Girth and Dual Girth Parameters in Polynomial Association Schemes

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in memory of Frédéric Vanhove
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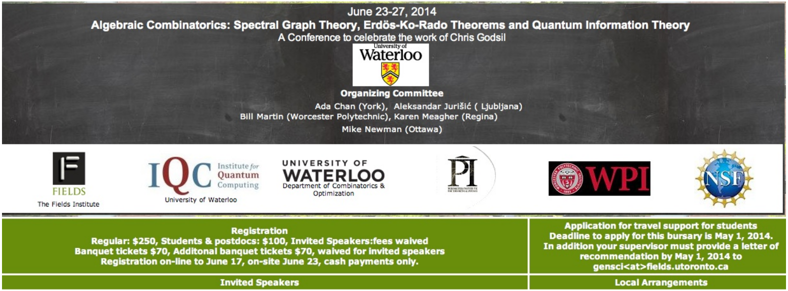

I wanted to call it "Wombats, Bilbies and Quolls"

Entrywise products of eigenvectors

Suppose we have a symmetric association scheme with primitive idempotents $\{E_j\}_{j=0}^{d}$ and eigenspaces $V_j = \text{colsp}\, E_j$.

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If $u \in V_i$ and $v \in V_j$ and $q_{ij}^k = 0$ then $u \circ v \perp V_k$.

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Here, $\circ$ denotes entrywise product of vectors and q_{ij}^k is the *Krein parameter* appearing in the expansion

$$E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{d} q_{ij}^k E_k$$

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P. J. Cameron, J.-M. Goethals, and J. J. Seidel, *The Krein condition, spherical designs, Norton algebras and permutation groups.* Proc. Kon. Nederl. Akad. Wetensch. (1978).

Application to Ovoids

Consider the 3-class association scheme whose vertices are the points $\mathcal{P}$ and planes $\mathcal{B}$ of $PG(3, q)$. The non-trivial relations are: *incidence*, *non-incidence* and *same type*.

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This is a Q-polynomial association scheme with

- V_0 spanned by the all-ones vector
- $V_0 + V_1$ spanned by all the characteristic vectors $\mathbf{1}_S$ where S is of the form $\{p \in \mathcal{P} \mid p \in \ell\} \cup \{\pi \in \mathcal{B} \mid \ell \subset \pi\}$ as ℓ ranges over the lines of $PG(3,q)$.

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One easily checks that, if $\mathcal{O}$ is any *ovoid* and S consists of the $q^2 + 1$ points on $\mathcal{O}$ together with the $q^2 + 1$ planes tangent to $\mathcal{O}$, then

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Since $q^3_{00} = q^3_{02} = q^3_{22} = 0$, $(\mathbf{1}_S \circ \mathbf{1}_{S'}) \perp V_3$ for any two "ovoids" S and S' in this geometry:

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One easily checks that, if $\mathcal{O}$ is any *ovoid* and S consists of the q^2+1 points on $\mathcal{O}$ together with the q^2+1 planes tangent to $\mathcal{O}$, then

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Since $q^3_{00} = q^3_{02} = q^3_{22} = 0$, $(\mathbf{1}_S \circ \mathbf{1}_{S'}) \perp V_3$ for any two "ovoids" S and S' in this geometry:

Corollary

Any two ovoids $\mathcal{O}$ and $\mathcal{O}'$ in $PG(3,q)$ have equally many points in common as they do tangent planes.

Frédéric: 2-Ovoids in Generalized Hexagons

Theorem (Theorem 6.4.20 in dissertation of Vanhove)

Any two distance-2-ovoids $\mathcal{O}$ and $\mathcal{O}'$ in a generalized hexagon of order (s, s^3) with $s > 1$ are either disjoint or have exactly $h(s^2 + s + 1)$ points in common for some $h > s^3 - s$.

A *distance-2-ovoid* $\mathcal{O}$ in a generalized hexagon is a set of pairwise non-collinear points hitting every line

Distance-Regular Graphs

A graph (X, R) is distance-regular if there exist scalars

$$\begin{array}{cccccc} b_0 = k & b_1 & b_2 & \cdots & b_{d-1} & \\ a_0 = 0 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\ & c_1 & c_2 & \cdots & c_{d-1} & c_d \end{array}$$

such that whenever $x, y \in X$ with $d(x, y) = i$, vertex y has

- c_i neighbors at distance $i - 1$ from x
- a_i neighbors at distance i from x
- b_i neighbors at distance $i + 1$ from x

We know very little about distance-regular graphs of large girth.

Girth of Distance-Regular Graphs

Aside from the polygons, we know no distance-regular graphs of large girth.

Conjecture[Suzuki, Koolen] Any distance-regular graph Γ of valency $k \geq 3$ has girth $g_1(\Gamma)$ at most 12.

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Examples:

- incidence graphs of generalized hexagons of order (s, s) have girth 12
- Foster graph has girth 10 $(k = 3)$
- Biggs-Smith graph has girth 9 $(k = 3)$
- incidence graphs of generalized quadrangles $GQ(s, s)$ have girth 8

Weiss (1985) proved that the only distance-**transitive** graphs with $k \geq 3$ and $g \geq 9$ those appearing in the first 3 items above.

