Finite Generalized Quadrangles

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Intensive Course on Galois Geometry and Generalized Polygons

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1 Axioms and definitions

A (finite) *generalized quadrangle* (GQ) is an *incidence structure* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{L}$ are disjoint (nonempty) sets of objects called *points* and *lines* respectively, and for which I is a symmetric point-line *incidence relation* satisfying the following axioms.

(i) Each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line.

(ii) Each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

(iii) If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{L}$ for which $x \mathrm{~I~} M \mathrm{~I~} y \mathrm{~I~} L$.

Generalized quadrangles were introduced by Tits [73] in his celebrated work on triality.

The integers s and t are the *parameters* of the generalized quadrangle and $\mathcal{S}$ is said to have *order* (s,t); if $s=t$, $\mathcal{S}$ is said to have *order* s. There is a point-line duality for generalized quadrangles (of order (s,t)) for which in any definition or theorem the words "point" and "line" are interchanged and the parameters s and t are interchanged. Hence, we assume without further notice that the dual of a given theorem or definition has also been given.

A *grid* is an incidence structure $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ with $\mathcal{P}=\{x_{ij}|i=0,1,\cdots,s_1$ and $j=0,1,\cdots,s_2\}$, $s_1>0$ and $s_2>0$, $\mathcal{L}=\{L_0,L_1,\cdots,L_{s_1},M_0,M_1,\cdots,M_{s_2}\}$, $x_{ij}\ \mathrm{I}\ L_k$ if and only if $i=k$, and $x_{ij}\ \mathrm{I}\ M_k$ if and only if $j=k$. A grid with parameters s_1,s_2 is a GQ if and only if $s_1=s_2$; such a GQ has order (s,t) with $s=s_1=s_2$ and $t=1$. Evidently any GQ with $t=1$ is a grid. *Dual grids* are defined dually.

Let $\mathcal{S}$ be a GQ, a grid, or a dual grid. Given two (not necessarily distinct) points x,y of $\mathcal{S}$, we write $x\sim y$ and say that x and y are *collinear*, provided that there is some line L for which $x\ \mathrm{I}\ L\ \mathrm{I}\ y$. And $x\not\sim y$ means that x and y are not collinear. Dually, for $L,M\in\mathcal{L}$, we write $L\sim M$ or $L\not\sim M$ according as L and M are *concurrent* or nonconcurrent, respectively. When $x\sim x'$ we also say that x is *orthogonal* or *perpendicular* to x'; similarly for $L\sim L'$. The line incident with distinct collinear points x and x' is denoted xx', and the point incident with distinct concurrent lines L and L' is denoted either $L\cap L'$ or LL'.

The *trace* of a pair $\{x,y\}$ of distinct points is defined to be the set $x^\perp\cap y^\perp$ and is denoted $\mathrm{tr}(x,y)$ or $\{x,y\}^\perp$. We have $|\{x,y\}^\perp|=s+1$ or $t+1$ according as $x\sim y$ or $x\not\sim y$. More generally, if $A\subseteq\mathcal{P}$, A "perp" is defined by $A^\perp=\cap\{x^\perp\,|\,x\in A\}$. For $x\neq y$, the *span* of the pair $\{x,y\}$ is $\mathrm{sp}(x,y)=\{x,y\}^{\perp\perp}=\{u\in\mathcal{P}\,|\,u\in z^\perp\,\forall z\in x^\perp\cap y^\perp\}$. If $x\not\sim y$, then $\{x,y\}^{\perp\perp}$ is also called the *hyperbolic line* defined by x and y.

If $x\sim y$, $x\neq y$, or if $x\not\sim y$ and $|\{x,y\}^{\perp\perp}|=t+1$, we say that the pair $\{x,y\}$ is *regular*. The point x is *regular* provided $\{x,y\}$ is regular for all $y\in\mathcal{P}$, $y\neq x$. A point x is *coregular* provided each line incident with x is regular.

The pair $\{x,y\}$, $x\not\sim y$, is *antiregular* provided $|z^\perp\cap\{x,y\}^\perp|\leq 2$ for all $z\in\mathcal{P}\setminus\{x,y\}$. A point x is *antiregular* provided $\{x,y\}$ is antiregular for all $y\in\mathcal{P}\setminus x^\perp$. If $1<s<t$, then one can show that no pair of distinct points is regular or antiregular, see Payne and Thas [55]; if a GQ of order s admits an antiregular pair of points, then one proves that s is odd, see Payne and Thas [55].

A *triad* (of points) is a triple of pairwise noncollinear points. Given a triad $T=\{x,y,z\}$, a *center* of T is just a point of $T^\perp$. We say T is *acentric, centric*, or *unicentric* according as $|T^\perp|$ is zero, positive, or equal to 1.

2 Restrictions on the parameters

Theorem 2.1
If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a GQ of order (s,t), then

(i) $|\mathcal{P}| = v = (s+1)(st+1)$;

(ii) $|\mathcal{L}| = b = (t+1)(st+1)$.

Proof. Let L be a fixed line of $\mathcal{S}$ and count in different ways the number of ordered pairs $(x, M) \in \mathcal{P} \times \mathcal{L}$ with $x \not\mathrm{I}\, L$, $x \,\mathrm{I}\, M$ and $L \sim M$. There arises $v - s - 1 = (s+1)ts$ or $v = (s+1)(st+1)$. Dually, $b = (t+1)(st+1)$. $\square$

If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a GQ of order (s,t), then the point graph $\Gamma(\mathcal{S})$ of $\mathcal{S}$ is a

$$\mathrm{srg}((s+1)(st+1), s(t+1), s-1, t+1),$$

while the point graph $\Gamma(\mathcal{S}^D)$ of the dual $\mathcal{S}^D$ of $\mathcal{S}$ is a

$$\mathrm{srg}((t+1)(st+1), t(s+1), t-1, s+1).$$

Hence we have the following theorem.

Theorem 2.2
If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a GQ of order (s,t) then the eigenvalues of the adjacency matrix A of $\Gamma(\mathcal{S})$ are $k = s(t+1)$, $r = s-1$, $l = -t-1$; also, r has multiplicity $f = st(s+1)(t+1)/(s+t)$, hence $s+t$ divides $st(s+1)(t+1)$.

Also the Krein condition for the graph $\Gamma(\mathcal{S})$ yields a necessary condition. For a GQ this condition is known as the inequality of Higman [27]. We will give a proof of this inequality using the so-called *variance trick*.

Theorem 2.3 (Inequality of Higman [27])
Assume $\mathcal{S}$ is a GQ of order (s,t). If $s > 1$ and $t > 1$, then $t \le s^2$, and dually $s \le t^2$.

Proof. Let x, y be two noncollinear points of $\mathcal{S}$. Put $V = \{z \in \mathcal{P} \mid z \not\sim x$ and $z \not\sim y\}$, so $|V| = d = (s+1)(st+1) - 2 - 2(t+1)s + (t+1)$. Denote the elements of V by $z_1, \ldots, z_d$ and let $t_i = |\{u \in \{x,y\}^\perp \mid u \sim z_i\}|$. Count in different ways the number of ordered pairs $(z_i, u) \in V \times \{x,y\}^\perp$ with $u \sim z_i$ to obtain

$$\sum_i t_i = (t+1)(t-1)s. \qquad (1)$$

Next count in two different ways the number of ordered triples $(z_i, u, u') \in V \times \{x,y\}^\perp \times \{x,y\}^\perp$ with $u \ne u'$, $u \sim z_i$, $u' \sim z_i$, to obtain

$$\sum_i t_i(t_i - 1) = (t+1)t(t-1). \tag{2}$$

From (1) and (2) it follows that $\sum_i t_i^2 = (t+1)(t-1)(s+t)$.

With $d\bar{t} = \sum_i t_i$, the expression $0 \leq \sum_i(\bar{t}-t_i)^2$ simplifies to $d\sum_i t_i^2 - (\sum_i t_i)^2 \geq 0$, which implies $d(t+1)(t-1)(s+t) \geq (t+1)^2(t-1)^2s^2$, or $t(s-1)(s^2-t) \geq 0$, completing the proof. □

There is an immediate corollary of the proof.

Theorem 2.4 (Bose and Shrikhande [10])
Let $\mathcal{S}$ be a generalized quadrangle of order $(s,t), s \neq 1 \neq t$. Then $t = s^2$ if and only if for each triad $\{x,y,z\}$ there is a constant number of points collinear with x, y and z. This constant number of points is $s+1$.

Remark

Higman [27] first proved Theorem 2.3 by a complicated matrix-theoretic method. The argument given here was used by Bose and Shrikhande [10] to show that in case $t = s^2 > 1$ each triad has $1+s$ centers, and Cameron [13] apparently first observed that the above technique also provides the inequality.

Theorem 2.5
If $s \neq 1, t \neq 1, s \neq t^2$ and $t \neq s^2$, then $t \leq s^2 - s$ and dually $s \leq t^2 - t$.

Proof. Suppose $s \neq 1$ and $t \neq s^2$. By Theorem 2.3 we have $t = s^2 - x$ with $x > 0$. By Theorem 2.2 $(s+s^2-x)|s(s^2-x)(s+1)(s^2-x+1)$. Hence modulo $s+s^2-x$ we have $0 \equiv x(-s)(-s+1) \equiv x(x-2s)$. If $x < 2s$, then $s+s^2-x \leq x(2s-x)$ forces $x \in \{s, s+1\}$. Consequently $x = s, x = s+1$, or $x \geq 2s$, from which it follows that $t \leq s^2 - s$. □

3 The classical finite generalized quadrangles

We give a brief description of three families of generalized quadrangles, known as the *classical generalized quadrangles*, all of which are associated with classical groups and were first recognized as generalized quadrangles by Tits; see, e.g., Dembowski [15].

(i) Consider a nonsingular quadric $Q^+(3,q)$ (respectively, $Q(4,q)$ and $Q^-(5,q)$) of Witt index 2, that is, of projective index 1, in the projective space $\mathrm{PG}(3,q)$ (respectively, $\mathrm{PG}(4,q)$ and $\mathrm{PG}(5,q)$). Then the points of the quadric together with the lines of the quadric form a generalized quadrangle with parameters

$$s = q, t = 1, v = (q+1)^2, b = 2(q+1), \text{ for } Q^+(3,q),$$

$$s = t = q, v = b = (q+1)(q^2+1), \text{ for } Q(4,q),$$

$$s = q, t = q^2, v = (q+1)(q^3+1), b = (q^2+1)(q^3+1), \text{ for } Q^-(5,q).$$

In each of the three cases for the quadric and the corresponding GQ the same notation will be used.

The GQ $Q^+(3,q)$ is a grid. Further, recall that these quadrics have the following canonical equations:

$$\begin{aligned} Q^+(3,q) &: \quad X_0X_1 + X_2X_3 = 0, \\ Q(4,q) &: \quad X_0^2 + X_1X_2 + X_3X_4 = 0, \\ Q^-(5,q) &: \quad F(X_0,X_1) + X_2X_3 + X_4X_5 = 0, \end{aligned}$$

where F is an irreducible homogeneous polynomial in X_0, X_1 over $\mathrm{GF}(q)$.

(ii) Let $H(d,q^2)$ be a nonsingular Hermitian variety of the projective space $\mathrm{PG}(d,q^2)$, $d = 3$ or 4. Then the points of $H(d,q^2)$, together with the lines on $H(d,q^2)$, form a generalized quadrangle with parameters:

$$s = q^2, t = q, v = (q^2+1)(q^3+1), b = (q+1)(q^3+1), \text{ when } d = 3,$$

$$s = q^2, t = q^3, v = (q^2+1)(q^5+1), b = (q^3+1)(q^5+1), \text{ when } d = 4.$$

Remember that $H(d,q^2)$ has the canonical equation

$$X_0^{q+1} + X_1^{q+1} + \cdots + X_d^{q+1} = 0.$$

In each of these two cases for the Hermitian variety and the corresponding GQ the same notation will be used.

(iii) The points of $\mathrm{PG}(3,q)$, together with the totally isotropic lines with respect to a symplectic polarity, form a generalized quadrangle $W_3(q)$, shortly $W(q)$, with parameters

$$s = t = q, v = b = (q+1)(q^2+1).$$

Remember that the lines of $W_3(q)$ are the elements of a general linear complex of lines of $\mathrm{PG}(3,q)$, see e.g. Hirschfeld [29], and that a symplectic polarity of $\mathrm{PG}(3,q)$ has the following canonical bilinear form :

$$X_0Y_1 - X_1Y_0 + X_2Y_3 - X_3Y_2.$$

Theorem 3.1 (Payne and Thas [55])

(a) $Q(4,q)$ is isomorphic to the dual of $W(q)$. Moreover, $Q(4,q)$ (or $W(q)$) is selfdual if and only if q is even.

(b) $Q^-(5,q)$ is isomorphic to the dual of $H(3,q^2)$.

Proof. (a) Let $Q^+(5,q)$ be the Klein quadric of the lines of $\mathrm{PG}(3,q)$; see Hirschfeld [29]. Then $Q^+(5,q)$ is a hyperbolic quadric of $\mathrm{PG}(5,q)$. The image of $W(q)$ on $Q^+(5,q)$ is the intersection of $Q^+(5,q)$ with a nontangent hyperplane $\mathrm{PG}(4,q)$ of $\mathrm{PG}(5,q)$. The nonsingular quadric $Q^+(5,q) \cap \mathrm{PG}(4,q)$ of $\mathrm{PG}(4,q)$ is denoted by $Q(4,q)$. The lines of $W(q)$ which are incident with a given point form a flat pencil of lines, hence their images on $Q^+(5,q)$ form a line of $Q(4,q)$. Now it easily follows that $W(q)$ is isomorphic to the dual of $Q(4,q)$.

