now it is crucial to prove that in case of equality, not only the projection of $r^\perp \cap \mathcal{K}$, but also the set $r^\perp \cap \mathcal{K}$ itself is a truncated cone.

Lemma 6.3.5. *Suppose that r is a point $r \in \mathrm{H}(2n, q^2) \setminus \mathcal{K}$, with $|r^\perp \cap \mathcal{K}| = b_{n-1}$. If $\overline{\beta}$ is a hyperplane of $\overline{\alpha}_r$ on r not containing the vertex π_{n-2} of the cone $\overline{\alpha}_r \cap \mathrm{H}(2n, q^2)$, then the points of $\overline{\beta} \cap \mathcal{K}$ lie in an $(n-1)$-dimensional subspace β of $\overline{\beta}$, $r \notin \beta$.*

Proof. Since $\overline{\beta}$ is a hyperplane of $\overline{\alpha}_r$ on r not containing the vertex π_{n-2}^r of the cone $\overline{\alpha}_r \cap \mathrm{H}(2n, q^2) = \pi_{n-2}^r \mathrm{H}^r(2, q^2)$, $\overline{\beta} \cap \mathrm{H}(2n, q^2)$ is a cone with base $\mathrm{H}^r(2, q^2)$ and vertex $\pi_{n-3}^{\overline{\beta}}$, an $(n-3)$-dimensional subspace on r. When $n = 3$, this subspace is the point r itself. The properties of the polarity associated to $\mathrm{H}(2n, q^2)$ imply that $\overline{\beta}^\perp \cap \mathrm{H}(2n, q^2) = \pi_{n-3}^{\overline{\beta}} \mathrm{H}^{\overline{\beta}}(1, q^2)$, and this cone meets the space $\overline{\alpha}_r$ in the space π_{n-2}^r. Thus there must exist a line L of $\mathrm{H}(2n, q^2)$ contained in $\overline{\beta}^\perp$ such that $L \cap \overline{\alpha}_r = \{r\}$. Since $L \subset \overline{\beta}^\perp$, we find $\overline{\beta} = L^\perp \cap \overline{\alpha}_r$. By Lemma 6.3.2, L does not meet $\mathcal{K}$.

Since $L^\perp \cap \mathcal{K} \subseteq r^\perp \cap \mathcal{K} \subseteq \overline{\alpha}_r$, it is clear that $L^\perp \cap \mathcal{K} = \overline{\beta} \cap \mathcal{K}$. Lemma 6.3.2 implies that $|L^\perp \cap \mathcal{K}| = b_{n-2}$. Suppose that p is a point of $L \setminus \{r\}$. Lemma 6.3.3 implies that $|p^\perp \cap \mathcal{K}| = b_{n-1}$. By Lemma 6.3.2, there exists an $(n+1)$-dimensional subspace $\overline{\alpha}_p$ that meets $\mathrm{H}(2n, q^2)$ in a cone $\pi_{n-2}^p \mathrm{H}^p(2, q^2)$ and $p^\perp \cap \mathcal{K} \subset \overline{\alpha}_p$. Furthermore, $\overline{\alpha}_p$ contains b_{n-1} points of $\mathcal{K}$, while $L^\perp$ contains

b_{n-2} points of $\mathcal{K}$, hence $L^\perp$ intersects $\overline{\alpha}_p$ in a hyperplane $\overline{\beta}'$ of $\overline{\alpha}_p$, with $p \in \overline{\beta}'$. We conclude that $L^\perp \cap \mathcal{K}$ is a subset of $\overline{\beta}$ and $\overline{\beta}'$. The spaces $\overline{\beta}$ and $\overline{\beta}'$ are different since $\overline{\beta}$ does not contain the line L and hence the point $p \notin \overline{\beta}$. Hence $L^\perp \cap \mathcal{K}$ lies in the $(n-1)$-dimensional subspace $\overline{\beta} \cap \overline{\beta}'$, and cannot lie in a subspace of lower dimension by Lemma 6.3.2. It is impossible that $r \in \beta = \overline{\beta} \cap \overline{\beta}'$; or else r projects the points of $\beta \cap \mathcal{K}$ onto an $(n-2)$-dimensional space, but these projected points form a truncated cone $\pi_{n-4}\mathrm{H}^r(2, q^2)$ which lies in a space of dimension $n-1$. Since $L^\perp \cap \mathcal{K} = \overline{\beta} \cap \mathcal{K}$, the lemma is proved. ■

Lemma 6.3.6. *Suppose that r is a point $r \in \mathrm{H}(2n, q^2) \backslash \mathcal{K}$ such that $|r^\perp \cap \mathcal{K}| = b_{n-1}$. Then there exists an n-dimensional subspace α_r, $r \notin \alpha_r$, such that $\alpha_r \cap \mathrm{H}(2n, q^2) = \pi_{n-3}\mathrm{H}(2, q^2)$, and such that the truncated cone $\pi^*_{n-3}\mathrm{H}(2, q^2)$ is equal to the set $r^\perp \cap \mathcal{K}$.*

Proof. Consider the $(n+1)$-dimensional space $\overline{\alpha}_r$ with $\overline{\alpha}_r \cap \mathrm{H}(2n, q^2) = \pi_{n-2}\mathrm{H}(2, q^2)$. Suppose that $\overline{\beta}_1$ is a hyperplane of $\overline{\alpha}_r$, not containing π_{n-2} and containing the point r. By Lemma 6.3.5, $\overline{\beta}_1$ contains an $(n-1)$-dimensional subspace β_1, $r \notin \beta_1$, such that $\beta_1 \cap \mathrm{H}(2n, q^2) = \pi^{\beta_1}_{n-4}\mathrm{H}^{\beta_1}(2, q^2)$ and $\overline{\beta}_1 \cap \mathcal{K} = \beta_1 \cap \mathcal{K} = \pi^{\beta_1 *}_{n-4}\mathrm{H}^{\beta_1}(2, q^2)$. Choose a tangent line T to $\mathrm{H}^{\beta_1}(2, q^2)$ in the plane of $\mathrm{H}^{\beta_1}(2, q^2)$. We can find a hyperplane $\overline{\beta}_2$ of $\overline{\alpha}_r$, $\overline{\beta}_2 \neq \overline{\beta}_1$, $r \in \overline{\beta}_2$, $\pi_{n-2} \not\subseteq \overline{\beta}_2$, $\beta_1 \not\subseteq \overline{\beta}_2$, but $\pi^{\beta_1}_{n-4}T \subseteq \overline{\beta}_2$. Again, by Lemma 6.3.5, we find an $(n-1)$-dimensional subspace β_2, $r \notin \beta_2$, $\beta_2 \cap \mathrm{H}(2n, q^2) = \pi^{\beta_2 *}_{n-4}\mathrm{H}^{\beta_2}(2, q^2)$. Necessarily, $\pi^{\beta_1}_{n-4} = \pi^{\beta_2}_{n-4}$, and T is a tangent line to $\mathrm{H}^{\beta_2}(2, q^2) \neq \mathrm{H}^{\beta_1}(2, q^2)$.

Define $\pi_1 := \langle \mathrm{H}^{\beta_1}(2, q^2) \rangle$ and $\pi_2 := \langle \mathrm{H}^{\beta_2}(2, q^2) \rangle$. Consider the n-dimensional space $\gamma = \langle \pi^{\beta_1}_{n-4}, \pi_1, \pi_2 \rangle$. The two planes π_1 and π_2 are skew to π_{n-2}, hence, $\pi_{n-2} \not\subseteq \gamma$. Furthermore, $r \notin \gamma$, since then γ would be an n-dimensional subspace on r, not containing π_{n-2}, spanned by points of $r^\perp \cap \mathcal{K}$, a contradiction with Lemma 6.3.5. We conclude that $\gamma \cap \mathrm{H}(2n, q^2) = \pi^\gamma_{n-3}\mathrm{H}^\gamma(2, q^2)$.

Choose an arbitrary Hermitian line $\mathrm{H}'(1, q^2) \subset \mathrm{H}^{\beta_1}(2, q^2)$, $\mathrm{H}'(1, q^2)$ containing the point $T \cap \mathrm{H}^{\beta_1}(2, q^2)$. Consider the $q^2 + 1$ $(n-1)$-dimensional subspaces δ_i of γ through the $(n-2)$-dimensional subspace $\langle \mathrm{H}'(1, q^2), \pi^{\beta_1}_{n-4} \rangle$. One δ_i, say δ_1, is the space $\langle \pi^\gamma_{n-3}\mathrm{H}'(1, q^2) \rangle$. Consider now a space δ_i, $i \neq 1$. It is clear that δ_i is spanned by points of $\mathcal{K}$, since the Hermitian curves $\mathrm{H}^{\beta_j}(2, q^2) \subset \mathcal{K}$, $j = 1, 2$, and δ_i intersects the spaces π_j in secants to $\mathrm{H}^{\beta_j}(2, q^2)$, $j = 1, 2$, or contains $\mathrm{H}^{\beta_1}(2, q^2)$. If p is a point of $\mathrm{H}(2n, q^2)$, $p \in \delta_i \setminus (\beta_1 \cup \beta_2 \cup \pi^\gamma_{n-3})$ such that $p \notin \mathcal{K}$, then by Lemma 6.3.2, the line $\langle r, p \rangle$ meets $\mathcal{K}$ in exactly one point t. But then the space $\langle t, \delta_i \rangle \subseteq \overline{\alpha}_r$ is an n-dimensional subspace on r, not containing π_{n-2} and spanned by points of $r^\perp \cap \mathcal{K}$, a contradiction with Lemma 6.3.5. We conclude that every point $p \in (\gamma \cap \mathrm{H}(2n, q^2)) \setminus \langle \mathrm{H}'(1, q^2), \pi^\gamma_{n-3} \rangle$ lies in $\mathcal{K}$.

Letting vary the Hermitian line $\mathrm{H}'(1, q^2)$, we can reach every point $p \in (\gamma \cap \mathrm{H}(2n, q^2)) \setminus \pi^{\gamma}_{n-3}$, since the intersection of all these Hermitian lines is empty. Hence, $\gamma \cap \mathcal{K} = \pi^{\gamma *}_{n-3}\mathrm{H}^{\gamma}(2, q^2)$, and the space γ is the space α_r. ■

Theorem 6.3.7. *Let* $\mathcal{K}$ *be a minimal blocking set of* $\mathrm{H}(2n, q^2)$, $n > 2$, $|\mathcal{K}| \leqslant q^{2n-2}(q^3 + 1)$. *Then there exists an* $(n - 2)$*-dimensional subspace* $\pi_{n-2} \subset \mathrm{H}(2n, q^2)$ *such that* $\mathcal{K}$ *is the truncated cone* $\pi^*_{n-2}\mathrm{H}(2, q^2) \subseteq \pi^{\perp}_{n-2}$*, and* $|\mathcal{K}| = q^{2n-2}(q^3 + 1)$.

