Ovoids and spreads of polar spaces and generalized polygons

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Intensive Course on Galois Geometry and Generalized Polygons

Contents

Part I: Ovoids and spreads of generalized polygons

1 Generalized polygons: definition and properties

1.1 Definition of generalized polygons

Let $n \geq 1$ be a natural number. A *weak generalized n-gon* is a geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ with $\mathcal{P} \neq \emptyset, \mathcal{B} \neq \emptyset$, such that the following two axioms are satisfied:

(GP1) $\mathcal{S}$ contains no ordinary k-gon (as a subgeometry), for $2 \leq k < n$.

(GP2) Any two elements $x, y \in \mathcal{P} \cup \mathcal{B}$ are contained in some ordinary n-gon in $\mathcal{S}$, a so-called *apartment*.

A *generalized n-gon* is a weak generalized n-gon $\mathcal{S}$ which satisfies also the following axiom:
(GP3) there exists an ordinary $(n+1)$-gon in $\mathcal{S}$.
Instead of 4-gons, 5-gons, 6-gons, 8-gons, n-gons we shall also speak of quadrangles, pentagons, hexagons, octagons, and polygons.
We will use the distance function of the corresponding incidence graph. Hence, from now on, we also use the notation Γ (instead of $\mathcal{S}$) for a generalized polygon. This way, we might write $\Gamma(v)$ for the set of elements incident with $v \in \mathcal{P}$, or $\Gamma_i(v)$ for the set of elements of Γ at distance i from v.
Note that the dual of a (weak) generalized n-gon is again a (weak) generalized n-gon. Obviously, a geometry can be a weak generalized n-gon for at most one value of n. Two elements of a weak generalized n-gon are said to be *opposite*, if they have the maximal distance n. Two flags are *opposite* if each element of one flag is opposite an element of the other flag. If for two points x, y there exists a unique point collinear with both, then we denote that point by $x \bowtie y$.
The concept of generalized polygons is due to Tits [60].

The following lemma says that the additional axiom (iii) holds precisely if Γ is thick. We shall see in Corollary 1.2 that every generalized n-gon has an order (s, t) (with $s, t \geq 2$).

Lemma 1.1
A weak generalized n-gon Γ is a generalized n-gon if and only if Γ is thick.

Proof. Assume that Γ is thick, and let $x_0, x_1, \ldots, x_{2n} = x_0$ be the points and lines of an ordinary n-gon, arranged in a closed path of length $2n$. Then $\delta(x_0, x_n) = n$ and $\delta(x_1, x_n) = n - 1 = \delta(x_{2n-1}, x_n)$. As Γ is thick, we can choose $y \in \Gamma(x_n) \setminus \{x_{n-1}, x_{n+1}\}$. Then $\delta(x_0, y) = n \pm 1$, hence $\delta(x_0, y) = n - 1$. Let $x_0, \ldots, y', y$ be a path of length $n - 1$, and choose

$z \in \Gamma(y) \backslash \{y', x_n\}$. Since $\delta(x_1, y) = n$, we have $\delta(x_1, z) = n-1$. Thus, a path from x_1 to z of length $n-1$ completes the path $z, y, x_n, x_{n+1}, \dots, x_{2n-1}, x_0$ to an ordinary $(n+1)$-gon.
Conversely, let $x_0, x_1, \dots, x_{2n+2} = x_0$ be an ordinary $(n+1)$-gon, arranged in a closed path of length $2n+2$. Then $\delta(x_0, x_{n+1}) \equiv n+1 \pmod 2$, hence $\delta(x_0, x_{n+1}) < n$. Let $x_0, x', \dots, x_{n+1}$ be a shortest path from x_0 to x_{n+1}. If $x' \in \{x_1, x_{2n+1}\}$, then we have an ordinary k-gon with $k < n$, a contradiction. Thus $\Gamma(x_0)$ contains the three distinct elements x_1, x_{2n+1}, x'. Of course, this argument shows that $|\Gamma(x_i)| \geq 3$ for $0 \leq i \leq 2n+1$.
Now let x be an arbitrary point or line of Γ. Then $\delta(x, x_i) = n$ for some $i \in \{0, 1, \dots, 2n+1\}$ (otherwise, we have $\delta(x, x_i) = \delta(x, x_{i+2}) \leq n-2$ for some i, hence an ordinary k-gon with $k < n$, a contradiction). The map $\theta : \Gamma(x) \to \Gamma(x_i)$ mapping $w \in \Gamma(x)$ to the unique element of $\Gamma(x_i)$ not opposite w is clearly a bijection. Thus $|\Gamma(x)| = |\Gamma(x_i)| \geq 3$. ∎

For two non-opposite elements x, y in a (weak) generalized polygon, there exists a unique element u incident with x and nearest to y. We will denote $u = \mathrm{proj}_x y$ and call it the *projection of y onto x*.

