COLLOQUIUM
ON GALOIS GEOMETRY

Friday, May 27, 2011

Abstracts

On arcs and caps in projective spaces

Daniele Bartoli

Università degli Studi di Perugia

(Joint work with A. A. Davydov, F. Pambianco and S. Marcugini)

Let $\mathrm{PG}(2, q)$ be the projective plane over the Galois field F_q. An k-arc is a set of k points no three of which are collinear. An k-arc is called complete if it is not contained in an $(k+1)$-arc of $\mathrm{PG}(2, q)$.
In particular, a complete arc in a plane $\mathrm{PG}(2, q)$, points of which are treated as 3-dimensional q-ary columns, defines a parity check matrix of a q-ary linear code with codimension 3, Hamming distance 4, and covering radius 2. Arcs can be interpreted as linear maximum distance separable (MDS) codes.
One of the main problems in the study of projective planes, which is also of interest in coding theory, is to find the spectrum of possible sizes of complete arcs.
In this work we give estimations for $t_2(2, q)$, the smallest size of a complete arc in $\mathrm{PG}(2, q)$. Moreover we propose new constructions of complete arcs in $\mathrm{PG}(2, q)$ and consider the spectrum of their possible sizes.

Daniele Bartoli, Università degli Studi di Perugia, Dipartimento di Matematica e Informatica, via Vanvitelli 1, I-06123 Perugia, Italy
daniele275@gmail.com

Small tight sets of hyperbolic quadrics

Linda Beukemann

Justus Liebig Universität Gießen

(Joint work with Klaus Metsch)

A *tight set* of a hyperbolic quadric $\mathcal{Q}^+(2n+1,q)$ is defined as a set M of points with the property that the average number of points of M in the tangent hyperplanes of points of M is as big as possible. It was shown in [4] that this number is bounded above by

$$q^n + |M|\frac{q^n-1}{q^{n+1}-1}.$$

Also if equality occurs, then all tangent hyperplanes of points in M have this many points in M, and the tangent hyperplanes of points not in M have q^n points less in M. Such a set has necessarily $x(q^{n+1}-1)/(q-1)$ points for an integer $x \geq 0$ and then it is common to call it an *x-tight set*.

A union of x mutually skew generators of $\mathcal{Q}^+(2n+1,q)$ provides an example for an x-tight set. For even n, this only gives examples when $x \leq 2$, since $\mathcal{Q}^+(2n+1,q)$, n even, does not possess three mutually skew generators. Historically, the tight sets of $\mathcal{Q}^+(5,q)$ play a particular role, since they can be translated to Cameron-Liebler line classes of $\mathrm{PG}(3,q)$ using Klein-correspondence.

We show that an x-tight set of $\mathcal{Q}^+(2n+1,q)$ is necessarily the union of x mutually disjoint generators, if $1 \leq n \leq 3$ and $x \leq q$, or if $n > 3$, $x < q$ and $q \geq 71$. This unifies and generalizes many results on x-tight sets that are presently known, see [1–3,5,6].

References

[1] J. Bamberg, S. Kelly, M. Law, and T. Penttila, *Tight sets and m-ovoids of finite polar spaces*, J. Combin. Theory Ser. A, 114 (2007), pp. 1293–1314.

[2] J. De Beule, P. Govaerts, A. Hallez, and L. Storme, *Tight sets, weighted m-covers, weighted m-ovoids, and minihypers*, Des. Codes Cryptogr., 50 (2009), pp. 187–201.

[3] J. De Beule, A. Hallez, and L. Storme, *A non-existence result on Cameron-Liebler line classes*, J. Combin. Des., 16 (2008), pp. 342–349.

[4] K. W. Drudge, *Extremal sets in projective and polar spaces*, ProQuest LLC, Ann Arbor, MI, 1998. Thesis (Ph.D.)–The University of Western Ontario (Canada).

[5] P. Govaerts and L. Storme, *On Cameron-Liebler line classes*, Adv. Geom., 4 (2004), pp. 279–286.

[6] K. Metsch, *The non-existence of Cameron-Liebler line classes with parameter* $2 < x \leqslant q$, Bull. Lond. Math. Soc., 42 (2010), pp. 991–996.