Proof. From Lemma 6.3.4, we find a point $r \in \mathrm{H}(2n, q^2) \setminus \mathcal{K}$ satisfying $|r^{\perp} \cap \mathcal{K}| = b_{n-1}$. The n-dimensional subspace α_r from Lemma 6.3.6 meets $\mathrm{H}(2n, q^2)$ in a cone $\pi^r_{n-3}\mathrm{H}^r(2, q^2)$. Choose the base $H = \mathrm{H}(2n - 2, q^2)$ of the cone $r^{\perp} \cap \mathrm{H}(2n, q^2)$ in such a way that $\langle H \rangle$ contains the cone $\pi^r_{n-3}\mathrm{H}^r(2, q^2)$. Let L be a line of $\mathrm{H}(2n, q^2)$ on r such that $L \not\subseteq \pi^{r\perp}_{n-3}$, which implies that $L^{\perp}$ does not contain the vertex π^r_{n-3} of α_r. Thus $L^{\perp}$ meets α_r in a hyperplane of α_r, and this hyperplane meets $\mathrm{H}(2n, q^2)$ in a cone $\pi_{n-4}\mathrm{H}(2, q^2)$. Note that $n \geqslant 3$. If $n = 3$, then this hyperplane meets $\mathrm{H}(2n, q^2)$ in a Hermitian curve $\mathrm{H}(2, q^2)$.

As $L^{\perp} \cap \mathcal{K}$ is contained in $r^{\perp} \cap \mathcal{K} = \alpha_r \cap \mathcal{K}$, it follows that $L^{\perp} \cap \mathcal{K}$ is a truncated cone $\pi^*_{n-4}\mathrm{H}(2, q^2)$. Hence, $|L^{\perp} \cap \mathcal{K}| = b_{n-2}$. By Lemma 6.3.3, $|s^{\perp} \cap \mathcal{K}| = b_{n-1}$ for all points $s \in L$. Every point $s \in L$ gives rise to a truncated cone $s^{\perp} \cap \mathcal{K} = \pi^{s*}_{n-3}\mathrm{H}(2, q^2)$, and all these truncated cones share the truncated cone $L^{\perp} \cap \mathcal{K} = \pi^*_{n-4}\mathrm{H}(2, q^2)$. Denote the subspace spanned by $L^{\perp} \cap \mathcal{K}$ by β_L.

Every point of $\mathcal{K}$ is collinear with a point of L, which implies that $\mathcal{K}$ is the union of these $q^2 + 1$ cones. It follows that $|\mathcal{K}| = b_n$, and that $\mathcal{K}$ is contained in the union of the $q^2 + 1$ n-dimensional subspaces α_s, $s \in L$, that share the $(n - 1)$-dimensional subspace β_L.

Consider now a second line L' of $\mathrm{H}(2n, q^2)$ on r such that $L'^{\perp} \not\subseteq \pi^{r\perp}_{n-3} \cap \mathrm{H}(2n, q^2)$. and choose it in such a way that $\beta_L \not\subseteq L'^{\perp}$. This is possible since $\langle \beta_L, r\rangle^{\perp}$ has only dimension $n - 1$. Then, as for L, the subspace $\beta_{L'} := \langle L'^{\perp} \cap \mathcal{K}\rangle$ has dimension $n-1$, and is contained in α_s for all $s \in L'$. We have $\beta_{L'} \neq \beta_L$, since $\beta_L \not\subseteq L'^{\perp}$. Let p be a point of L' with $p \neq r$. Then α_p has dimension n and meets α_r in $\beta_{L'}$. Furthermore, $\beta_{L'} \cap \mathcal{K} = \pi^{L'*}_{n-4}\mathrm{H}^{L'}(2, q^2)$ and $|\mathrm{H}^{L'}(2, q^2) \cap \mathrm{H}^{L}(2, q^2)| \geqslant 1$.

Varying the point $t \in L'$, the tangent hyperplanes $t^{\perp}$ vary over the hyperplanes through $L'^{\perp}$, hence, every point of the $(n - 3)$-dimensional spaces π^s_{n-3}, $s \in L$, lies in some $t^{\perp}$, $t \in L'$. Every point of π^s_{n-3}, $s \in L$, lies on lines with q^2 points of $\mathcal{K}$ to the points of $\mathrm{H}^L(2, q^2) \cap \mathrm{H}^{L'}(2, q^2)$, and hence belongs to one of the vertices π^t_{n-3}, $t \in L'$.

Consider a fixed point $s \in L \setminus \{r\}$, two fixed points $p_1 \in \pi^r_{n-3}$, $p_2 \in \pi^s_{n-3}$, $p_1, p_2 \notin \pi^r_{n-3} \cap \pi^s_{n-3}$. Consider a fixed point $u \in \pi^{r*}_{n-3}\mathrm{H}^r(2, q^2)$, then it is possible to select a line L'', satisfying the conditions of L', for which $u \in L''^{\perp}$. Then the preceding arguments show that the set $\langle u, p_2\rangle \setminus \{p_2\}$ is contained in $\mathcal{K}$.

Consider an arbitrary line M of $\pi^{r*}_{n-3}\mathrm{H}^r(2, q^2)$ passing through p_1 and containing q^2 points of $\mathcal{K}$. The q^4 points of $\langle M, p_2\rangle \setminus \langle p_1, p_2\rangle$ all lie in $\mathcal{K}$; this implies that the truncated cone $\langle \pi^r_{n-3}, \pi^s_{n-3}\rangle^*\mathrm{H}^r(2, q^2)$ lies in $\mathcal{K}$. Since $|\mathcal{K}| = |\langle \pi^r_{n-3}, \pi^s_{n-3}\rangle^*\mathrm{H}^r(2, q^2)| = b_n$, this cone must be equal to $\mathcal{K}$. ■

Bijlage A

Nederlandstalige samenvatting

In dit proefschrift onderzoeken we voornamelijk partiële spreads van bepaalde eindige veralgemeende vierhoeken en blokkerende verzamelingen van bepaalde eindige klassieke polaire ruimten. We beschouwen steeds eindige veralgemeende vierhoeken en de eindige klassieke polaire ruimten die opgebouwd zijn uit deelstructuren van de eindige projectieve ruimten $\mathrm{PG}(n, q)$. We starten deze samenvatting met een kort overzicht van de belangrijkste definities en gekende resultaten uit Hoofdstuk 1. Daarna beschrijven we de resultaten uit de opeenvolgende hoofdstukken, alsook de belangrijkste conclusies.

A.1 Inleiding

In Hoofdstuk 1 wordt het onderzoek gesitueerd en worden basisbegrippen uit eindige projectieve meetkunde ingevoerd. We verwijzen naar enkele belangrijke referentiewerken [59, 60, 61, 81], voor een uitvoerige inleiding tot en beschrijving van eindige projectieve ruimten, eindige veralgemeende vierhoeken en eindige klassieke polaire ruimten. We herhalen hier kort de belangrijkste definities.

Definitie A.1.1. Een *blokkerende verzameling* van $\mathrm{PG}(2, q)$ is een verzameling $\mathcal{B}$ van punten van $\mathrm{PG}(2, q)$, zodat elke rechte van $\mathrm{PG}(2, q)$ minstens 1 punt van $\mathcal{B}$ bevat. Een blokkerende verzameling $\mathcal{B}$ wordt *triviaal* genoemd als ze een rechte van $\mathrm{PG}(2, q)$ bevat; ze wordt *minimaal* genoemd als $\mathcal{B} \setminus \{p\}$ geen blokkerende verzameling is voor elk punt $p \in \mathcal{B}$.

Definitie A.1.2. Een *t-spread* van $\mathrm{PG}(n, q)$ is een verzameling S van t-dimensionale deelruimten van $\mathrm{PG}(n, q)$ zodat elk punt van $\mathrm{PG}(n, q)$ bevat is in juist 1 element van S.

Definitie A.1.3. De *eindige klassieke polaire ruimten* zijn:

(i) De niet-singuliere kwadrieken in oneven dimensie, $Q^+(2n+1,q)$, $n \geqslant 1$, en $Q^-(2n+1,q)$, $n \geqslant 2$, samen met de deelruimten erin bevat; dit zijn polaire ruimten van rang $n+1$ en n.

(ii) De niet-singuliere parabolische kwadriek in even dimensie, $Q(2n,q)$, $n \geqslant 2$, samen met de deelruimten erin bevat; dit is een polaire ruimte van rang n.

(iii) De punten van $PG(2n+1,q)$, $n \geqslant 1$, samen met de totaal isotrope deelruimten van een niet-singuliere symplectische polariteit van $PG(2n+1,q)$; dit is een polaire ruimte van rang $n+1$.

(iv) De niet-singuliere Hermitische variëteit in $PG(2n,q)$, samen met de deelruimten erin bevat, $n \geqslant 2$ (respectievelijk, $PG(2n+1,q)$, $n \geqslant 1$); dit is een polaire ruimte van rang n (respectievelijk rang $n+1$).

Zij $\mathcal{S}$ een polaire ruimte van rang n, dan worden de deelruimten van $\mathcal{S}$ van dimensie $n-1$ ook *generatoren* genoemd.

Definitie A.1.4. Een *eindige veralgemeende vierhoek van de orde* (s,t) is een punt-rechte meetkunde $(\mathcal{P},\mathcal{B},\mathrm{I})$, $\mathcal{P}$ en $\mathcal{B}$ disjuncte verzamelingen, $\mathrm{I} \subseteq (\mathcal{P}\times\mathcal{B}) \cup (\mathcal{B}\times\mathcal{P})$, waarbij I voldoet aan de volgende axioma's:

(i) Elk punt is incident met $1+t$ rechten ($t \geqslant 1$) en twee verschillende rechten zijn incident met ten hoogste 1 punt.

(ii) Elke rechte is incident met $1+s$ punten ($s \geqslant 1$) en twee verschillende rechten zijn incident met ten hoogste 1 punt.

(iii) Als x een punt is en L een rechte niet incident met x, dan bestaat er een uniek paar $(y,M) \in \mathcal{P}\times\mathcal{B}$, zodat $x \mathrm{~I~} M \mathrm{~I~} y \mathrm{~I~} L$.

De natuurlijke getallen s en t zijn de *parameters* van de veralgemeende vierhoek $\mathcal{S}$ en $\mathcal{S}$ is een veralgemeende vierhoek van orde (s,t). Wanneer $s=t$, dan is $\mathcal{S}$ een veralgemeende vierhoek van de orde s.