Now consider the nonsingular quadric $Q(4,q)$ of $\mathrm{PG}(4,q)$. Let L_0 and L_1 be nonconcurrent lines of $Q(4,q)$. Then the 3-space L_0L_1 intersects $Q(4,q)$ in a hyperbolic quadric having reguli $\{L_0, L_1, \ldots, L_q\}$ and $\{M_0, M_1, \ldots, M_q\}$. In $Q(4,q)$ we have $L_i \sim M_j$, $i,j = 0,\ldots,q$, so $\{L_0, L_1\}$ is a regular pair of lines of $Q(4,q)$. Hence all lines of $Q(4,q)$ are regular. As for q odd no point of $Q(4,q)$ is regular, $Q(4,q)$ (and also $W(q)$) is not selfdual. So let q be even. The tangent 3-spaces of $Q(4,q)$ all meet in one point n, the nucleus of $Q(4,q)$; see Hirschfeld and Thas [30]. From n we project $Q(4,q)$ onto a $\mathrm{PG}(3,q)$ not containing n. This yields a bijection of the point set of $Q(4,q)$ onto $\mathrm{PG}(3,q)$, mapping the $(q+1)(q^2+1)$ lines of $Q(4,q)$ onto $(q+1)(q^2+1)$ lines of $\mathrm{PG}(3,q)$. Since the $q+1$ lines of $Q(4,q)$ which are incident with a given point are contained in a tangent 3-space of $Q(4,q)$, they are mapped onto the elements of a flat pencil of lines of $\mathrm{PG}(3,q)$. Hence the images of the lines of $Q(4,q)$ constitute a general linear complex of lines of $\mathrm{PG}(3,q)$, i.e. they are the totally isotropic lines with respect to a symplectic polarity of $\mathrm{PG}(3,q)$. It follows that $Q(4,q) \cong W(q)$, and consequently $Q(4,q)$ (and $W(q)$) are selfdual.

(b) Let $Q^-(5,q)$ be an elliptic quadric, i.e. a nonsingular quadric of projective index 1, in $\mathrm{PG}(5,q)$. Extend $\mathrm{PG}(5,q)$ to $\mathrm{PG}(5,q^2)$. Then the extension of $Q^-(5,q)$ is a hyperbolic quadric $Q^+(5,q^2)$, i.e. a nonsingular quadric of projective index 2, in $\mathrm{PG}(5,q^2)$. Hence $Q^+(5,q^2)$ is the Klein quadric of the lines of $\mathrm{PG}(3,q^2)$. So to $Q^-(5,q)$ in $Q^+(5,q^2)$ there corresponds a set $\mathcal{V}$ of lines in $\mathrm{PG}(3,q^2)$. To a given line L of the GQ $Q^-(5,q)$ there correspond $q+1$ lines of $\mathrm{PG}(3,q^2)$ that all lie in a plane and pass through a point x. Let $\mathcal{H}$ be the set of points on the lines of $\mathcal{V}$. Then with each point of $Q^-(5,q)$ there corresponds a line of $\mathcal{V}$, and with each line L of $Q^-(5,q)$ there corresponds a point x of $\mathcal{H}$. With distinct lines L, L' of $Q^-(5,q)$ correspond distinct points x, x' of $\mathcal{H}$ (a plane of $Q^+(5,q^2)$ contains at most one line of $Q^-(5,q)$). Since a point y of $Q^-(5,q)$ is on q^2+1 lines of $Q^-(5,q)$, these q^2+1 lines are mapped onto the q^2+1 points of the image of y. Hence we obtain an antiisomorphism from $Q^-(5,q)$ onto the structure $(\mathcal{H}, \mathcal{V}, \mathrm{I})$, where I is the natural incidence relation. So $(\mathcal{H}, \mathcal{V}, \mathrm{I})$ is a GQ of order (q^2, q) embedded in $\mathrm{PG}(3, q^2)$. But now by a celebrated result of Buekenhout and Lefèvre [12] (see also Section 6) the GQ $(\mathcal{H}, \mathcal{V}, \mathrm{I})$ must be $H(3, q^2)$. $\square$

The proof just given is in Payne and Thas [55]. An algebraic proof of the same theorem was given by Bruen and Hirschfeld [11].

Remark

Tits [74] proves that $W(q)$ is selfpolar if and only if $q = 2^{2h+1}, h \geq 0$.

4 Nonclassical finite generalized quadrangles

4.1 The trivial nonclassical generalized quadrangles

In Section 1 grids and dual grids were defined. The GQ $\mathcal{S}$ of order (s, t) has $t = 1$ if and only if it is a grid for which all lines are incident with the same number $s + 1$ of points; the GQ $\mathcal{S}$ of order (s, t) has $s = 1$ if and only if it is a dual grid for which all the points are incident with the same number $t + 1$ of lines.

4.2 The nonclassical examples of Tits

The earliest known nontrivial nonclassical examples of GQ were discovered by Tits and first appeared in Dembowski [15].

Let $d = 2$ (respectively, $d = 3$) and let O be an oval (respectively, an ovoid) of $\mathrm{PG}(d, q)$; for the definitions of oval and ovoid, see Hirschfeld [28], [29] and Thas [66]. Further, let $\mathrm{PG}(d, q)$ be embedded as a hyperplane in $\mathrm{PG}(d + 1, q)$. Define points as

(i) the points of $\mathrm{PG}(d + 1, q) \setminus \mathrm{PG}(d, q)$,

(ii) the hyperplanes X of $\mathrm{PG}(d + 1, q)$ for which $|X \cap O| = 1$, and

(iii) one new symbol (∞).

Lines are defined as

(a) the lines of $\mathrm{PG}(d+1, q)$ which are not contained in $\mathrm{PG}(d, q)$ and meet O (necessarily in a unique point), and

(b) the points of O.

Incidence is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of $\mathrm{PG}(d + 1, q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of O in it. The point (∞) is incident with no line of type (a) and all lines of type (b). It is an easy exercise to show that the incidence structure so defined is a GQ with parameters

$$s = t = q, v = b = (q+1)(q^2+1), \text{ when } d = 2,$$

$$s = q, t = q^2, v = (q+1)(q^3+1), b = (q^2+1)(q^3+1), \text{ when } d = 3.$$

If $d = 2$, the GQ is denoted by $T_2(O)$; if $d = 3$, the GQ is denoted by $T_3(O)$. If no confusion is possible, these quadrangles are also denoted by $T(O)$.

4.3 The examples of Ahrens and Szekeres; Hall, Jr. and Payne

For each prime power q, Ahrens and Szekeres [1] constructed a GQ of order $(q-1, q+1)$. For q even, these examples were found independently by Hall [26]. Then a construction was found by Payne [42] which included all these examples and for q even produced some additional ones (see also Payne [43], Payne [47] and Payne and Thas [55]). These examples yield the only known cases, with $s > 1$ and $t > 1$, in which s and t are not powers of the same prime.

4.3.1 The construction of Ahrens and Szekeres and Hall [1, 26], for q even

Let O be a hyperoval (see Hirschfeld [28] and Thas [66]), i.e. a $(q+2)$-arc, of the projective plane $\mathrm{PG}(2, q)$, $q = 2^h$, and let $\mathrm{PG}(2, q)$ be embedded as a plane in $\mathrm{PG}(3, q)$. Define an incidence structure $T_2^*(O)$ by taking for points all points of $\mathrm{PG}(3, q)$ not in $\mathrm{PG}(2, q)$ and for lines just those lines of $\mathrm{PG}(3, q)$ which are not contained in $\mathrm{PG}(2, q)$ and meet O (necessarily in a unique point). The incidence is that of $\mathrm{PG}(3, q)$. It is clear that the incidence structure $T_2^*(O)$ so defined is a GQ with parameters

$$s = q-1, t = q+1, v = q^3, b = q^2(q+2).$$

4.3.2 The construction of Ahrens and Szekeres [1], for q odd

Let the elements of $\mathcal{P}$ be the points of the affine 3-space $\mathrm{AG}(3, q)$ over $\mathrm{GF}(q)$, q odd. Elements of $\mathcal{L}$ are the following curves of $\mathrm{AG}(3, q)$:

(i) $x = \sigma, y = a, z = b$,

(ii) $x = a, y = \sigma, z = b$,

(iii) $x = c\sigma^2 - b\sigma + a, y = -2c\sigma + b, z = \sigma$.

Here the parameter σ ranges over $\mathrm{GF}(q)$ and a, b, c are arbitrary elements of $\mathrm{GF}(q)$. The incidence I is the natural one. Then $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a GQ of order $(q-1, q+1)$. It will be denoted by $AS(q)$.

4.3.3 The construction of Payne [42]

Let x be a regular point of the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order s, $s > 1$. Then $\mathcal{P}'$ is defined to be the set $\mathcal{P} \setminus x^\perp$. The elements of $\mathcal{L}'$ are of two types: the elements of type (a) are the lines of $\mathcal{L}$ which are not incident with x; the elements of type (b) are the hyperbolic lines $\{x, y\}^{\perp\perp}$, $y \not\sim x$. Now let us define the incidence I'. If $y \in \mathcal{P}'$ and $L \in \mathcal{L}'$ is of type (a), then $y \mathrm{I}' L$ if and only if $y \mathrm{I} L$; if $y \in \mathcal{P}'$ and $L \in \mathcal{L}'$ is of type (b), then $y \mathrm{I}' L$ if and only if $y \in L$. The incidence structure $P(\mathcal{S}, x) = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ so defined is a GQ of order $(s-1, s+1)$.

A quick look at the examples of order s in 3 and 4.2 reveals that regular points and regular lines arise in the following cases (see Payne and Thas [55]):

- all lines of $Q(4, q)$ are regular;
- the points of $Q(4, q)$ are regular if and only if q is even;
- all points of $W(q)$ are regular;
- the lines of $W(q)$ are regular if and only if q is even;
- the unique point (∞) of type (iii) of $T_2(O)$ is regular if and only if q is even;
- all lines of type (b) of $T_2(O)$ are regular.

It is easily seen that for q even the GQ $P(T_2(O), (\infty))$ is the GQ $T_2^*(\overline{O})$, with $\overline{O}$ the hyperoval containing the oval O. Further, $P(W(q), x)$, with x any point of $\mathrm{PG}(3, q)$, has the following nice description: points are the points of $\mathrm{PG}(3, q)$ not in the plane $\pi_x = x^\zeta$, with ζ the symplectic polarity defining $W(q)$; lines are the lines of $W(q)$ not in π_x together with the lines of $\mathrm{PG}(3, q)$ through x but not in π_x; incidence is the natural one.

4.4 Generalized quadrangles as group coset geometries

The following construction method for GQ was introduced by Kantor [35].

Let G be a finite group of order $s^2t, 1 < s, 1 < t$, together with a family $J = \{A_i \mid 0 \leq i \leq t\}$ of $1 + t$ subgroups of G, each of order s. Assume furthermore that for each $A_i \in J$, there exists a subgroup A_i^* of G, of order st, containing A_i. Put $J^* = \{A_i^* \mid 0 \leq i \leq t\}$ and define as follows an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I}) = \mathcal{S}(G, J)$.

Points are of three kinds:

(i) the elements of G;

(ii) the right cosets $A_i^* g$, $A_i^* \in J^*$, $g \in G$;

(iii) a symbol (∞).

Lines are of two kinds:

(a) the right cosets $A_i g$, $A_i \in J$, $g \in G$;

(b) the symbols $[A_i]$, $A_i \in J$.

A point g of type (i) is incident with each line $A_i g$, $A_i \in J$; a point $A_i^* g$ of type (ii) is incident with $[A_i]$ and with each line $A_i h$ contained in $A_i^* g$; the point (∞) is incident with each line $[A_i]$ of type (b).

Then Kantor [35] proved that the following holds: $\mathcal{S}(G, J)$ is a GQ of order (s, t) provided

K1. $A_i A_j \cap A_k = \{1\}$, for i, j, k distinct, and

K2. $A_i^* \cap A_j = \{1\}$, for $i \neq j$.

If the conditions K1 and K2 are satisfied, then one easily sees that

$$A_i^* = \cup\{A_i g \mid A_i g = A_i \text{ or } A_i g \cap A_j = \emptyset \text{ for all } A_j \in J\},$$

so that A_i^* is uniquely defined by A_i.

Suppose K1 and K2 are satisfied. For any $h \in G$ let us define θ_h by

$$g^{\theta_h} = gh, (A_i g)^{\theta_h} = A_i gh,$$

$$(A_i^* g)^{\theta_h} = A_i^* gh, [A_i]^{\theta_h} = [A_i], (\infty)^{\theta_h} = (\infty),$$

with $g \in G, A_i \in J$, $A_i^* \in J^*$. Then θ_h is an automorphism of $\mathcal{S}(G, J)$ which fixes the point (∞) and all lines of type (b). If $G' = \{\theta_h \mid h \in G\}$, then clearly $G' \cong G$ and G' acts regularly on the points of type (i).

