On the Maximum Size of M-Cliques of Generators on Hermitian Polar Spaces

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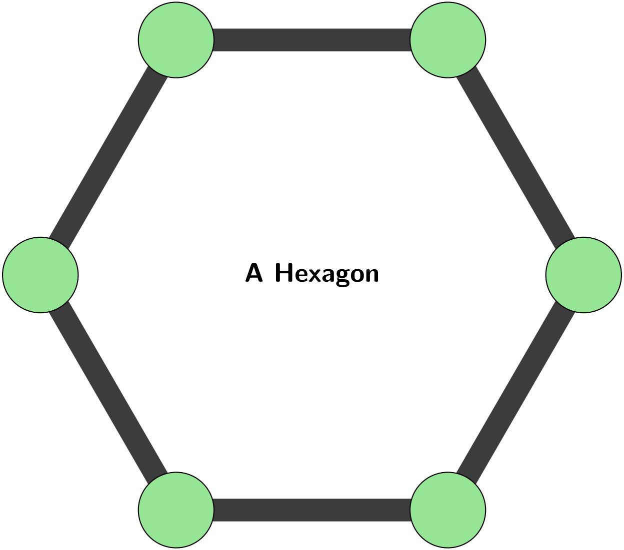
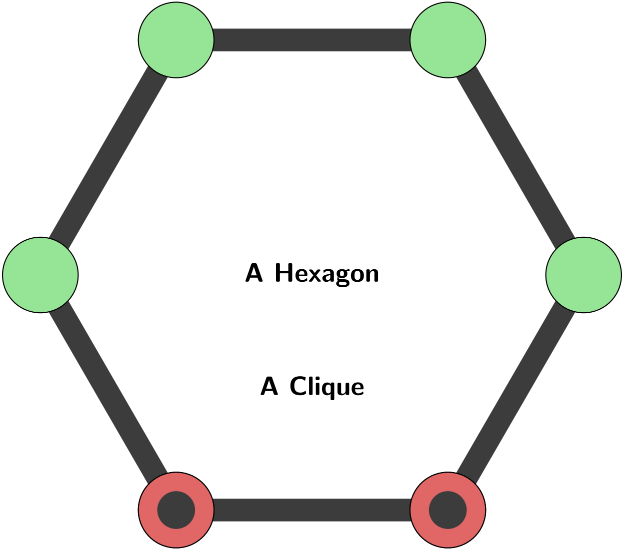
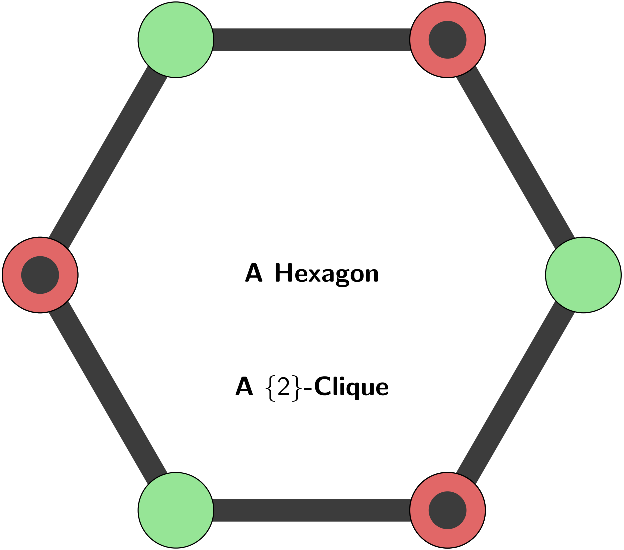
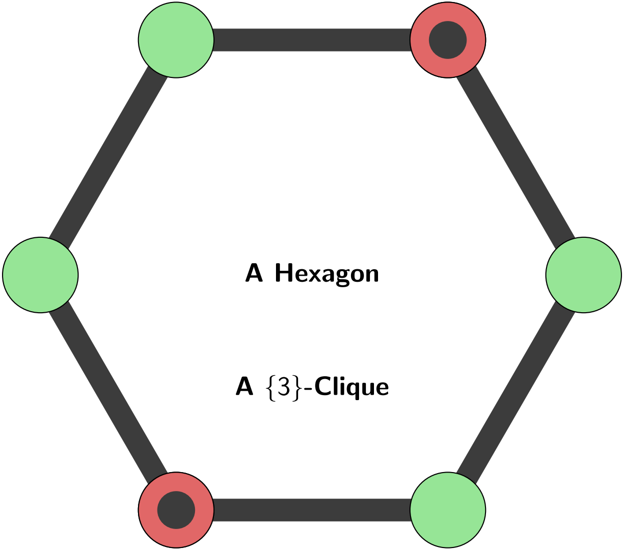