We merken tenslotte op dat eindige klassieke polaire ruimten van rang 2 veralgemeende vierhoeken zijn.

Definitie A.1.5. Zij $\mathcal{G} = (\mathcal{P},\mathcal{B},\mathrm{I})$ een eindige veralgemeende vierhoek.

(i) Een *ovoïde* is een verzameling $\mathcal{O}$ van punten van $\mathcal{G}$ zodat elke rechte van $\mathcal{G}$ juist 1 punt van $\mathcal{O}$ bevat.

(ii) Een *blokkerende verzameling* is een verzameling $\mathcal{B}$ van punten van $\mathcal{G}$ zodat elke rechte van $\mathcal{G}$ ten minste 1 punt van $\mathcal{B}$ bevat.

(iii) Een *spread* is een verzameling S van rechten van $\mathcal{G}$ zodat elk punt van $\mathcal{G}$ bevat is in juist 1 element van S.

(iv) Een *bedekking* is een verzameling $\mathcal{C}$ van rechten van $\mathcal{G}$ zodat elk punt van $\mathcal{G}$ bevat is in ten minste 1 element van $\mathcal{C}$.

Definitie A.1.6. Zij $\mathcal{G} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ een eindige klassieke polaire ruimte van rang n, $n \geqslant 3$.

(i) Een *ovoïde* is een verzameling $\mathcal{O}$ van punten van $\mathcal{G}$ zodat elke generator van $\mathcal{G}$ juist 1 punt van $\mathcal{O}$ bevat.

(ii) Een *t-blokkerende verzameling* is een verzameling $\mathcal{B}$ van punten van $\mathcal{G}$ zodat elke t-dimensionale deelruimte van $\mathcal{G}$ ten minste 1 punt van $\mathcal{B}$ bevat. Als t de dimensie van de generatoren is, dan spreken wij ook kortweg van *blokkerende verzameling*.

(iii) Een *t-spread* is een verzameling S van t-dimensionale deelruimten van $\mathcal{G}$ zodat elk punt van $\mathcal{G}$ bevat is in juist 1 element van S.

(iv) Een *t-bedekking* is een verzameling $\mathcal{C}$ van t-dimensionale deelruimten van $\mathcal{G}$ zodat elk punt van $\mathcal{G}$ bevat is in ten minste 1 element van $\mathcal{C}$.

Voor een aantal basisresultaten omtrent de hierboven gedefinieerde structuren, verwijzen we naar Hoofdstuk 1.

Tot slot vermelden we de definities van een aantal begrippen gerelateerd aan een blokkerende verzameling en een bedekking van een veralgemeende vierhoek.

Beschouw een veralgemeende vierhoek $\mathcal{G} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$. Stel dat $\mathcal{B}$ een blokkerende verzameling is van $\mathcal{G}$. Een *meervoudige rechte met betrekking tot $\mathcal{B}$* is een rechte van $\mathcal{G}$ die minstens 2 punten van $\mathcal{B}$ bevat. De *surplus van een rechte van $\mathcal{G}$ met betrekking tot $\mathcal{B}$* is 1 minder dan het aantal punten van $\mathcal{B}$ bevat in deze rechte.

Stel dat $\mathcal{C}$ een bedekking is van $\mathcal{G}$. Een *meervoudig punt met betrekking tot $\mathcal{C}$* is een punt van $\mathcal{G}$ dat op minstens twee rechten van $\mathcal{C}$ ligt. De *surplus van een punt met betrekking tot $\mathcal{C}$* is 1 minder dan het aantal rechten van $\mathcal{C}$ door dit punt.

A.2 Maximale partiële spreads van translatie veralgemeende vierhoeken

Voor een willekeurige veralgemeende vierhoek is het bestaan van spreads geenszins gegarandeerd. Voor een aantal gekende klassen echter is het bestaan of niet bestaan van spreads aangetoond; we zullen verderop de details omtrent de beschouwde veralgemeende vierhoeken vermelden.

Hoofdstuk 2 is gebaseerd op gezamelijk werk met M. R. Brown en L. Storme [21].

Definitie A.2.1. Zij $\mathcal{G} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ een eindige veralgemeende vierhoek.

(i) Een partiële spread is een verzameling S van rechten van $\mathcal{G}$ zodat elk punt van $\mathcal{G}$ bevat is in ten hoogste 1 element van S. Een partiële spread heet *maximaal* als en slechts als S niet kan uitgebreid worden tot een grotere partiële spread.

(ii) Een bedekking is een verzameling $\mathcal{C}$ van rechten van $\mathcal{G}$ zodat elk punt van $\mathcal{G}$ bevat is in ten minste 1 element van $\mathcal{C}$. Een bedekking $\mathcal{C}$ heet *minimaal* als geen enkele echte deelverzameling van $\mathcal{C}$ nog een bedekking is.

Probleem 1. Wanneer de veralgemeende vierhoek $\mathcal{G} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ een spread heeft, dan kan men onderzoeken wanneer partiële spreads uitbreidbaar zijn tot spreads of wanneer bedekkingen een spread bevatten, en dus herleidbaar zijn tot spreads. Met andere woorden, we trachten een bovengrens, respectievelijk ondergrens, te vinden voor het aantal elementen in een maximale partiële spread, respectievelijk een minimale bedekking van $\mathcal{G}$. Men kan eveneens in het geval van niet bestaan van een spread deze boven- en ondergrens onderzoeken.

Probleem 1 is ook een klassiek probleem voor de ruimte $\mathrm{PG}(n, q)$. Recent werden er een aantal resultaten beschreven in [45]. Gebruikmakend van analoge technieken zullen we het bestaande probleem beschouwen voor een belangrijke klasse van eindige veralgemeende vierhoeken, namelijk translatie veralgemeende vierhoeken, verderop genoteerd met *TGQ*. We zullen TGQs definiëren aan de hand van een model in de projectieve ruimte $\mathrm{PG}(N, q)$. Voor een alternatieve definitie en belangrijke gekende resultaten hieromtrent, verwijzen we naar Hoofdstuk 2 en Sectie 2.1. Voorafgaand aan de definitie van TGQs, geven we een definitie van drie belangrijke structuren in $\mathrm{PG}(n, q)$.

Definitie A.2.2. (i) Een *ovaal* van $\mathrm{PG}(2, q)$ is een verzameling $\mathcal{O}$ van $q + 1$ punten van $\mathrm{PG}(2, q)$, zodat geen drie punten van $\mathcal{O}$ collineair zijn.

(ii) Een *ovoïde* van $\mathrm{PG}(3, q)$, $q > 2$, is een verzameling $\mathcal{O}$ van q^2+1 punten van $\mathrm{PG}(3, q)$, zodat geen drie punten van $\mathcal{O}$ collineair zijn.

(iii) Een *egg* is een verzameling $\mathcal{E}$ van q^m+1 $(n-1)$-dimensionale deelruimten α_i van $P = \mathrm{PG}(2n+m-1, q)$, zodat elke drie deelruimten een $(3n-1)$-dimensionale deelruimte van P opspannen, en zodat elke α_i bevat is in een $(n + m - 1)$-dimensionale deelruimte β_i, scheef aan elke α_j, $j \neq i$. We noemen β_i de *raakruimte aan $\mathcal{E}$ in het element α_i.*

We merken op dat het begrip "egg" een veralgemening is van de begrippen "ovaal" en "ovoïde". We vermelden kort enkele belangrijke gekende resultaten over ovalen en ovoïden.

Stelling A.2.3. (i) **(Segre [83, 84])** *In* $\mathrm{PG}(2, q)$*, q oneven, is elke ovaal een kegelsnede* $\mathrm{Q}(2, q)$*.*

(ii) **(Barlotti [7], Panella [77])** *Elke ovoïde van* $\mathrm{PG}(3, q)$*, q oneven, is een elliptische kwadriek* $\mathrm{Q}^-(3, q)$*.*

(iii) **(Brown [20])** *Als een ovoïde $\mathcal{O}$ van* $\mathrm{PG}(3, q)$*, $q > 2$, q even, een kegelsnede* $\mathrm{Q}(2, q)$ *bevat, dan is $\mathcal{O}$ een elliptische kwadriek* $\mathrm{Q}^-(3, q)$*.*

We definiëren nu de structuur $T(n, m, q)$.

Definitie A.2.4. Zij $\mathcal{E}$ een egg in $P = \mathrm{PG}(2n + m - 1, q)$. We bedden de ruimte P in $\mathrm{PG}(2n+m, q)$ in als hypervlak, en we definiëren een punt-rechte meetkunde $(\mathcal{P}, \mathcal{B}, \mathrm{I})$ als volgt. De verzameling $\mathcal{P}$ bestaat uit de volgende elementen:

(i) de punten van $\mathrm{PG}(2n + m, q) \setminus P$,

(ii) de $(n+m)$-dimensionale deelruimten van $\mathrm{PG}(2n+m, q)$ die P snijden in 1 van de raakruimten aan $\mathcal{E}$,

(iii) het symbool (∞).

De verzameling $\mathcal{B}$ bestaat uit de volgende elementen:

(a) de n-dimensionale deelruimten van $\mathrm{PG}(2n+m, q)$ die P snijden in een element van $\mathcal{E}$,

(b) de elementen van $\mathcal{E}$.

Een punt van type (i) is enkel incident met rechten van type (a); de incidentie is de natuurlijke incidentie van de ruimte $\mathrm{PG}(2n+m, q)$. Een punt van type (ii) is enkel incident met alle rechten van type (a) die erin bevat zijn en met het unieke element van $\mathcal{E}$ dat erin bevat is. Het punt (∞) is incident met alle rechten van type (b) en met geen enkele rechte van type (a).

De meetkunde $T(n,m,q)$ is een veralgemeende vierhoek van de orde (q^n, q^m).

De klasse van de TGQs is per definitie de klasse van alle structuren $T(n,m,q)$. We noteren $T(n,m,q)$ in wat volgt als $T_{n,m}(\mathcal{E})$. De TGQ $T(1,1,q)$ noteren we ook met $T_2(\mathcal{O})$, $\mathcal{O}$ is nu een ovaal van $\mathrm{PG}(2,q)$. De TGQ $T(1,2,q)$ noteren we ook met $T_3(\mathcal{O})$, $\mathcal{O}$ is nu een ovoïde van $\mathrm{PG}(3,q)$. We merken ook op dat $T_2(\mathcal{O}) \cong \mathrm{Q}(4,q)$ als en slechts als $\mathcal{O}$ een kegelsnede $\mathrm{Q}(2,q)$ is en dat $T_3(\mathcal{O}) \cong \mathrm{Q}^-(5,q)$ als en slechts als $\mathcal{O}$ een elliptische kwadriek $\mathrm{Q}^-(3,q)$ is. Omdat alle ovalen kegelsneden zijn als q oneven is, geldt ook $T_2(\mathcal{O}) \cong \mathrm{Q}(4,q)$ als q oneven is.

