Quantum Walks and Distance-Regular Graphs

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28 February, 2014

Outline

Cosmology

Quote

"Hydrogen is a colorless, odorless gas which given sufficient time, turns into people." (Henry Hiebert)

Axioms

Quote

"The axioms of quantum physics are not as strict as those of mathematics"

Linear Algebra

(Speaking of vectors in $\mathbb{R}^n$.)

Quote

"However a vector is not the same thing as the list of its components. The vector has a . . . meaning."

Outline

1. The Problems
   - Physics 101
   - Transitions
   - More Complex Behavior
2. Schemes and DRGs
   - Mixing
   - Transfer
3. Aftermath

A Transition Operator

Let A be the adjacency matrix of the graph X.

Definition

The transition operator $U(t)$ is defined by

$$U(t) := \exp(itA)$$

It determines a continuous quantum walk.

An Example: P_2

If $X = P_2$, then

$$U(t) = \begin{pmatrix} \cos(t) & i\sin(t) \\ i\sin(t) & \cos(t) \end{pmatrix}.$$

Properties of $U(t)$

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3. $U(t)$ is unitary.

What We Observe

The entries of $U(t)$ are not observable. What we can measure are the values of the following:

Definition

The mixing matrix is

$$M(t) := U(t) \circ \overline{U(t)} = U(t) \circ U(-t).$$

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$$M(t) = \begin{pmatrix} \cos^2(t) & \sin^2(t) \\ \sin^2(t) & \cos^2(t) \end{pmatrix}.$$

Our Problems

We are interested in the behaviour of the entries of $U(t)$, in particular:

perfect state transfer: If a and b are vertices of X then $|U(t)_{a,b}| \leq 1$. Can we get equality?

uniform mixing: Is there a time t such that all entries of $U(t)$ have the same absolute value? (Is $U(t)$ flat?)

Example: P_2

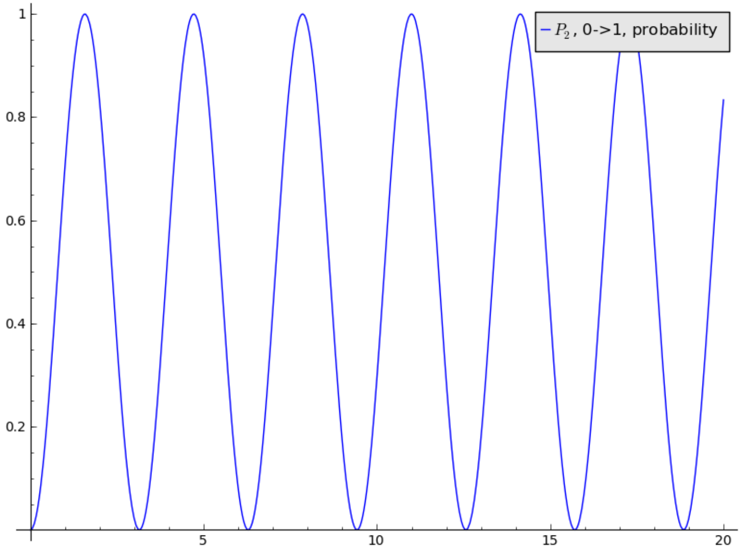
For P_2, we have

$$U(\pi/4) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

and

$$U(\pi/2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

A Plot: P_2



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We have both uniform mixing and perfect state transfer.

Composite Systems

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- The Cartesian product of X and Y has adjacency matrix $A_X \otimes I + I \otimes A_Y$. Since $A_X \otimes I$ and $I \otimes A_Y$ commute,