Girth of Distance-Regular Graphs

Related Facts:

- (Ivanov, 1983): If Γ has valency $k \geq 3$, girth g then its diameter is bounded by $d < g \cdot 2^{2k-3}$
- (BCN): Only three distance-regular graphs known with $d \geq 2k$ (all with $k = 3$)
- some graphs of small numerical girth can still have large "geometric girth"
- (Tanaka-WJM): The only distance-regular graphs we know with $k > 2$ and a splitting field which is more than a degree two extension of $\mathbb{Q}$ are the Biggs-Smith graph, the above generalized hexagons, and their line graphs

Polynomial association schemes

A *(symmetric) association scheme* consists of a finite set X together with a partition $\{R_0, \ldots, R_d\}$ of $X \times X$ into symmetric binary relations whose adjacency matrices $A_0, \ldots, A_d$ span a real vector space closed under matrix multiplication and containing I.

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P-**polynomial:** For some ordering, A_i is a polynomial of degree i in A_1 (This occurs iff (X, R_1) is a *distance-regular graph* of diameter d.)

Q-**polynomial:** For some ordering of primitive idempotents (orthogonal projections onto maximal common eigenspaces) $E_0, E_1, \ldots, E_d$, each E_i is an entrywise polynomial of degree i in E_1.

Example – An association scheme and its E_1

Consider the 3-cube. $A_0 = I, A_1 = \begin{bmatrix} 0&1&1&0&1&0&0&0\\ 1&0&0&1&0&1&0&0\\ 1&0&0&1&0&0&1&0\\ 0&1&1&0&0&0&0&1\\ 1&0&0&0&0&1&1&0\\ 0&1&0&0&1&0&0&1\\ 0&0&1&0&1&0&0&1\\ 0&0&0&1&0&1&1&0 \end{bmatrix}$,

$$A_2 = \begin{bmatrix} 0&0&0&1&0&1&1&0\\ 0&0&1&0&1&0&0&1\\ 0&1&0&0&1&0&0&1\\ 1&0&0&0&0&1&1&0\\ 0&1&1&0&0&0&0&1\\ 1&0&0&1&0&0&1&0\\ 1&0&0&1&0&1&0&0\\ 0&1&1&0&1&0&0&0 \end{bmatrix}, A_3 = \begin{bmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{bmatrix}$$

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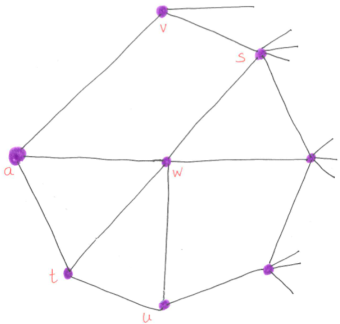
For the 3-cube,

$$E_1 = \frac{1}{8}\begin{bmatrix} 3 & 1 & 1 & -1 & 1 & -1 & -1 & -3 \\ 1 & 3 & -1 & 1 & -1 & 1 & -3 & -1 \\ 1 & -1 & 3 & 1 & -1 & -3 & 1 & -1 \\ -1 & 1 & 1 & 3 & -3 & -1 & -1 & 1 \\ 1 & -1 & -1 & -3 & 3 & 1 & 1 & -1 \\ -1 & 1 & -3 & -1 & 1 & 3 & -1 & 1 \\ -1 & -3 & 1 & -1 & 1 & -1 & 3 & 1 \\ -3 & -1 & -1 & 1 & -1 & 1 & 1 & 3 \end{bmatrix}$$

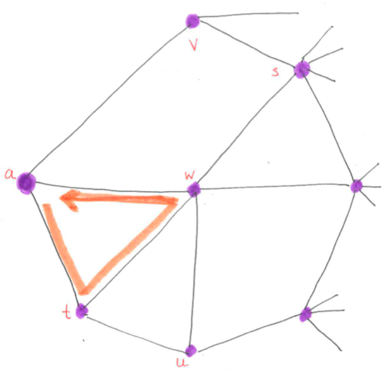
An Excursion into Homotopy

The following idea appears in the thesis work of Heather Lewis (*Discrete Math.* (2000))
under the supervision of Paul Terwilliger.



Consider equivalence classes of closed walks in Γ starting and ending at basepoint *a*.

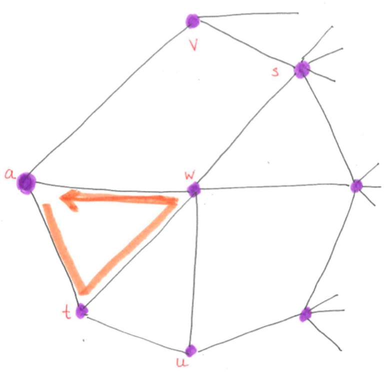
Discrete Homotopy on a Graph



Closed walk *atwa* is in the same equivalence class as *atwswa* and

avswsvatutwa

Discrete Homotopy on a Graph



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These all have "essential length" 3.