1.2 Examples of generalized polygons

The weak generalized 2-gons are precisely the geometries $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ with trivial incidence relation $\mathrm{I} = \mathcal{P} \times \mathcal{L}$; their incidence graphs are the complete bipartite graphs.
Thus generalized n-gons are interesting geometries only for $n \geq 3$.
The generalized 3-gons are precisely the projective planes. This important observation can be easily proved: axiom (GP2) implies that two distinct points (lines) are incident with at least one line (point), axiom (GP1) says that they are uniquely determined, and (GP3) is the usual non-degeneracy axiom. So we see that generalized polygons can (and should) be regarded as natural generalizations of projective planes.
Generalized quadrangles are close relatives of projective planes. See the previous lectures on generalized quadrangles.
Define the double of a geometry $\mathcal{S}$ as the geometry $2\mathcal{S}$ the points of which are the points and lines of $\mathcal{S}$, the lines of which are the flags of $\mathcal{S}$, and the incidence relation of which is the obvious one.
Each ordinary n-gon is also a weak generalized n-gon of order $(1,1)$. Furthermore, the double 2Γ of a weak generalized n-gon Γ is a weak generalized $2n$-gon, which is not thick (each line of 2Γ carries precisely 2 points). But one can do better: there exist (thick) generalized n-gons for every n. Examples are provided by a free construction process due to Tits [62].
On the other hand, there are several results which indicate that the more interesting generalized n-gons have $n = 3, 4, 6$, or 8, see Theorem 1.3. Indeed, it seems that for $n = 5, 7$ or $n \geq 9$, no generalized n-gon has been

constructed yet without invoking the free construction process of Tits at some stage.

1.3 Some important results

Theorem 1.2
Every generalized n-gon has an order (s,t) (with $s,t \geq 2$); if n is odd, then $s = t$. ■

Theorem 1.3 (Feit & Higman [23])
Let Γ be a weak generalized n-gon of order (s,t) with $n \geq 3$. If Γ is finite, then one of the following holds:

(i) $s = t = 1$, *and Γ is an ordinary n-gon;*

(ii) $n = 3$, $s = t > 1$, *and Γ is a projective plane;*

(iii) $n = 4$ *and the number*
$$\frac{st(1+st)}{s+t}$$
is an integer;

(iv) $n = 6$, *and if $s,t > 1$, then st is a perfect square. In that case, we put $u = \sqrt{st}$ and $w = s+t$. The number*
$$\frac{u^2(1+w+u^2)(1 \pm u + u^2)}{2(w-u)}$$
is an integer for both choices of signs;

(v) $n = 8$, *and if $s,t > 1$, then $2st$ is a perfect square; in particular $s \neq t$. If we put $u = \sqrt{\frac{st}{2}}$ and $w = s+t$, then the number*
$$\frac{u^2(1+w+2u^2)(1+2u^2)(1 \pm 2u + 2u^2)}{2(w \pm 2u)}$$
is an integer for both choices of signs;

(vi) $n = 12$ *and $s = 1$ or $t = 1$.* ■

Theorem 1.4
Let Γ be a finite generalized n-gon of order (s,t), $n \geq 4$. Then one of the following holds.

(i) **(Higman [27]).** $n = 4$ *and $s \leq t^2$; dually $t \leq s^2$;*

(ii) **(Haemers & Roos [25]).** $n = 6$ *and $s \leq t^3$; dually $t \leq s^3$;*

(iii) **(Higman [27]).** $n = 8$ *and $s \leq t^2$; dually $t \leq s^2$.* ■

A very important corollary to the previous results is the following fact.

