# Extremal Theorems in polar spaces

Valentina Pepe

La Sapienza University of Rome
–
Joint work with Leo Storme and Frédéric Vanhove

February 28th 2014

# Extremal Combinatorics

It studies discrete structures whose characteristic parameters meet extreme values.
Typically, the parameter is the size.
It started in the 1930's with the work of Erdős and Turán.

# Extremal Combinatorics

It studies discrete structures whose characteristic parameters meet extreme values.
Typically, the parameter is the size.
It started in the 1930's with the work of Erdős and Turán.
Extremal combinatorics problems can originate in different areas, such as geometry, graph theory, analysis, number theory, and they have remarkable applications on computer science and information theory.

# A classical problem in extremal Set Theory

$S :=$ set with $n$ elements

# A classical problem in extremal Set Theory

$S :=$ set with $n$ elements
$\mathcal{F} =$ family of subsets of $S$ of size $k$, $2k \leq n$, pairwise intersecting

# A classical problem in extremal Set Theory

$S :=$ set with $n$ elements
$\mathcal{F} =$ family of subsets of $S$ of size $k$, $2k \leq n$, pairwise intersecting
What is the maximum M for $|\mathcal{F}|$? Is it possible to characterize the families $\mathcal{F}$ such that $|\mathcal{F}| =$M?

# The first Erdős-Ko-Rado Theorem

## E.K.R. [1961]

If $S$ is a set with $n$ elements and $\mathcal{F}$ is a family of subsets of size $k$ of $S$, with $n \geq 2k$, such that the elements of $\mathcal{F}$ are pairwise intersecting, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

# The first Erdős-Ko-Rado Theorem

## E.K.R. [1961]

If $S$ is a set with $n$ elements and $\mathcal{F}$ is a family of subsets of size $k$ of $S$, with $n \geq 2k$, such that the elements of $\mathcal{F}$ are pairwise intersecting, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

## Characterization of the families of maximum size

If $|\mathcal{F}| = \binom{n-1}{k-1}$, then:

- $2k < n$ and $\mathcal{F}$ is the family of subsets of size $k$ containing a fixed element of $S$.
- $2k = n$ and $\mathcal{F}$ is either the family of subsets of size $k$ containing a fixed element of $S$ or it consists of the representatives of all the complementary pairs.

# Theorems of *EKR* type

Several different variants of this theorem have been proved.

# Theorems of *EKR* type

Several different variants of this theorem have been proved.

- B.M.I. Rands [1982], for blocks of $t-(v,k,\lambda)$ designs
- P.Frankl and R.M.Wilson [1986]/ M.W.Newman [2004] for subspaces of vector spaces
- D.Stanton [1980] for Chevalley groups
- ....

An upper bound for the size of the intersecting family is found and the family reaching it is, most of the times, a "point pencil" or "star".

# Classical finite polar spaces

Classical finite polar spaces are incidence structures consisting of the lattices of subspaces of a finite projective space totally isotropic with respect to a certain non-degenerate sesquilinear form.

- the parabolic quadric $\mathcal{Q}(2n, q)$: $n$-dimensional generators,
- the hyperbolic quadric $\mathcal{Q}^+(2n+1, q)$: $n$-dimensional generators,
- the elliptic quadric $\mathcal{Q}^-(2n+1, q)$: $(n-1)$-dimensional generators,
- the symplectic space $W(2n+1, q)$: $n$-dimensional generators,
- the hermitian variety $\mathcal{H}(2n, q^2)$: $(n-1)$-dimensional generators,
- the hermitian variety $\mathcal{H}(2n+1, q^2)$: $n$-dimensional generators.

In case we have a quadric or a hermitian variety, they are just the subspaces contained in them.

The analogue problem in this setting is finding the largest size for a set of pairwise intersecting subspaces of a polar space and characterizing the sets meeting the bound.

The analogue problem in this setting is finding the largest size for a set of pairwise intersecting subspaces of a polar space and characterizing the sets meeting the bound.
We deal with the case of generators of polar spaces, when their dimension is at least two.