A Hexagon

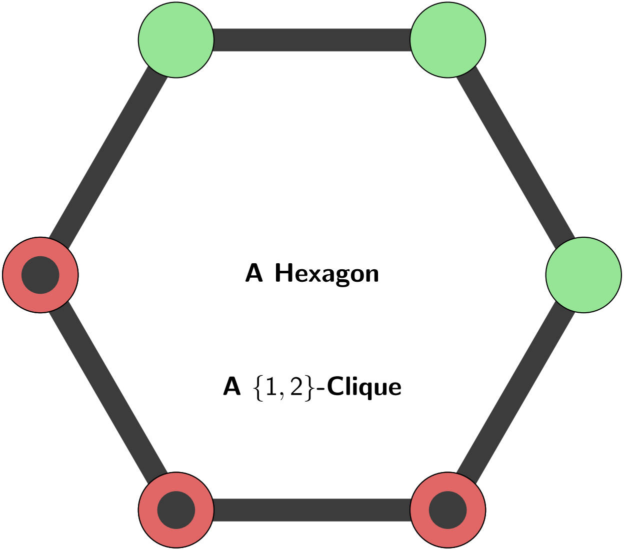

A Hexagon
A Clique



A Hexagon

A $\{2\}$-Clique


A Hexagon
A $\{3\}$-Clique


A Hexagon
A $\{1, 2\}$-Clique

Polar Spaces

Finite classical polar spaces are incidence geometries (points, lines, . . . , generators):

- $\mathbf{Q}^-(2d+1,q)/\Omega^-(2d+2,q)$: Elliptic quadric.
- $\mathbf{Q}(2d,q)/\Omega(2d+1,q)$: Parabolic quadric.
- $\mathbf{Q}^+(2d-1,q)/\Omega^+(2d,q)$: Hyperbolic quadric.
- $\mathbf{W}(2d-1,q)/Sp(2d,q)$: Symplectic polar space.
- $\mathbf{H}(2d-1,q^2)/U(2d,q^2)$: Hermitian polar space.
- $\mathbf{H}(2d,q^2)/U(2d+1,q^2)$: Hermitian polar space.

In this talk:

- All polar spaces are classical and finite.
- Focus on $\mathbf{H}(2d-1,q^2)$.

(Distance-)Regular Graphs

Definition

Let $G_M = (X, \sim_M)$ be a graph, where

- the **vertices** X are the **generators** (d-spaces) of $\mathbf{H}(2d-1, q^2)$,
- $M \subseteq \{1, \dots, d\}$,
- the **adjacency relation** $\sim_M$ is defined by $x \sim_M y$ if and only if $\mathrm{codim}(x \cap y) \in M$.

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- This defines a **regular graph**: The number of generators meeting a fixed generator x in an i-space for some $i \in M$ is independent of x.
- **Very regular**: the $\sim_{\{i\}}$ are the relations of an **association scheme**.

M-Cliques

Problem

Let $M \subseteq \{1, \ldots, d\}$. Let Y be a set of generators such that $x, y \in Y$, $x \neq y$, implies codim$(x \cap y) \in M$. Classical questions:

- *What is the maximum size of Y?*
- *How does an example of maximum size look like?*

The set Y would be a clique of G_M. In this talk: an M-**clique**.

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Examples

- If $M = \{d\}$, then Y is a **(partial) spread** (of generators).
- If $M = \{1, \ldots, t\}$, then Y is an **Erdős-Ko-Rado set** (often only $t = d - 1$).
- If $M = \{t\}$, then Y is a **constant-distance subspace code**.
- If $M = \{t + 1, \ldots, d\}$, then Y is a **subspace code** with minimum distance t.