Corollary 1.5
Finite generalized n-gons exist only for $n \in \{3, 4, 6, 8\}$.

Theorem 1.6
Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite (weak) generalized n-gon of order (s, t), with $n \in \{3, 4, 6, 8\}$, then we have

$$|\mathcal{P}| = \begin{cases} s^2 + s + 1 & \text{if } n = 3, \\ (1+s)(1+st) & \text{if } n = 4, \\ (1+s)(1+st+s^2t^2) & \text{if } n = 6, \\ (1+s)(1+st)(1+s^2t^2) & \text{if } n = 8. \end{cases}$$

Dually,

$$|\mathcal{L}| = \begin{cases} s^2 + s + 1 & \text{if } n = 3, \\ (1+t)(1+st) & \text{if } n = 4, \\ (1+t)(1+st+s^2t^2) & \text{if } n = 6, \\ (1+t)(1+st)(1+s^2t^2) & \text{if } n = 8. \end{cases}$$

■

Clearly, a generalized polygon of order (s, t) is finite if and only if s and t are finite.
Without proof we mention the following results (we omit the results on quadrangles, which can be found in the previous chapter).
All projective planes of order ≤ 10 are classified. For the orders $2, 3, 4, 5, 7$ and 8, there is just one projective plane in each case, see e.g. Stevenson [48], page 71. For order 9 there are exactly 4 isomorphism classes of projective planes, see Lam, Kolesova & Thiel [37] (a computer result). For the orders 6 and 10, there are no projective planes at all.
A finite generalized hexagon of order (s, t) with $s = 2$ has necessarily $t = 2$ or $t = 8$. In the first case there are exactly two mutually dual examples, namely $\mathsf{H}(2)$ and its dual $\mathsf{H}(2)^D$ (for the construction, see next section); in the second case there is a unique hexagon which is the dual of the triality hexagon $T(8, 2)$ (see Cohen & Tits [13]). Nothing on uniqueness is known for other orders. All known examples of finite generalized hexagons have, up to duality, order (s, t) with either $s = t$, or $s = t^3$.
For generalized octagons, nothing is known concerning uniqueness and non-existence. The smallest example has order $(2, 4)$ and dually $(4, 2)$. It is believed to be unique, but that is not yet proved. Actually, all known examples (the Ree–Tits octagons) have, up to duality, order $(2^{2e+1}, 2^{4e+2})$.

1.4 A class of generalized hexagons

We will now construct a class of generalized hexagons $\mathsf{H}(q)$ of order (q, q), q any prime power. Therefore, we consider the quadric $\mathsf{Q}(6, q)$ in $\mathsf{PG}(6, q)$

given by the equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$. The points of $\mathsf{H}(q)$ are all points of $\mathsf{Q}(6,q)$. The lines of $\mathsf{H}(q)$ are certain lines of $\mathsf{Q}(6,q)$. For our purposes, the exact definition does not matter[1]. We only need some properties of this hexagon.

Theorem 1.7

(*i*) *Two points of $\mathsf{H}(q)$ are opposite in $\mathsf{H}(q)$ if and only if they are not collinear on $\mathsf{Q}(6,q)$.*

(*ii*) *The lines of $\mathsf{H}(q)$ through any point x of $\mathsf{H}(q)$ are all lines of $\mathsf{Q}(6,q)$ through x lying in a certain plane[2] $x^\perp$ of $\mathsf{Q}(6,q)$.*

(*iii*) *The order of $\mathsf{H}(q)$ is (q,q).* ■

We can now prove the following interesting regularity property of $\mathsf{H}(q)$:

Theorem 1.8
Let x be any point of $\Gamma := \mathsf{H}(q)$ and let y,z be two points opposite x. Let $i = 2,3$. If $S_i(x;y,z) := \Gamma_i(x) \cap \Gamma_{6-i}(y) \cap \Gamma_{6-i}(z)$ contains at least two elements, then it contains $q+1$ elements.