$$U_{X \square Y}(t) = U_X(t) \otimes U_Y(t).$$

An Example: Cartesian Product

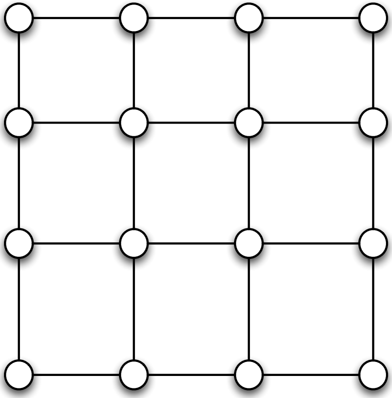


Figure : $P_4 \,\square\, P_4$

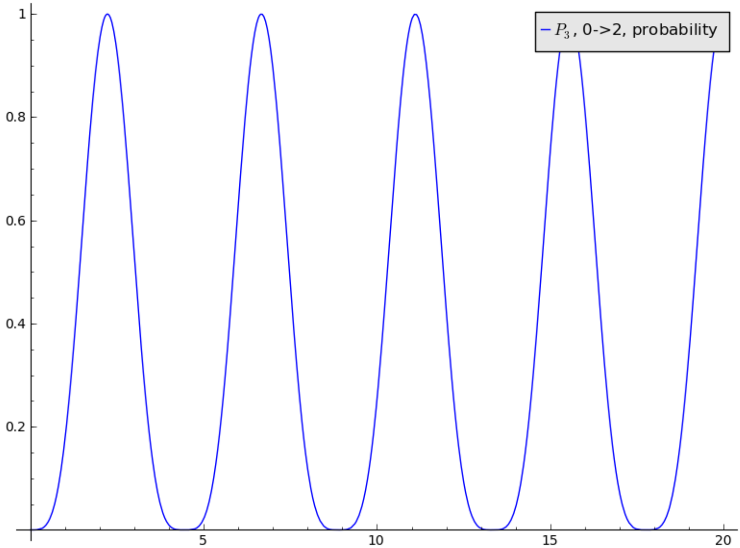
d-Cubes: Mixing and State Transfer

The d-cube Q_d is the Cartesian product of d copies of P_2.

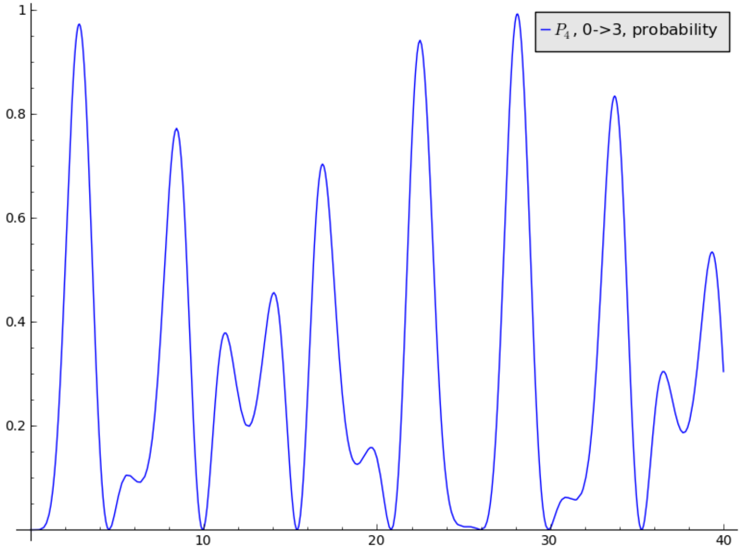
Hence we have uniform mixing on Q_d (at time $\pi/4$) and perfect state transfer (at time $\pi/2$), from any vertex u to the unique vertex at distance d from u.