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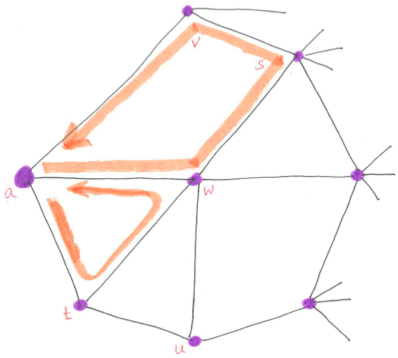
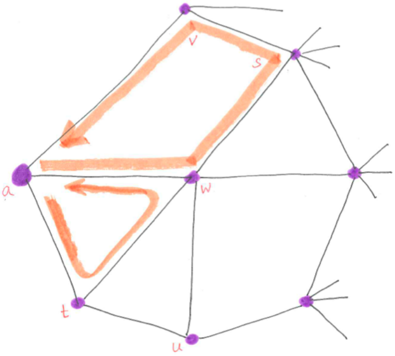


Closed walk *awsva* represents the same group element as

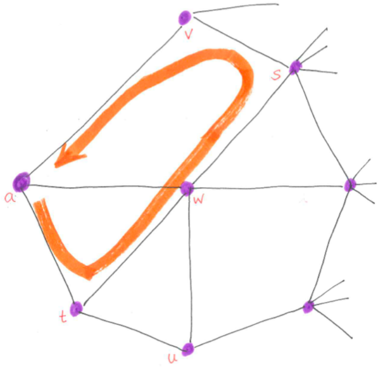
awtuwutwsva

Discrete Homotopy on a Graph



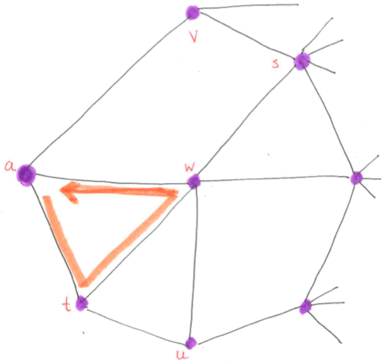
Our group operation is concatenation of walks. Of course, the concatenation of these two walks is represented by another cycle.

Homotopy: the group operation



$$atwa \;\star\; awsva \;=\; atwsva$$

Homotopy: the group operation



In this way, larger cycles are built from smaller ones. For example, take our first walk *atwa*

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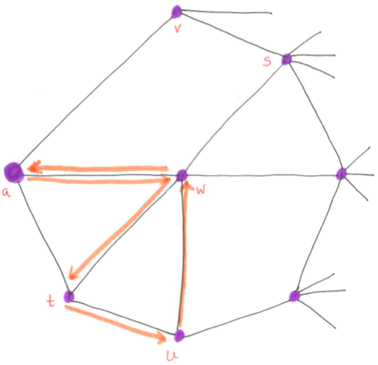

. . . and concatenate with the walk *awtuwa*

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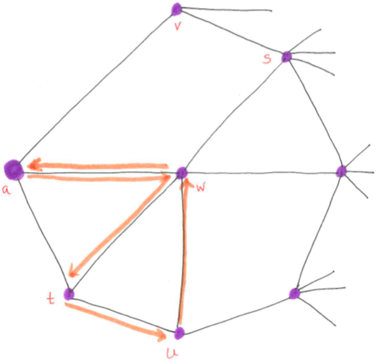
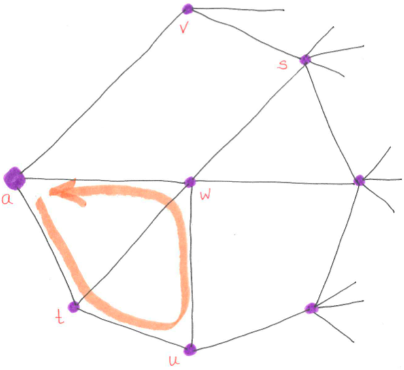


...and concatenate with the walk *awtuwa*
which also has *essential length* 3 as it has form $\mathbf{pqp}^{-1}$ for a walk $\mathbf{q} = wtuw$ of length three and a path $\mathbf{p}$.

Homotopy: the group operation



In our fundamental group, we have $atwa \star awtuwa = atuwa$

Subgroups of the Fundamental Group

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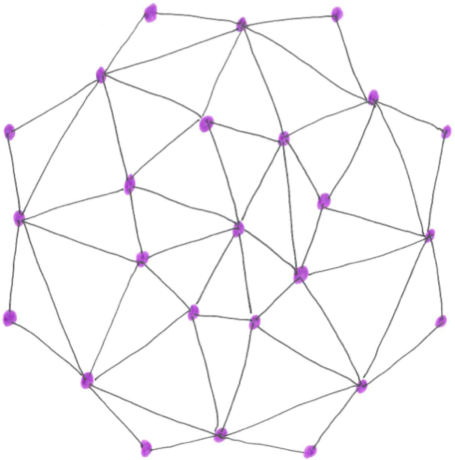
For example, if Γ is a simple graph, $\pi_k(\Gamma, a) = 1$ for $k = 0, 1, 2$.