The Adjacency Matrix

Definition

The **adjacency matrix** A of G_M is defined as follows:

$$(A)_{xy} = \begin{cases} 1 & \text{if } \operatorname{codim}(x \cap y) \in M \\ 0 & \text{if } \operatorname{codim}(x \cap y) \notin M \end{cases}$$

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The matrix A has up to $d+1$

- **eigenvalues** $\theta_0, \theta_1, \dots, \theta_d$, (in the same order)
- **eigenspaces** $V_0, \dots, V_d \subseteq \mathbb{R}^n$ where $n := |X|$,
- **multiplicities** $f_0 = \dim(V_0), \dots, f_d = \dim(V_d)$.

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The graph G_M is k-**regular** for some k, so w.l.o.g.

- $\theta_0 = k$,
- $V_0 = \langle j \rangle$, j is the all-one vector.

Erdős-Ko-Rado Sets

Definition

Let $n \geq 2k$. Consider $X = \{1, \ldots, n\}$. An **Erdős-Ko-Rado set** (EKR set) of X is a set Y of k-subsets of X such that the elements of Y meet pairwise in at least t elements.

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Examples ($t = 1$)

1. All k-sets that contain 1. For $n = 4$, $k = 2$:

$$\{1, 2\}, \{1, 3\}, \{1, 4\}.$$

2. $n = 2k$: All k-sets that do not contain n. For $n = 4$, $k = 2$:

$$\{1, 2\}, \{1, 3\}, \{2, 3\}.$$

Maximum size and complete classification by Erdős, Ko, Rado (1961), Frankl, Wilson (1986), and Ahlswede, Khachatrian (1997).

Erdős-Ko-Rado Sets of Generators on Polar Spaces

Definition

An **EKR set** Y **of generators** on a polar space is a $\{1, \ldots, t\}$-clique. (Hence, the elements of Y meet pairwise in a subspace of at most codimension t.)

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Examples ($t = d - 1$)

1. All generators on a fixed point.
2. All generators which meet a fixed generator in at most codimension $t/2$.

Example

All generators on a fixed $(d - t)$-space.

Some Results for $t = d - 1$ resp. $M = \{1, \ldots, d - 1\}$

Theorem (Stanton (1980))

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Theorem (I., Metsch (2013))

An EKR set of $\mathbf{H}(2d-1, q^2)$, *d odd, has at most size* $\approx q^{(d-1)^2+1}$. *(The largest known example for* $d > 3$ *has size* $\approx q^{(d-1)^2}$*.)*

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Theorem (Pepe, Storme, Vanhove (2011))

The classification of all EKR sets of maximum size for all polar spaces except $\mathbf{H}(2d-1, q^2)$, $d > 3$ odd.

The Hoffman Bound

Nearly all mentioned results for EKR sets ($\{1, \ldots, t\}$-cliques) use the **(weighted) Hoffman bound**.

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Theorem (Hoffman Bound)

Let Y be an M-clique. Let $\mathcal{C}M := \{1, \ldots, d\} \setminus M$. Let θ_{min} be the smallest eigenvalue of the adjacency matrix A of $G_{\mathcal{C}M}$. Then

$$|Y| \leq \frac{-n\theta_{\text{min}}}{k - \theta_{\text{min}}}$$

with equality **if and only if** $\chi \in \langle j \rangle + V_{\text{min}}$*, where* χ *is the characteristic vector of* Y*.*

The Proof

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$$A = \frac{k}{n}J + \theta_1 E_1 + \ldots + \theta_d E_d.$$