Linda Beukemann, Justus Liebig Universität Gießen, Mathematisches Institut, Arndstraße 2, D-35392 Gießen, Germany
Linda.Beukemann@math-uni.giessen.de

Regular graphs arising from projective planes – a closer look at Brown's construction method

Tamás Héger

Eötvös Loránd University

(joint work with András Gács *and* Zsuzsa Weiner)

The study of k regular graphs of girth (the length of the shortest cycle) g, called (k, g)-*graphs*, began with the papers of Tutte (1947) and Kárteszi (1960), and became intensive after Erdős and Sachs proved the existence of (k, g)-graphs for all k and g in 1963. The focus has been on finding the smallest possible (k, g)-graphs, called (k, g)-*cages*.

The incidence graph (or Levi graph) of a projective plane of order q is a $(q + 1, 6)$-cage. In 1967, W. G. Brown deleted some points and lines from such a plane to make the incidence graph of the rest a $(q + 1 - t, 6)$-graph. Thus Brown's method in general is to look for (small) $(k - t, g)$-graphs as *induced* subgraphs of a (k, g) cage. It is easy to see that in a projective plane a point set $\mathcal{P}_0$ and line set $\mathcal{L}_0$ that are proper to delete have the following property:

- $\forall P \notin \mathcal{P}_0$ there are exactly t lines in $\mathcal{L}_0$ through P,
- $\forall l \notin \mathcal{L}_0$ there are exactly t points in $\mathcal{P}_0$ on l.

We call such a pair $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ a t-*good structure* and refer to the elements of $\mathcal{P}_0$ and $\mathcal{L}_0$ as *deleted* points and lines, respectively. To review the known constructions of t-good structures, we need a definition first.
DEFINITION. *A point P (or a line l) is* completely deleted *if P and all the lines through P (l and all the points on l) are deleted.*

Essentially two types of t-good structures are known when $t < \sqrt{q}$.

- **Construction 1: completely deleted subplanes.** Take a (possibly degenerate) subplane which has t points and t lines, and delete all its points and lines completely. This is obviously t-good.
- **Construction 2: disjoint Baer-subplanes.** Delete the points and the lines of t disjoint Baer-subplanes.

The following theorem describes t-good structures in $\mathrm{PG}(2, q)$, when t is not too large compared to q.

THEOREM. *Let p be a prime, let $\mathcal{T}$ be a t-good structure in $\mathrm{PG}(2, q)$, $q = p^h$, and let (roughly) $t < \min\{p+1, q^{1/6}/8\}$. Then $\mathcal{T}$ is one of the two constructions above.*

In the talk we sketch the proofs of the above theorems, which use combinatorial and algebraic tools like the standard equations, a recent lemma of Szőnyi and Weiner, the Combinatorial Nullstellensatz with multiplicities, and also rely on results on weighted multiple blocking sets.

Tamás Héger, Eötvös Loránd University, Department of Computer Science, Pázmány P. sétány 1/C, H-1117 Budapest, Hungary
hetamas@cs.elte.hu

On ovoids and spreads of $\mathcal{Q}^+(7,q)$

Valentina Pepe

Universiteit Gent

Ovoids and spreads are among the most investigated objects in finite classical polar spaces and it is still an open question whether they exist or not for some polar spaces and fields. Ovoids and spreads of $\mathcal{Q}^+(7,q)$ are related by the triality map introduced in the fundamental paper of Tits [7].They are interesting in their own right, but they also give rise to translation planes, symplectic spreads and Kerdock sets ([3–5]). It is still an open question whether they exist for $q \equiv 1 \bmod 6$ and q not a prime but non–existence results haven't been proven so far. In order to get an indication where to look for, there is an attempt to characterize the known ones: first from a group theoretic point of view ([1, 2]) and now from a new one ([6]).

References

[1] J. Bamberg and T. Penttila, *A classification of transitive ovoids, spreads, and m-systems of polar spaces*, Forum Math., 21 (2009), pp. 181–216.

[2] A. Gunawardena, *Primitive ovoids in $O_8^+(q)$*, J. Combin. Theory Ser. A, 89 (2000), pp. 70–76.

[3] W. M. Kantor, *Ovoids and translation planes*, Canad. J. Math., 34 (1982), pp. 1195–1207.

[4] W. M. Kantor, *Spreads, translation planes and Kerdock sets. I*, SIAM J. Algebraic Discrete Methods, 3 (1982), pp. 151–165.