De TGQ $T_2(\mathcal{O})$ heeft steeds een ovoïde, terwijl de TGQ $T_3(\mathcal{O})$ steeds een spread heeft. Als $\mathcal{O}$ een translatieovaal van $\mathrm{PG}(2,q)$, q even, is, dan is $T_2(\mathcal{O})$ zelfduaal, dus dan heeft $T_2(\mathcal{O})$ een spread. Voor de definitie van translatieovaal verwijzen we naar Sectie 2.1.2; we vermelden dat de kegelsnede $\mathrm{Q}(2,q)$, q even, een translatieovaal is. Tenslotte heeft de TGQ $\mathrm{Q}(4,q)$, q oneven geen spread ([81]). Voor een willekeurige egg $\mathcal{E}$ is er geen criterium voor het bestaan of niet bestaan van spreads van $T_{n,m}(\mathcal{E})$. In Sectie 2.3 wordt een voorbeeld van een TGQ $T_{n,m}(\mathcal{E})$ gegeven, verschillend van $T_2(\mathcal{O})$ en verschillend van $T_3(\mathcal{O})$, die een spread bezit.

Resultaten over het vermelde spreadprobleem omtrent partiële t-spreads en t-bedekkingen van projectieve ruimten $\mathrm{PG}(n,q)$ en eindige klassieke polaire ruimten werden in [45] bereikt door karakteriseringen van bepaalde minihypers te gebruiken. We geven de definitie van minihypers en aanverwante structuren, en we vermelden de belangrijkste karakterisatieresultaten.

Definitie A.2.5. Een $\{f,m;N,q\}$-*minihyper* is een koppel (F,w), F een deelverzameling van de puntenverzameling van $\mathrm{PG}(N,q)$, en w een gewichtsfunctie $w : \mathrm{PG}(N,q) \to \mathbb{N} : x \mapsto w(x)$, die voldoet aan de volgende voorwaarden:

(i) $w(x) > 0 \iff x \in F$,

(ii) $\sum_{x\in F} w(x) = f$, en

(iii) $\min\{\sum_{x\in H} w(x) \| H \in \mathcal{H}\} = m$, met $\mathcal{H}$ de verzameling van alle hypervlakken van $\mathrm{PG}(N,q)$.

Karakterisatiestellingen omtrent minihypers gebruiken vaak de volgende structuur.

Definitie A.2.6. Stel dat $\mathcal{A}$ de verzameling is van alle t-dimensionale deelruimten van $\mathrm{PG}(N,q)$. Een *som van t-dimensionale deelruimten* is een gewichtsfunctie w: $\mathcal{A} \to \mathbb{N}$: $\pi_t \mapsto w(\pi_t)$. Een dergelijke som van t-deelruimten induceert een gewichtsfunctie op de deelruimten van kleinere dimensie als volgt. Als π_r een deelruimte is van dimensie $r < t$, dan definiëren we $w(\pi_r) = \sum_{\pi \in \mathcal{A}, \pi \supset \pi_r} w(\pi)$. Meer bepaald is het gewicht van een punt de som van de gewichten van alle t-dimensionale deelruimten door dat punt. Een som van t-dimensionale deelruimten heet een *som van n t-dimensionale deelruimten* als de som van de gewichten van alle t-dimensionale deelruimten gelijk is aan n.

We vermelden nu onmiddellijk de volgende karakterisatiestelling. We gebruiken $\theta_\mu = \frac{q^{\mu+1}-1}{q-1}$.

Stelling A.2.7. *Stel dat $q > 2$ en $\delta < \epsilon$, met $q + \epsilon$ de grootte van de kleinste niet triviale blokkerende verzameling van* PG$(2, q)$*. Als (F, w) een $\{\delta\theta_\mu, \delta\theta_{\mu-1}; N, q\}$-minihyper is, zodat $\mu \leqslant N - 1$, dan is w de gewichtsfunctie op de punten van* PG(N, q) *geïnduceerd door een som van δ μ-dimensionale deelruimten.*

A.2.1 Algemene resultaten

Stel nu dat S een partiële spread is van $T_{n,m}(\mathcal{E})$ met deficiëntie δ, dit is, $|S| = q^{n+m} + 1 - \delta$. Een *gat met betrekking tot S* is een punt x van $T_{n,m}(\mathcal{E})$, zodat x niet bevat is in een rechte van S. Voor het vervolg noteren we met π_0 de $(2n + m - 1)$-dimensionale deelruimte die $\mathcal{E}$ bevat. Stel dat $n > 1$, m.a.w., $\mathcal{E}$ is geen ovaal of ovoïde, dan stellen we voor elk punt $x \in \pi_0$ dat $x \in \mathcal{E}$ als en slechts als $x \in \alpha$ voor een element $\alpha \in \mathcal{E}$.

Definitie A.2.8. Stel dat $\alpha \in \mathcal{E}$. Definieer $A_\alpha = q^n$ als $\alpha \in S$ en definieer A_α als het aantal rechten van type (a), incident in PG$(2n + m, q)$ met α, als $\alpha \notin S$. We definiëren de *lokale deficiëntie* van α met betrekking tot S als $\Delta_\alpha = q^n - A_\alpha$. We definiëren de *lokale deficiëntie* van $x \in \alpha$, $\alpha \in \mathcal{E}$, met betrekking tot S, als $\delta_x = \Delta_\alpha$.

Als $n = 1$, m.a.w., als $\mathcal{E}$ een ovaal of een ovoïde is, dan zijn de elementen van $\mathcal{E}$ reeds punten van π_0, dus dan valt het onderscheid tussen Δ_x en δ_x weg. De volgende definitie is geldig in alle gevallen.

Definitie A.2.9. Stel dat S een partiële spread is van $T_{n,m}(\mathcal{E})$, $|S| = q^{n+m} + 1 - \delta$. We definiëren een gewichtsfunctie $w_S :$ PG$(2n + m, q) \to \mathbb{N}$ als volgt:

(i) Als $x \in$ PG$(2n + m, q) \setminus \pi_0$ en x is een gat met betrekking tot S, dan $w_S(x) = 1$, zoniet, $w_S(x) = 0$.

(ii) Als $x \in \mathcal{E}$, definieer $w_S(x) = \delta_x$.

(iii) $w_S(x) = 0, \forall x \in \pi_0 \setminus \mathcal{E}$.

Deze gedefinieerde gewichtsfunctie bepaalt een minihyper (F, w_S) van $\mathrm{PG}(2n+m, q)$. We vermelden het volgende lemma, dat onmiddellijk tot de eerste stelling leidt.

Lemma A.2.10. *Stel dat S een partiële spread is van $T_{n,m}(\mathcal{E})$, met deficiëntie $\delta < q$, zodat het punt (∞) bedekt wordt. Dan is w_S de gewichtsfunctie van een $\{\delta\theta_n, \delta\theta_{n-1}; 2n+m, q\}$-minihyper (F, w_S).*

Gebruikmakend van de karakterisatie van dergelijke minihypers (Stelling A.2.7), bekomen we de volgende stelling.

Stelling A.2.11. *Stel dat S een partiële spread is van $T_{n,m}(\mathcal{E})$, met deficiëntie $\delta < \epsilon$, met $q + \epsilon$ de grootte van de kleinste niet triviale blokkerende verzameling van $\mathrm{PG}(2, q)$, en zodat het punt (∞) bedekt wordt, dan kan S uitgebreid worden tot een spread van $T_{n,m}(\mathcal{E})$.*

A.2.2 Verbetering voor q een kwadraat

Wanneer q een kwadraat is, dan is de kleinste niet triviale blokkerende verzameling van $\mathrm{PG}(2, q)$ een Baer deelvlak. Niet zozeer de verschillen in ondergrenzen voor de kleinste niet triviale blokkerende verzamelingen, maar echter het vinden van deze Baer deelmeetkundes stelt ons in staat Stelling A.2.11 te verbeteren voor $T_2(\mathcal{O})$ wanneer q een kwadraat is. We bekomen het volgende resultaat.

Stelling A.2.12. *Stel dat q een kwadraat is en dat S een partiële spread is van $T_2(\mathcal{O})$, met deficiëntie $\delta \leqslant \frac{q}{4}$, zodat elke blokkerende verzameling van $\mathrm{PG}(2, q)$ van grootte ten hoogste $q + \delta$ een rechte of een Baer deelvlak bevat, en zodat (∞) bedekt wordt. Dan kan S uitgebreid worden tot een spread van $T_2(\mathcal{O})$.*

Er bestaat een niet triviale blokkerende verzameling van $\mathrm{PG}(2, q)$, van grootte $q + \frac{q}{4} + 1$, q even en een kwadraat, die geen Baer deelvlak bevat. Daardoor kunnen de grenzen van Stelling A.2.12 de waarde $\frac{q}{4}$ niet overschrijden.

A.2.3 Het resultaat voor $Q(4, q)$, q even

Wanneer we veronderstellen dat S een partiële spread is, zodat (∞) niet bedekt wordt, dan wordt in zekere zin de symmetrie verstoord; de verzameling gaten met betrekking tot S vormt dan geen $\{\delta\theta_n, \delta\theta_{n-1}; 2n+m, q\}$-minihyper.

Door gebruik te maken van het model $T_2(\mathcal{O})$ en de uitbreidbaarheid van $(q-1)$-bogen van $\mathrm{PG}(2,q)$, kunnen we interessante resultaten bekomen. We vermelden hierbij het volgende resultaat.

Stelling A.2.13. *Stel dat q even is, en stel dat S een maximale partiële spread van $T_2(\mathcal{O})$ is, met deficiëntie $\delta \leqslant q-1$. Dan moet het punt (∞) bedekt worden.*

Wanneer we nu veronderstellen dat $\mathcal{O}$ een kegelsnede $\mathrm{Q}(2,q)$ is, m.a.w., $T_2(\mathcal{O}) \cong \mathrm{Q}(4,q)$, dan vinden we de volgende stelling.

Stelling A.2.14. *Stel dat q even is. Als S een maximale partiële spread is van $\mathrm{Q}(4,q)$ met positieve deficiëntie, dan geldt $|S| \leqslant q^2-q+1$.*

Deze laatste stelling verbetert punt (ii) van de volgende stelling.