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If Y is an M-clique, then the characteristic vector $\chi \in \mathbb{R}^n$ of Y satisfies

$$\chi^T A \chi = 0.$$

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The vector χ can be decomposed into eigenvectors:

$$\chi = \frac{k}{n} j + E_1 \chi + \ldots + E_d \chi.$$

The Hoffman Bound (Part 2)

$$A = \frac{k}{n}J + \theta_1 E_1 + \ldots + \theta_d E_d,$$

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Hence,

$$\begin{aligned} 0 = \chi^T A\chi &= \chi^T(\frac{k}{n}J + \theta_1 E_1 + \ldots + \theta_d E_d)\chi \\ &= \frac{k}{n}|Y|^2 + \theta_1|E_1\chi|^2 + \ldots + \theta_d|E_d\chi|^2 \\ &\geq \frac{k}{n}|Y|^2 + \theta_{\min}|E_{\min}\chi|^2. \end{aligned}$$

Here $\theta_{\min} < 0$ is the smallest eigenvalue of A.

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with equality **if and only if**

$$\chi = \frac{k}{n}j + E_{\min}\chi \in \langle j \rangle + V_{\min}.$$

The Weighted Hoffman Bound

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This bound is sufficient for the results on EKR sets by Stanton (1980), Pepe, Storme, Vanhove (2011).
The proof never uses that A is the adjacency matrix! Only

$$\chi^T A \chi \leq 0 \text{ if } \chi \text{ is the characteristic vector of an } M\text{-clique}$$

is necessary.

Linear Programming and the Hoffman Bound

Problem

How does one find matrices A' satisfying the following?

$$\chi^T A' \chi \leq 0 \text{ if } \chi \text{ is the characteristic vector of an } M\text{-clique} \quad (1)$$

Solution (**Delsarte's LP bound**): Consider linear combinations A' of $J, E_1, \ldots, E_d$ with $A'_{ij} \leq 0$ if $A_{ij} = 0$.

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Example ($\mathbf{H}(5, q^2)$, $\{1, 2\}$-cliques)

- The adjacency matrix A for the disjointness graph has the eigenvalues $q^9, q^3, -q^4, -q^6$.
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- The adjacency matrix A for the disjointness graph has the eigenvalues $q^9, q^3, -q^4, -q^6$.
- The Hoffman bound yields approximately $|Y| \leq q^6$.
- There exists an A' as in (1) that has $-q^5$ as its smallest eigenvalue.
- The weighted Hoffman bound yields approximately $|Y| \leq q^5$.

A variant of this technique was used to prove better upper bounds for $\mathbf{H}(2d-1, q^2)$, d odd, by I., Metsch (2013).

What if $t < d - 1$?

Some geometrical results on EKR sets with pairwise intersections in at least codimension t:

Theorem (Brouwer, Hemmeter (1992))

A classification of all $\{1, 2\}$-cliques in non-Hermitian polar spaces.

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- *A classification of examples of maximum size for $t \leq c\sqrt{d}$ for some constant c.*
- *Estimates of the (non-weighted) Hoffman bound for all t.*

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Theorem (De Boeck)

- *Classification of all EKR sets of planes (not necessarily generators) in nearly all polar spaces.*
- *Classification of EKR sets on $Q^+(4n+1, q)$ for $t = d - 1$.*

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Let Y be a EKR set ($\{1, \ldots, t\}$-clique) of $\mathbf{H}(2d-1, q^2)$. Let P be a point of $\mathbf{H}(2d-1, q^2)$. Let Y_1 be subset of Y on P, and let $Y_2 := Y \setminus Y_1$.

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- The projected elements of Y_1 meet all elements of Y_2 in at least codimension t.
- The projected elements of Y_2 meet all elements of Y_1 in at least codimension t.

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- The projected elements of Y_1 meet all elements of Y_2 in at least codimension t.
- The projected elements of Y_2 meet all elements of Y_1 in at least codimension t.

Can this be used to improve results on EKR sets?

Maximum Size?

Definition

A **cross-intersecting EKR set** is a pair of sets of generators Y_1, Y_2 such that

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- The sum: $|Y_1| + |Y_2|$.
- Some linear combination: $|Y_1| + q|Y_2|$.
- Something silly: $e^{|Y_1|} \cdot |Y_2| + \log(|Y_1|)$.

In this talk: $|Y_1| \cdot |Y_2|$.

The (weighted) Hoffman Bound

The Hoffman bound for cross-intersecting sets was used by ...