Discrete Homotopy on a Graph

In this example, $\pi(\Gamma, a) = \pi_3(\Gamma, a)$



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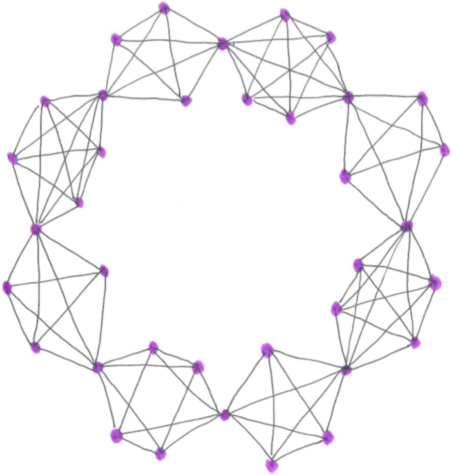
In this example, $\pi_3(\Gamma, a) = \pi_4(\Gamma, a) = \pi_5(\Gamma, a) \neq \pi(\Gamma, a)$



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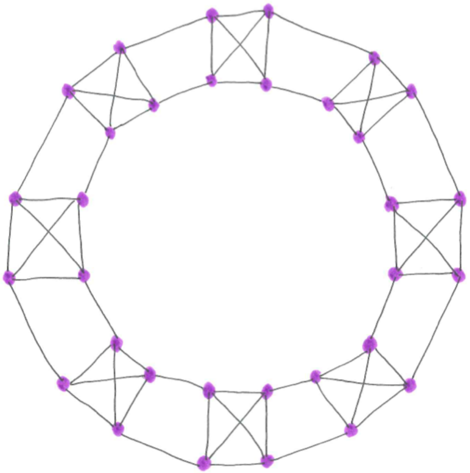

Some results of Heather Lewis

- $\pi_0(\Gamma, x) = \pi_1(\Gamma, x) = \pi_2(\Gamma, x) \subseteq \pi_{2d+1}(\Gamma, x) = \pi(\Gamma, x)$
- a Q-polynomial distance-regular graph has girth at most 6
- For any Q-polynomial distance-regular graph, $\pi_6(\Gamma, x) \neq \{e\}$
- and either $\pi_6(\Gamma, x) = \pi(\Gamma, x)$ or
  - Γ is a "pseudoquotient" with $D \in \{2d, 2d+1\}$ and
  - $\pi_6(\Gamma, x) = \pi_{D-1}(\Gamma, x) \neq \pi_D(\Gamma, x) = \pi(\Gamma, x)$

Girth Parameters

So it makes sense to consider not only the *numerical girth*

$$g_1(\Gamma)$$

of a distance-regular graph, but also

$$g_2(\Gamma) = \min\{k \mid \pi_k(\Gamma, x) = \pi(\Gamma, x)\}$$

It's Important to have Open Problems

"It's fine to discover cures, but, remember, chronic conditions are our bread and butter."

The Ideal of a Cometric Scheme

Consider a cometric association scheme $(X, \mathcal{R})$ with Q-polynomial ordering

$$E_0, E_1, \ldots, E_d$$

of its primitive idempotents.

We consider the columns of E_1 as $|X|$ vectors in $\mathbb{R}^|X|$ and wish to determine the ideal of all polynomials in $|X|$ variables which vanish on all of these points.

If $m = \operatorname{rank} E_1$, we may instead find a matrix U with $|X$ rows and m columns satisfying $E_1 = UU^\top$ and find the ideal of all polynomials in m variables that vanish on each row of U.

In fact, we will identify X with this set (or something equivalent) and denote by $\mathcal{I}(X)$ this ideal.

Ideal of a finite set

Let X be a finite subset of $\mathbb{R}^m$. For $a \in X$, write

$$a = (a_1, \ldots, a_m).$$

Now consider polynomials in m variables $F(Y) = F(Y_1, \ldots, Y_m)$ from the polynomial ring $\mathcal{R} = \mathbb{C}[Y_1, \ldots, Y_m]$.

We wish to study the ideal

$$\mathcal{I}(X) = \{F \in \mathcal{R} \mid F(a_1, \ldots, a_m) = 0 \ \forall \ a \in X\}$$

of all polynomials in m variables that vanish at every point of X.

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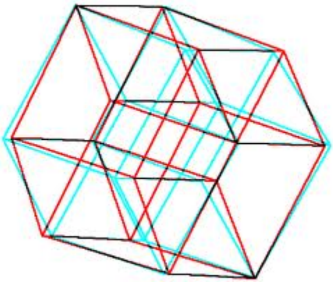
Cubes

Let X consist of the 2^m points $a = (\pm 1, \pm 1, \ldots, \pm 1)$



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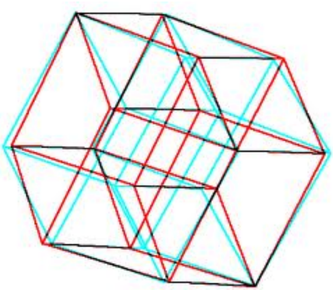

- For each i, the quadratic $Y_i^2 - 1$ vanishes at a for each $a \in X$
- The ideal $\mathsf{I} = \langle Y_1^2 - 1, \ldots, Y_m^2 - 1 \rangle$ clearly has exactly X as its zero set: $\mathcal{Z}(\mathsf{I}) = X$