Outline

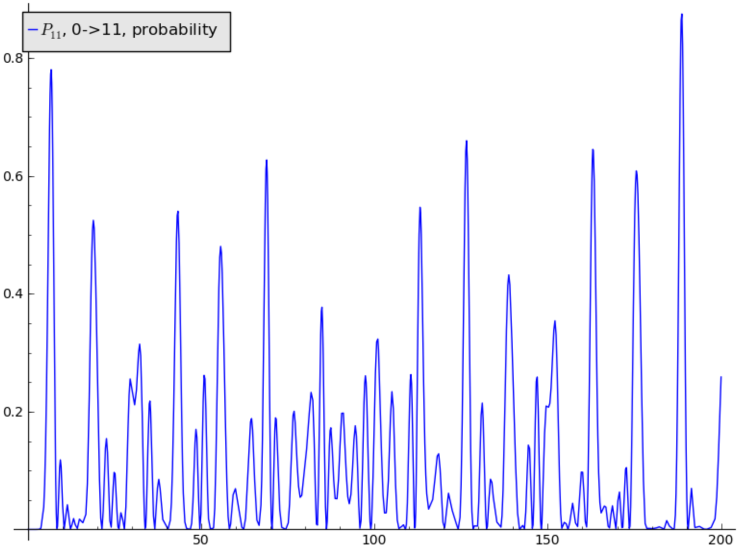
P_3



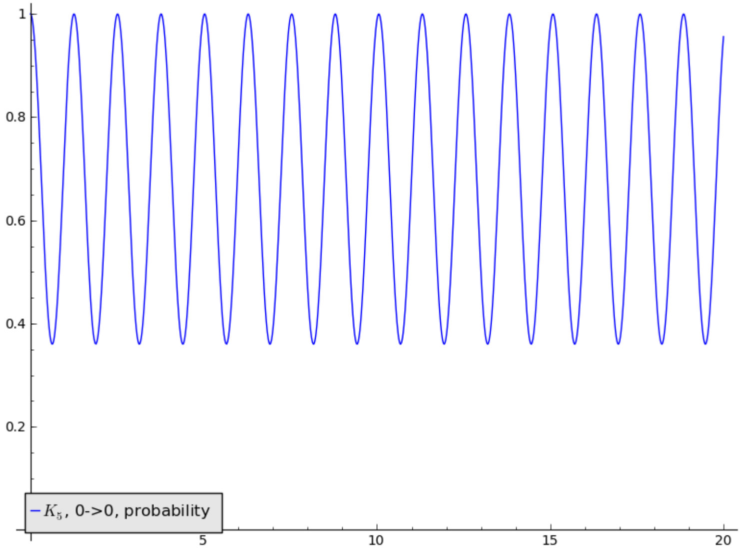
Things Become Complicated: P_4



And Yet More Complicated: P_{11}



Even Complete Graphs have their Quirks



Outline

Complete and Strongly Regular Graphs

Uniform mixing occurs on K_2, K_3, K_4 and their Cartesian powers, but not on K_n when $n \geq 5$.

We have uniform mixing on the Paley graph on 9 vertices, and on graphs constructed from regular symmetric Hadamard matrices with constant diagonal. No other srgs admit uniform mixing. (Joint work with Aidan Roy and Natalie Mullin, depends on work of Ada Chan.)

Gelfond-Schneider

Theorem

Let α and β be algebraic numbers. If $\alpha \neq 0, 1$ and β is not rational, then α^β is transcendental.

Using Gelfond-Schneider

Theorem

If the entries of $U(t)$ are algebraic, the ratios of the non-zero eigenvalues of X are rational.

Proof.

If the entries of $U(t)$ are algebraic numbers, so are its eigenvalues. If λ and μ are distinct non-zero eigenvalues of X, the corresponding eigenvalues of $U(t)$ are $e^{it\lambda}$ and $e^{it\mu}$. But

$$e^{it\mu} = \left(e^{it\lambda}\right)^{\mu/\lambda}$$

Now apply GS with $\alpha = e^{it\lambda}$ and $\beta = \mu/\lambda$. □

Transition Matrices of Bipartite Graphs

If X is bipartite, there is a diagonal matrix D with diagonal entries ± 1 such that $DAD = -A$. Then

$$DU(t)D = D\exp(itA)D = \exp(-itA) = \overline{U(t)}.$$

So there are real matrices $C_1(t)$, $C_2(t)$ and $K(t)$ such that

$$U(t) = \begin{pmatrix} C_1(t) & iK(t) \\ iK(t)^T & C_2(t) \end{pmatrix}.$$

Hadamard Matrices

If X is bipartite and $U(t)$ is flat, then $U(t)$ is a complex Hadamard matrix, and is similar to a real Hadamard matrix (modulo some scaling). Hence:

Lemma

If X is bipartite and uniform mixing occurs then $|V(X)| = 2$ or is divisible by four. If X is bipartite and regular and uniform mixing occurs, $|V(X)|$ is the sum of the squares of two integers.