If K1 and K2 are satisfied, then J is called a 4-*gonal family* for G.

Let $\mathbb{F} = \mathrm{GF}(q)$, q any prime power. Let $f : \mathbb{F}^2 \times \mathbb{F}^2 \to \mathbb{F}$ be a fixed symmetric, nonsingular biadditive map. Put $G = \{(\alpha, c, \beta) \mid \alpha, \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$. Define a binary operation on G by:

$$(\alpha, c, \beta).(\alpha', c', \beta') = (\alpha + \alpha', c + c' + f(\beta, \alpha'), \beta + \beta').$$

This makes G into a group whose center is $C = \{(0, c, 0) \in G \mid c \in \mathbb{F}\}$. Suppose that for each $u \in \mathbb{F}$ there is an additive map $\delta_u : \mathbb{F}^2 \to \mathbb{F}^2$ and a map $g_u : \mathbb{F}^2 \to \mathbb{F}$ for which

$$g_u(\alpha + \beta) - g_u(\alpha) - g_u(\beta) = f(\alpha^{\delta_u}, \beta) = f(\beta^{\delta_u}, \alpha),$$

for all $\alpha, \beta \in \mathbb{F}^2, u \in \mathbb{F}$. With such a setup, we can define a family of subgroups of G by:

$$A(u) = \{(\alpha, g_u(\alpha), \alpha^{\delta_u}) \mid \alpha \in \mathbb{F}^2\},\ u \in \mathbb{F},$$

and

$$A(\infty) = \{(0, 0, \beta) \in G \mid \beta \in \mathbb{F}^2\}.$$

Then put $J = \{A(u) \mid u \in \mathbb{F} \cup \{\infty\}\}$ and $J^* = \{A^*(u) \mid u \in \mathbb{F} \cup \{\infty\}\}$, with $A^*(u) = A(u)C$. So

$$A^*(u) = \{(\alpha, c, \alpha^{\delta_u}) \mid \alpha \in \mathbb{F}^2\}, \ u \in \mathbb{F},$$

and

$$A^*(\infty) = \{(0, c, \beta) \in G \mid \beta \in \mathbb{F}^2\}.$$

Necessary and sufficient conditions were worked out in Payne [46] (or see Section 10.4 of Payne and Thas [55]) for J to be a 4-gonal family.

Theorem 4.1
J is a 4-gonal family for G if and only if

(i) $\delta(u, r) : \alpha \to \alpha^{\delta_u} - \alpha^{\delta_r}$ is bijective for $u \neq r$,

(ii) $g_u(\alpha) = g_r(\alpha)$, $u \neq r$, implies $\alpha = 0$,

(iii) if u, r, and v are distinct, then $\gamma = 0$ is the only solution to

$$g_u(\gamma^{\delta^{-1}(u,v)}) - g_v(\gamma^{\delta^{-1}(u,v)}) + g_v(-\gamma^{\delta^{-1}(v,r)}) - g_r(-\gamma^{\delta^{-1}(v,r)}) = 0.$$

Let $\mathcal{C} = \{A_u \mid u \in \mathbb{F}\}$ be a set of q distinct 2×2-matrices over $\mathbb{F}$. Then $\mathcal{C}$ is called a *q-clan* provided $A_u - A_r$ is anisotropic whenever $u \neq r$, i.e. $\alpha(A_u - A_r)\alpha^T = 0$ has only the trivial solution $\alpha = (0, 0)$. For $A_u \in \mathcal{C}$, put $K_u = A_u + A_u^T$, and then define $g_u(\alpha) = \alpha A_u \alpha^T$ and $\alpha^{\delta_u} = \alpha K_u$ for $\alpha \in \mathbb{F}^2$. Then necessarily $f(\alpha, \beta) = \alpha\beta^T$ for all $\alpha, \beta \in \mathbb{F}^2$. With G, $A(u)$, $A^*(u)$, J as above, the following theorem is a combination of results of Payne [46], Payne [48] and Kantor [37].

Theorem 4.2
The set J is a 4-gonal family for G if and only if $\mathcal{C}$ is a q-clan.

In particular, let $\mathcal{C} = \{A_u \mid u \in \mathbb{F}\}$ be a set of q upper triangular 2×2-matrices over $\mathbb{F}$, with

$$A_u = \begin{pmatrix} x_u & y_u \\ 0 & z_u \end{pmatrix},$$

$x_u, y_u, z_u, u \in \mathbb{F}$. For q odd, $\mathcal{C}$ is a q-clan if and only if

$$-\det(K_u - K_r) = (y_u - y_r)^2 - 4(x_u - x_r)(z_u - z_r) \tag{3}$$

is a nonsquare of $\mathbb{F}$ whenever $r, u \in \mathbb{F}, r \neq u$. For q even, $\mathcal{C}$ is a q-clan if and only if

$$y_u \neq y_r \text{ and } \operatorname{tr}((x_u + x_r)(z_u + z_r)(y_u + y_r)^{-2}) = 1 \tag{4}$$

whenever $r, u \in \mathbb{F}, r \neq u$. (Here $\mathrm{tr}(w)$ is the trace of w over the prime subfield, so that $\mathrm{tr}(w) = 1$ if and only if $X^2 + X + w = 0$ has no solution in $\mathbb{F}$.)

In Thas [65] it was shown that (3) and (4) are exactly the conditions for the planes

$$x_u X_0 + z_u X_1 + y_u X_2 + X_3 = 0$$

of $\mathrm{PG}(3, q)$ to define a *flock* of the quadratic cone $X_0 X_1 = X_2^2$, that is, to intersect the cone in q mutually disjoint nonsingular conics.

Theorem 4.3
To any flock of the quadratic cone of $\mathrm{PG}(3, q)$ *there corresponds a* GQ *of order* (q^2, q).

The last examples of 4-gonal families were deduced by Payne [49], [50] from the semifield planes of Ganley [22], but more than just the ideas of Thas [65] is required. These are the first example of GQ of order (q^2, q) known to arise from 4-gonal families but not from q-clans via the procedure just described (that is, via the group $G = \{(\alpha, c, \beta) | \alpha, \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$, the map f, etc.).

Let $q = 3^h$ and let n be a nonsquare of $\mathrm{GF}(q)$. With the notations of this section, put

$$g_u(\gamma) = \gamma \begin{pmatrix} u & 0 \\ 0 & -nu \end{pmatrix} \gamma^T + \left(\gamma \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \gamma^T\right)^{\frac{1}{3}} + \left(\gamma \begin{pmatrix} 0 & 0 \\ 0 & -n^{-1}u \end{pmatrix} \gamma^T\right)^{\frac{1}{9}},$$

$$\alpha^{\delta_u} = u\alpha,$$

and

$$f(\alpha, \beta) = \alpha \begin{pmatrix} -1 & 0 \\ 0 & -n \end{pmatrix} \beta^T + \left(\alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \beta^T\right)^{\frac{1}{3}} + \left(\alpha \begin{pmatrix} 0 & 0 \\ 0 & n^{-1} \end{pmatrix} \beta^T\right)^{\frac{1}{9}}.$$

Then Payne [49], [50] shows that the corresponding set J is a 4-gonal family, thus giving a GQ of order (q^2, q). As these results were obtained while he was visiting the University of Rome, Payne calls these GQ the *Roman generalized quadrangles*.

The construction of Knarr [40]

Let $\mathcal{F} = \{C_1, C_2, \cdots, C_q\}$ be a flock of the quadratic cone K with vertex x_0 of $\mathrm{PG}(3, q)$, with **q odd**. The plane of C_i is denoted by $\pi_i, i = 1, 2, \cdots, q$. Let K be embedded in the nonsingular quadric $Q(4, q)$ of $\mathrm{PG}(4, q)$. The polar line of π_i with respect to $Q(4, q)$ is denoted by L_i; let

$$L_i \cap Q(4, q) = \{x_0, x_i\}, i = 1, 2, \cdots, q.$$

Then no point of $Q(4,q)$ is collinear with all three of $x_0, x_i, x_j, 1 \leq i < j \leq q$. In Bader, Lunardon and Thas [6] it is proved that it is also true that no point of $Q(4,q)$ is collinear with all three of $x_i, x_j, x_k, 0 \leq i < j < k \leq q$. Such a set U of $q+1$ points of $Q(4,q)$ will be called a BLT-*set* in $Q(4,q)$, following a suggestion of Kantor [39]. Since the GQ $Q(4,q)$ is isomorphic to the dual of the GQ $W(q)$, to a BLT-set in $Q(4,q)$ corresponds a set V of $q+1$ lines of $W(q)$ with the property that no line of $W(q)$ is concurrent with three distinct lines of V; such a set V will also be called a BLT-*set*.

To $\mathcal{F}$ corresponds a GQ $\mathcal{S}$ of order (q^2, q). Knarr [40] proves that $\mathcal{S}$ is isomorphic to the following incidence structure.

Start with a symplectic polarity ζ of $\mathrm{PG}(5,q)$. Let $p \in \mathrm{PG}(5,q)$ and let $\mathrm{PG}(3,q)$ be a 3-dimensional subspace of $\mathrm{PG}(5,q)$ for which $p \notin \mathrm{PG}(3,q) \subset p^\zeta$. In $\mathrm{PG}(3,q)$ ζ induces a symplectic polarity ζ', and hence a GQ $W(q)$. Let V be a BLT-set of the GQ $W(q)$ and construct a geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ as follows.

Points:

- (i) p;
- (ii) lines of $\mathrm{PG}(5,q)$ not containing p but contained in one of the planes $\pi_t = pL_t$, with L_t a line of the BLT-set V;
- (iii) points of $\mathrm{PG}(5,q)$ not in p^ζ.

Lines:

- (a) planes $\pi_t = pL_t$, with $L_t \in V$;
- (b) totally isotropic planes of ζ not contained in p^ζ and meeting some π_t in a line (not through p).

The incidence relation I is just the natural incidence inherited from $\mathrm{PG}(5,q)$.

Then Knarr [40] proves that $\mathcal{S}$ is a GQ of order (q^2, q) isomorphic to the GQ arising from the flock $\mathcal{F}$.

The constructions of Thas [70]

Recently, Thas found a pure geometrical construction for the GQ $\mathcal{S}$ arising from a flock $\mathcal{F}$, which works for any q.

4.5 Isomorphisms

For a proof of the following theorem we refer to Payne [50] and Payne and Thas [55].

Theorem 4.4

(a) The GQ $T_2(O)$ *is isomorphic to the classical* GQ $Q(4,q)$ *if and only if* O *is an irreducible conic; it is isomorphic to* $W(q)$ *if and only if* q *is even and* O *is a conic.*

(b) The GQ $T_2^*(O)$ *and* $AS(q)$ *are isomorphic to the respective* GQ $P(T_2(O'), (\infty))$*, with* $O' = O \setminus \{x\}$ *and* $x \in O$*, and* $P(W(q), y)$*, with* y *any point of* $W(q)$*.*

(c) The GQ $T_3(O)$ *is isomorphic to* $Q^-(5,q)$ *if and only if* O *is an elliptic quadric of* $\mathrm{PG}(3,q)$*.*

(d) Apart from a few small values of q*, no nonclassical* GQ *arising from a flock is isomorphic to a previously known* GQ*.*

Remark

If q is odd, then the oval O is a conic (see Hirschfeld [28]), implying $T_2(O) \cong Q(4,q)$. In such a case $T_2(O)$ is not selfdual. If q is even and O is a conic, then $T_2(O)$, which is isomorphic to $Q(4,q)$, is selfdual. The problem of determining all ovals for which $T_2(O)$ is selfdual has been solved (cf. Eich and Payne [18], Thas and Payne [71], Payne and Thas [55]). For a survey of the known ovals of $\mathrm{PG}(2,q)$, q even, see Thas [66].

If q is odd, then the ovoid O is an elliptic quadric (see Hirschfeld [29]), implying $T_3(O) \cong Q^-(5,q)$. For q even, one other class of ovoids is known; these ovoids only exist for $q = 2^{2e+1}$, $e \geq 1$, and are called *Tits-Suzuki ovoids* (see Hirschfeld [29]).

5 Generalized quadrangles with small parameters

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite GQ of order (s,t), $1 < s \leq t$.

5.1 $s = 2$

By Theorems 2.2 and 2.3 $s+t$ divides $st(s+1)(t+1)$, and $t \leq s^2$. Hence $t \in \{2,4\}$.

Theorem 5.1

Up to isomorphism there is only one GQ *of order* 2 *and only one* GQ *of order* $(2,4)$*.*

Corollary

The GQ $W(2)$ *and* $Q(4,2)$ *are selfdual and mutually isomorphic. The* GQ $AS(3)$ *is isomorphic to the* GQ $Q^-(5,2)$*.*

It is easy to show that the GQ of order 2 is unique. The uniqueness of the GQ of order $(2,4)$ was proved independently at least five times, by Dixmier and Zara [17], Seidel [60], Shult [61], Thas [63] and Freudenthal [20].

Two interesting models of the GQ *of order* 2

(1) Let O be a hyperoval, i.e. a 6-arc of $\mathrm{PG}(2,4)$. Points of the GQ are the points of $\mathrm{PG}(2,4)$ not on O, lines of the GQ are the lines of $\mathrm{PG}(2,4)$ intersecting O, incidence is the natural one.