# The bounds

Stanton [1980]:

| Polar space | Upper bound | Example of set meeting the bound |
|---|---|---|
| $\mathcal{Q}(2n,q)$ | $\prod_{i=1}^{n-1}(q^i+1)$ | generators through a point |
| $\mathcal{Q}^+(2n+1,q)$, $n$ odd | $\prod_{i=0}^{n-1}(q^i+1)$ | generators through a point |
| $\mathcal{Q}^+(2n+1,q)$, $n$ even | $\prod_{i=1}^{n}(q^i+1)$ | generators of one system |
| $\mathcal{Q}^-(2n+1,q)$ | $\prod_{i=2}^{n}(q^i+1)$ | generators through a point |
| $W(2n+1,q)$ | $\prod_{i=1}^{n}(q^i+1)$ | generators through a point |
| $\mathcal{H}(2n,q^2)$ | $\prod_{i=1}^{n-1}(q^{2i+1}+1)$ | generators through a point |
| $\mathcal{H}(2n+1,q^2)$, $n$ odd | $\prod_{i=0}^{n-1}(q^{2i+1}+1)$ | generators through a point |
| $\mathcal{H}(2n+1,q^2)$, $n$ even | $\prod_{i=0,i\neq\frac{n}{2}}^{n}(q^{2i+1}+1)$ | No examples known |

## Graph theoretic approach

We consider the regular graph $\Gamma$ s.t.

# Graph theoretic approach

We consider the regular graph $\Gamma$ s.t.
$\Omega :=$ set of vertices $=$ the generators of a polar space

# Graph theoretic approach

We consider the regular graph $\Gamma$ s.t.
$\Omega :=$ set of vertices $=$ the generators of a polar space
$\pi \sim \pi'$ iff $\pi \cap \pi' = \emptyset$.

# Graph theoretic approach

We consider the regular graph $\Gamma$ s.t.
$\Omega :=$ set of vertices $=$ the generators of a polar space
$\pi \sim \pi'$ iff $\pi \cap \pi' = \emptyset$.
$\Gamma$ is regular with valency $val =$ number of generators skew with a given one.

# Graph theoretic approach

We consider the regular graph $\Gamma$ s.t.
$\Omega :=$ set of vertices $=$ the generators of a polar space
$\pi \sim \pi'$ iff $\pi \cap \pi' = \emptyset$.
$\Gamma$ is regular with valency $val =$ number of generators skew with a given one.
An intersecting family $S$ corresponds to a coclique of the graph.

# Graph theoretic approach

We consider the regular graph $\Gamma$ s.t.
$\Omega :=$ set of vertices $=$ the generators of a polar space
$\pi \sim \pi'$ iff $\pi \cap \pi' = \emptyset$.
$\Gamma$ is regular with valency $val =$ number of generators skew with a given one.
An intersecting family $S$ corresponds to a coclique of the graph.
If $\tau$ is the least eigenvalue, then

$$|S| \leq \frac{|\Omega|}{1 - \frac{val}{\tau}}$$

and if $|S|$ meets the bound, then its characteristic vector $\chi_S$ is such that $\chi_S = \frac{|S|}{|\Omega|}\mathbf{1} + u$, where $u$ is an eigenvector with eigenvalue $\tau$.

# Association schemes

A $d$-class *association scheme* on a finite set $\Omega$ is a pair $(\Omega, \mathcal{R})$ with $\mathcal{R}$ a set of symmetric relations $\{R_0, R_1, \ldots, R_d\}$ on $\Omega$ such that the following axioms hold:

# Association schemes

A $d$-class *association scheme* on a finite set $\Omega$ is a pair $(\Omega, \mathcal{R})$ with $\mathcal{R}$ a set of symmetric relations $\{R_0, R_1, \ldots, R_d\}$ on $\Omega$ such that the following axioms hold:

(i) $R_0$ is the identity relation,

(ii) $\mathcal{R}$ is a partition of $\Omega^2$,

# Association schemes

A $d$-class *association scheme* on a finite set $\Omega$ is a pair $(\Omega, \mathcal{R})$ with $\mathcal{R}$ a set of symmetric relations $\{R_0, R_1, \ldots, R_d\}$ on $\Omega$ such that the following axioms hold:

(i) $R_0$ is the identity relation,

(ii) $\mathcal{R}$ is a partition of $\Omega^2$,

(iii) there are *intersection numbers* $p_{ij}^k$ such that for $(x, y) \in R_k$, the number of elements $z$ in $\Omega$ for which $(x, z) \in R_i$ and $(z, y) \in R_j$ equals $p_{ij}^k$.

All the relations $R_i$ are symmetric regular relations with valency $p_{ii}^0$, and hence define regular graphs on $\Omega$.

# Association scheme on generators

$\pi :=$ generator of the polar space and $\dim \pi = n$.

# Association scheme on generators

$\pi :=$ generator of the polar space and $\dim \pi = n$.
On $\Omega$ we can define a set of $n$ relations $R_i, i = 0, \cdots, n+1$ such that $\pi \sim \pi'$ with respect to $R_i$ iff $\dim \pi \cap \pi' = n - i$.
$R_0 =$ identity relation
$R_{n+1} =$ disjointness relation

# Association scheme on generators

$\pi :=$ generator of the polar space and $\dim \pi = n$.
On $\Omega$ we can define a set of $n$ relations $R_i, i = 0, \cdots, n+1$ such that $\pi \sim \pi'$ with respect to $R_i$ iff $\dim \pi \cap \pi' = n - i$.
$R_0 =$ identity relation
$R_{n+1} =$ disjointness relation
These relations give rise to an association scheme.

# Most of the cases

For the following polar spaces:

- $\mathcal{Q}(2n, q)$, $n$ even
- $\mathcal{Q}^{-}(2n+1, q)$
- $W(2n+1, q)$, $n$ odd
- $\mathcal{H}(2n, q^2)$ and $\mathcal{H}(2n+1, q^2)$, $n$ odd

if $u$ is an eigenvector for the disjointness relation $R_{n+1}$, then it is a an eigenvector for $R_i, i = 0, \cdots, n$.

# Most of the cases

For the following polar spaces:

- $\mathcal{Q}(2n, q)$, $n$ even
- $\mathcal{Q}^-(2n+1, q)$
- $W(2n+1, q)$, $n$ odd
- $\mathcal{H}(2n, q^2)$ and $\mathcal{H}(2n+1, q^2)$, $n$ odd

if $u$ is an eigenvector for the disjointness relation $R_{n+1}$, then it is a an eigenvector for $R_i, i = 0, \cdots, n$.
If $S$ is a intersecting set of maximum size, then $\chi_S = h\mathbf{1} + u$ and $u$ is an eigenvector w.r.t $R_i, \forall i$.

# Most of the cases

What is the geometric consequence?

# Most of the cases

What is the geometric consequence?

$S =$ intersecting family of maximum size, $\pi \in S$

$v_{\pi,S} =$ the vector of length $n$ such that $(v_{\pi,S})_i$ is the number of elements of $S$ meeting $\pi$ in a space of codimension $i$.

# Most of the cases

What is the geometric consequence?

$S$ = intersecting family of maximum size, $\pi \in S$

$v_{\pi,S}$ = the vector of length $n$ such that $(v_{\pi,S})_i$ is the number of elements of $S$ meeting $\pi$ in a space of codimension $i$.

$v_{\pi,S}$ does not depend on the geometrical configuration of $S$.

# Most of the cases

What is the geometric consequence?

$S$ = intersecting family of maximum size, $\pi \in S$

$v_{\pi,S}$ = the vector of length $n$ such that $(v_{\pi,S})_i$ is the number of elements of $S$ meeting $\pi$ in a space of codimension $i$.

$v_{\pi,S}$ does not depend on the geometrical configuration of $S$.

Known example of maximum intersecting family in these polar spaces: $S_0$ = point pencil.