Stelling A.2.15. (Tallini [88]) *Beschouw $\mathrm{Q}(4,q)$:*

(i) *als q oneven is, dan heeft $\mathrm{Q}(4,q)$ geen spreads, en als S een partiële spread is, dan geldt: $|S| \leqslant q^2-q+1$.*

(ii) *als q even is, $q \geqslant 4$, en als S een maximale partiële spread met positieve deficiëntie is, dan geldt: $|S| < q^2 - \frac{q}{2}$.*

A.2.4 Voorbeelden van maximale partiële spreads van $T_2(\mathcal{O})$ en $T_3(\mathcal{O})$

Op het einde van Sectie 2.3 volgen voorbeelden van maximale partiële spreads van $T_2(\mathcal{O})$ en $T_3(\mathcal{O})$. We vermelden het volgende resultaat.

Stelling A.2.16. (i) *Als $T_2(\mathcal{O})$, q even, een spread heeft, dan heeft $T_2(\mathcal{O})$ een maximale partiële spread van grootte q^2-q+1, zodat (∞) bedekt is.*

(ii) *De GQ $T_3(\mathcal{O})$ heeft een maximale partiële spread van grootte q^3-q+1, zodat (∞) bedekt wordt.*

(iii) *Als $\mathcal{O}$ de Tits ovoïde is, dan heeft $T_3(\mathcal{O})$ een maximale partiële spread van grootte q^3-q+2, zodat (∞) bedekt wordt.*

Uit Stelling A.2.16 volgt dat de gevonden grens uit Stelling A.2.14 scherp is.

A.3 De kleinste minimale blokkerende verzamelingen van $\mathbf{Q(6,q)}$, q even

Hoofdstuk 3 behandelt de karakterisering van de kleinste blokkerende verzamelingen van $Q(6,q)$, q even, $q \geqslant 32$. Het is een gekend resultaat dat $Q(6,q)$, q even, geen ovoïde heeft [92]. Het is bijgevolg een natuurlijk probleem om de kleinste minimale blokkerende verzamelingen van $Q(6,q)$ te onderzoeken. We zullen veronderstellen dat $\mathcal{K}$ een minimale blokkerende verzameling is van $Q(6,q)$, zodat $q^3+1 < |\mathcal{K}| \leqslant q^3+q$. De bovengrens is zo gekozen omdat we een eenvoudig voorbeeld van een minimale blokkerende verzameling van $Q(6,q)$ kunnen vinden. Kies een willekeurig punt $p \in Q(6,q)$. Beschouw de basis $Q(4,q)$ van de kegel $T_p(Q(6,q)) \cap Q(6,q)$, en kies een ovoïde $\mathcal{O}$ van deze basis $Q(4,q)$. Beschouw de kegel met top p en basis $\mathcal{O}$, dan vormen de punten van deze kegel zonder de top p, een minimale blokkerende verzameling van $Q(6,q)$, van grootte q^3+q.

Hoofdstuk 3 is gebaseerd op gezamenlijk werk met L. Storme [37].

Een belangrijk idee omtrent projecties van ovoïden van polaire ruimten vinden we terug in [92]. De informatie is beperkt tot ovoïden van $Q(2n,q)$, maar het idee kan uitgebreid worden naar blokkerende verzamelingen van polaire ruimten in het algemeen.

Lemma A.3.1. *Als de kwadriek $Q(2n,q)$, $n \geqslant 2$, ovoïden heeft, dan heeft elke kwadriek $Q(2m,q)$, $n \geqslant m \geqslant 2$, een ovoïde.*

In het lemma wordt de projectie van de ovoïde van $Q(2n,q)$ beschouwd, vanuit een punt van de kwadriek niet op deze ovoïde. Dit idee zullen we ook gebruiken om gegevens over blokkerende verzamelingen van $Q(4,q)$, q even, te gebruiken om eigenschappen van de blokkerende verzamelingen $\mathcal{B}$ van $Q(6,q)$, q even, te bewijzen.

De volgende stelling uit [42] geeft belangrijke informatie over blokkerende verzamelingen van $Q(4,q)$, q even, $q \geqslant 32$.

Stelling A.3.2. *Stel dat $\mathcal{B}$ een blokkerende verzameling is van $Q(4,q)$, q even, $q \geqslant 32$, met als grootte q^2+1+r, met $0 < r \leqslant \sqrt{q}$. Dan is $\mathcal{B}$ de unie van een ovoïde met een verzameling van r extra punten buiten deze ovoïde. Bijgevolg heeft een minimale blokkerende verzameling van $Q(4,q)$, q even, $q \geqslant 32$, verschillend van een ovoïde, een grootte van q^2+1+r, $r > \sqrt{q}$.*

Omdat $Q(4,q)$, q even, zelfduaal is, kan informatie over bedekkingen van $Q(4,q)$ vertaald worden naar informatie over blokkerende verzamelingen. Het blijkt dat een uitgebreide versie van de volgende stelling nuttig is om de informatie over de grootte van een blokkerende verzameling van $Q(6,q)$ te verkrijgen. We vermelden eerst de basisstelling.

Stelling A.3.3. *Stel dat $\mathcal{C}$ een minimale bedekking is van* Q$(4, q)$*. Stel dat $|\mathcal{C}| = q^2 + 1 + r$, met $q + r$ kleiner dan de grootte van de kleinste niet triviale blokkerende verzameling in* PG$(2, q)$*. De meervoudige punten van $\mathcal{C}$ vormen een som van rechten, bevat in* Q$(4, q)$*, waarbij het gewicht van een rechte in deze som gelijk is aan het gewicht van deze rechte met betrekking tot de bedekking, en waarbij de som van de gewichten van de rechten gelijk is aan r.*

In Sectie 3.3 wordt een uitgebreide versie van deze stelling bewezen. We vermelden deze uitgebreide versie.

Lemma A.3.4. *Stel dat $\mathcal{C}$ een minimale bedekking is van* Q$(4, q)$*, met grootte $q^2 + 1 + r$, $0 < r \leqslant q - 1$. Als elk meervoudig punt minstens surplus $\sqrt{q}$ heeft, dan is de verzameling van meervoudige punten een som van rechten, met de som van de gewichten van de rechten gelijk aan r.*

Dit lemma geeft aanleiding tot het volgende lemma, waarvan we het bewijs geven in Sectie 3.3.

Lemma A.3.5. *Voor een minimale blokkerende verzameling van* Q$(4, q)$*, q even, met grootte $q^2 + 1 + r$, $r > 0$, zodat er enkel punten zijn met positieve excess van tenminste $\sqrt{q}$, geldt dat $r \geqslant \frac{q+4}{6}$.*

Door gebruik te maken van het model $T_2(\mathcal{O})$ voor Q$(4, q)$ bekomen we ook het volgende resultaat.

Lemma A.3.6. *Stel dat $\mathcal{C}$ een minimale bedekking is van* Q$(4, q)$*, met grootte $q^2 + 1 + r$, $0 < r < q$, zodat er een rechte $L \in$* Q$(4, q) \setminus \mathcal{C}$ *is zodat elk punt van L op $r + 1$ rechten van $\mathcal{C}$ ligt, en zodat alle andere punten van* Q$(4, q)$ *op 1 rechte van $\mathcal{C}$ liggen, dan geldt $(r + 2)|q$, of, $r = q - 1$. Daarenboven geldt dat $r \leqslant \frac{q}{2} - 2$ onmogelijk is.*

We vermelden nu de belangrijkste stappen om tot de karakterisatie te komen. We veronderstellen dat $\mathcal{K}$ een minimale blokkerende verzameling is van Q$(6, q)$, $|\mathcal{K}| = q^3 + 1 + \delta$, $0 < \delta \leqslant q - 1$. Het projectieargument, samen met de veronderstellingen omtrent $\mathcal{K}$, leidt tot de volgende twee lemma's.

Lemma A.3.7. *Voor elk punt $p \in$* Q$(6, q)$*, $p \in \mathcal{K}$, geldt $|T_p($*Q$(6, q)) \cap \mathcal{K}| \leqslant 1 + \delta$.

Lemma A.3.8. *Voor elk punt $p \in$* Q$(6, q) \setminus \mathcal{K}$ *vinden we dat p de punten van $T_p($*Q$(6, q)) \cap \mathcal{K}$ *projecteert op een minimale blokkerende verzameling van* Q$(4, q)$*, met* Q$(4, q)$ *de basis van de kegel $T_p($*Q$(6, q)) \cap$* Q$(6, q)$*.*

Telargumenten, samen met de resultaten van Stellingen A.3.2 en A.3.3, en Lemma's A.3.4, A.3.5 en A.3.6 leiden tot het volgende lemma.

Lemma A.3.9. *Stel dat L een rechte is van $Q(6, q)$, zodat L minstens 2 punten van $\mathcal{K}$ bevat. Dan bevat L minstens $\frac{q+10}{6}$ punten van $\mathcal{K}$.*

Deze informatie stelt ons dan in staat de volgende karakterisatiestelling te bewijzen. De voorwaarde $q \geqslant 32$ volgt rechtstreeks uit de voorwaarde $q \geqslant 32$ bij Stelling A.3.2.

Stelling A.3.10. *Stel dat $\mathcal{K}$ een minimale blokkerende verzameling is van $Q(6, q)$, q even, $q \geqslant 32$, zodat $|\mathcal{K}| \leqslant q^3 + q$. Dan bestaat er een punt $p \in Q(6, q) \setminus \mathcal{K}$, zodat $T_p(Q(6, q)) \cap Q(6, q) = pQ(4, q)$ en de punten van $\mathcal{K}$ zijn de punten van de rechten L door p die $Q(4, q)$ snijden in een ovoïde $\mathcal{O}$, behalve het punt p zelf. Daarenboven geldt $|\mathcal{K}| = q^3 + q$.*

Stelling A.3.10 werd onafhankelijk bewezen werd door K. Metsch [70]. We vermelden zijn resultaat.

Stelling A.3.11. *Elke blokkerende verzameling van $W(2n + 1, q)$ bevat minstens $q^{n+1} + q^{n-1}$ punten. Er kan enkel gelijkheid zijn als q even is en dan bestaat deze verzameling uit de punten van een kegel met top een $(n-2)$-dimensionale ruimte π en basis een ovoïde van $W(3, q)$, bevat in de poolruimte van π, behalve de punten van de top zelf.*

Omdat voor even q geldt dat $W(5, q) \cong Q(6, q)$ kan onze karakterisatiestelling hieruit afgeleid worden. Wanneer we de twee bewijzen vergelijken, dan zien we echter vrij belangrijke verschillen. Ons bewijs steunt essentieel op resultaten omtrent minimale blokkerende verzamelingen van $Q(4, q)$, q even, terwijl het bewijs van K. Metsch voornamelijk combinatorische argumenten over $W(2n+1, q)$ gebruikt en er geen grenzen nodig zijn voor minimale blokkerende verzamelingen van $W(3, q)$ verschillend van een ovoïde.