(2) The following construction was apparently first discovered by Sylvester [62]. A *duad* is an unordered pair $ij = ji$ of distinct integers from among $1, 2, \ldots, 6$. A *syntheme* is a set $\{ij, kl, mn\}$ of three duads for which i, j, k, l, m, n are distinct. It is routine to verify that Sylvester's syntheme-duad geometry with duads playing the role of points, synthemes playing the role of lines, and containment as the incidence relation, is the unique GQ of order 2.

Two interesting models of the GQ *of order* $(2,4)$

(1) In Payne and Thas [55] the following construction of the GQ of order $(2,4)$ is given. In addition to the duads and synthemes given above, let $1, 2, \ldots, 6$ and $1', 2', \ldots, 6'$ denote twelve additional points, and let $\{i, ij, j'\}$, $1 \leq i, j \leq 6$, $i \neq j$, denote thirty additional lines. It is easy to verify that the 27 points and 45 lines just constructed yield a representation of the unique GQ of order $(2,4)$.

(2) Points of the GQ are the 27 lines on a general cubic surface V in $\mathrm{PG}(3,\mathbb{C})$, lines of the GQ are the 45 tritangent planes of V, and incidence is inclusion (for the properties of the general cubic surface in $\mathrm{PG}(3,\mathbb{C})$ we refer, e.g., to Baker [7]; see also Hirschfeld [29] where a full chapter is devoted to cubic surfaces).

5.2 $s = 3$

Again by Theorems 2.2 and 2.3 we have $t \in \{3, 5, 6, 9\}$.

Theorem 5.2
Any GQ *of order* $(3,5)$ *must be isomorphic to the* GQ $T_2^*(O)$ *arising from a hyperoval in* $\mathrm{PG}(2,4)$*, any* GQ *of order* $(3,9)$ *must be isomorphic to* $Q^-(5,3)$*, and a* GQ *of order* 3 *is isomorphic either to* $W(3)$ *or to its dual* $Q(4,3)$*. Finally, there is no* GQ *of order* $(3,6)$*.*

The uniqueness of the GQ of order $(3,5)$ was proved by Dixmier and Zara [17], the uniqueness of the GQ of order $(3,9)$ was proved independently by Dixmier and Zara [17] and Cameron (see Thas and Payne [71]), the determination of all GQ of order 3 is due independently to Dixmier and Zara [17] and Payne [44]. Dixmier and Zara [17] proved that there is no GQ of order $(3, 6)$. All these proofs, some of them simplified or streamlined, are also contained in Payne and Thas [55].

5.3 $s = 4$

Using Section 2 it is easy to check that $t \in \{4, 6, 8, 11, 12, 16\}$. Nothing is known about $t = 11$ or $t = 12$. In the other cases unique examples are known, but the uniqueness question is settled only in the case $t = 4$.

Theorem 5.3
A GQ *of order* 4 *must be isomorphic to* $W(4)$.

The proof of this fact that appears in Payne and Thas [55] is that of Payne [45], with a gap filled in by Tits.

6 Generalized quadrangles in finite projective and affine spaces

6.1 Generalized quadrangles in finite projective spaces

A (finite) *projective* GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a GQ for which $\mathcal{P}$ is a subset of the pointset of some projective space $\mathrm{PG}(d, q)$, $\mathcal{L}$ is a set of lines of $\mathrm{PG}(d, q)$, $\mathcal{P}$ is the union of all members of $\mathcal{L}$, and the incidence relation I is the one induced by that of $\mathrm{PG}(d, q)$. We also say that $\mathcal{S}$ is *embedded* in $\mathrm{PG}(d, q)$. If $\mathrm{PG}(d', q)$ is the subspace of $\mathrm{PG}(d, q)$ generated by all points of $\mathcal{P}$, then we say that $\mathrm{PG}(d', q)$ is the *ambient space* of $\mathcal{S}$.
The following fundamental and beautiful theorem is due to Buekenhout and Lefèvre.

Theorem 6.1 (Buekenhout and Lefèvre [12])
A projective GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *with ambient space* $\mathrm{PG}(d, q)$ *must be obtained in one of the following ways:*

(i) there is a nonsingular quadric $Q(d, q)$ *of Witt index* 2 *in* $\mathrm{PG}(d, q)$, $d = 3, 4$ *or* 5, *such that* $\mathcal{P}$ *is the set of points of* $Q(d, q)$ *and* $\mathcal{L}$ *is the set of lines on* $Q(d, q)$*;*

(ii) there is a nonsingular Hermitian variety $H(d, q^2)$ *in* $\mathrm{PG}(d, q^2)$, $d = 3$ *or* 4, *such that* $\mathcal{P}$ *is the set of points of* $H(d, q^2)$ *and* $\mathcal{L}$ *is the set of lines on* $H(d, q^2)$*;*

(iii) $d = 3$, $\mathcal{P}$ *is the set of all points of* $\mathrm{PG}(3, q)$ *and* $\mathcal{L}$ *is the set of all totally isotropic lines with respect to some symplectic polarity of* $\mathrm{PG}(3, q)$.

Hence $\mathcal{S}$ must be one of the classical examples described in Section 3.

6.2 Generalized quadrangles in finite affine spaces

All GQ embedded in AG(d, q) were determined by Thas [64]; the theorem on the embedding in AG$(3, q)$ was proved independently by Bichara [8].

We note that in contrast with the projective case, there arise five non-trivial "sporadic" cases in the finite affine case.

7 Elation generalized quadrangles

A collineation σ of a generalized quadrangle $\mathcal{S}$ is called an *elation about the point* x provided either σ is the identity or it fixes each line incident with x and no point of $\mathcal{P} \setminus x^\perp$. Dually we can define an *elation about the line* L.

If σ acts as the identity over $x^\perp$, we say that σ is a *symmetry* with *center* x. A *symmetry* with *axis* L is an elation about L which fixes all the lines concurrent with L.

Theorem 7.1
Let σ and θ be nonidentity symmetries with distinct axes L and M respectively. Then $\sigma\theta = \theta\sigma$ if and only if L and M are concurrent.

Let $\mathcal{S}$ be a finite generalized quadrangle of order (s, t) which is neither a grid nor a dual grid. Suppose that $\mathcal{S}$ has a collineation group E such that:

(1) E fixes a point x linewise;

(2) E is sharply transitive on $\mathcal{P} \setminus x^\perp$.

We say that E is an *elation group* with *base point* x and $\mathcal{S}$ is an *elation generalized quadrangle* . Throughout this section we will always denote by E an elation group of $\mathcal{S}$ with base point x.

Let $\{L_i \mid i = 0, 1, \ldots, t\}$ be the set of lines of $\mathcal{S}$ incident with x, and let y be a given point not collinear with x. For each $i = 0, 1, \ldots, t$ let z_i be the point of L_i collinear with y, and M_i the line joining y and z_i. Define

$$\begin{aligned} A_i^* &= \{g \in E \mid z_i^g = z_i\}, \\ A_i &= \{g \in E \mid M_i^g = M_i\}. \end{aligned}$$

Theorem 7.2 (Payne and Thas [55] Chapter 8)
$J = \{A_i \mid i = 0, 1, \ldots t\}$ is a $4-$gonal family.

The map θ defined by

$$\begin{aligned} \theta &: (\infty) \mapsto x, \\ \theta &: [A_i^*] \mapsto L_i, \\ \theta &: A_i^* g \mapsto z_i^g, \\ \theta &: A_i g \mapsto M_i^g, \\ \theta &: g \mapsto y^g \end{aligned}$$

is an isomorphism between $\mathcal{S}(E, J)$ and $\mathcal{S}$.

For each element h of E, the right multiplication by h defines a collineation of $\mathcal{S}(E, J)$ such that $g \mapsto gh$, $A_i g \mapsto A_i gh$, $A_i^* g \mapsto A_i^* gh$, $[A_i^*] \mapsto [A_i^*]$ and $(\infty) \mapsto (\infty)$. Then E acts (by right multiplication) as an elation group with base point (∞). Thus we have proved

Theorem 7.3
$\mathcal{S}(E, J)$ is isomorphic to $\mathcal{S}$. The group E acts by right multiplication as an elation group with base point (∞).

Theorem 7.4
The elements of A_i are symmetries with axis L_i if and only if A_i is a normal subgroup of E.

Proof. An element h of A_i fixes all the points of L_i if and only if $A_i^* gh = A_i^* g$ for all $g \in E$, i.e. $g^{-1} A_i g < A_i^*$.

The lines incident with z_i different from L_i are the lines M_i^g with $g \in A_i^*$. Then an element h of A_i fixes all the lines incident with z_i if and only if $M_i ghg^{-1} = M_i$ for all $g \in A_i^*$, if and only if $A_i \lhd A_i^*$.

If N is a line incident with L_i, with $x \not\mathrm{I}\, N$, then there is an element $g \in E$ such that $M_i^g = N$. Then $h \in E$ is a symmetry with axis L_i if and only if $M_i^{gh} = M_i^g$ for all g in E. This is equivalent to saying that $A_i gh = A_i g$, i.e. $ghg^{-1} \in A_i$ for all $g \in E$ and $h \in A_i$. Then the elements of A_i are symmetries with axis L_i if and only if A_i is a normal subgroup of E. ∎

If each element of A_i is a symmetry with axis L_i for each line L_i incident with x, then we say that $\mathcal{S}$ is a *translation generalized quadrangle* and E is a *translation group* with base point x. Section 10 deals with translation generalized quadrangles. As a corollary of Theorem 7.4 we have

Theorem 7.5
An abelian elation group is a translation group.

We now prove that the finite classical generalized quadrangles have an elation group with base point x for arbitrary x.

Example I

Let $f(X_2, X_3)$ be a polynomial over $\mathbb{F} = \mathrm{GF}(q)$ such that either q is odd and $f(X_2, X_3) = X_2^2 + \alpha X_3^2$ where $-\alpha$ is a non-square, or q is even and $f(X_2, X_3) = X_2^2 + X_2 X_3 + m X_3^2$ where m has trace 1.

Let $Q^-(5, q)$ be the non-singular elliptic quadric of $\mathrm{PG}(5, q)$ with equation

$$X_0 X_5 - X_1 X_4 - f(X_2, X_3) = 0.$$

If $x = (0, 0, 0, 0, 0, 1)$, then $X_0 = 0$ is the tangent hyperplane to $Q^-(5, q)$ at x.

The collineation $M(a,b,c,d)$ of $\mathrm{PG}(5,q)$ defined by the matrix

$$\begin{pmatrix} 1 & a & b & c & d & ad+f(b,c) \\ 0 & 1 & 0 & 0 & 0 & d \\ 0 & 0 & 1 & 0 & 0 & c \\ 0 & 0 & 0 & 1 & 0 & b \\ 0 & 0 & 0 & 0 & 1 & a \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

fixes the quadric $Q^-(5,q)$ and all the lines on $Q^-(5,q)$ incident with the point x. Put $E = \{M(a,b,c,d) \mid a,b,c,d \in \mathbb{F}\}$. Then E is abelian. As E is transitive on the points of $Q^-(5,q)$ not on the tangent hyperplane to $Q^-(5,q)$ at x, E is a translation group with base point x. As the orthogonal group is transitive on the points of $Q^-(5,q)$ (see [16]), we have proved that $Q^-(5,q)$ is a translation quadrangle with base point y, where y is arbitrary point of $Q^-(5,q)$.

The hyperplane H with equation $X_3 = 0$ of $\mathrm{PG}(5,q)$ is non-tangent and $Q(4,q) = H \cap Q^-(5,q)$. If $\bar{E}$ is the stabilizer of H in E, then it is an easy to prove that $\bar{E}$ is sharply transitive on the points of $Q(4,q)$ not on the three-dimensional quadratic cone $x^\perp \cap Q(4,q)$ and acts as the identity on this quadratic cone. Then $\bar{E}$ is a translation group of $Q(4,q)$ with base point x. Also each point of $Q(4,q)$ is the base point of a translation group of $Q(4,q)$. Hence $Q(4,q)$ is a translation generalized quadrangle with arbitrary base point.

Example II

Let $\perp$ be the symplectic polarity of $\mathrm{PG}(3,q)$ associated by the alternating form

$$\langle (x_0,x_1,x_2,x_3),(y_0,y_1,y_2,y_3)\rangle = x_0y_3 - x_3y_0 + x_1y_2 - x_2y_1.$$

Denote with $W(q)$ the generalized quadrangle defined by $\perp$.

The collineation $S(a,b,c)$ of $\mathrm{PG}(3,q)$ defined by the matrix

$$\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

commutes with the polarity $\perp$ and fixes all the lines contained in the plane $X_0 = 0$ and incident with $x = (0,0,0,1)$. Then $E = \{S(a,b,c) \mid a,b,c \in \mathbb{F}\}$ is an elation group of the symplectic quadrangle $W(q)$ with base point x. As the symplectic group is transitive on the points of $W(q)$ (see [16]) we have proved that each point of $W(q)$ is a base point of an elation group of $W(q)$.