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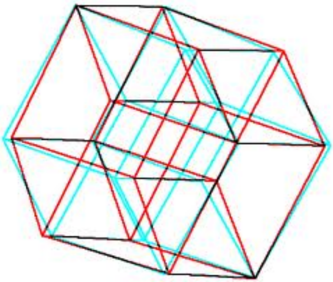

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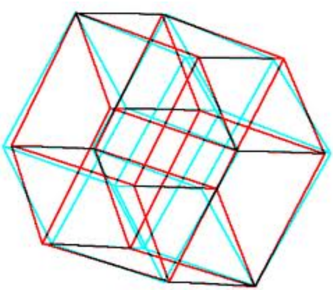

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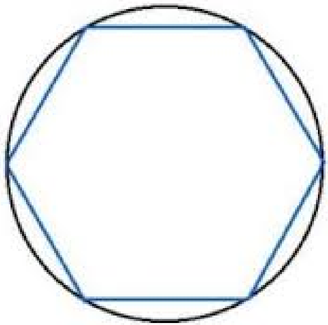
- The ideal $\mathsf{I} = \langle Y_1^2 - 1, \ldots, Y_m^2 - 1 \rangle$ clearly has exactly X as its zero set: $\mathcal{Z}(\mathsf{I}) = X$
- Since each zero is a simple zero of I, the ideal is *radical*
- So $\mathsf{I} = \mathcal{I}(X)$, the ideal of all polynomials that vanish on X.

Polygons

Let $m = 2$ and consider the regular n-gon X on the unit circle, with equation
$Y_1^2 + Y_2^2 = 1$



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By Bezout's Theorem, any polynomial in $\mathcal{I}(X)$ which is not a multiple of Nm has degree $\lceil n/2 \rceil$ or larger.

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Let $m = 2$ and consider the regular n-gon X on the unit circle.

$Y_1^2 + Y_2^2 - 1 \in \mathcal{I}(X)$. We call this Nm and consider it "trivial".

By Bezout's Theorem, any polynomial in $\mathcal{I}(X)$ which is not a multiple of Nm has degree $\lceil n/2 \rceil$ or larger.

We can choose two zonal polynomials F and G and see that

$$\mathcal{I}(X) = \langle \mathrm{Nm}, F, G \rangle$$

E.g., for the regular hexagon, we may choose
$F(Y) = Y_2(Y_2^2 - 3/4)$ and

$$G(Y) = (\sqrt{3}Y_1 + Y_2)(\sqrt{3}Y_1 + Y_2 - \sqrt{3})(\sqrt{3}Y_1 + Y_2 + \sqrt{3})$$

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  where $\mathrm{Rad}(\mathrm{J}) = \{F \mid \exists n \; F^n \in \mathrm{J}\}$
- If $\mathcal{Z}(J)$ is finite and J is radical (i.e., $\mathrm{J} = \mathrm{Rad}(\mathrm{J})$), then any ideal containing J is radical

Two "dual girth" parameters

For interesting structures represented by subsets X of Euclidean space, we are interested in two measures of complexity:

- $\gamma_1(X)$: the smallest degree of a non-trivial polynomial in $\mathcal{I}(X)$
- $\gamma_2(X)$: the smallest k for which $\mathcal{I}(X)$ admits a generating set of polynomials of degree k or less

Observe: These two values are invariant under invertible affine transformation.

Two Dual Girth Parameters

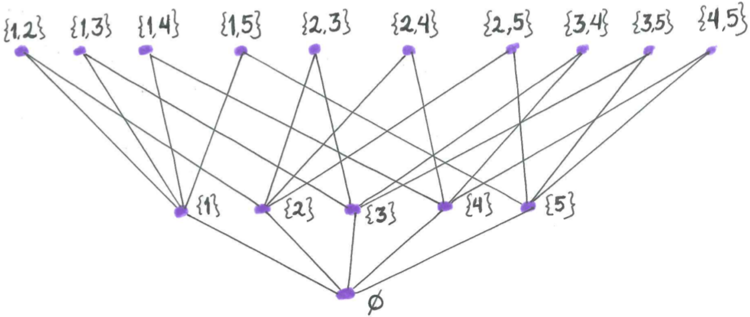
Recap: Trivial polynomials don't depend on X in $\mathcal{X}$. In our case, we will take $\mathcal{X}$ to be the set of all spherical codes in $\mathbb{R}^m$.

If $\mathcal{T}$ denotes the ideal of trivial polynomials, a principal ideal in this case, we may write

$$\gamma_1(X) := \min \{\deg F \mid F \in \mathcal{I}(X),\ F \notin \mathcal{T}\}$$

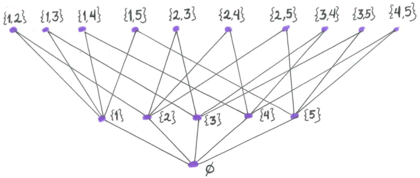
$$\gamma_2(X) := \min \{\max\{\deg F \mid F \in \mathcal{G}\} \ \mid \langle \mathcal{G} \rangle = \mathcal{I}(X)\}$$

Truncated Boolean Lattice



For $n = 5$, $\Omega = \{1, 2, 3, 4, 5\}$ and $k = 2$, we take all subsets of Ω of size at most k, ordered by inclusion.

