COLLOQUIUM
ON GALOIS GEOMETRY

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Abstracts

Some problems of Paul Erdős I solved, partly solved, or just wanted to solve. Few Distance Sets, Isosceles sets, Maximal k-cliques and blocking sets with few points on every line.

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In the style of Paul Erdős I will discuss some of his favourite problems. The topic of my thesis was Few-Distance sets, and my first mathematical result was an improvement of the bound for two-distance sets in Euclidean Space. After hearing this result Paul Erdős asked me about isosceles point sets: How many points can you have in Euclidean n-space such that every triangle is isosceles (or equilateral of course). It was a good question, and I could solve it. Another favourite Erdős problem concerns maximal k-cliques: What is the minimal size of a maximal k-uniform intersecting hypergraph (so how many k-sets that mutually intersect must you have at least so that the system cannot be enlarged). It was (my contribution to) this problem that started my contacts with 'the Hungarians', notably Zoltan Füredi and Imre Bárány. It also was one of the first nice applications of (affine) blocking sets to a problem in combinatorial set theory. A very annoying problem finally, that he asked me everytime we met: Does every projective plane have a blocking set with at most 10.000.000 points on a line? We know that there are arbitrarily large planes with a blocking set with at most some constant number of points on a line, but they must have 'small characteristic'. In general I don't know what to think.

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Erdős-Ko-Rado theorems in geometrical settings

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The original Erdős-Ko-Rado problem was posed in [6] and asks for the maximal size of an Erdős-Ko-Rado set, a set of k-subsets in a finite set such that every two subsets have a non-empty intersection. This question has been generalised in many ways, and especially to geometrical settings, e.g. to projective and polar spaces and to designs. An Erdős-Ko-Rado set of k-spaces in a projective or polar space is defined as a set of k-dimensional subspaces such that every two of them have a non-empty intersection. An Erdős-Ko-Rado set of a design is a set of mutually intersecting blocks. An Erdős-Ko-Rado set is called maximal if it is non-extendable regarding this intersection condition. The general Erdős-Ko-Rado problem asks for the classification of the (large) maximal Erdős-Ko-Rado sets.

In this talk we will give a survey of Erdős-Ko-Rado theorems in geometrical settings. This includes recent results on projective spaces, which can be found in [1–3,8], recent results on polar spaces, which can be found in [3,7] and recent results on designs, which can be found in [4]. This talk is based on the survey article [5].

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On sets with few intersection numbers in finite projective and affine spaces

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A set X of points of $\mathrm{PG}(n, q)$ (or of $\mathrm{AG}(n, q)$) is of *class* $[m_1, m_2, \ldots, m_t]_1$ if it intersects every line of the geometry either in m_1 or in m_2 or $\ldots$ or in m_t points (where m_i are non-negative integers in increasing order). In the last thirty years many classes of sets of points with few intersection numbers w.r.t. lines mainly in finite projective spaces (but sometimes also in finite affine spaces) have been studied (see e.g. [1] for a list of results).
In connection with a study of codes in Johnson graphs [2] by Liebler and Praeger, subsets X of points of $\mathrm{PG}(n, q)$ and $\mathrm{AG}(n, q)$ with at most three possible line-intersection sizes arise, such that the stabiliser in the relevant collineation group acts transitively on pairs of points (x, y) with $x \in X$ and $y \notin X$, [2, Section 5.2]. With the notation established above these are sets of class $[0, m, q + 1]_1$ for $\mathrm{PG}(n, q)$ and class $[0, m, q]_1$ for $\mathrm{AG}(n, q)$. Various examples were given in [2, Sections 6, 7], such as subspaces and their complements in both (projective and affine) spaces, and the 2-transitive hyperoval and its complement in the projective case. The analysis in [2] showed that, apart from known examples, the value of m was restricted to $m \in \{2, \sqrt{q}\}$ in $\mathrm{PG}(n, q)$, and in $\mathrm{AG}(n, q)$ the pair (q, m) was restricted to one of $(4, 2)$ or $(16, 4)$. Classification in these exceptional cases was left open, and at her plenary lecture in Ferrara at the Conference in Finite Geometry in honor of Frank De Clerck, Praeger asked what was known about subsets of class $[0, m, q + 1]_1$ in $\mathrm{PG}(n, q)$ and of class $[0, m, q]_1$ in $\mathrm{AG}(n, q)$, regardless of the symmetry restrictions.