For every $S$ intersecting family of maximum size and $\pi \in S$, $v_{\pi,S} = v_{\pi,S_0}$.

# Most of the cases

What is the geometric consequence?
$S$ = intersecting family of maximum size, $\pi \in S$
$v_{\pi,S}$ = the vector of length $n$ such that $(v_{\pi,S})_i$ is the number of elements of $S$ meeting $\pi$ in a space of codimension $i$.
$v_{\pi,S}$ does not depend on the geometrical configuration of $S$.
Known example of maximum intersecting family in these polar spaces: $S_0$ = point pencil.
For every $S$ intersecting family of maximum size and $\pi \in S$, $v_{\pi,S} = v_{\pi,S_0}$.

**Theorem**

For the polar spaces $\mathcal{Q}(2n, q)$, $n$ even, $\mathcal{Q}^-(2n+1, q)$, $W(2n+1, q)$, $n$ odd, $\mathcal{H}(2n, q^2)$ and $\mathcal{H}(2n+1, q^2)$, $n$ odd, the largest intersecting set of generators is the set of generators through a fixed point.

# Remaining cases

For the remaining cases, the algebraic combinatorics techniques, still very useful, are less powerful.
We need more (finite) geometry and to use the peculiar properties of the different polar spaces.

# Remaining cases

For the remaining cases, the algebraic combinatorics techniques, still very useful, are less powerful.
We need more (finite) geometry and to use the peculiar properties of the different polar spaces.
We needed to introduce the definition of nucleus of a generator.
$S =$ maximal intersecting family of generators, $\pi \in S$.

$\pi_s :=$ nucleus of $\pi$ defined as $\displaystyle \bigcap_{\pi' \in S|_{\mathrm{codim}\,\pi \cap \pi' = 1}} \pi' \cap \pi$

# Remaining cases

For the remaining cases, the algebraic combinatorics techniques, still very useful, are less powerful.
We need more (finite) geometry and to use the peculiar properties of the different polar spaces.
We needed to introduce the definition of nucleus of a generator.
$S$ = maximal intersecting family of generators, $\pi \in S$.

$\pi_s :=$ nucleus of $\pi$ defined as $\bigcap_{\pi' \in S|_{\mathrm{codim}\pi \cap \pi' = 1}} \pi' \cap \pi$

In the remaining cases, we have that $s \in \{-1, 0, \dim \pi - 1\}$. For $s = 0$, we have the point pencil.

# Hyperbolic quadric $\mathcal{Q}^+(2n+1,q)$

In $\mathcal{Q}^+(2n+1,q)$ there are two system of generators, $\Omega_1$ and $\Omega_2$ of the same size, such that two generators $\pi_1$ and $\pi_2$ are in the same system iff $\dim \pi_1 \cap \pi_2$ has the same parity as $\dim \pi$.

# Hyperbolic quadric $\mathcal{Q}^+(2n+1, q)$

In $\mathcal{Q}^+(2n+1, q)$ there are two system of generators, $\Omega_1$ and $\Omega_2$ of the same size, such that two generators $\pi_1$ and $\pi_2$ are in the same system iff $\dim \pi_1 \cap \pi_2$ has the same parity as $\dim \pi$.

## Even $n$

The generators of $\Omega_i$ pairwise intersect in a non–empty space.
The size of $\Omega_i$ meets the Stanton bound.
It is the only possible intersecing set meeting the bound.

# Hyperbolic quadric $\mathcal{Q}^+(2n+1,q)$

In $\mathcal{Q}^+(2n+1,q)$ there are two system of generators, $\Omega_1$ and $\Omega_2$ of the same size, such that two generators $\pi_1$ and $\pi_2$ are in the same system iff $\dim \pi_1 \cap \pi_2$ has the same parity as $\dim \pi$.

## Even $n$

The generators of $\Omega_i$ pairwise intersect in a non–empty space.
The size of $\Omega_i$ meets the Stanton bound.
It is the only possible intersecing set meeting the bound.