Truncated Boolean Lattice

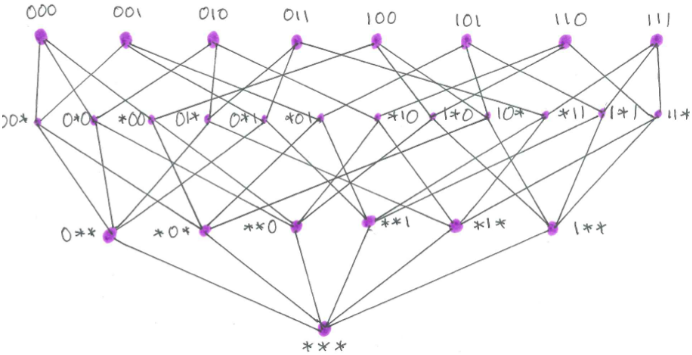


Incidence matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

X consists of 10 points in $\mathbb{R}^5$ and $\mathcal{I}(X)$ is generated by the obvious quadratics (trivial polynomials for designs).

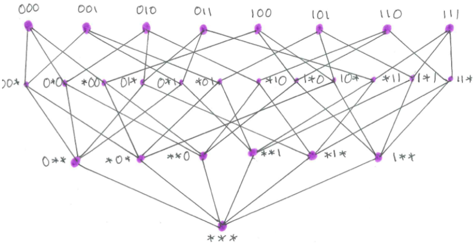
Hamming Lattice



For $n = 3$ and $q = 2$, we consider all "partial" n-tuples over $\mathbb{Z}_q$, marking unspecified entries with '*'. Partial order relation is:

$$a \preceq b \text{ if } a_i = b_i \text{ whenever } a_i \neq *$$

Hamming Lattice



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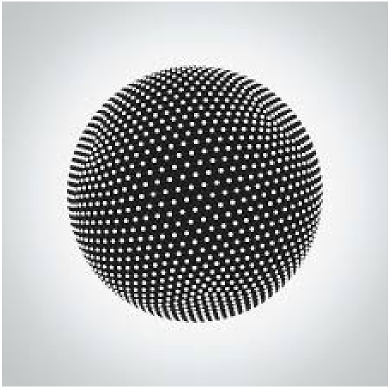
X consists of 8 points in $\mathbb{R}^6$ and $\mathcal{I}(X)$ is generated by trivial polynomials together with

$$Y_1 + Y_6 - 1, \quad Y_2 + Y_5 - 1, \quad Y_3 + Y_4 - 1.$$

Similar ideas work for the Grassmann scheme and the bilinear forms scheme.

Spherical designs

From now on, we will assume that X is contained in some sphere, of radius r say, centered at the origin.



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So the ideal $\mathcal{I}(X)$ contains the "norm" polynomial

$$\mathrm{Nm} = Y_1^2 + Y_2^2 + \cdots + Y_m^2 - r^2$$

as well as every polynomial in the principal ideal $\langle \mathrm{Nm} \rangle$. In this setting, these polynomials are called "trivial".

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A subset X of the unit sphere is a *spherical t-design* if the average over X of any polynomial F of degree $\leq t$ in m variables is exactly equal to the average of F over the sphere.

Lower Bound on $\gamma_1(X)$ for Spherical t-Designs

The following observation is due to Bannai (probably also known in cubature community).

Lemma

If X is a spherical t-design, then every polynomial in $\mathcal{I}(X)$ of degree $t/2$ or less is trivial.

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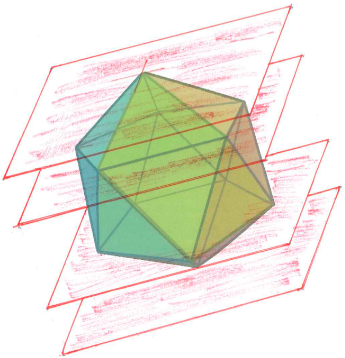
Proof: If $F \in \mathcal{I}(X)$, then F^2 is zero on X, so F^2 averages to zero on the sphere. But F^2 is a non-negative polynomial function, so F^2 is identically zero on the sphere.

Zonal Polynomials

For a single-variable polynomial $f(t)$ and a point $a \in \mathbb{R}^m$, we define the *zonal polynomial*

$$Z_{f,a}(Y) = f(a \cdot Y)$$



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For example, if $f(t) = \prod_{h=0}^{d}(t - \omega_h)$, then

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is a polynomial of degree $d+1$ in m variables.

In this case, if $\{a \cdot b \mid b \in X\} \subseteq \{\omega_0, \ldots, \omega_d\}$, then $Z_{f,a} \in \mathcal{I}(X)$.