Example III
Let $x \mapsto \bar{x}$ be the involutorial automorphism of $\mathrm{GF}(q^2)$. Define $tr(x) = x+\bar{x}$. Let $\perp$ be the unitary polarity of $\mathrm{PG}(4,q^2)$ associated with the hermitian form

$$\langle (x_0, x_1, x_2, x_3, x_4), (y_o, y_1, y_2, y_3, y_4) \rangle = \sum_{i=0}^{4} x_i \bar{y}_{4-i}.$$

The collineation $H(a,b,c,d)$ of $\mathrm{PG}(4,q^2)$ defined by the matrix

$$\begin{pmatrix} 1 & a & b & c & d \\ 0 & 1 & 0 & 0 & -\bar{c} \\ 0 & 0 & 1 & 0 & -\bar{b} \\ 0 & 0 & 0 & 1 & -\bar{a} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad b\bar{b} + tr(d + a\bar{c}) = 0$$

preserves the unitary polarity $\perp$ while acting as the identity on the lines incident with $x = (0,0,0,0,1)$ and contained in $x^{\perp}$. As $x^{\perp}$ is the hyperplane $X_0 = 0$, the points of $H(4,q^2)$ not contained in $x^{\perp}$ have homogeneous coordinates $(1, a, b, c, -a\bar{c} - b\bar{b})$ and the group $E = \{H(a,b,c,d) \mid a,b,c,d \in GF(q^2), b\bar{b} + tr(d + a\bar{c}) = 0\}$ is sharply transitive on the points of $H(4,q^2)$ not in $x^{\perp}$. Then E is an elation group with base point x. As the unitary group is transitive on the points of $H(4,q^2)$ (see [16]), we have proved that each point of $H(4,q^2)$ is the base point of an elation group.

We remark that the point $(0,0,1,0,0)$ is not absolute with respect to $\perp$. As the polar hyperplane $X_2 = 0$ of $(0,0,1,0,0)$ with respect to $\perp$ is not tangent, we have $H(3,q^2) = H(4,q^2) \cap (0,0,1,0,0)^{\perp}$. Then the stabilizer $\bar{E}$ of $(0,0,1,0,0)$ in E is an elation group of $H(3,q^2)$ with base point x.

Example IV
The generalized quadrangles $T_2(O)$ and $T_3(O)$ are translation generalized quadrangles. If H is the hyperplane containing O, then the group of all the elations of $\mathrm{PG}(d+1,q)$ $(d = 2,3)$ with axis H and center incident with H is a translation group with base point (∞).

In Section 4.4 we have proved that a $4-$gonal family J of a finite group G defines a generalized quadrangle. Then:

Theorem 7.6 (Payne and Thas [55])
Let G be a finite group. If J is a $4-$gonal family of G, then $\mathcal{S}(G,J)$ is a generalized quadrangle, and G acts by right multiplication as an elation group with base point (∞).

In [21], D. E. Frohardt has proved that a finite group of order s^2t, which has a $4-$gonal family, is a $p-$group if $s \leq t$. This implies that the generalized quadrangle $T_2^*(O)$ with parameters $(q-1, q+1)$ is not an elation generalized quadrangle.

If $\mathcal{S}$ is an elation generalized quadrangle of order (s,t) with base point (∞) and elation group E, for which E contains a group of t symmetries with center (∞), we say $\mathcal{S}$ is a *skew-translation* generalized quadrangle. If s,t are the parameters of a skew-translation generalized quadrangle, then $t \leq s$ (see [55]). In [25], D. Hachenberger has proved that the parameters of a skew-translation generalized quadrangle are powers of the same prime number.

8 Flock quadrangles

8.1 $q-$clans

Here we **always** put $\mathbb{F} = \mathrm{GF}(q)$ and $\tilde{\mathbb{F}} = \mathrm{GF}(q) \cup \{\infty\}$.

In the sequel, denote by E the group whose elements are those of $\mathbb{F}^2 \times \mathbb{F} \times \mathbb{F}^2$, and whose product is defined by

$$(\alpha, c, \beta)(\alpha', c', \beta') = (\alpha + \alpha', c + c' + \alpha' \cdot \beta, \beta + \beta')$$

where $\alpha, \beta, \alpha', \beta' \in \mathbb{F}^2$, $c, c' \in \mathbb{F}$, and $\alpha' \cdot \beta = \alpha' \beta^T$. The center of E is the set $Z = \{(0, c, 0) \mid c \in \mathbb{F}\}$ and the group $\bar{E} = E/Z$ is elementary abelian. Moreover, we can regard $\bar{E}$ as a $4-$dimensional vector space over $\mathbb{F}$. For each element g of E, let g^* be the preimage in E of the 1-space of $\bar{E}$ spanned by $\bar{g} = gZ$. If $\bar{g}, \bar{h}$ are elements of $\bar{E}$, note that $(\bar{g}, \bar{h}) = [g, h]$ defines a non-singular alternating $\mathbb{F}$-bilinear form on $\bar{E}$; if q is even, $\bar{h} \mapsto h^2$ defines a quadratic form associated with $(\ ,\ )$. Hence, $\bar{E}$ is equipped with a symplectic or orthogonal geometry.

If $[g, h] = 1$, then $[g^*, h^*] = 1$. Hence maximal elementary abelian subgroups of E are preimages of totally isotropic (or singular) 2-spaces of $\bar{E}$. Therefore a maximal elementary abelian subgroup of E has order q^3.

A $2 \times 2-$matrix D is *anisotropic* if $\alpha D \alpha^T = 0$, where $\alpha \in \mathbb{F}^2$, implies $\alpha = 0$. Hence, if $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then D is anisotropic if and only if the polynomial $ax^2 + (b+c)x + d$ is irreducible over $\mathbb{F}$. We note that:

- when q is odd, D is anisotropic if and only if $(b+c)^2 - 4ad$ is a non-square in $\mathbb{F}$;
- when q is even, D is anisotropic if and only if $b+c \neq 0$ and the polymial $x^2 + x + ad(b+c)^{-1}$ is irreducible over $\mathbb{F}$, i.e. the element $ad(b+c)^{-1}$ has trace 1 (see [28]).

Recall that a $q-$clan is a family $\mathcal{C} = \{A_t = \begin{pmatrix} a_t & b_t \\ 0 & c_t \end{pmatrix} \mid t \in \mathbb{F}\}$ such that the matrix $A_t - A_u$ is anisotropic for all distinct elements t and u of $\mathbb{F}$. Let $K_t = A_t + A_t^T$ for all t in $\mathbb{F}$. By Theorem 4.2, the $4-$gonal family

$J = J(\mathcal{C})$ of E associated with $\mathcal{C}$ is defined by the following subgroups of E :

$$\begin{aligned} A(t) &= \{(\alpha, \alpha A_t \alpha^T, \alpha K_t) \mid \alpha \in \mathbb{F}^2\}, \\ A^*(t) &= \{(\alpha, b, \alpha K_t) \mid \alpha \in \mathbb{F}^2, b \in \mathbb{F}\}, \\ A(\infty) &= \{(0, 0, \gamma) \mid \gamma \in \mathbb{F}^2\}, \\ A^*(\infty) &= \{(0, b, \gamma) \mid \gamma \in \mathbb{F}^2, b \in \mathbb{F}\}. \end{aligned}$$

Let s be an element of $\mathbb{F}$. Then the map

$$\tau_s : (\alpha, c, \beta) \mapsto (\alpha, c - \alpha A_s \alpha^t, \beta - \alpha K_s)$$

is an automorphism of E which maps the $4-$gonal family associated with $\mathcal{C}$ to the $4-$gonal family associated with the new $q-$clan $\mathcal{C}' = \{A_t - A_s \mid t \in \mathbb{F}\}$. Thus we can suppose that $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ belongs to $\mathcal{C}$.

8.2 Flocks

Let K be a quadratic cone of $\mathrm{PG}(3, q)$ with vertex v. A *flock* of K is a partition in conics of the points of K different from the vertex, i.e. a set of q mutually disjoint non-singular conics contained in K.

The easiest example of flock is the *linear flock* defined in the following way. Let L be an exterior line with respect to K : the linear flock (with axis L) is the set of all the conics $K \cap \pi$, where π is a plane through L not incident with the vertex of K.

For $i = 1, 2$ let $\mathcal{F}_i$ be a flock of the quadratic cone K_i. The flocks $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be *isomorphic* when there is a collineation σ of $\mathrm{PG}(3, q)$ such that $K_2 = K_1^\sigma$ and $\mathcal{F}_2 = \mathcal{F}_1^\sigma$, i.e. $\mathcal{F}_2^\sigma = \{C^\sigma \mid C \text{ is a conic of} \mathcal{F}_1\}$.

We can choose the homogeneous coordinates of $\mathrm{PG}(3, q)$ in such a way that $v = (0, 0, 0, 1)$ and K has equation $X_2^2 - X_0 X_1 = 0$. Hence, a plane of $\mathrm{PG}(3, q)$ not incident with v has equation $aX_0 + bX_1 + cX_2 + X_3 = 0$, where the elements a, b, c of $\mathbb{F}$ are uniquely defined.

Let $\mathcal{C} = \{A_t \mid t \in \mathbb{F}\}$ be a $q-$clan. For each matrix $A_t = \begin{pmatrix} a_t & b_t \\ 0 & c_t \end{pmatrix} \in \mathcal{C}$, let π_t be the plane $a_t X_0 + b_t X_1 + c_t X_2 + X_3 = 0$ and let $C_t = \pi_t \cap K$. Then C_t is a non-singular conic.

Theorem 8.1 (Thas [65])
(a) If $\mathcal{C}$ is a $q-$clan, then $\mathcal{F}(\mathcal{C}) = \{C_t \mid t \in \mathbb{F}\}$ is a flock of K.

(b) For each flock $\mathcal{F}$ of the quadratic cone K, there is a $q-$clan $\mathcal{C}$ such that $\mathcal{F} = \mathcal{F}(\mathcal{C})$.

When $\mathcal{F}$ is the flock $\mathcal{F}(\mathcal{C})$, we will denote by $\mathcal{S}(\mathcal{F}) = \mathcal{S}(E, J(\mathcal{C}))$ the generalized quadrangle associated with the $q-$clan. We will call a generalized quadrangle $\mathcal{S}$ a *flock quadrangle* if $\mathcal{S}$ is isomorphic to $\mathcal{S}(\mathcal{F})$ for some flock $\mathcal{F}$.

Theorem 8.2 (Thas [65])
If two flocks $\mathcal{F}_1$ and $\mathcal{F}_2$ of a quadratic cone have more than $\frac{q-1}{2}$ conics in common, then $\mathcal{F}_1 = \mathcal{F}_2$.

Proof. Let $\mathcal{F}_1 = \{C_1, C_2, ...C_q\}$ and $\mathcal{F}_2 = \{C'_1, C'_2, ...C'_q\}$, and let $\mathcal{F}_1 \cap \mathcal{F}_2 = \{C_1, C_2, ...C_s\}$. Assume that $C'_i \notin \mathcal{F}_1$. Then $|C'_i \cap C_j| \leq 2$ with $j = s+1, ..., q$. Hence $|C'_i| \leq 2(q-s)$ because the conics of $\mathcal{F}_1$ are disjoint. So $q+1 \leq 2(q-s)$, i.e. $s \leq \frac{q-1}{2}$, a contradiction. ■

By Theorem 8.2 the quadratic cone of PG(3, 3) has only the linear flock.

Theorem 8.3 (Thas [65])
Let $\mathcal{F}$ be a flock of the quadratic cone K. The generalized quadrangles $\mathcal{S}(\mathcal{F})$ and $H(3, q^2)$ are isomorphic if and only if the flock is linear.

Let $\mathcal{F}$ be a flock of the quadratic cone K of PG$(3, q)$ and let π_t be the plane $a_tX_0 + b_tX_1 + c_tX_2 + X_3 = 0$. Note that the map $t \mapsto a_t$ is a bijection. Let f and g be the two functions of $\mathbb{F}$ to itself defined by $f(t) = b_{a_t}$ and $g(t) = c_{a_t}$. If C_t is the intersection of K with the plane $tX_0 - f(t)X_1 + g(t)X_2 + X_3 = 0$, then $\mathcal{F}(f, g) = \{C_t \mid t \in \mathbb{F}\}$ is a flock isomorphic to $\mathcal{F}$ (see [24]). We will also write $\mathcal{F} = \mathcal{F}(f, g)$. Further we can suppose that $f(0) = g(0) = 0$.

8.3 Flock spreads

Associated with a flock of the quadratic cone there is a spread of PG$(3, q)$ (hence, a translation plane of order q^2 and kernel containing $\mathbb{F}$), as noted by Fisher and Thas [19] and by Walker [75], using the Klein correspondence. Following Johnson and Gevaert [24], we can construct the spread associated with $\mathcal{F}$ in the following way.