Proof. We first prove the statement for $i = 2$. The set $S_2(x;y,y)$ is contained in the intersection of $x^\perp$ with $y^{\perp\!\!\!\perp}$ (the latter is the notation for the set of points of $\mathsf{H}(q)$ not opposite y; it is hence the set of points of $\mathsf{Q}(6,q)$ lying in the tangent hyperplane at y). Consequently $S_2(x;y,y)$ is equal to that intersection (because it contains exactly $q+1$ points). Similarly $S_2(x;z,z) = x^\perp \cap z^{\perp\!\!\!\perp}$. Now clearly $S_2(x;y,z) = S_2(x;y,y) \cap S_2(x;z,z)$, and the result follows easily.
Now let $i = 3$. Suppose that $S_3(x;y,z)$ contains at least two elements L and M. We must show that every line N at distance 3 from both x and z is also at distance 3 from y. We may assume that $y \neq z$. Let u and w be the points on N at distance 2 from x and z, respectively. By the previous paragraph, we have $u \in S_2(x;y,z)$. So $\delta(y,u) = 4$. Similarly $\delta(y,w) = 4$. This is only possible if $\delta(y,N) = 3$. The result follows. ■

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[1] The lines of $\mathsf{H}(q)$ are the lines of $\mathsf{Q}(6,q)$ whose *Grassmannian coordinates* satisfy the equations $p_{01} = p_{36}$, $p_{12} = p_{34}$, $p_{20} = p_{35}$, $p_{03} = p_{56}$, $p_{13} = p_{64}$ and $p_{23} = p_{45}$.

[2] If x has coordinates $(a_0,a_1,a_2,a_3,a_4,a_5,a_6)$, then the plane $x^\perp$ has equations (and one must choose 4 independent equations out of the following list of eight):

$$\begin{aligned}
a_1X_0 - a_0X_1 + a_6X_3 - a_3X_6 &= 0\\
a_2X_0 - a_0X_2 - a_5X_3 + a_3X_5 &= 0\\
a_2X_1 - a_1X_2 + a_4X_3 - a_3X_4 &= 0\\
a_3X_3 - a_0X_4 - a_1X_5 - a_2X_6 &= 0\\
a_3X_0 - a_0X_3 + a_6X_5 - a_5X_6 &= 0\\
a_3X_1 - a_1X_3 - a_6X_4 + a_4X_6 &= 0\\
a_3X_2 - a_2X_3 + a_5X_4 - a_4X_5 &= 0\\
a_4X_0 + a_5X_1 + a_6X_2 - a_3X_3 &= 0
\end{aligned}$$

The property in the previous theorem is called *distance-i-regularity* and we will refer to it as such.
Finally, we mention the following important property. It can be proved geometrically using the previous theorem. It can also be checked algebraically using the definitions of the lines of $\mathsf{H}(q)$. We omit the proof.

Theorem 1.9
Every pair of opposite points x, y of $\Gamma = \mathsf{H}(q)$ is contained in a (necessarily unique) weak subhexagon $\Gamma(x, y)$ of order $(1, q)$.

It is easy to see that every weak generalized hexagon of order $(1, q)$ is the double of a projective plane of order q. For the subhexagons of $\Gamma = \mathsf{H}(q)$, this plane is the usual Desarguesian one over the Galois field $\mathsf{GF}(q)$. We will denote by $\Gamma^+(x, y)$ the set of points of $\Gamma(x, y)$ at distance 0 or 4 from x. The structure $(\Gamma^+(x, y), \Gamma^+(y, x), \perp)$ is a projective plane whose double is isomorphic to $\Gamma(x, y)$.
There is one other class of finite generalized hexagons known; the members of it have order (q, q^3) (or dually (q^3, q)), but it will be of no importance to us, since we will show that they cannot have ovoids, nor spreads.

2 General constructions of ovoids and spreads; polarities

2.1 Definitions

Let Γ be a generalized n-gon and let $\mathcal{O}$ be a set of points of Γ. Then we call $\mathcal{O}$ an *ovoid* of Γ if all points of $\mathcal{O}$ are mutually opposite, and if every element of Γ lies at distance at most $n/2$ from a point of $\mathcal{O}$. It follows that n is even since otherwise all elements of Γ lie at distance at most $n/2$ from a single point, a contradiction. Dually, one defines a *spread*.
Again let $\Gamma = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a generalized n-gon. A *polarity* is an involution in $\mathcal{P} \cup \mathcal{B}$ mapping $\mathcal{P}$ to $\mathcal{B}$ and vice versa, and preserving incidence. An *absolute point* is a point which is mapped onto a line incident with it under the given polarity. Dually, one defines an *absolute line*.