Ideal of zonals

Now suppose X has inner product set

$$\{a \cdot b \mid a, b \in X\} = \{\omega_0, \ldots, \omega_d\}.$$

Then, for each $a \in X$, the ideal $\mathcal{I}(X)$ contains the zonal polynomial $Z_{f,a}(Y)$ where

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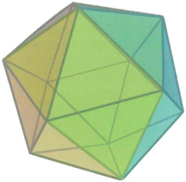
Observe that, if X spans $\mathbb{R}^m$, then the tangent space at every solution to this system of equations is zero-dimensional.
So this ideal is radical, as is any ideal that contains it.
We can also show that every solution to this system of polynomial equations lies in $\mathbb{R}^m$
(except for the orthoplex! E.g. the octahedron.)

Sliced zonals

Consider the icosahedron



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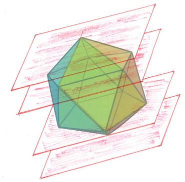

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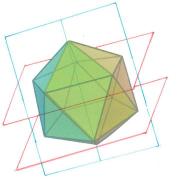

Not only does our ideal contain the degree four zonal polynomials, but it also contains many degree three *"sliced zonal polynomials"*. If $\omega_0 = 1 > \omega_1 > \cdots > \omega_d = -1$, replace

$$Z_{f,a}(Y) = (a \cdot Y - \omega_0)(a \cdot Y - \omega_1) \cdots (a \cdot Y - \omega_d) \qquad \text{by}$$

$$S_{f,a,b}(Y) = (b \cdot Y)(a \cdot Y - \omega_1) \cdots (a \cdot Y - \omega_{d-1})$$

for any $b \perp a$ in $\mathbb{R}^m$ and this also vanishes at each point in X, including a and $-a$.

The Icosahedron and Famous Lattices

These sliced zonal polynomials generate $\mathcal{I}(X)$ in these cases:

Name	Dim	strength	$\gamma_1(X)$	$\gamma_2(X)$
icos.	3	5	3	3
E_6	6	5	3	3
E_7	7	5	3	3
E_8	8	7	4	4
Leech	24	11	6	6

(joint with Corre Love Steele `arXiv:1310.6626`)

These are examples of Q-bipartite association schemes, where $\gamma_1(E_1) \leq d$ is shown using sliced zonals.

Basic Inequalities

$$2 \leq \gamma_1(E_1) \leq \gamma_2(E_1) \leq d + 1$$

The zonal ideal is not always equal to the full ideal (e.g., non-maximal sets of real mutually unbiased bases), but we need only throw in some polynomials of degree d to "shave off" phantom vertices.

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- When is $\mathcal{I}(X)$ generated by its small-degree polynomials?
- If the polynomials of degree $\leq k$ do <u>not</u> generate $\mathcal{I}(X)$, what is the variety of the ideal they do generate?

A dual pair of association schemes

The Q-polynomial association scheme (it's just a strongly regular graph) $K_{3,3}$ has eigenmatrix $P = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 0 & -1 \\ 1 & -3 & 2 \end{bmatrix}$ with inverse $\frac{1}{6}\begin{bmatrix} 1 & 4 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$.

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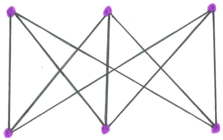

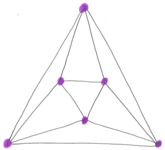

The dual association scheme is the one coming from the octahedron. It has eigenmatrix $\begin{bmatrix} 1 & 4 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$ with inverse $\frac{1}{6}P$.

Claim: ideal of $K_{3,3}$ is closely linked to homotopy of octahedron.

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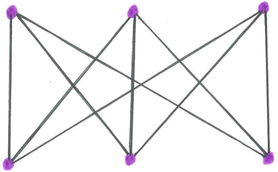

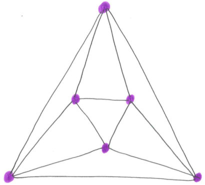

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Association Scheme Duality

Association scheme $(X, \{R_0, \ldots, R_d\})$ has eigenmatrix P whose columns are the eigenvalues of the graphs (X, R_i). (These $d+1$ adjacency matrices are simultaneously diagonalizable. Take my word for it.)

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Recall that a finite abelian group Γ has $|G|$ linear characters (homomorphisms $\chi : G \to \mathbb{C}^*$) and these form a group under multiplication of functions.

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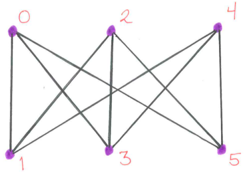
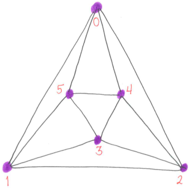
Recall that a finite abelian group Γ has $|G|$ linear characters (homomorphisms $\chi : G \to \mathbb{C}^*$) and these form a group under multiplication of functions. This group $G^\dagger$ is isomorphic to Γ.

Association Scheme Duality

Both $K_{3,3}$ and the octahedron are translation association schemes: the abelian group $\mathbb{Z}_6$ acts on each as a regular group of automorphisms.





Connection set for $K_{3,3}$ is $\{1, 3, 5\}$; connection set for octahedron is $\{1, 2, 4, 5\}$.

Association Scheme Duality

Connection set for the octahedron is $\{1, 2, 4, 5\}$.