- **Vector spaces**: "The eigenvalue method for cross t-intersecting families.", Tokushige (2013).
- **Permutations**: "Intersecting families of permutations.", Ellis, Friedgut, Pilpel (2011).
- **Coding Theory**: "Scalable secure storage when half the system is faulty.", Alon, Kaplan, Krivelevich, Malkhi, Stern (2000).

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Theorem (Hoffman bound for cross-intersecting EKR sets)

Let Y_1, Y_2 be an cross-intersecting EKR set. Let $\theta_{2\,\max}$ be a second largest **absolute** *eigenvalue of the adjacency matrix A of $G_{\{t+1,\dots,d\}}$.*

$$\sqrt{|Y_1| \cdot |Y_2|} \leq \frac{n \cdot |\theta_{2\,\max}|}{k + |\theta_{2\,\max}|}$$

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with equality **if and only if** $\chi_i \in \langle j \rangle + V_- + V_+$*, where*

- χ_i *is the characteristic vector of Y_i,*
- V_+ *is the eigenspace corresponding to $|\theta_{2\,\max}|$ (if it exists),*
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- The proof is the same. Only with $0 = \chi_1^T A \chi_2$ instead of $0 = \chi^T A \chi$.
- Again, A can be replaced with other matrices A' with $0 \geq \chi_1^T A' \chi_2$.
- Hence, everything works the same as in the “normal” case.

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The proof reveals some more details:

- If $\chi_1 \in \langle j \rangle + V_-$, then $Y_1 = Y_2$ is an EKR set.
- If $\chi_1 = \alpha j + v_- + v_+$ (with $v_- \in V_-$, $v_+ \in V_+$), then $\chi_2 = \alpha j + v_- - v_+$.

Some Results for $M = \{1, \ldots, d-1\}$

Example

The matrix A that belongs to $\mathbf{Q}^-(5, q)$ has the eigenvalues

$$q^9 \qquad \mathbf{-q^5} \qquad q^3 \qquad -q^3.$$

The absolute second largest eigenvalue is the smallest eigenvalue.

Some Results for $M = \{1, \dots, d-1\}$

Example

The matrix A that belongs to $\mathbf{Q}^-(5, q)$ has the eigenvalues

$$q^9 \qquad \mathbf{-q^5} \qquad q^3 \qquad -q^3.$$

The absolute second largest eigenvalue is the smallest eigenvalue. Hence, $Y_1 = Y_2$: the classification of all EKR sets by Pepe, Storme, and Vanhove is sufficient.

Theorem

For all polar spaces except $\mathbf{H}(2d-1, q^2)$, $\mathbf{Q}^+(2d-1, q)$ (if d even), $\mathbf{Q}(2d, q)$ (if d even), and $\mathbf{W}(2d-1, q)$ (d, q both even) the cross-intersecting EKR sets of maximum size are EKR sets.

$\mathbf{Q}^+(2d-1,q)$ and $\mathbf{Q}(2d,q)$, d even

Example ($\mathbf{Q}^+(7,q)$)

$$q^6 \qquad -\mathbf{q^3} \qquad q^2 \qquad -\mathbf{q^3} \qquad q^6.$$

- The absolute second largest eigenvalue is the second largest eigenvalue.
- Y_1 are the **latins** of $\mathbf{Q}^+(7,q)$, Y_2 are the **greeks** of $\mathbf{Q}^+(7,q)$.

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Example ($\mathbf{Q}(8,q)$)

$$q^{10} \qquad -\mathbf{q^6} \qquad q^4 \qquad -q^4 \qquad q^6.$$

- The absolute second largest eigenvalue is the second largest eigenvalue **as well** as the smallest eigenvalue.
- Either $Y_1 = Y_2$ or the $\mathbf{Q}^+(7,q)$ example.

$\mathbf{H}(2d-1,q^2)$

Example

$$\begin{array}{llllll} \mathbf{H}(5,q^2): & q^9 & -q^4 & q^3 & -\mathbf{q^6} & \text{resp.} \\ \mathbf{H}(7,q^2): & q^{16} & -\mathbf{q^9} & q^6 & -q^7 & q^{12}. \end{array}$$

- The blue eigenvalues belong to nice EKR sets.
- The **bold** eigenvalues are the smallest.
- The red eigenvalues are the absolute second largest.