A.4 De kleinste minimale blokkerende verzamelingen van $Q(6, q)$, q oneven priem

In Hoofdstuk 4 worden de kleinste blokkerende verzamelingen van $Q(6, q)$, q oneven, onderzocht. In tegenstelling tot $Q(6, q)$, q even, is het bestaan of niet bestaan van ovoïden van $Q(6, q)$, q oneven, niet volledig gekend. Het is bekend dat $Q(6, q)$, $q = 3^r$, $r \geqslant 1$, ovoïden heeft. Daarenboven werd recent aangetoond dat $Q(4, q)$, q oneven priem, enkel elliptische kwadrieken $Q^-(3, q)$ als ovoïde bezit [6]. Dit heeft als gevolg dat $Q(6, q)$, $q > 3$, priem, geen ovoïden bezit [76]. Het bestaan of niet bestaan van ovoïden voor alle andere waarden van q is een open probleem; er is echter een conjectuur die stelt dat Q$(6, q)$ ovoïden heeft als en slechts als $q = 3^r$, $r \geqslant 1$.

Hoofdstuk 4 is gebaseerd op gezamelijk werk met L. Storme [36] en gezamelijk werk met K. Metsch [34].

In dit hoofdstuk gaan we na in hoeverre de resultaten uit Hoofdstuk 3 vertaald kunnen worden naar Q$(6, q)$, q oneven. Als eerste invalshoek besluiten we om dezelfde weg te bewandelen als in Hoofdstuk 3, daar de combinatorische argumenten en de projectieargumenten onafhankelijk zijn van q. Het is echter ook duidelijk dat Stelling A.3.2 zeer belangrijk is, en een versie voor q oneven is dan ook noodzakelijk als we dezelfde technieken willen gebruiken. Doordat een even sterke versie van Stelling A.3.2 voor oneven q voorlopig echter niet gekend is, kunnen we met de gekende technieken uit Hoofdstuk 3 enkel resultaten bekomen voor $q = 3, 5, 7$. We starten met een overzicht van deze resultaten, waarna we beschrijven hoe we een karakterisatieresultaat verkrijgen voor $q > 3$, q priem. We veronderstellen voor de rest van deze sectie dat $\mathcal{K}$ een minimale blokkerende verzameling is van Q$(6, q)$, q oneven, $|\mathcal{K}| = q^3 + 1 + \delta$, $0 < \delta < q$.

A.4.1 De karakterisatie voor $q = 3, 5, 7$

Zoals vermeld is een versie van Stelling A.3.2 nodig voor q oneven. Uit [42] beschikken we over het volgende resultaat.

Stelling A.4.1. *Stel dat $\mathcal{C}$ een bedekking is van de klassieke veralgemeende vierhoek $\mathcal{S}$ van orde (q, t). Stel dat $|\mathcal{C}| = qt + r + 1$, met $q + r$ kleiner dan de grootte van de kleinste niet triviale blokkerende verzameling van* PG$(2, q)$*. Dan zijn de meervoudige punten met betrekking tot $\mathcal{C}$ de punten van een som van rechten van* PG(n, q)*, waarbij deze punten alle bevat zijn in $\mathcal{S}$, en zodat het gewicht van een rechte in deze som gelijk is aan het gewicht van deze rechte met betrekking tot de bedekking, en met de som van de gewichten gelijk aan r.*

Uit deze stelling kunnen we een belangrijk gevolg halen omtrent blokkerende verzamelingen van Q$(4, q)$, q oneven.

Stelling A.4.2. *Stel dat $\mathcal{B}$ een blokkerende verzameling is van* Q$(4, q)$*, q oneven, $|\mathcal{B}| = q^2 + 1 + r$, $q + r$ kleiner dan de grootte van de kleinste niet triviale blokkerende verzameling van* PG$(2, q)$*. Als $r = 1$, dan hebben alle meervoudige rechten juist 1 punt $p \in$* Q$(4, q) \setminus \mathcal{B}$ *gemeenschappelijk. Als $r = 2$, en alle meervoudige rechten hebben excess 2, dan hebben alle juist 1 punt $p \in$* Q$(4, q) \setminus \mathcal{B}$ *gemeenschappelijk.*

Hieruit kunnen we onmiddellijk het volgende lemma bewijzen.

Lemma A.4.3. *Als $\mathcal{B}$ een minimale blokkerende verzameling is van* $\mathrm{Q}(4,3)$*, verschillend van een ovoïde, dan geldt* $|\mathcal{B}| > 11$.

Opnieuw gebruikmakend van Stelling A.4.1, vinden we, met behulp van een computer, de volgende resultaten voor $q = 5, 7$.

Lemma A.4.4. *Stel dat $\mathcal{B}$ een minimale blokkerende verzameling is van* $\mathrm{Q}(4,q)$*,* $q = 5, 7$*, verschillend van een ovoïde van* $\mathrm{Q}(4,q)$*. Dan geldt* $|\mathcal{B}| > q^2 + 2$.

Lemma A.4.5. *Er bestaat geen minimale blokkerende verzameling $\mathcal{B}$ van* $\mathrm{Q}(4,7)$*, met* $|\mathcal{B}| = 52$*, zodat er een punt* $p \in \mathrm{Q}(4,7) \setminus \mathcal{B}$ *bestaat met de eigenschap dat alle rechten van* $\mathrm{Q}(4,7)$ *door p juist* 3 *punten van $\mathcal{B}$ bevatten.*

Deze drie lemma's zijn voldoende als vervanging van Stelling A.3.2 om, met behulp van dezelfde technieken als in Hoofdstuk 3, de kleinste minimale blokkerende verzamelingen verschillend van een ovoïde van $\mathrm{Q}(6,q)$, $q = 3, 5, 7$, te classificeren. We vermelden dat de Lemma's A.3.7 en A.3.8 ook hier een belangrijke rol spelen.

Stelling A.4.6. *Stel dat $\mathcal{K}$ een minimale blokkerende verzameling is van* $\mathrm{Q}(6,q)$*,* $q = 3, 5, 7$*, verschillend van een ovoïde van* $\mathrm{Q}(6,q)$*,* $|\mathcal{K}| \leqslant q^3 + q$*. Dan bestaat er een punt* $p \in \mathrm{Q}(6,q) \setminus \mathcal{K}$ *met de volgende eigenschap:* $p^\perp \cap \mathrm{Q}(6,q) = p\mathrm{Q}(4,q)$ *en $\mathcal{K}$ bestaat uit de punten van de rechten L van* $\mathrm{Q}(6,q)$ *door p die* $\mathrm{Q}(4,q)$ *snijden in een elliptische kwadriek* $\mathrm{Q}^-(3,q)$*, behalve het punt p zelf. Daarenboven geldt* $|\mathcal{K}| = q^3 + q$.

A.4.2 De karakterisatie voor $q > 3$, q priem

Wanneer we terug het projectieargument, met name de Lemma's A.3.7 en A.3.8, beschouwen, dan kunnen we nagaan of we de classificatie van ovoïden van $\mathrm{Q}(4,q)$, q oneven priem, niet kunnen aanwenden om de kleinste blokkerende verzamelingen van $\mathrm{Q}(6,q)$, q oneven priem, te karakteriseren. Immers, stel dat voor een punt $p \in \mathrm{Q}(6,q) \setminus \mathcal{K}$ geldt dat $|p^\perp \cap \mathcal{K}| = q^2 + 1$. Lemma A.3.8 impliceert dan onmiddellijk dat de punten van $p^\perp \cap \mathcal{K}$ door p op een elliptische kwadriek $\mathrm{Q}^-(3,q)$ geprojecteerd worden.

Deze observatie leidt vrijwel onmiddellijk tot het volgende lemma.

Lemma A.4.7. *Beschouw een punt* $p \in \mathrm{Q}(6,q) \setminus \mathcal{K}$*, dan geldt* $|p^\perp \cap \mathcal{K}| \geqslant q^2 + 1$*. Als er gelijkheid optreedt, dan bestaat er een* 4*-dimensionale deelruimte* α_p *door p die* $\mathrm{Q}(6,q)$ *snijdt in een kegel met top p en basis een elliptische kwadriek, en zodat elke rechte van* $\mathrm{Q}(6,q) \cap \alpha_p$ *door p de blokkerende verzameling $\mathcal{K}$ in een uniek punt snijdt.*

We vermelden daarbij ook het volgende lemma.

Lemma A.4.8. *Veronderstel dat L een rechte is van* $\mathrm{Q}(6,q)$*, scheef aan* $\mathcal{K}$*, en zodat L twee punten p_1 en p_2 bevat met de eigenschap* $|p_i^\perp \cap \mathcal{K}| = q^2+1$*,* $i = 1, 2$*. Dan bestaat er een vlak β dat* $\mathrm{Q}(6,q)$ *snijdt in een kegelsnede* $\mathrm{Q}(2,q)$ *en zodat* $L^\perp \cap \mathcal{K} = \mathrm{Q}(2,q)$*.*

Met behulp van een telargument kunnen we eenvoudig een groot aantal punten $p \in \mathrm{Q}(6,q) \setminus \mathcal{K}$ vinden met de eigenschap $|p^\perp \cap \mathcal{K}| = q^2+1$. Daarenboven impliceert Lemma A.4.8 het bestaan van een groot aantal kegelsneden in $\mathcal{K}$. De speciale eigenschap dat alle ovoïden elliptische kwadrieken $\mathrm{Q}^-(3,q)$ zijn, levert dus onmiddellijk een speciale structuur voor $\mathcal{K}$. We vermelden hierbij twee lemma's.

Lemma A.4.9. *Veronderstel dat voor een punt* $p \in \mathrm{Q}(6,q) \setminus \mathcal{K}$ *geldt dat* $|p^\perp \cap \mathcal{K}| = q^2+1$*, en dat het punt r behoort tot* $\alpha_p \cap \mathcal{K}$*. Dan is het mogelijk de verzameling* $\alpha_p \cap \mathcal{K}$ *te schrijven als de unie van q kegelsneden* $\mathrm{Q}(2,q)$*, die twee aan twee enkel het punt r gemeenschappelijk hebben. Er zijn daarenboven ten minste* $\frac{1}{2}(q+1)$ *verschillende manieren om deze verzameling te beschrijven.*

De eerste voorwaarde in het volgende lemma ontstaat juist omdat we zullen gebruiken dat $\frac{1}{2}(q+1) > 2$.