The corresponding characters of $\mathbb{Z}_6$ form a basis for the first eigenspace of $K_{3,3}$. Let ω be a primitive sixth root of unity in $\mathbb{C}$. Then we have

$$\chi_1 = \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ -1 \\ \omega^4 \\ \omega^5 \end{bmatrix}, \ \chi_2 = \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ 1 \\ \omega^8 \\ \omega^{10} \end{bmatrix}, \ \chi_4 = \begin{bmatrix} 1 \\ \omega^4 \\ \omega^8 \\ 1 \\ \omega^{16} \\ \omega^{20} \end{bmatrix}, \ \chi_5 = \begin{bmatrix} 1 \\ \omega^5 \\ \omega^4 \\ -1 \\ \omega^2 \\ \omega \end{bmatrix}$$

Characters yield spherical representation of $K_{3,3}$

We put these characters together in a matrix and map vertex i to the i^{th} column of

$$\begin{bmatrix} 1 & \omega & \omega^2 & -1 & \omega^4 & \omega^5 \\ 1 & \omega^2 & \omega^4 & 1 & \omega^2 & \omega^4 \\ 1 & \omega^4 & \omega^2 & 1 & \omega^4 & \omega^2 \\ 1 & \omega^5 & \omega^4 & -1 & \omega^2 & \omega \end{bmatrix}$$

to obtain

$$X = \{(1,1,1,1), (\omega,\omega^2,\omega^4,\omega^5), (\omega^2,\omega^4,\omega^2,\omega^4),$$
$$(-1,1,1,-1), (\omega^4,\omega^2,\omega^4,\omega^2), (\omega^5,\omega^4,\omega^2,\omega)\}$$

Discrete Homotopy on a Graph

Now we observe that every closed walk in the octahdron gives us a nice polynomial in $\mathcal{I}(X)$. E.g. $w = 0450$ gives $Y_4 Y_1 Y_1 - 1$ in $\mathcal{I}(X)$.

Homotopy, Ideals and Duality

The upshot of all this is the following theorem:

If a translation distance-regular graph G has its fundamental group generated by small cycles, then the dual (Q-polynomial) scheme has its ideal generated by small degree polynomials.

$$\pi_k(\Gamma, 0) = \pi(\Gamma, 0) \quad \Rightarrow \quad \gamma_2(E_1) \le \lceil k/2 \rceil \ .$$

Homotopy, Ideals and Duality

Theorem

Let $(X, \{R_0, \ldots, R_d\})$ be a cometric translation association scheme defined on abelian group X and let $(X^\dagger, \{R_0', \ldots, R_d'\})$ be the (metric) dual association scheme defined on the group of characters $X^\dagger$. Let $\Gamma = (X^\dagger, R_1')$ denote the underlying translation distance-regular graph. Let E_1 denote the first primitive idempotent in the corresponding Q-polynomial ordering for the original scheme.

If the homotopy group $\pi(\Gamma, \mathbf{1})$ is generated by closed walks of essential length k or less,then the ideal $\mathcal{I}(E_1)$ is generated by polynomials of degree k or less.

$$\pi_k(\Gamma, 0) = \pi(\Gamma, 0) \quad \Rightarrow \quad \gamma_2(E_1) \le \lceil k/2 \rceil \ .$$

Conjectures

In all six statements, exclude polygons.

- [P] **Conj (Suzuki/Koolen):** Any distance-regular graph has girth at most 12
- [Q] **Conj (WJM):** For any Q-polynomial scheme, $\gamma_2(E_1) \leq 6$
- [P] **Thm (Lewis):** A Q-polynomial drg has girth at most 6
- [Q] **Thm:** For any P- and Q-poly scheme, $\gamma_1(E_1) \leq 3$
- [P] **Thm (Lewis):** When Γ is a Q-poly drg, and not a pseudo-quotient, $\pi_6(\Gamma, a) = \pi(\Gamma, a)$
- [Q] **Conj (WJM):** For any P- and Q-poly scheme, $\gamma_2(E_1) \leq 3$

Consequences and Partial Results

- If we prove $\gamma_1(E_1) \leq 6$ for cometric schemes, we get a new proof of a result of Bannai and Damerell showing the non-existence of tight spherical t-designs
- Note that $\gamma_1 \leq 3$ for a Q-polynomial distance-regular graph with $k > 2$, so these have spherical strength at most five
- Likewise, we would rule out t-designs with $t \geq 12$ in the regular semilattices that induce Q-polynomial schemes
- (WJM & Williford): For Q-polynomial association schemes, $\gamma_2(E_1)$ is bounded by a function of $m_1 := \operatorname{rank} E_1$

Open Problems

- Prove $\gamma_1(E_1) \leq 6$ for important classes of Q-poly schemes
- ... real MUBs, hemisystems in GQs, relative hemisystems, ...
- For regular semilattices, $\gamma_2(E_1) \leq 2$?
- What happens for other (Euclidean) lattices?
- Finite bounds for special classes of schemes would be of value
- as would any finite bounds on γ_2 (or γ_1) even if not at the conjectured optimum
- Can we close the gap between $t/2$ and t for **block designs** or **nonlinear codes**?

Thank You

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