2.2 Some easy properties

Theorem 2.1
A set of mutually opposite points in a finite generalized $2m$-gon Γ of order (s, t) is an ovoid if and only if it contains

(i) $1 + st$ *elements for* $m = 2$;

(ii) $\dfrac{(1+s)(1+st+s^2t^2)}{1+s+st}$ *elements for* $m = 3$;

(iii) $1+s^2t^2$ *elements for* $m=4$.

A set of mutually opposite points in a finite generalized $2m$*-gon* Γ *of order* s *is an ovoid if and only if it contains* $1+s^m$ *elements.* ■

Proof. The last assertion follows from the previous one by putting $s=t$. So we only have to prove (i), (ii) and (iii).
Suppose first that $\mathcal{O}$ is an ovoid. If $m=2$, then there are $|\mathcal{O}|\cdot(1+t)$ lines at distance 1 from some element of $\mathcal{O}$. Since every line arises in this way, we have $|\mathcal{O}|\cdot(1+t)=(1+t)(1+st)$. If $m=3$, then there are $|\mathcal{O}|$ points at distance 0 from some element of $\mathcal{O}$ and $|\mathcal{O}|\cdot(1+t)s$ points at distance 2 from some (unique) element of $\mathcal{O}$. Again, no other points are available and hence $|\mathcal{O}|\cdot(1+s+st)=(1+s)(1+st+s^2t^2)$. If $m=4$, then there are $|\mathcal{O}|\cdot(1+t)$ lines at distance 1 from a unique element of $\mathcal{O}$ and there are $|\mathcal{O}|\cdot(1+t)st$ lines at distance 3 from a unique element of $\mathcal{O}$. We obtain $|\mathcal{O}|\cdot(1+t)(1+st)=(1+t)(1+st)(1+s^2t^2)$. Hence the result.
Now suppose $\mathcal{O}$ is a set of mutually opposite points of the given size. For m even, we count the number of lines at distance $<m$ from some (unique) point of $\mathcal{O}$; for m odd, we count the number of points at distance $<m$ from some (unique) element of $\mathcal{O}$. This counting is in fact carried out in the previous paragraph and we always obtain the full number of lines (m even) or points (m odd) of Γ. Hence all lines, respectively, points lie at distance $<m$ from some point of $\mathcal{O}$. Hence every line lies at distance $\leq m$ from an element of $\mathcal{O}$. ■

The existence problem of ovoids in generalized polygons seems only to be interesting in the finite case. For instance, for infinite quadrangles of order (s,t), the method of transfinite induction (in the countable case this is just ordinary induction) guarantees the existence of ovoids and spreads if $s=t$, see Cameron [11].

Theorem 2.2
Let ρ *be a polarity of a generalized* n*-gon* Γ. *If* $n=2m$ *is even, then the set of all absolute points with respect to* ρ *forms an ovoid of* Γ.

Proof. Let x and y be two absolute points and suppose they are not opposite each other. Since $\delta(x,y)=\delta(x^\rho,y^\rho)$ and $x \mathrm{\ I\ } x^\rho$ and $y \mathrm{\ I\ } y^\rho$, we may assume upon interchanging x and y that x^ρ is the projection of y onto x. Indeed, otherwise $\delta(x,y)=n$ or $\delta(x,y)=n-1$. The first possibility contradicts our assumption, the second implies that n is odd. Let p be the projection of y onto x^ρ, then $x \mathrm{\ I\ } p^\rho$. But $\delta(x,y)-2=\delta(p,y)=\delta(p^\rho,y^\rho)=\delta(p^\rho,x^\rho)+\delta(x^\rho,y^\rho)=2+\delta(x,y)$, a contradiction. Hence all absolute points are mutually opposite.
Let v be any element of Γ. Since n is even, $\delta(v,v^\rho)<n$. Now the projection v_1 of v onto v^ρ must be mapped onto the projection v_1' of v^ρ onto v; the

projection of v_1 onto v_1' must be mapped onto the projection of v_1' onto v_1, etc. Continuing like this, remembering that $\delta(v, v^\rho)$ is odd, we obtain an element w, $\delta(v, w) + \delta(w, v^\rho) = \delta(v, v^\rho) < n$, incident with w^ρ. Hence either w or w^ρ is an absolute point. Suppose that w is a point. Since $\delta(v, w) = \delta(v^\rho, w^\rho)$ and $\delta(v, w) + \delta(v^\rho, w^\rho) = \delta(v, v^\rho) \pm 1$, we conclude that $\delta(v, w) \leq m$. The theorem is proved. ∎