[5] G. Lunardon, L. Parlato, V. Pepe, and R. Trombetti, *Slices of the unitary spread*, J. Algebraic Combin., 33 (2011), pp. 37–56.

[6] T. Penttila and V. Pepe, *A characterization of the Desarguesian and unitary ovoids of $\mathcal{Q}^+(7,q)$.* preprint.

[7] J. Tits, *Sur la trialité et certains groupes qui s'en déduisent.*, Inst. Hautes Etudes Sci. Publ. Math., 2 (1959), pp. 13–60.

Valentina Pepe, Ghent University, Department of Mathematics, Krijgslaan 281 S22, B-9000 Gent, Belgium
valepepe@cage.ugent.be

The complete $(k, 3)$-arcs of $\mathrm{PG}(2, q)$, $q \leq 13$

Heide Sticker

Universiteit Gent

(joint work with Kris Coolsaet)

We obtained a full classification (up to equivalence) of all complete $(k, 3)$-arcs in the Desarguesian projective planes of order q, $q \leq 13$. This was done by computer. The algorithm used is an application of isomorph-free backtracking using canonical augmentation, an adaptation of our earlier algorithms for the generation of $(k, 2)$-arcs. [1,2]. We explain the general techniques of the algorithm and those parts of the algorithm that are specific to the particular problem of $(k, 3)$-arcs. For each of the complete arcs we have computed the automorphism group, the results are listed in tables. Several of the computer results can be generalized to other values of q. We describe some of these constructions.

References

[1] K. Coolsaet and H. Sticker, *A full classification of the complete k-arcs of* $\mathrm{PG}(2, 23)$ *and* $\mathrm{PG}(2, 25)$, J. Combin. Des., 17 (2009), pp. 459–477.

[2] ——, *The complete k-arcs of* $\mathrm{PG}(2, 27)$ *and* $\mathrm{PG}(2, 29)$, J. Combin. Des., 19 (2011), pp. 111–130.

Heide Sticker, Ghent University, Department of Applied Mathematics and Computer Science, Krijgslaan 281 S9, B-9000 Gent, Belgium
Heide.Sticker@UGent.be

LDPC codes from finite geometries

Geertrui Van de Voorde

Vrije Universiteit Brussel

Apart from their nice geometric properties, the motivation for the study of codes from finite geometries is their possible application when viewed as an *LDPC code*. If the 0-1 parity check matrix of a code C is sparse, roughly speaking if there are 'few' 1s and 'many' 0s, then we say that C is a Low Density Parity Check code (or LDPC) code. LDPC codes were introduced by Gallager [2], who invented an easy decoding method for these codes in the early 1960s. These codes were forgotten for more than 30 years due to the fact that the computer power in those days was unsufficient to decode codes with a useful length. They were rediscovered in the 1990s by MacKay and Neal [3], who showed that their empirical performance is excellent. The problem remains, however, to give explicit constructions for good LDPC codes. One of the methods is to construct LDPC codes using the incidence matrix of some finite incidence structure. Determining the performance of an LDPC code under iterative decoding is done by simulations. But over the binary erasure channel, the performance is entirely defined by combinatorial structures, called stopping sets (see [1]).

In this talk, I will give an introduction to LDPC codes of finite geometries and stopping sets. I will focus on two particular codes: the well-known dual code of points and lines in $\mathrm{PG}(2,q)$, and the dual code of $\mathcal{Q}^+(5,q)$.

References

[1] C. Di, D. Proietti, I. E. Telatar, T. J. Richardson, and R. L. Urbanke, *Finite-length analysis of low-density parity-check codes on the binary erasure channel*, IEEE Trans. Inform. Theory, 48 (2002), pp. 1570–1579. Special issue on Shannon theory: perspective, trends, and applications.

[2] R. G. Gallager, *Low-density parity-check codes*, IRE Trans., IT-8 (1962), pp. 21–28.

[3] D. J. C. MacKay and R. M. Neal, *Near shannon limit performance of low density parity check codes*, Electron. Lett., 32 (1996), pp. 1645–1646.

[4] G. Van de Voorde, *On sets without tangents and exterior sets of a conic.* preprint.

Geertrui Van de Voorde, Vrije Universiteit Brussel, Department of Mathematics, Pleinlaan 2, B-1050 Elsene, Belgium
gvdevoor@vub.ac.be