Let $\mathcal{F} = \mathcal{F}(f, g)$ be a flock of the quadratic cone K. Define

$$L_{t,u} = \{(a, b, c, d) \mid (c, d) = (a, b) \begin{pmatrix} u + g(t) & f(t) \\ t & u \end{pmatrix}; a, b \in \mathbb{F}\}$$

for t and u in $\mathbb{F}$ and

$$L_\infty = \{(0, 0, c, d) \mid c, d \in \mathbb{F}\}.$$

The spread $S(\mathcal{F})$ of PG$(3, q)$ associated with $\mathcal{F}$ is defined by

$$S(\mathcal{F}) = \{L_{t,u} \mid t, u \in \mathbb{F}\} \cup \{L_\infty\}.$$

The set $\mathcal{R}_t = \{L_{t,u} \mid u \in \mathbb{F}\} \cup \{L_\infty\}$ with t in $\mathbb{F}$ is a regulus of PG$(3, q)$ and $S(\mathcal{F})$ is the union of the q reguli $\mathcal{R}_t$. The spread $S(\mathcal{F})$ is called a *flock spread.*

Theorem 8.4 (Gevaert and Johnson [24])
A spread of PG$(3, q)$ *is a flock spread if and only if it is the union of q reguli which share a line.*

8.4 BLT-sets

We recall that the 4−gonal family of E associated with the flock $\mathcal{F}(f,g)$ is defined by the following subgroups of E :

$$\begin{aligned} A(t) &= \{(\alpha, \alpha A_t \alpha^T, \alpha K_t) \mid \alpha \in \mathbb{F}^2\}, \\ A^*(t) &= \{(\alpha, b, \alpha K_t) \mid \alpha \in \mathbb{F}^2, b \in \mathbb{F}\}, \\ A(\infty) &= \{(0,0,\gamma) \mid \gamma \in \mathbb{F}^2\}, \\ A^*(\infty) &= \{(0,b,\gamma) \mid \gamma \in \mathbb{F}^2, b \in \mathbb{F}\}. \end{aligned}$$

where $A_t = \begin{pmatrix} t & g(t) \\ 0 & -f(t) \end{pmatrix}$ and $K_t = A_t + A_t^T$. We have $A^*(t) = A(t)Z$ for all $t \in \tilde{\mathbb{F}}$.

Suppose q is an **odd** prime power. Then $\bar{E} = E/Z$ is a 4−dimensional vector space equipped with a symplectic polarity. Denote by $W(q)$ the associated generalized quadrangle. We have noted in §8.1 that $A^*(t)/Z$ is a totally isotropic line of $W(q)$.

Theorem 8.5 (Kantor [39])
For q odd, the set $\{A^*(t)/Z \mid t \in \tilde{\mathbb{F}}\}$ is the BLT-set of lines of $W(q)$ associated with the flock $\mathcal{F}$.

8.5 Herds

Suppose q is **even.** Then $\bar{E} = E/Z$ is a 4−dimensional vector space equipped with the quadratic form $\bar{h} \mapsto h^2$. Denote by $Q^+(3,q)$ the associated non-singular hyperbolic quadric of PG$(3,q)$. For each flock $\mathcal{F} = \mathcal{F}(f,g)$ the set $\mathcal{R} = \{A^*(t)/Z \mid t \in \tilde{\mathbb{F}}\}$ is one of the reguli contained in $Q^+(3,q)$.

Let $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For each α in $\mathbb{F}^2$ define $E_\alpha = \{(a\alpha, c, b\alpha P) \mid a,b,c \in \mathbb{F} = \mathrm{GF}(q)\}$. Then E_α is an abelian subgroup of E of order q^3, and $\mathcal{R}' = \{E_\alpha/Z \mid \alpha \in \mathbb{F}^2\}$ is the opposite regulus to $\mathcal{R}$.

Let $B^*(t) = A^*(t) \cap E_\alpha$ and $B(t) = A(t) \cap E_\alpha$.

The family $\mathcal{O}_\alpha = \{B(t) \mid t \in \tilde{\mathbb{F}}\}$ is a 4−gonal family of E_α. As E_α, $B(t)$ and $B^*(t)$ are vector spaces over $\mathbb{F}$, $\mathcal{O}_\alpha$ defines an oval of PG$(2,q) =$ PG$(E_\alpha, \mathbb{F})$.

The set of the $q+1$ ovals $\mathcal{O}_\alpha$ is called a *herd* of ovals. It has been proved in [14] that the flock is uniquely defined by its herd of ovals.

8.6 Some examples of flocks

In this section we describe two infinite families of flocks. We will introduce in the sequel the other known families of flocks.

8.6.1 Kantor-Payne flocks

Let $q \equiv \pm 2$ $(mod.\ 5)$ and let β and γ be two elements of $\mathbb{F} = \mathrm{GF}(q)$ such that $\beta^2 = 5\gamma$. If $f(t) = -\gamma t^5$ and $g(t) = -\beta t^3$, then $\mathcal{F}(f,g)$ is a (non-linear) flock ([38], [48]).

8.6.2 Generalized Subiaco flocks

Let $q = 2^e$. For $t \in \mathbb{F}$, let Π_t be the plane with equation

$$F(t)X_0 + G(t)X_1 + t^{\frac{1}{2}}X_2 + X_3 = 0$$

where

$$\begin{aligned}
F(t) &= \alpha(t)/\gamma(t) + Ht^{\frac{1}{2}} \quad , \quad 0 \neq H \in \mathbb{F}, \\
G(t) &= \beta(t)/\gamma(t) + Kt^{\frac{1}{2}} \quad , \quad 0 \neq K \in \mathbb{F}, \\
\alpha(t) &= \sum_1^4 a_i t^i \quad ; \quad \beta(t) = \sum_1^4 b_i t^i; \\
\gamma(t) &= t^4 + c_2 t^2 + c_0.
\end{aligned}$$

Note that $t^2 + \sqrt{c_2}t + \sqrt{c_0}$ must be irreducible over $\mathbb{F}$ so that $k(t) \neq 0$ for all $t \in \mathbb{F}$. Hence $tr(c_0/c_2^2) = 1$, where $tr(x)$ denotes the trace of x for $x \in \mathbb{F}$.

Each *Subiaco flock* consists of the non-singular conics $\Pi_t \cap K$ for $t \in \mathbb{F}$ where $\alpha(t)$, $\beta(t)$, $\gamma(t)$, H, K are suitably chosen.

Using the above notation, we can now give the Subiaco flocks originally presented in [14]:

Construction I. Let e be odd (so $t^2+t+1 \neq 0$ for all $t \in \mathbb{F}$) and $e \geq 5$ (to obtain new generalized quadrangles). Put $\alpha(t) = t^2 + t$, $\beta(t) = t^4 + t^3$, $\gamma(t) = t^4 + t^2 + 1$, $H = K = 1$.

Construction II. Let $e = 2r \geq 6$, r odd. Then $\mathbb{F}$ contains an element w for which $\omega^2+\omega+1 = 0$. Put $\alpha(t) = t^4+\omega t^3+\omega t^2$, $\beta(t) = \omega^2 t^3+\omega^2 t^2+\omega t$, $\gamma(t) = t^4 + \omega^2 t^2 + 1$, $H = \omega^2$, $K = 1$.

Construction III. Let $e \geq 4$ and choose $\delta \in \mathbb{F}$ for which both $\delta^2+\delta+1 \neq 0$ and $tr(1/\delta) = 1$ (so that $t^2 + \delta t + 1 \neq 0$ for all $t \in \mathbb{F}$). Then put

$$\begin{aligned}
\alpha(t) &= \delta^2 t^4 + (\delta^2+\delta^3+\delta^4)t^3 + (\delta^2+\delta^3+\delta^4)t^2 + \delta^2 t \ , \ H = 1 \\
\beta(t) &= \frac{\delta^3}{\delta^2+\delta+1}t^4 + (\delta^2+\delta^3+\delta^4)t^3 + (\frac{\delta^2+\delta^4}{\delta^2+\delta+1})t, \ K = (\delta^{\frac{1}{2}}+\delta^{\frac{3}{2}}+\delta^{\frac{5}{2}})^{-1} \\
\gamma(t) &= t^4 + \delta^2 t^2 + 1.
\end{aligned}$$

9 Transitive flocks

A flock $\mathcal{F}$ is called *transitive* if there is a collineation group of $\mathrm{PG}(3,q)$ acting regularly on $\mathcal{F}$. By Gevaert and Johnson [24], $\mathcal{F}$ is a transitive flock if and only if there is a collineation group of $\mathrm{PG}(3,q)$ of order q^2, which fixes $S(\mathcal{F})$ and l_∞, and acts regularly on $S(\mathcal{F}) \setminus \{l_\infty\}$. Using the properties of the translation planes with a collineation group inducing such a group on the line at infinity, one can prove the following theorem

Theorem 9.1 (Gevaert and Johnson [24])
If $\mathcal{F} = \mathcal{F}(f,g)$ is a transitive flock, then either

(a) f, g are additive maps, or

(b) q is a power of a prime number $p \neq 3$, $g(t) = -t^2$ and $f(t) = -\frac{1}{3}t^3 + J(t)$ where J is an additive map of $\mathbb{F}$ to itself such that $u^2 = t^2u - \frac{1}{3}t^4 + tJ(t) \Rightarrow t = u = 0$.

The flock $\mathcal{F}$ is called a *semifield flock* in case (a), and a *likeable flock* in case (b). When q is even, all transitive flocks are known. More precisely

Theorem 9.2 (Johnson [31])
For q even, the only semifield flock is the linear flock.

Theorem 9.3 (Ganley [23])
For q even, the only likeable flock is the Fisher-Thas-Walker flock.

We will now list the known examples of transitive flocks.

9.1 Linear flocks

Let z^2+bz-c be an irreducible polynomial over $\mathbb{F}$. If $g(t) = bt$ and $f(t) = ct$, then $\mathcal{F} = \mathcal{F}(f,g)$ is a flock, which is *linear.* The line $X_0 - cX_1 + bX_2 = 0 = X_3$ is contained in all the planes $\pi_t's$. The translation plane associated with $S(\mathcal{F})$ is desarguesian (see [24]).

9.2 Kantor-Knuth semifield flocks

Let q be an odd prime power. Denote by σ a fixed non-trivial automorphism of $\mathbb{F}$ and by m a fixed non-square of $\mathbb{F}$. If $f(t) = mt^\sigma$ and $g(t) = 0$, then $\mathcal{F} = \mathcal{F}(f,g)$ is a flock, which is called the *Kantor-Knuth semifield flock*, because the spread $S(\mathcal{F})$ is associated with a Knuth semifield (see [24]).

Here, all the planes π_t are incident with the point $(0,0,1,0)$. Using a result of Blokhuis, Thas has proved the following

Theorem 9.4 (Thas [65])
If all the planes of a flock $\mathcal{F}$ are incident with a common point, then $\mathcal{F}$ is either linear or a Kantor-Knuth semifield flock.

We recall that the spread $S(\mathcal{F})$ associated with the flock $\mathcal{F}$ is the union of the reguli $\mathcal{R}_t$. If m is a line of PG$(3,q)$ not in $S(\mathcal{F})$, put $\mathcal{D}_m = \{l \in S(\mathcal{F}) \mid l \cap m \neq \emptyset\}$. If $\mathcal{D}_m$ is not a regulus, then $\mathcal{D}_m$ has at most one transversal line different from m.

Theorem 9.5 (Johnson and Lunardon [33])
Suppose that m is not a transversal of any of the reguli $\mathcal{R}_t$. If $\mathcal{D}_m$ has a transversal line different from m, then $\mathcal{F}$ is either linear or a Kantor-Knuth semifield flock.

9.3 Ganley flocks

Let $q = 3^r$ with $r > 2$, and let n be a fixed non-square in GF(q). If $f(t) = n^{-1}t + nt^9$ and $g(t) = t^3$, then $\mathcal{F} = \mathcal{F}(f,g)$ is a semifield flock called the *Ganley flock* because $S(\mathcal{F})$ is associated with a Ganley semifield (see [24] and [50]).

9.4 A semifield flock of order 243

Let $q = 243 = 3^5$ and let $f(t) = t^9$ and $g(t) = t^{27}$. Then $\mathcal{F} = \mathcal{F}(f,g)$ is a semifield flock (see [5]).

9.5 Fisher-Thas-Walker flocks

Let $q = -1$ $(mod\ 3)$. If $f(t) = -\frac{1}{3}t^3$ and $g(t) = -t^2$, then $\mathcal{F} = \mathcal{F}(f,g)$ is a likeable flock called the *Fisher-Thas-Walker flock*. We note that here $J(t) = 0$ for all t in $\mathbb{F}$.

The Fisher-Thas-Walker flock has the following geometric construction. The twisted cubic $T = \{(u^3, ut^2, tu^2, t^3) \mid t, u \in \mathbb{F},\ (t,u) \neq (0,0)\}$ is contained in the quadratic cone K and the vertex $v = (0,0,0,1)$ of K belongs to T.

Let π_t be the osculating plane to T at the point $(1, t^2, t, t^3)$ of T. As v does not belong to π_t, the conic $C_t = \pi_t \cap K$ is irreducible. Then by [19] $\mathcal{F} = \{C_t \mid t \in \mathrm{GF}(q)\}$ is a flock if and only if $q = -1$ $(mod\ 3)$.

The generalized quadrangle associated with the Fisher-Thas-Walker flock was constructed by Kantor [35] before the relationship between flocks, $q-$clans and generalized quadrangle was pointed out.

9.6 Kantor likeable flocks

Let $q = 5^r$ with $r > 1$, and let k be a fixed non-square in GF(q). Define $J(t) = -kt^5 - k^{-1}t$. If $f(t) = -\frac{1}{3}t^3 + J(t)$ and $g(t) = -t^2$, then $\mathcal{F} = \mathcal{F}(f,g)$ is a likeable flock (see [38]).

9.7 Sporadic transitive flocks

Suppose q is an odd prime power. Let $\mathcal{F} = \mathcal{F}(f, g)$ be a transitive flock. If n is a fixed non-square of $\mathbb{F}$, then there is a polynomial $h(t)$ with coefficients in $\mathbb{F}$ such that

$$g^2(t) + 4tf(t) = nh^2(t)$$

for all $t \in \mathbb{F}$.