In the finite case, there is an obvious restriction on the order (s, t) of a generalized polygon to admit a polarity. Indeed, s must certainly be equal to t, since a self-polar polygon is in particular self-dual. In fact, one can show more:

Theorem 2.3 (Payne [42], Ott [40])
If there is a polarity in a finite generalized $2m$-gon Γ of order (s, t), then $s = t$ and ms is a perfect square.

The proof of this uses some elementary eigenvalue technique; see Van Maldeghem [63] for a uniform treatment.
A similar technique is used to show the following result on finite projective planes, which we include for completeness:

Theorem 2.4
Suppose that Γ is a finite projective plane of order q and θ is a polarity of Γ. Then the number N_θ of absolute points satisfies $q + 1 \leq N_\theta \leq q\sqrt{q} + 1$. If $N_\theta = q + 1$, then the set of absolute points is an oval <u>if</u> q is odd, and all absolute points are collinear <u>if</u> q is even; if $N_\theta = q\sqrt{q} + 1$, then the set of absolute points furnished with the sets of points incident with the secant lines forms a unital. Also, if q is not a square, then $N_\theta = q + 1$. ∎

For a proof, see for instance Hughes & Piper [29].
Another general construction of ovoids applies to quadrangles which are subquadrangles of others. We state the theorem in full generality.

Theorem 2.5
Let Γ be a subquadrangle of a generalized quadrangle Γ' having the property that $\Gamma(L) = \Gamma'(L)$, for every line of Γ. Let p be a point of Γ' outside Γ. Then the set of points of Γ collinear with p in Γ' constitutes an ovoid $\mathcal{O}$ of Γ.

Proof. Each line of Γ is incident with a unique point collinear with p; hence $\mathcal{O}$ is an ovoid of Γ. ∎

Let Γ have order (s, t) and let Γ' have order (s', t'), with Γ a subquadrangle of Γ'. The situation of the previous theorem occurs in the finite case if and only if $s = s'$. Also, one can show that, whenever Γ' is a classical quadrangle (or more generally, a so-called *Moufang quadrangle*), then the

ovoid $\mathcal{O}$ admits a doubly transitive group inside the automorphism group of Γ.
Finally, we present some general nonexistence results.

Theorem 2.6
No finite generalized quadrangle of order (s, s^2) has an ovoid. Also, no finite generalized hexagon of order (s, s^3) or (s^3, s) has an ovoid.

Proof. Let Γ be a generalized quadrangle of order (s, s^2) with an ovoid $\mathcal{O}$. Let $x, y \in \mathcal{O}$ be two fixed points. We count, in two ways, the pairs (u, z) with $z \in \mathcal{O} \setminus \{x, y\}$ and $u \in \{x, y, z\}^\perp$. Since $|\{x, y, z\}^\perp| = s + 1$ (see the chapter on generalized quadrangles), we obtain $(s^2 + 1)(s^2 - 1) = (s^3 - 1)(s + 1)$, a contradiction.
Now consider an ovoid in a finite generalized hexagon of order (s, t). If $t = s^3$ and if the number

$$\frac{(1 + s)(1 + st + s^2t^2)}{1 + s + st}$$

is an integer, then (carrying out the division) $(1 + s)(1 + s + s^2)$ is zero (this is the remainder if $s > 2$; for $s = 2$ the above number is clearly not an integer), a contradiction. If $s = t^3$, then a similar calculation shows that the remainder $2t^3 + t^2 - 2t + 2$ must be zero. But this number is always positive for non-negative t. The theorem is proved. ∎

3 Existence and nonexistence of ovoids and spreads in finite classical polygons

3.1 Finite generalized octagons

We start by mentioning that almost nothing is known for the classical finite generalized octagons. A computer result of Coolsaet (unpublished) tells us that there is no ovoid in the smallest known octagon (of order $(2, 4)$).