## Odd $n$

$S$ is a maximum intersecting set iff $S = S_1 \cup S_2$, $S_i$ is a maximum intersecting set in the half dual polar graphs arising from $\Omega_i, i = 1, 2$.

# $\mathcal{Q}^+(2n+1,q)$, $n$ odd

We can focus on only one system of generators $\Omega_i$.

**Theorem**

If $n > 3$ is odd, then $S_i$ is the set of elements of $\Omega_i$ through a point. If $n = 3$, then $S_i$ is either the set of elements of $\Omega_i$ through a point or it is the set of elements of $\Omega_i$ meeting a fixed element of $\Omega_j$ in a plane.

The union of any two $S_i \subset \Omega_i$ is an intersecting set of maximum size.

# Parabolic quadric $\mathcal{Q}(2n, q)$, $n$ odd

Embed $\mathcal{Q}(2n, q)$, $n$ odd, as a hyperplane section in a $\mathcal{Q}^+(2n+1, q)$: every generator of $\mathcal{Q}(2n, q)$ is contained in a unique generator of a fixed system $\Omega_i$ of $\mathcal{Q}^+(2n+1, q)$.

# Parabolic quadric $\mathcal{Q}(2n, q)$, $n$ odd

Embed $\mathcal{Q}(2n, q)$, $n$ odd, as a hyperplane section in a $\mathcal{Q}^+(2n+1, q)$: every generator of $\mathcal{Q}(2n, q)$ is contained in a unique generator of a fixed system $\Omega_i$ of $\mathcal{Q}^+(2n+1, q)$.

An intersecting set $S$ of maximum size of $\mathcal{Q}(2n, q)$ gives rise to intersecting set $S'$ of maximum size of $\Omega_i$.

# Parabolic quadric $\mathcal{Q}(2n, q)$, $n$ odd

Embed $\mathcal{Q}(2n, q)$, $n$ odd, as a hyperplane section in a $\mathcal{Q}^+(2n+1, q)$: every generator of $\mathcal{Q}(2n, q)$ is contained in a unique generator of a fixed system $\Omega_i$ of $\mathcal{Q}^+(2n+1, q)$.

An intersecting set $S$ of maximum size of $\mathcal{Q}(2n, q)$ gives rise to intersecting set $S'$ of maximum size of $\Omega_i$.

**Theorem**

If $S$ is a maximum intersecting sets of generators of $\mathcal{Q}(2n, q)$, then one the following possibilities can occur:

- $S$ is a point pencil
- $S$ is the set of generators of one system of a $\mathcal{Q}^+(2n-1, q)$ embedded in $\mathcal{Q}(2n, q)$.
- $n = 3$ and $S$ consists of a plane $\pi$ and all the planes meeting $\pi$ in a line

# $W(2n+1, q)$, $n$ and $q$ even

If $q$ is even, then:
$W(2n+1, q) \cong Q(2n+2, q)$
There is a $\mathcal{Q}^+(2n+1, q)$ inducing the symplectic polarity

# $W(2n+1, q)$, $n$ and $q$ even

If $q$ is even, then:
$W(2n+1, q) \cong Q(2n+2, q)$
There is a $\mathcal{Q}^+(2n+1, q)$ inducing the symplectic polarity

**Theorem**

An intersecting set of maximum size $S$ is

- a point pencil or
- the set of generators of one system of a $\mathcal{Q}^+(2n+1, q)$ or
- $n = 2$ and it consists of the plane $\pi$ and the planes meeting $\pi$ in a line

## $W(2n+1, q)$, $n$ even and $q$ odd

Let $v_{\pi,S}$ be the vector of length $n$ such that $(v_{\pi,S})_i$ is the number of elements of $S$ meeting $\pi$ in a space of codimension $i$, then:

$$v = hv_1 + (1-h)v_2$$

where

## $W(2n+1, q)$, $n$ even and $q$ odd

Let $v_{\pi,S}$ be the vector of length $n$ such that $(v_{\pi,S})_i$ is the number of elements of $S$ meeting $\pi$ in a space of codimension $i$, then:

$$v = hv_1 + (1-h)v_2$$

where $v_1$ arises from the point pencil construction

## $W(2n+1, q)$, $n$ even and $q$ odd

Let $v_{\pi,S}$ be the vector of length $n$ such that $(v_{\pi,S})_i$ is the number of elements of $S$ meeting $\pi$ in a space of codimension $i$, then:

$$v = hv_1 + (1-h)v_2$$

where $v_1$ arises from the point pencil construction and $v_2$ from the construction of the elements of one system of a hyperbolic quadric.