The flock $\mathcal{F}$ is called *non-sporadic* when $g^2(t)+4tf(t) = nh^2(t)$ holds as a polynomial identity [1].

We notice that if $\mathcal{F}$ is non-sporadic, the polynomials f, g and h respectively define functions f_r, g_r, h_r of $\mathrm{GF}(q^r)$ to itself such that

$$g_r^2(t) + 4tf_r(t) = nh_r^2(t)$$

Also, $\mathcal{F}(f_r, g_r)$ is a flock of the quadratic cone $X_0X_1 - X_2^2 = 0$ of $\mathrm{PG}(3, q^r)$ if and only if r is odd. Moreover, if $\mathcal{F}$ is a semifield flock, then $\mathcal{F}(f_r, g_r)$ is a semifield flock, and, if $\mathcal{F}$ is a likeable flock, then $\mathcal{F}(f_r, g_r)$ is a likeable flock.

Theorem 9.6 (Bader and Lunardon [3])
If $\mathcal{F}$ is a non-sporadic transitive flock, then $\mathcal{F}$ is isomorphic to a linear flock or to a Kantor-Knuth semifield flock or to a Ganley flock or to a Fisher-Thas-Walker flock or to a Kantor likeable flock.

Theorem 9.7 (Bader, Lunardon and Pinneri [5])
The semifield flock of order $3^5 = 243$ described in §9.4 is sporadic.

10 Translation generalized quadrangles

10.1 Eggs

An *egg*, $\mathcal{E}$, is a partial $(n-1)$-spread of $\mathrm{PG}(2n+m-1, s)$ such that:

a) $\mathcal{E}$ contains $s^m + 1$ elements;

b) every three elements of $\mathcal{E}$ generate a $(3n-1)$-dimensional subspace of $\mathrm{PG}(2n+m-1, s)$;

c) each element X of $\mathcal{E}$ is contained in a $(n+m-1)$-dimensional subspace T_X having no point in common with any element of $\mathcal{E}$ different from X. The subspace T_X is called the *tangent space* of $\mathcal{E}$ at X.

Embed $\mathrm{PG}(2n+m-1, s)$ in $\mathrm{PG}(2n+m, s)$ as a hyperplane, and define an incidence structure $T(\mathcal{E})$ as follows.

Points are:

(i) the points of $\mathrm{PG}(2n+m, s) \setminus \mathrm{PG}(2n+m-1, s)$,

(ii) the $(n+m)$-dimensional subspaces X of $\mathrm{PG}(2n+m, s)$ for which $X \cap \mathrm{PG}(2n+m-1, s)$ is an element of $\mathcal{E}$, and

[1] In [3] sporadic flocks appear under the name *non-hyperelliptic* flocks.

(iii) a new symbol (∞).

Lines are:

(a) the n-dimensional subspaces of $\mathrm{PG}(2n+m, s)$ which are not contained in $\mathrm{PG}(2n+m-1, s)$ and meet $\mathrm{PG}(2n+m-1, s)$ in an element of $\mathcal{E}$, and

(b) the elements of $\mathcal{E}$.

Incidence is defined as follows: A point of type *(i)* is incident only with lines of type *(a)* (here the incidence is that of $\mathrm{PG}(2n+m, s)$). A point of type *(ii)* is incident with all the lines of type *(a)* contained in it and with the unique element of $\mathcal{E}$ contained in it. The point (∞) is incident with no line of type *(a)* and all lines of type *(b)*.

Payne and Thas ([55] 8.7.1) proved that $T(\mathcal{E})$ is a translation generalized quadrangle of order (s^n, s^m) with base point (∞), and conversely every translation generalized quadrangle, $\mathcal{S}$, of order (r, t), with base point x and translation group T, is isomorphic to some $T(\mathcal{E})$ in the following way: fix a point y of $\mathcal{S}$ not collinear with x, and let $L_0, L_1, \ldots, L_t$ be the lines of $\mathcal{S}$ incident with x. For $i = 0, 1, \ldots, t$ let z_i be the point of L_i collinear with y, and denote by M_i the line of $\mathcal{S}$ joining y and z_i. Define $T_i = \{\tau \in T \mid M_i^\tau = M_i\}$, $T_i^* = \{\tau \in T \mid z_i^\tau = z_i\}$, $\mathcal{E} = \{T_i \mid i = 0, 1, \ldots, t\}$.Then T_i and T_i^* are subgroups of T such that T_i has order t, T_i^* has order rt and $T_i < T_i^*$. The *kernel* K of $\mathcal{S}$ is the set of all endomorphisms α of T such that $T_i^\alpha \subset T_i$, $0 \leq i \leq t$. By [55] (8.5.1) K is a field, T is a vector space over the subfield $\mathrm{GF}(s)$ of K, and T_i and T_i^* are $\mathrm{GF}(s)$-vector subspaces for all $i = 0, 1, \ldots, t$. If $r = s^n$ and $t = s^m$, then $\mathcal{E}$ is an egg of $\mathrm{PG}(2n+m-1, s) = \mathrm{PG}(T, \mathrm{GF}(s))$ and $\mathcal{S}$ is canonically isomorphic to $T(\mathcal{E})$. As T is transitive on the points not collinear with x, the construction of $\mathcal{E}$ does not depend on y.

As a corollary of the above construction it has been proved in [5] that

Theorem 10.1
Let $\mathcal{E}_1$, $\mathcal{E}_2$ be two eggs of $\mathrm{PG}(2n+m, s)$*. There is an isomorphism from $T(\mathcal{E}_1)$ to $T(\mathcal{E}_2)$, which maps the point (∞) onto the point (∞) if and only if there is a collineation α of* $\mathrm{PG}(2n+m, s)$ *such that $\mathcal{E}_1^\alpha = \mathcal{E}_2$.* ■

If $\mathcal{E}$ is either an oval of $\mathrm{PG}(2, q)$ or an ovoid of $\mathrm{PG}(3, q)$, then $\mathcal{E}$ is an egg. If this is the case, we will use the standard notations $T_2(\mathcal{E})$ when $\mathcal{E}$ is an oval, and $T_3(\mathcal{E})$ when $\mathcal{E}$ is an ovoid.

Let $\mathcal{E}$ be an egg in $\mathrm{PG}(4n-1, s)$. By 8.7.2 of [55] the $s^{2n}+1$ tangent spaces of $\mathcal{E}$ form an egg $\mathcal{E}^*$ in the dual space to $\mathrm{PG}(4n-1, s)$ with the same parameters: note that the tangent space to $\mathcal{E}^*$ at the element T_X is the element X of $\mathcal{E}$. The translation generalized quadrangle $T(\mathcal{E}^*)$ is called the *translation dual* of $T(\mathcal{E})$.

10.2 Translation generalized quadrangles and semifield flocks

Theorem 10.2 (Johnson [31])
$\mathcal{F}$ is a semifield flock if and only if $\mathcal{S}(\mathcal{F})$ is a translation generalized quadrangle with base line $[A(\infty)]$.

An egg, $\mathcal{E}$, of $\mathrm{PG}(4n-1, s)$ is *good* at the element X of $\mathcal{E}$ if any $(3n-1)$-dimensional subspace containing X and at least two other elements of $\mathcal{E}$ contain exactly s^n+1 elements of $\mathcal{E}$.

Theorem 10.3 (Thas [70])
Let s be an odd prime power. If $\mathcal{E}$ is a good egg of $\Sigma = \mathrm{PG}(4n-1, s)$, then its translation dual $T(\mathcal{E}^*)$ is isomorphic to the point line dual of a semifield flock quadrangle.

If $\mathcal{F}$ is a semifield flock, the associated generalized quadrangle $\mathcal{S}(\mathcal{F})$ is a translation quadrangle with respect to the line $[A(\infty)]$, hence its point-line dual $\mathcal{S}^*(\mathcal{F})$ is isomorphic to the generalized quadrangle $T(\mathcal{E}_\mathcal{F})$ for a suitable egg $\mathcal{E}_\mathcal{F}$. Recall that if $\mathcal{E}$ is an elliptic quadric of $\mathrm{PG}(3,q)$ then so is $\mathcal{E}^*$, hence $T(\mathcal{E})$ and $T(\mathcal{E}^*)$ are isomorphic to the classical generalized quadrangle $Q^-(5,q)$ associated with an elliptic quadric of $\mathrm{PG}(5,q)$ and the corresponding flock is linear. S. E. Payne in [50] has shown that if $\mathcal{F}$ is a Kantor-Knuth flock, then $T(\mathcal{E}_\mathcal{F})$ is isomorphic to its translation dual $T(\mathcal{E}^*_\mathcal{F})$, while if $\mathcal{F}$ is a Ganley flock, then $T(\mathcal{E}^*_\mathcal{F})$ is the so-called Roman generalized quadrangle, which is not isomorphic to the point-line dual of a flock quadrangle.

Theorem 10.4 (Bader, Lunardon and Pinneri [5])
If $\mathcal{F}$ is neither linear nor a Kantor semifield flock, then $T(\mathcal{E}^*_\mathcal{F})$ is not the point-line dual of a flock generalized quadrangle.

Proof. We recall that a flock generalized quadrangle has a collineation group, of order q^2, fixing the special point (∞), the line $[A(\infty)]$ and acting sharply transitively on the points of $[A(\infty)]$ different from (∞), because it is an elation generalized quadrangle with base point (∞). Let $\mathcal{E}_\mathcal{F}$ be the egg of $PG(4n-1, s)$, $s^n = q$, associated with the semifield flock generalized quadrangle $\mathcal{S}(\mathcal{F})$. By duality, the line $[A(\infty)]$ of $\mathcal{S}(\mathcal{F})$ corresponds to the point (∞) of $T(\mathcal{E}_\mathcal{F})$ and the point (∞) of $\mathcal{S}(\mathcal{F})$ corresponds to an element Y of $\mathcal{E}_\mathcal{F}$.

By Theorem 10.1, there is a collineation group G of $PG(4n-1, s)$, of order q^2, stabilizing $\mathcal{E}_\mathcal{F}$, fixing Y, and acting transitively on the elements of $\mathcal{E}_\mathcal{F}$ different from Y. Therefore, G stabilizes $\mathcal{E}^*_\mathcal{F}$, fixes T_Y, and acts transitively on the elements of $\mathcal{E}^*_\mathcal{F} \setminus \{T_Y\}$.

By way of contradiction, suppose that $T(\mathcal{E}^*_\mathcal{F})$ is the point-line dual of a flock generalized quadrangle $\mathcal{S}$.

If $T(\mathcal{E}^*_\mathcal{F})$ is not classical, then the point (∞) of $\mathcal{S}$ corresponds to an element T_Z of $\mathcal{E}^*_\mathcal{F}$ ([67] and [68]). If $T(\mathcal{E}^*_\mathcal{F})$ is classical, then we can suppose that the point (∞) of $\mathcal{S}$ corresponds to an element T_Z of $\mathcal{E}^*_{\mathcal{F}}$ because the collineation group of $T(\mathcal{E}^*_{\mathcal{F}})$ is transitive on the points. By Theorem 6.7 of [68] , $\mathcal{E}_{\mathcal{F}}$ is a good egg at Z. Also, there is a collineation group H of $PG(4n-1, s)$, of order q^2, stabilizing $\mathcal{E}^*_{\mathcal{F}}$ and fixing T_Z. Hence, H stabilizes $\mathcal{E}_{\mathcal{F}}$ and fixes the element Z.

If $Y \neq Z$, the collineation group $\langle H, G\rangle$ of $PG(4n-1, s)$, spanned by H and G, acts 2-transitively on $\mathcal{E}_{\mathcal{F}}$. Hence, $\mathcal{E}_{\mathcal{F}}$ is a good egg at all elements. By Section 8.7 of [55], $T(\mathcal{E}_{\mathcal{F}}) \cong Q^-(5, q)$, and $\mathcal{F}$ is a linear flock.

If $Y = Z$, by Corollary 4.2 of [68], $\mathcal{F}$ is a Kantor semifield flock. ∎

If $\mathcal{F}$ is the sporadic semifield flock of order 243 then $\mathcal{S}(\mathcal{F})$ is not isomorphic to the generalized quadrangle associated with a Kantor-Knuth semifield flock or with a Ganley flock (see [5]). Let $\mathcal{E}_{\mathcal{F}}$ be the egg of PG(19, 3) associated with $\mathcal{F}$. Then the translation dual, $T(\mathcal{E}^*_{\mathcal{F}})$, is not isomorphic to the Roman generalized quadrangle (see [5]).

10.3 Translation ovoids and semifield flocks

Suppose q is **odd.**

An ovoid of $Q(4, q)$ is a set of q^2+1 points which has exactly one point in common with each line of $Q(4, q)$. Using the Klein mapping ([29]) from the lineset of PG$(3, q)$ onto the pointset of the hyperbolic quadric $Q^+(5, q)$ of PG$(5, q)$, with each ovoid $\mathcal{O}$ of $Q(4, q)$ there corresponds a line spread $S(\mathcal{O})$ of PG$(3, q)$, whose lines are totally isotropic with respect to the symplectic polarity of PG$(3, q)$ defined by $Q(4, q)$ via the Klein correspondence. If the translation plane associated with $S(\mathcal{O})$ is a semifield plane, we say that $\mathcal{O}$ is a *translation ovoid* of $Q(4, q)$.