$\mathbf{H}(2d-1, q^2)$

Example

$\mathbf{H}(5, q^2)$:	q^9	$-q^4$	q^3	$\mathbf{-q^6}$	resp.
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The bounds for $\sqrt{|Y_1| \cdot |Y_2|}$.

- The cross-intersecting Hoffman bound yields $\approx q^{d(d-1)}$.
- The cross-intersecting Hoffman bound with LP yields $\approx q^{(d-1)^2+1}$.
- The largest known examples have size $\approx q^5$ for $d = 3$, $q^{19/2}$ for $d = 4$, and $q^{(d-1)^2}$ for $d > 4$.

Largest Known Examples on $\mathbf{H}(2d-1, q^2)$

Example ($\mathbf{H}(5, q^2)$)

$Y_1 = Y_2$ is the set of all generators meeting a fixed plane in at least a line: $\sqrt{|Y_1| \cdot |Y_2|} \approx q^5$.

Example ($\mathbf{H}(7, q^2)$)

Y_1 is the set of all generators meeting a fixed generator G in at least a line, Y_2 the set of all generators meeting G in at least a plane: $\sqrt{|Y_1| \cdot |Y_2|} \approx q^{19/2}$.

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Example ($\mathbf{H}(9, q^2)$)

$Y_1 = Y_2$ is the set of all generators on a fixed point: $\sqrt{|Y_1| \cdot |Y_2|} \approx q^{16}$.

Example ($\mathbf{H}(11, q^2)$)

$Y_1 = Y_2$ is the set of all generators on a fixed point: $\sqrt{|Y_1| \cdot |Y_2|} \approx q^{25}$.

Summary for $\mathbf{H}(2d-1, q^2)$

Theorem (I., Metsch (2013))

Let Y be an EKR set, d odd. Then

$$|Y| \lessapprox q^{(d-1)^2+1}.$$

Theorem

Let Y_1, Y_2 be a cross-intersecting EKR set. Then

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Examples

The largest known examples:

- $\mathbf{H}(5, q^2)$: $\sqrt{|Y_1| \cdot |Y_2|} \approx q^{(d-1)^2+1}$.
- $\mathbf{H}(7, q^2)$: $\sqrt{|Y_1| \cdot |Y_2|} \approx q^{(d-1)^2+1/2}$.
- $\mathbf{H}(2d-1, q^2)$: $\sqrt{|Y_1| \cdot |Y_2|} \approx q^{(d-1)^2}$.

Spreads

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- If Y partitions the points, then it is a **spread**.

History:

- In 1981 J. A. Thas publishes "Ovoids and spreads of finite classical polar spaces.", a first complete survey of spreads on polar spaces.
- Upper bounds for the size of **partial spreads** and sets reaching these bounds were investigated since the 70's.

Theorem (J. A. Thas (1981/1990))

The Hermitian polar space $\mathbf{H}(2d-1, q^2)$ *has no spread.*

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A partial spread of $\mathbf{H}(2d-1, q^2)$*,* d *even, has at most*

- $\frac{1}{2}(q^3+q+2)$ *elements if* $d=2$ *(sharp for* $q=2,3$*),* *(Dye (*$q=2$*, 1992), Ebert, Hirschfeld (*$q=3$*, 1999))*
- $q^{2d-1} - q^{3d/2}(\sqrt{q}-1)$ *elements if* $d>2$*.*

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- $q^{2d-1}-q^{3d/2}(\sqrt{q}-1)$ *elements if* $d>2$.