Lemma A.4.10. *Veronderstel dat $q > 3$. Veronderstel dat voor een punt* $p \in \mathrm{Q}(6,q) \setminus \mathcal{K}$ *geldt dat* $|p^\perp \cap \mathcal{K}| = q^2+1$*. Dan is de verzameling* $p^\perp \cap \mathcal{K}$ *een elliptische kwadriek* $\mathrm{Q}^-(3,q)$*.*

Dit lemma stelt ons nu in staat om, met behulp van een drietal andere lemma's, de volgende karakterisatiestelling te bewijzen.

Stelling A.4.11. *Stel dat $\mathcal{K}$ een minimale blokkerende verzameling is van* $\mathrm{Q}(6,q)$*, $q > 3$ priem, zodat* $|\mathcal{K}| \leqslant q^3+q$*. Dan bestaat er een punt* $p \in \mathrm{Q}(6,q) \setminus \mathcal{K}$ *met de volgende eigenschap:* $p^\perp \cap \mathrm{Q}(6,q) = p\mathrm{Q}(4,q)$ *en $\mathcal{K}$ bestaat uit de punten op de rechten van* $\mathrm{Q}(6,q)$ *door p die* $\mathrm{Q}(4,q)$ *snijden in een elliptische kwadriek* $\mathrm{Q}^-(3,q)$*, behalve het punt p zelf. Daarenboven geldt* $|\mathcal{K}| = q^3+q$*.*

A.5 De kleinste minimale blokkerende verzamelingen van $\mathrm{Q}(2n,q)$, q oneven priem

Hoofdstuk 5 is gebaseerd op gezamelijk werk met L. Storme [36]. In dit hoofdstuk gebruiken we de resultaten van Hoofdstuk 4 om de kleinste blokkerende verzamelingen van $\mathrm{Q}(2n,q)$, $n \geqslant 4$, q oneven priem, te karakteriseren. Met betrekking tot ovoïden vermelden we het volgende resultaat.

Stelling A.5.1. (Gunawardena and Moorhouse [49]) *De polaire ruimte* $\mathrm{Q}(8,q)$, *q oneven, heeft geen ovoïden.*

Door Lemma A.3.1 kunnen we besluiten dat $\mathrm{Q}(2n,q)$, q oneven, $n \geqslant 4$, geen ovoïden heeft. Net zoals voor $\mathrm{Q}(6,q)$ kunnen we voor $\mathrm{Q}(2n,q)$ vrij eenvoudig voorbeelden van minimale blokkerende verzamelingen vinden. We gebruiken daarvoor de volgende definitie.

Definitie A.5.2. Stel dat $\alpha\mathcal{O}$ een kegel is, met top α, een k-dimensionale ruimte, en basis $\mathcal{O}$, met $\mathcal{O}$ een puntenverzameling gelegen in een ruimte π, $\pi \cap \alpha = \emptyset$. Dan definiëren we de *afgeknotte kegel* $\alpha^*\mathcal{O}$ als $\alpha\mathcal{O} \setminus \alpha$, met andere woorden, de punten van $\alpha^*\mathcal{O}$ zijn de punten van $\alpha\mathcal{O}$ met uitzondering van de punten van α. Als α de ledige deelruimte is, dan geldt bij definitie $\alpha^*\mathcal{O} = \mathcal{O}$.

Stel nu dat $\mathrm{Q}(6,q)$, q oneven, geen ovoïden heeft. Dan is $\pi^*_{n-3}\mathcal{O}$, $\pi_{n-3} \subseteq \mathrm{Q}(2n,q)$, een $(n-3)$-dimensionale ruimte, $\pi^{\perp}_{n-3} \cap \mathrm{Q}(2n,q) = \pi_{n-3}\mathrm{Q}(4,q)$, $\mathcal{O}$ een ovoïde van $\mathrm{Q}(4,q)$, een minimale blokkerende verzameling van $\mathrm{Q}(2n,q)$, $n \geqslant 4$, met als grootte $q^n + q^{n-2}$.

Stel nu dat $\mathrm{Q}(6,q)$, q oneven, een ovoïde heeft. Dan is $\pi^*_{n-4}\mathcal{O}$, $\pi_{n-4} \subseteq \mathrm{Q}(2n,q)$ een $(n-4)$-dimensionale ruimte, $\pi^{\perp}_{n-4} \cap \mathrm{Q}(2n,q) = \pi_{n-4}\mathrm{Q}(6,q)$, $\mathcal{O}$ een ovoïde van $\mathrm{Q}(6,q)$, een minimale blokkerende verzameling van $\mathrm{Q}(2n,q)$, $n \geqslant 4$, met als grootte $q^n + q^{n-3}$.

Voor bepaalde waarden van q oneven kennen de kleinste minimale blokkerende verzameling (verschillend van een ovoïde) van $\mathrm{Q}(6,q)$, we zullen deze informatie vervolgens aanwenden door opnieuw projectieargumenten te gebruiken.

We veronderstellen eerst dat $\mathrm{Q}(6,q)$, q oneven, geen ovoïden heeft, en dat de kleinste minimale blokkerende verzameling van $\mathrm{Q}(6,q)$, q oneven, een afgeknotte kegel $p^*\mathrm{Q}^-(3,q)$ is, $\mathrm{Q}^-(3,q) \subseteq \mathrm{Q}(4,q)$, met $\mathrm{Q}(4,q)$ de basis van de kegel $p^{\perp} \cap \mathrm{Q}(6,q)$. Deze veronderstelling is waar voor $q > 3$, q priem.

We zullen de volgende stelling bewijzen.

Stelling A.5.3. *De kleinste minimale blokkerende verzamelingen van de kwadriek* $\mathrm{Q}(2n,q)$, *$q > 3$ priem, $n \geqslant 4$, zijn afgeknotte kegels* $\pi^*_{n-3}\mathrm{Q}^-(3,q)$, $\pi_{n-3} \subseteq \mathrm{Q}(2n,q)$, $\mathrm{Q}^-(3,q) \subseteq \pi^{\perp}_{n-3} \cap \mathrm{Q}(2n,q)$.

We veronderstellen dat $\mathcal{K}$ een minimale blokkerende verzameling is van $\mathrm{Q}(2n,q)$, $n \geqslant 4$, $|\mathcal{K}| = q^n + \delta$, $1 < \delta \leqslant q^{n-2}$.

We zullen de karakterisatie bewijzen met behulp van een inductiehypothese; we veronderstellen dat Stelling A.5.3 bewezen is voor $\mathrm{Q}(2n-2,q)$, $n \geqslant 4$. Deze veronderstelling is geldig voor $n = 4$. De eerste twee lemma's zijn een uitbreiding van Lemma's A.3.7 en A.3.8.

Lemma A.5.4. *Veronderstel dat* $p \in \mathcal{K}$, *dan geldt* $|p^\perp \cap \mathcal{K}| \leqslant \delta$.

Lemma A.5.5. *Stel dat* $p \in \mathrm{Q}(2n, q) \setminus \mathcal{K}$, $n \geqslant 4$, *dan worden de punten van* $p^\perp \cap \mathcal{K}$ *door* p *geprojecteerd op* $\mathcal{K}_p$, *een minimale blokkerende verzameling van* $Q = \mathrm{Q}(2n-2, q)$, *de basis van de kegel* $p^\perp \cap \mathrm{Q}(2n, q)$.

Met behulp van de veronderstelling omtrent $\mathcal{K}$, en een telargument, bewijzen we het volgende lemma.

Lemma A.5.6. *Er bestaat een punt* $r \in \mathrm{Q}(2n, q) \setminus \mathcal{K}$, *zodat* $|r^\perp \cap \mathcal{K}| = q^{n-1} + q^{n-3}$.

Stel dat voor een punt $r \in \mathrm{Q}(2n, q) \setminus \mathcal{K}$ geldt dat $|r^\perp \cap \mathcal{K}| = q^{n-1} + q^{n-3}$. Door de inductiehypothese en Lemma A.5.5 volgt onmiddellijk dat de verzameling $r^\perp \cap \mathcal{K}$ geprojecteerd wordt op een afgeknotte kegel. Het is essentieel om te bewijzen dat niet alleen de projectie een afgeknotte kegel is, maar de verzameling $r^\perp \cap \mathcal{K}$ zelf een afgeknotte kegel is. In drie stappen bekomen we het volgende lemma.

Lemma A.5.7. *Veronderstel dat* r *een punt is van* $\mathrm{Q}(2n, q) \setminus \mathcal{K}$, *met* $|r^\perp \cap \mathcal{K}| = q^{n-1} + q^{n-3}$, *dan geldt* $r^\perp \cap \mathcal{K} = \pi^*_{n-4}\mathrm{Q}^-(3, q)$.

Met behulp van dit lemma kunnen we nu de verzameling $\mathcal{K}$ karakteriseren. We merken op dat het feit dat alle ovoïden van $\mathrm{Q}(4, q)$, q oneven priem, elliptische kwadrieken zijn bepaalde vereenvoudigingen oplevert. We vermelden het laatste lemma.

Lemma A.5.8. *De verzameling* $\mathcal{K}$ *is een afgeknotte kegel* $\pi^*_{n-3}\mathrm{Q}^-(3, q)$.

Daarmee kunnen we Stelling A.5.3 besluiten.

We veronderstellen nu dat $\mathrm{Q}(6, q)$, q oneven, ovoïden heeft, en dat de kleinste minimale blokkerende verzamelingen van $\mathrm{Q}(6, q)$, afgeknotte kegels $p^*\mathcal{O}$ zijn, $\mathcal{O}$ een ovoïde van $\mathrm{Q}(4, q)$. Deze veronderstelling is waar voor $q = 3$. Daarenboven zijn alle ovoïden van $\mathrm{Q}(4, 3)$ elliptische kwadrieken, maar we zullen deze eigenschap niet gebruiken.

In tegenstelling tot de vorige situatie, kunnen we deze informatie niet op dezelfde manier vertalen naar $\mathrm{Q}(2n, q)$, $n \geqslant 4$. We kunnen wel de classificatie van de kleinste minimale blokkerende verzamelingen van $\mathrm{Q}(8, q)$ bekomen, met behulp van technieken vergelijkbaar met de technieken uit de Hoofdstukken 3 en 4. Omdat we de kleinste minimale blokkerende verzamelingen van $\mathrm{Q}(6, 3)$ kennen (en niet alleen informatie hebben over de grootte, cfr. Hoofdstuk 4), vinden we in enkele stappen de volgende stelling.

Stelling A.5.9. *Stel dat* Q$(6, q)$, *q oneven, een ovoïde heeft en dat de kleinste minimale blokkerende verzamelingen van* Q$(6, q)$, *verschillend van een ovoïde, afgeknotte kegels $p^*\mathcal{O}$ zijn, $\mathcal{O}$ een ovoïde van* Q$(4, q)$. *Dan zijn de kleinste minimale blokkerende verzamelingen van* Q$(8, q)$, *q oneven, afgeknotte kegels $p^*\mathcal{O}'$, $\mathcal{O}'$ een ovoïde van* Q$(6, q)$.