The known translation ovoids of $Q(4, q)$ are ([59]):

(1) the elliptic quadric $Q^-(3, q)$, which is the intersection of $Q(4, q)$ with a non-tangent hyperplane;

(2) the Kantor ovoid constructed using a Knuth semifield ([36] §5);

(3) the Thas-Payne ovoid defined in [72] for $q = 3^r$;

(4) the Penttila-Williams ovoid defined in [59] for $q = 3^5 = 243$.

By [69], [41] and [9] a translation ovoid[2] defines a semifield flock of the quadratic cone of PG$(3, q)$. We briefly review the construction.

Let $\mathcal{F} = \mathcal{F}(f, g)$ be a semifield flock of the quadratic cone K with equation $X_0X_1 - X_2^2 = 0$ and vertex $(0, 0, 0, 1)$.

Denote by $[a, b, c, d]$ the plane of PG$(3, q)$ with equation $aX_0 + bX_1 + cX_2 + dX_3 = 0$. Let δ be the polarity of $PG(3, q)$ defined by the map $(a, b, c, d) \mapsto [a, b, c, d]$. Dualizing by δ, the vertex $(0, 0, 0, 1)$ of K is mapped

[2] In [41] translation ovoids appear as *semifield* ovoids.

to the plane π with equation $X_3 = 0$. The $q+1$ lines of K are mapped to the tangent lines to the conic C of π with equations $4X_0X_1 - X_2^2 = 0 = X_3$. The plane π_t of the flock $\mathcal{F}(f,g)$ is mapped to the point $(t, -f(t), g(t), 1)$ which does not belong to π. As we suppose q is odd, $\mathcal{F}(f,g)$ is a flock if and only if the line of $PG(3,q)$ joining $(t,-f(t),g(t),1)$ and $(u,-f(u),g(u),1)$, $t \neq u$, intersects π in an internal point to C. Then *$\mathcal{F}(f,g)$ is a semifield flock if and only if f and g are additive and $(t,-f(t),g(t),0)$ is an internal point to C for all t in $\mathbb{F}$.* Let $\pi = \mathrm{PG}(V,\mathbb{F})$. If $\mathcal{F} = \mathcal{F}(f,g)$ is a semifield flock, then let $\mathrm{GF}(s)$ be a subfield of $\mathbb{F}$ for which f and g are $\mathrm{GF}(s)-$linear maps of $\mathbb{F}$ to itself. If $q = s^n$, then $L(\mathcal{F}) = \{(t,-f(t),g(t),0) \mid t \in \mathbb{F}\}$ is a $(n-1)-$subspace of $\mathrm{PG}(3n-1,s) = \mathrm{PG}(V,\mathrm{GF}(s))$.

Regarding V as a vector space of dimension $3n$ over $\mathrm{GF}(s)$, each point x of π defines a $(n-1)$-dimensional subspace $P(x)$ of the projective space $\mathrm{PG}(V,\mathrm{GF}(s)) = \mathrm{PG}(3n-1,s)$, and each line l of π defines a $(2n-1)$-dimensional subspace $P(l)$ of $\mathrm{PG}(3n-1,s)$. Then $\mathcal{E} = \{P(x) \mid x \in C\}$ is an egg whose tangent spaces are the subspaces $P(l)$ where l is a tangent line to C. Moreover $T(\mathcal{E})$ is a generalized quadrangle isomorphic to $T_2(C) \simeq Q(4,s^n)$ ([55] 8.5 and 8.7.1).

Let γ be the polarity of $\mathrm{PG}(3n-1,s)$ defined by the quadratic form $tr(4X_0X_1 - X_2^2)$. Then $\mathcal{E}^\gamma = \mathcal{E}^* = \{P(l) \mid l \textit{ is tangent to } C\}$. Let U be the polar space of $L(\mathcal{F})$ with respect to γ. Let x be a fixed point of $\mathrm{PG}(3n,s)$ not incident with $\mathrm{PG}(3n-1,s)$ (i.e. a point of type (i) of $T(\mathcal{E})$). Define

$$O(\mathcal{F}) = (\langle U, x\rangle \setminus U) \cup \{\infty\}$$

Theorem 10.5 (Thas [68])
$O(\mathcal{F})$ is an ovoid of $T(\mathcal{E}) \simeq Q(4,q)$.

Theorem 10.6 (Lunardon [41], Bloemen [9])
If $\mathcal{F}$ is a semifield flock then $O(\mathcal{F})$ is a translation ovoid.
If $\mathcal{O}$ is a translation ovoid, then there is a semifield flock $\mathcal{F}$ such that $\mathcal{O}$ and $O(\mathcal{F})$ are isomorphic.

Theorem 10.7 (Lunardon [41])
If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two semifield flocks, then $O(\mathcal{F}_1)$ is isomorphic to $O(\mathcal{F}_2)$ if and only if $\mathcal{F}_1$ is isomorphic to $\mathcal{F}_2$.

Moreover:

- $O(\mathcal{F})$ is an elliptic quadric $\Leftrightarrow$ $\mathcal{F}$ is linear ([69] Appendix II);
- $O(\mathcal{F})$ is a Kantor ovoid $\Leftrightarrow$ $\mathcal{F}$ is a Kantor-Knuth semifield flock ([69] Appendix II);
- $O(\mathcal{F})$ is a Thas-Payne ovoid $\Leftrightarrow$ $\mathcal{F}$ is a Ganley flock ([69] Appendix II);
- $O(\mathcal{F})$ is the Penttila-Williams ovoid $\Leftrightarrow$ $\mathcal{F}$ is the sporadic flock of order 243 ([5]).

10.4 Semifield flocks of order s^3

Suppose $\mathcal{F} = \mathcal{F}(f, g)$ is a semifield flock of order $q = s^3$ such that f and g are GF(s)-linear maps. Then there are three possibilities:

(1) $L(\mathcal{F})$ is a point $\Leftrightarrow$ $\mathcal{F}$ is linear;

(2) $L(\mathcal{F})$ is contained in a line $\Leftrightarrow$ $\mathcal{F}$ is a Kantor-Knuth semifield flock;

(3) $L(\mathcal{F})$ is a subplane of order s of $\pi = \mathrm{PG}(2, s^3)$ whose points are all internal to the non-singular conic C.

The Ganley flock of order 27 is the only known example with property (3).

Conversely suppose $L(\mathcal{F})$ is a subplane of $\pi = \mathrm{PG}(2, s^3)$. For $s = 3$, then we can prove, with the aid of a computer, that $\mathcal{F}$ is a Ganley flock. If $s \neq 3$, then the flock $\mathcal{F}$ is sporadic. Therefore the existence of a sporadic flock of order s^3, $s \neq 3$, is equivalent to finding *a subplane of* $\pi = \mathrm{PG}(2, s^3)$, *whose points are all internal to* C .

11 BLT-sets

11.1 Derivation

Let $\mathcal{F}_\infty = \{C_t \mid t \in \mathbb{F}\}$ be a flock of the quadratic cone K_∞ with vertex x_∞ of $\mathrm{PG}(3, q)$, with q **odd**. As usual, the plane of C_t is denoted by $\pi_t, t \in \mathbb{F}$. Let K_∞ be embedded in the non-singular quadric $Q(4, q)$ of $\mathrm{PG}(4, q)$. The polar line of π_t with respect to $Q(4, q)$ is denoted by L_t. For $t \in \mathbb{F}$, let

$$L_t \cap Q(4, q) = \{x_\infty, x_t\}.$$

Then no point of $Q(4, q)$ is collinear with all three of x_∞, x_t, x_u, with $t, u \in \mathbb{F}$, and $t \neq u$. Rephrasing Theorem 1 of [6] it is also true that no point of $Q(4, q)$ is collinear with all three of x_t, x_u, x_v, with $t, u, v \in \tilde{\mathbb{F}}$, and $t \neq u \neq v \neq t$. Such a set B of $q+1$ points of $Q(4, q)$ is called a BLT-*set* of points of $Q(4, q)$.

For all $t, u \in \tilde{\mathbb{F}}$ put $\pi_{t,u} = x_t^\perp \cap x_u^\perp$ where $\perp$ is the polarity defined by $Q(4, q)$. Then $C_{t,u} = \pi_{t,u} \cap Q(4, q)$ is a non-singular conic. Note that $C_{\infty,u} = C_u$ for all $u \in \mathbb{F}$.

Theorem 11.1 (Bader, Lunardon and Thas [6])
For all $t \in \tilde{\mathbb{F}}$ let K_t be the quadratic cone which is the intersection of $x_t^\perp$ and $Q(4, q)$. Then $\mathcal{F}_t = \{C_{t,u} \mid u \in \tilde{\mathbb{F}}, u \neq t\}$ is a flock of K_t.

The flocks $\mathcal{F}_t$ are called the flocks *derived* from $\mathcal{F}_\infty$.

Theorem 11.2 (Payne and Rogers [54])
The generalized quadrangles $\mathcal{S}(\mathcal{F}_t)$ and $\mathcal{S}(\mathcal{F}_u)$ are isomorphic for all $t, u \in \tilde{\mathbb{F}}$.

If $\mathcal{F}_\infty$ is a transitive flock, then $\mathcal{F}_t \simeq \mathcal{F}_u$ for all $t, u \in \mathbb{F}$ (see [6]), but in general $\mathcal{F}_\infty$ is not isomorphic to $\mathcal{F}_t$ for $t \neq \infty$.

Theorem 11.3 (Johnson, Lunardon and Wilke [34])
If $\mathcal{F}_\infty$ is a transitive flock and $\mathcal{F}_t \simeq \mathcal{F}_\infty$ for all t, then $\mathcal{F}_\infty$ is linear or a Kantor-Knuth semifield flock or a Fisher-Thas-Walker flock.

The BLT-set associated with a Kantor-Payne flock produces two non-isomorphic flocks ([32] and [51]).

Two other examples of classes of BLT-sets are known: the BLT-sets associated with the Fisher flocks and a new class of BLT-sets recently constructed by Penttila [57]. We give a short description of these two examples.

11.2 Fisher BLT-sets

Let $Q(4,q)$ be the quadric $X_0X_1 - X_2^2 + X_3X_4 = 0$. Denote by Π and Π' the planes $X_0 - \alpha X_1 = X_2 = 0$ and $X_0 + \alpha X_1 = X_3 + cX_4 = 0$ respectively, where $-\alpha$ is a non-square in $\mathbb{F}$ and $-c$ is a square different from 0.

The polar line l (resp. l') of Π (resp. Π') with respect to $Q(4,q)$ is contained in Π' (resp. Π). The lines l and l' are exterior lines with respect to $Q(4,q)$. Denote by ω an involutorial collineation of $Q(4,q)$ interchanging the lines l and l' and fixing the point $(0,0,0,-c,1)$. Then ω interchanges Π and Π'.

Denote by ζ a generator of the multiplicative group of $\mathrm{GF}(q^2)$ and put $z = \zeta^{q-1}$. Define

$$B_1 = \{(\alpha t, t, 0, -\alpha t^2, 1) \mid t^2 - \frac{4}{N(1+z)} \textit{ is a square in } \mathbb{F}\}.$$

Then B_1 is contained in the conic $\Pi \cap Q(4,q)$, and $B = B_1 \cup B_1^\omega$ is a BLT-set ([2], [56], [6]). The flocks defined by B are all isomorphic to the Fisher flock constructed in [19].

11.3 Penttila BLT-sets

Let $V = \{(x,y,a) \mid x,y \in \mathrm{GF}(q^2), a \in \mathbb{F}\}$, and define a quadratic form Q over V by

$$Q(x,y,a) = N(x) + N(y) + a^2$$

where $N(x) = x^{1+q}$. Then Q is non-degenerate. Let $Q(4,q)$ be the non-singular quadric of $\mathrm{PG}(4,q) = \mathrm{PG}(V,\mathbb{F})$ with equation $Q(x,y,a) = 0$.

Suppose q is a prime power such that $q \equiv \pm 1$ $(mod.\ 10)$.

Let x, y be elements of $\mathrm{GF}(q^2)$ with $N(x) = -\frac{4}{5}$ and $N(y) = -\frac{1}{5}$. If

$$B = \{(\beta^2 x, \beta^3 y, 1) \mid N(\beta) = 1\},$$

then B is a BLT-set of $Q(4,q)$ (see [57]).

Let α be an element of $\mathrm{GF}(q^2)$ of order $q+1$ with $N(\alpha) = 1$. Then the cyclic group of order $q+1$ defined by $(x, y, a) \mapsto (\alpha^2 x, \alpha^3 y, a)$ preserves $Q(4, q)$, and it is transitive on B.

12 Final remarks

It has been proved in [6], [56], [51], [14], and [57] that the known families of BLT-sets produce non-isomorphic generalized quadrangles. Penttila and Royle have determined in [58], with the aid of a computer, all the BLT-sets for small values of q : there are some examples not belonging to any of the listed families.

Also, we have not discussed here the Fundamental Theorem of $q-$clan Geometry because we are not investigating collineation groups of flock quadrangles: the reader can find it in [4] and [52].

Finally, we recall that all known results on the Subiaco flocks have been collected in Payne's excellent Subiaco Notebook [53].

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