No Uniform Mixing on Even Cycles

If X is bipartite and $U(t)$ is flat, then its entries are algebraic and so the ratios of the eigenvalues of X are rational. Hence:

Lemma

If n is even and $n > 4$, uniform mixing does not occur on C_n.

Prime Cycles

- If $X = C_p$ (p prime) and $U(t)$ is flat then $U(t)$ is a cyclic p-root.

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- Flat unitary matrices are type-II matrices, and the set of type-II matrices in the Bose-Mesner algebra of a scheme is an algebraic variety, defined by polynomials with integer coefficients.
- The entries of a cyclic p-root are algebraic.
- By Gelfond-Schneider, if $p > 3$ then uniform mixing does not occur on C_p.

Outline

Perfect State Transfer

We have perfect state transfer from a to b in X if there is a complex number γ of norm 1 such that

$$U(t)e_a = \gamma e_b;$$

equivalently $U(t)_{a,b} = \gamma$. Since $U(t)$ is symmetric, $U(t)_{b,a} = \gamma$. We also have

$$U(2t)_{a,a} = U(2t)_{b,b} = \gamma;$$

we say X is periodic at a (and at b) at time $2t$.

Where Does PST Occur?

A (very) rough outline:

- P_2, P_3 and their Cartesian powers. Also $K_{2,n}$.
- Joins of K_2 or $2K_1$ with regular graphs.
- Cayley graphs for abelian groups.
- Double covers of K_n coming from symmetric regular Hadamard matrices with constant diagonal.

Cubelike Graphs

A cubelike graph is a Cayleyvgraph for $\mathbb{Z}_2^d$. So we choose a subset $\mathcal{C}$ of $\mathbb{Z}_2^d \setminus 0$ and define a graph with vertex set $\mathbb{Z}_2^d$, where two vectors are adjacent if there difference is in $\mathcal{C}$.

Theorem (Bernasconi, Godsil, Severini)

Suppose X is cubelike with connection set $\mathcal{C}$ and set $\sigma = \sum_c \in \mathcal{C}$. If $c \neq 0$, we have perfect state transfer from 0 to c at time $\pi/2$.

When Does It Fail?

Theorem

If in some graph X we have perfect state transfer involving a vertex a of valency k, then the radius of X relative to k is at most $2k$.

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Corollary

There are only finitely many connected graphs with maximum valency k on which perfect state transfer occurs.

Highly Regular Graphs

Theorem

Assume X is vertex transitive, or a union of classes in an association scheme. If we have perfect state transfer on X at time t, then $U(t) = \gamma P$ where P is a permutation matrix that represents a fixed-point free automorphism of X with order two.

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Remarks:

- $|V(X)|$ must be even.
- P lies in the centre of $\mathrm{Aut}(X)$.
- If X is distance-regular, it's antipodal and state transfer involves antipodal pairs.

Joint Work

Gabriel Couthino, Krystal Guo, Frédéric Vanhove and I have determined which known distance-regular graphs admit perfect state transfer. `http://arxiv.org/pdf/1401.1745.pdf`

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- There are three new infinite families, all based on Hadamard matrices.
- If we cheat and consider graphs that are unions of classes, there are even more examples.
- Direct products with K_2 (bipartite double) work well. Infinite families based on point graphs of GQs, orthogonal array graphs, and many graphs with classical parameters.

Problems, Questions,...

- If we have uniform mixing on X, must X be regular?
- Which cycles of odd length admit uniform mixing?
- Investigate uniform mixing on graphs in schemes.
- Which Cayley graphs for $\mathbb{Z}_2^m$ admit perfect state transfer? (At times less than $\pi/4$.)
- How fast can perfect state transfer occur?

The End(s)