# $W(2n+1, q)$, $n$ even and $q$ odd

Let $v_{\pi,S}$ be the vector of length $n$ such that $(v_{\pi,S})_i$ is the number of elements of $S$ meeting $\pi$ in a space of codimension $i$, then:

$$v = hv_1 + (1-h)v_2$$

where $v_1$ arises from the point pencil construction and $v_2$ from the construction of the elements of one system of a hyperbolic quadric. Further investigation on the related association scheme and with more geometric arguments, we get:

**Theorem**

- $S$ is a point pencil or
- $n = 2$ and $S$ consists of the plane $\pi$ and the planes meeting $\pi$ in a line.

## $\mathcal{H}(4n+1, q^2)$

**Theorem**

Intersecting set $|S| < \frac{|\Omega|}{1-\frac{k}{\tau}} = \frac{|\Omega|}{q^{2n+1}+1}$ (more than point-pencil).

# $\mathcal{H}(4n+1, q^2)$

**Theorem**

Intersecting set $|S| < \frac{|\Omega|}{1-\frac{k}{\tau}} = \frac{|\Omega|}{q^{2n+1}+1}$ (more than point-pencil).

The algebraic combinatorial techniques cannot be used.

## $\mathcal{H}(4n+1, q^2)$

**Theorem**

Intersecting set $|S| < \frac{|\Omega|}{1-\frac{k}{\tau}} = \frac{|\Omega|}{q^{2n+1}+1}$ (more than point-pencil).

The algebraic combinatorial techniques cannot be used.

**Theorem for planes in $\mathcal{H}(5, q^2)$**

- maximum size: $1+q+q^3+q^5 < \frac{|\Omega|}{q^3+1} = (q+1)(q^5+1)$,
- only construction: a fixed plane and all the those meeting it in line.

If $S$ is a point pencil, then $|S| = (q+1)(q^3+1) < 1+q+q^3+q^5$.

| Polar space | intersecting set of maximum size |
|---|---|
| $\mathcal{Q}(4n, q)$ | point pencil |
| $\mathcal{Q}(4n+2, q) n > 1$ | point pencil, generators of one system in a $\mathcal{Q}^+(4n+1, q)$ |
| $\mathcal{Q}(6, q)$ | point pencil, generators of one system in a $\mathcal{Q}^+(5, q)$<br>a fixed plane and the planes meeting it in a line |
| $\mathcal{Q}^+(4n+3, q)$,<br>$n > 1$ a fixed system | point pencil |
| $\mathcal{Q}^+(7, q)$ a fixed system | point pencil<br>solids meeting a fixed one of the other system in a plane |
| $\mathcal{Q}^+(4n+1, q)$ | generators of one system |
| $\mathcal{Q}^-(2n+1, q)$ | point pencil |
| $W(4n+3, q)$ | point pencil |
| $W(4n+1, q) n > 1$ | point pencil, generators of one system in $\mathcal{Q}^+(4n+1, q)$ $q$ even |
| $W(5, q)$ | point pencil, a fixed plane and the planes meeting it in a line<br>generators of one system in $\mathcal{Q}^+(5, q)$ $q$ even |
| $\mathcal{H}(2n, q^2), \mathcal{H}(4n+3, q^2)$ | point pencil |
| $\mathcal{H}(5, q^2)$ | a fixed plane and the planes meeting it in a line |
| $\mathcal{H}(4n+1, q^2) n > 1$ | ? |