Theorem (De Beule, Metsch ($d=3$, 2007)/Vanhove (2009))

A partial spread of $\mathbf{H}(2d-1, q^2)$, d *odd, has at most*

$$q^d+1$$

elements. This bound is sharp.
(Agulglia, Cossidente, Ebert ($d=3$*, 2003)/Luyckx (2008))*

More Results

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Theorem (Another Hoffman Bound)

Let Y be a $\{t\}$-clique. Let $\theta_{\min}$ be the smallest eigenvalue of the adjacency matrix A of $G_{\{t\}}$. Then

$$|Y| \leq 1 - \frac{k}{\theta_{\min}}$$

with equality **if and only if** *the characteristic vector χ of Y satisfies $\chi \in V_{\min}^{\perp}$.*

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Theorem (Godsil (1978))

Let f_i be the multiplicity of an eigenvalue of the adjacency matrix A of $G_{\{t\}}$ not equal to k. Then

$$|Y| \leq 1 + f_i$$

with equality **only if** *the Hoffman bound is sharp.*

Proof.

Let Y be a $\{t\}$-clique. The decomposition of A:

$$A = \frac{k}{n}J + \theta_1 E_1 + \ldots + \theta_d E_d.$$

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Proof.

Rearranging yields

$$|Y| \leq \frac{-\alpha}{\beta} \text{ resp. } |Y| \leq 1 - \frac{k}{\theta_i}$$

if $\beta < 0$ resp. $\theta_i < 0$. This proves the Hoffman bound.

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$$\operatorname{rank}(S) = \operatorname{rank}(\alpha I + \beta J) \leq \operatorname{rank}(E_i) = f_i.$$

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Now

$$\begin{aligned} \mathrm{rank}(S) &= |Y| - 1 && \text{if } \alpha = -\beta|Y|, \\ \mathrm{rank}(S) &= |Y| && \text{if } \alpha \neq -\beta|Y|, \end{aligned}$$

yields Godsil's bound. □

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Another application: **distance-2 ovoids** in the **generalized hexagon** with parameter (s, s^3) by Coolsaet, Van Maldeghem (2000).

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Some Comparsions for Codimension 2

Example ($\mathbf{H}(3, q^2)$, $t = 2$)

- Multiplicity bound: $q^3 - q^2 + q$.
- Best known bound for $q \neq 4$: $\frac{1}{2}(q^3 + q + 2)$.
- Largest examples: probably $\approx \alpha q^2$.

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Example ($\mathbf{H}(5, q^2)$, $t = 2$)

- Multiplicity bound: $q^5 - q^4 + q^3 - q^2 + q$.
- Sharp bound by Maarten De Boeck: $q^4 + q^2 + 2$.

Example ($\mathbf{H}(2d - 1, q^2)$, $t = 2$)

- Multiplicity bound: $\frac{q^{2d}-1}{q+1} + 1$.
- Largest example: $\frac{q^{2d}-1}{q^2-1}$.

Some Comparsions for Partial Spreads

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Example ($\mathbf{H}(7, q^2)$, $t = 4$)

- Multiplicity bound: $q^7 - q^6 + q^5 - q^4 + q^3 - q^2 + q$.
- Best known bound for $q > 3$: $q^7 - q^6(\sqrt{q} - 1)$.

Example ($\mathbf{H}(2d-1, q^2)$, $t = d > 4$ even)

- Multiplicity bound: $q^{2d-1} - q\frac{q^{2d-2}-1}{q+1}$.
- Previously best known bound: $q^{2d-1} - q^{3d/2}(\sqrt{q} - 1)$.

Theorem (Vanhove (2011))

A $\{d\}$-clique of $\mathbf{H}(2d-1,q^2)$, d odd, has at most

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Frédéric Vanhove also provided a second, geometrical proof.

Problem

Is there a better geometrical argument?

What is Missing?

Problem

The dual problem to $\{1, \ldots, t\}$-cliques resp. EKR sets:

- $\{t + 1, \ldots, d\}$*-cliques of polar spaces.*
- *analog problems for sets (***codes***) and vector spaces (***network codes***) are hard.*

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- $\{t+1, \dots, d\}$*-cliques of polar spaces.*
- *analog problems for sets (***codes***) and vector spaces (***network codes***) are hard.*

Problem

The dual problem to $\{t\}$-cliques resp. constant distance codes:

- $\{1, \dots, t-1, t+1, \dots, d\}$*-cliques of polar spaces.*
- *an alternative generalization of* $\{1, \dots, d-1\}$*-cliques.*

Thank You!