Question (C. Praeger 2012) *Is it possible to characterise the subsets of* $PG(n, q)$ *of class* $[0, m, q + 1]_1$ *and subsets of* $AG(n, q)$ *of class* $[0, m, q]_1$*?*

In this talk we answer positively to the previous question and hence give an answer to the open cases by Liebler and Praeger.

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The probabilistic method in Galois geometries

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On March 26, 2013, the hundred birthday of Pál Erdős is celebrated. His influence on mathematics is immense.

One of the methods he developed was the probabilistic method in Galois geometries [1]. With this method, the existence of certain substructures can be proven, without having to construct these substructures explicitly. This is of great value when only the existence of the substructure, but not the exact elements, is required.

This kind of results were used in the study of linear MDS codes, to prove the non-extendability of specific linear MDS codes to longer MDS codes [2,3].

In this talk, I will present the probabilistic result that was used, and then show how it was used in geometrical arguments to prove the non-extendability of linear MDS codes to longer MDS codes.

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Directions and Erdős problems

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Direction problems have played an important role in the research on finite (Galois) geometry for at least fourty years. They are interesting enough themselves, and they have many applications and connections to other fields.

The definition is quite innocent: let U be a point set in the n-dimensional affine space $\mathrm{AG}(n,q)$ over the finite field $\mathrm{GF}(q)$ of q elements. We say a direction d (i.e. a point in the hyperplane at infinity H_∞) is **determined** by U if there is a line with the point d at infinity, containing at least two points of U. The set of determined directions is usually denoted by D.

A typical direction problem puts some restriction on U and asks about the properties of D, or vice versa, under some condition on D, asks about the structure of U. E.g. Rédei and Megyesi proved that in $\mathrm{AG}(2,p)$, p prime, if $|U| = p$, then U determines either 1 or at least $\frac{p+3}{2}$ directions. Another example is a result of Bruen and Levinger, classifying all affine point sets $U \subset \mathrm{AG}(2,q)$ of size q, which determine directions corresponding to slopes being square elements of $\mathrm{GF}(q)$.

By the pigeon hole principle it is clear that if $|U| > q^{n-1}$ then U determines every direction. Most of the classical results consider the "extremal case", i.e. affine pointsets of size q^{n-1}.

In my talk I will give a summary of the history and of the most important results, including some basic facts about Rédei-polynomials, as their theory provided the key methods in this topic. I will also show some of the related non-geometrical problems from group theory, graph theory, etc.

After this survey-like part we will deal with several extensions and generalizations. We get interesting cases when changing the size of U, or if we alter the definition of determining a direction. Then we embed the topic into a broader context and we define a large class of problems, which resemble to the direction questions.

This leads to the last part of my talk where we will look at problems on "distances" occurring in point sets over finite fields, which are finite analogues of Euclidean Erdős problems (and they are quite difficult in general).

Some of the results are joint work with Jan De Beule, Szabolcs Fancsali and Marcella Takáts.

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Optimal blocking multisets

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(Joint work with I. Landjev)

Definition *An (f, m)-minihyper in $\mathrm{PG}(t, q)$ is an m-fold blocking multiset of size f, i.e. a multiset $\mathfrak{F}$ of f points in $\mathrm{PG}(t, q)$ s. t. every hyperplane contains at least m of these points.*

A natural question here would be: for given m, what is the least number f such that there exists an (f, m)-minihyper? For proper multisets, one has $f \geq \frac{v_t}{v_{t-1}} m$, with $v_i = \frac{q^i - 1}{q-1}$.

Definition *In $\mathrm{PG}(t, q)$, an optimal blocking multiset is an (xv_t, xv_{t-1})-minihyper, for some $x \in \mathbb{N}$.*

Hence, these parameters are a very particular choice of minihypers, and many examples of them are known . The study on minihypers was originally started by Hamada in the context of linear codes meeting the Griesmer bound [2], and coincidently, it turns out that such codes were most highly divisible, exactly when its parameters are (xv_t, xv_{t-1}).

When two different problems point to the same structure, it is no surprise that they have been studied before [1,4,5]. This study was however always performed from a purely combinatorial point of view.

Starting from a theorem by Landjev and Storme [5], I will present a simple natural problem in linear algebra, which turns out to have exactly these optimal blocking multisets as its set of solutions. So we now have 3 different natural problems pointing to this very same structure.

Using this new characterization, we could greatly extend and improve upon most known results on this structure [3–5]. In this talk, I will attempt to give some insight in this characterization and its strengths, to convince the audience that these optimal blocking multisets are very special combinatorial structures and that they are also an example of structures where purely combinatorial methods are really not the right tool for the job.

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