We beschikken nu over vergelijkbare informatie als in de vorige situatie. Het blijkt echter noodzakelijk meetkundige eigenschappen over ovoïden van Q$(6, q)$ kort te onderzoeken om alle lemma's aan te passen aan de nieuwe situatie. We vermelden de belangrijkste resultaten.

Stelling A.5.10. (Ball et al. [6]) *Een ovoïde van* Q$(6, q)$, *$q = p^h$, p priem, $h \geqslant 1$, heeft $1 \mod p$ punten gemeen met elke elliptische kwadriek* Q$^-(5, q)$ *bevat in* Q$(6, q)$.

Lemma A.5.11. *Beschouw een ovoïde $\mathcal{O}$ van* Q$(6, q)$. *Beschouw een hypervlak α van* PG$(6, q)$, *zodat $\alpha \cap$* Q$(6, q) =$ Q$^+(5, q)$. *Dan geldt $\langle \alpha \cap \mathcal{O} \rangle = \alpha$.*

We zullen de nu volgende stelling bewijzen, op een vergelijkbare manier als in de vorige situatie.

Stelling A.5.12. *Veronderstel dat* Q$(6, q)$ *ovoïden heeft en dat de kleinste minimale blokkerende verzamelingen van* Q$(6, q)$, *verschillend van een ovoïde, afgeknotte kegels $p^*\mathcal{O}$ zijn, $p \in$* Q$(6, q)$, *$\mathcal{O}$ een ovoïde van* Q$(4, q)$. *Dan zijn de kleinste minimale blokkerende verzamelingen van* Q$(2n, q)$, *$n \geqslant 5$, afgeknotte kegels $\pi^*_{n-4}\mathcal{O}'$, $\pi_{n-4} \subseteq$* Q$(2n, q)$, *$\mathcal{O}'$ een ovoïde van* Q$(6, q) \subseteq \pi^{\perp}_{n-4} \cap$ Q$(2n, q)$.

We veronderstellen dat $\mathcal{K}$ een minimale blokkerende verzameling is van Q$(2n, q)$, $n \geqslant 5$, $|\mathcal{K}| = q^n + \delta$, $1 < \delta \leqslant q^{n-3}$. Lemma's A.5.4 en A.5.5 blijven geldig in deze situatie, Lemma's A.5.6 en Lemma A.5.7 worden aangepast.

Lemma A.5.13. *Er bestaat een punt $r \in$* Q$(2n, q) \setminus \mathcal{K}$, *zodat $|r^{\perp} \cap \mathcal{K}| = q^{n-1} + q^{n-4}$.*

Lemma A.5.14. *Veronderstel dat r een punt is van* Q$(2n, q) \setminus \mathcal{K}$, *met $|r^{\perp} \cap \mathcal{K}| = q^{n-1} + q^{n-4}$, dan geldt $r^{\perp} \cap \mathcal{K} = \pi^*_{n-4}\mathcal{O}$, $\mathcal{O}$ een ovoïde van de basis* Q$(6, q)$ *van de kegel $\pi^{\perp}_{n-4} \cap$* Q$(2n, q)$.

Lemma A.5.15. *De verzameling $\mathcal{K}$ is een afgeknotte kegel $\pi^*_{n-4}\mathcal{O}$, $\mathcal{O}$ een ovoïde van de basis* Q$(6, q)$ *van de kegel $\pi^{\perp}_{n-4} \cap$* Q$(2n, q)$.

Stelling A.5.12 en de resultaten van Hoofdstuk 4 leiden tot de volgende stelling.

Stelling A.5.16. *De kleinste minimale blokkerende verzamelingen van* Q$(2n, q = 3)$, *$n \geqslant 4$, zijn afgeknotte kegels $\pi^*_{n-4}\mathcal{O}$, $\mathcal{O}$ een ovoïde van* Q$(6, q = 3)$, *met $\mathcal{O}$ bevat in de basis van de kegel $\pi^{\perp}_{n-4} \cap$* Q$(2n, q)$.

A.6 De kleinste minimale blokkerende verzamelingen van $\mathbf{H}(2n, q^2)$

Hoofdstuk 6 is gebaseerd op gezamelijk werk met K. Metsch [35].

Het is een gekend resultaat dat $\mathrm{H}(2n, q^2)$ geen ovoïden heeft [92]. Opnieuw kunnen we de kleinste minimale blokkerende verzamelingen van $\mathrm{H}(2n, q^2)$ onderzoeken. Zoals in Hoofdstukken 4 en 5 zullen we eerst de kleinste minimale blokkerende verzamelingen van $\mathrm{H}(4, q^2)$ karakteriseren. Nadien zullen we deze karakterisatie aanwenden om de kleinste minimale blokkerende verzamelingen van $\mathrm{H}(2n, q^2)$, $n \geqslant 3$, te karakteriseren. Ook in dit geval is het mogelijk om voorbeelden van minimale blokkerende verzamelingen te vinden.

Beschouw $\mathrm{H}(2n, q^2)$, $n \geqslant 2$, en een $(n-2)$-dimensionale deelruimte π_{n-2} bevat in $\mathrm{H}(2n, q^2)$. Dan is $\pi_{n-2}\mathrm{H}(2, q^2)$, $\mathrm{H}(2, q^2)$ de basis van de kegel $\pi_{n-2}^{\perp} \cap \mathrm{H}(2n, q^2)$, met $\perp$ de polariteit geassocieerd aan $\mathrm{H}(2n, q^2)$, een minimale blokkerende verzameling van $\mathrm{H}(2n, q^2)$.

We veronderstellen nu dat $\mathcal{K}$ een minimale blokkerende verzameling is van $\mathrm{H}(4, q^2)$, $|\mathcal{K}| = q^5 + \delta$, $1 \leqslant \delta \leqslant q^2$.

We starten met een klassiek lemma.

Lemma A.6.1. *Voor elk punt $p \in \mathcal{K}$ geldt $|p^{\perp} \cap \mathcal{K}| \leqslant \delta$.*

Het volgende lemma zal sterke implicaties hebben.

Lemma A.6.2. *Voor alle punten $r \in \mathrm{PG}(4, q^2) \setminus \mathcal{K}$ geldt dat $|r^{\perp} \cap \mathcal{K}| \geqslant q^3 + 1$.*

Dit lemma geeft aanleiding tot het volgende lemma, wat onmiddellijk een goede ondergrens geeft voor δ, en daarenboven het niet bestaan van ovoïden van $\mathrm{H}(4, q^2)$ tot gevolg heeft.

Lemma A.6.3. *Voor elk punt $p \in \mathcal{K}$ geldt dat $|p^{\perp} \cap \mathcal{K}| \geqslant q^2 - q + 1$.*

Deze informatie stelt ons nu in staat om, via een drietal lemma's, de karakterisatie te bewijzen. Net zoals in het geval van de kwadrieken zullen we trachten aan te tonen dat generatoren (rechten) van $\mathrm{H}(4, q^2)$ ofwel juist 1 punt, ofwel "veel" punten van $\mathcal{K}$ bevatten. We gebruiken de volgende notatie. Beschouw een punt $p \in \mathrm{H}(4, q^2)$, dan is $w_p + 1$ het minimaal aantal punten van $\mathcal{K}$ op alle rechten van $\mathrm{H}(4, q^2)$ door p. We bekomen de volgende lemma's.

Lemma A.6.4. *Voor elk punt $r \in \mathrm{H}(4, q^2) \setminus \mathcal{K}$, met $w_r = 0$, geldt dat $|r^{\perp} \cap \mathcal{K}| \leqslant q^3 - q^2 + q + \delta$.*

Lemma A.6.5. *Stel dat L een rechte is van $\mathrm{H}(4, q^2)$, die minstens 2 punten met $\mathcal{K}$ gemeen heeft, dan bestaat er een punt $s \in L \setminus \mathcal{K}$, met $w_s > 0$.*

Lemma A.6.6. *Veronderstel dat* $q > 2$*. Beschouw een punt* $p \in \mathrm{H}(4, q^2) \setminus \mathcal{K}$*, met* $w_p > 0$*, dan geldt* $w_p \geqslant q^2 - q$*.*

Dit lemma stelt ons in staat, om met behulp van een laatste lemma, de karakterisatie voor $q > 2$ te voltooien. In een afzonderlijk lemma behandelen we het geval $q = 2$. Daarin zullen we rechtstreeks gebruik maken van de verkregen ondergrens voor δ uit Lemma A.6.3. We bekomen de volgende stelling.

Stelling A.6.7. *De kleinste minimale blokkerende verzamelingen van de Hermitische variëteit* $\mathrm{H}(4, q^2)$ *zijn afgeknotte kegels* $p^*\mathrm{H}(2, q^2)$*,* $\mathrm{H}(2, q^2)$ *de basis van de kegel* $p^\perp \cap \mathrm{H}(4, q^2)$*.*

In de laatste sectie van Hoofdstuk 6 veralgemenen we de karakterisatie naar $\mathrm{H}(2n, q^2)$. We bereiken deze karakterisatie op een volkomen analoge manier als voor $\mathrm{Q}(2n, q)$, $q > 3$ priem. We vermelden de laatste stelling.

Stelling A.6.8. *De kleinste minimale blokkerende verzamelingen van de Hermitische variëteit* $\mathrm{H}(2n, q^2)$*,* $n \geqslant 2$*, zijn afgeknottekegels* $\pi^*_{n-2}\mathrm{H}(2, q^2)$*, met* $\mathrm{H}(2, q^2)$ *de basis van de kegel* $\pi^\perp_{n-2} \cap \mathrm{H}(2n, q^2)$*.*

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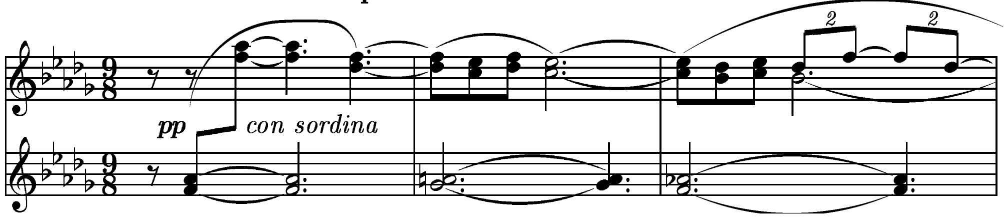
Andante très expressif



C. Debussy, *Clair de lune.* from “suite Bergamasque”