3.2 Ovoids arising from polarities in finite generalized polygons

For finite classical generalized n-gons of order (s, t), the necessary conditions of Theorem 2.3 are also sufficient.

Theorem 3.1
A classical quadrangle Γ is self-dual if and only if it is isomorphic to a symplectic quadrangle $\mathsf{W}(q)$, q even. Also, $\mathsf{W}(q)$ admits a polarity if and only if $2q$ is a perfect square, i.e., $q = 2^{2e+1}$.

Proof. For the statement about the self-duality, see the chapter on generalized quadrangles. We now supose that $q = 2^{2e+1}$. We must show that there is a polarity of $\mathsf{W}(q)$. Therefore, we first find a way of presenting $\mathsf{W}(q)$ where the points and lines play a similar role. This can be done by coordinatization. We will not give the general theory, but only coordinatize in this special case. This is done as follows. Let the symplectic polarity defining $\mathsf{W}(q)$ be given by the alternating bilinear form $B(x,y) = x_0y_1 + x_1y_0 + x_2y_3 + x_3y_2$ (the image of the point $y = (y_0, y_1, y_2, y_3)$ is the plane with equation $B(X,y) = 0$), and redefine coordinates on the points as follows:

$$\begin{array}{rcl} (1,0,0,0) & \mapsto & (\infty) \\ (a,0,1,0) & \mapsto & (a) \\ (b,0,k,1) & \mapsto & (k^{2^e}, b) \\ (l+aa', 1, a', a) & \mapsto & (a, l^{2^e}, a') \end{array}$$

We define coordinates for the lines as follows:

$$\begin{array}{rcl} \langle (1,0,0,0),(0,0,1,0)\rangle & \mapsto & [\infty] \\ \langle (1,0,0,0),(0,0,k,1)\rangle & \mapsto & [k^{2^e}] \\ \langle (a,0,1,0),(l,1,0,a)\rangle & \mapsto & [a, l^{2^e}] \\ \langle (b,0,k,1),(k',1,b,0)\rangle & \mapsto & [k^{2^e}, b, k'^{2^e}] \end{array}$$

Using the bilinear form given above, it is easy to check that all the above lines are indeed absolute lines of the symplectic polarity. A counting argument shows that we have all of them. Now we see that, using the new coordinates, the point (∞) is incident only with $[\infty]$ and $[k]$, for all $k \in \mathsf{GF}(q)$. Dually, $[\infty]$ is incident only with (∞) and (a), for all $a \in \mathsf{GF}(q)$. Also, the point (a) is incident only with $[\infty]$ and with $[a,l]$, for all $l \in \mathsf{GF}(q)$. Dually, $[k]$ is incident only with (∞) and with (k,b), for all $b \in \mathsf{GF}(q)$. Furthermore, the point (k,b) is incident only with $[k]$ and with $[k,b,k']$, for all $k' \in \mathsf{GF}(q)$. Dually, the line $[a,l]$ is incident only with the points (a) and (a,l,a'), for all $a' \in \mathsf{GF}(q)$. So we have easy and self-dual rules to decide whether two elements are incident whenever at least one of these elements have less than three coordinates. Suppose now both have three coordinates, i.e., we ask ourselves when (a,l,a') is incident with $[k,b,k']$. This will be the case if and only if $(l^{2^{e+1}} + aa', 1, a', a)$ can be written as a linear combination of $(b, 0, k^{2^{e+1}}, 1)$ and $(k'^{2^{e+1}}, 1, b, 0)$. Looking at the third and fourth coordinate, we see that this is equivalent with

$$(k'^{2^{e+1}}, 1, b, 0) + a \times (b, 0, k^{2^{e+1}}, 1) = (l^{2^{e+1}} + aa', 1, a', a).$$

Hence (a,l,a') is incident with $[k,b,k']$ if and only if

$$\left\{ \begin{array}{rcl} b & = & a' + ak^{2^{e+1}}, \\ l & = & k' + ka^{2^{e+1}}. \end{array} \right.$$

Now it is clear that the mapping which interchanges the parentheses with the brackets induces a polarity in $\mathsf{W}(q)$. ∎

The set of absolute points of the polarity mentioned in the above proof is obviously equal to $\{(\infty)\}\cup\{(a, a'+a^{2^{e+1}+1}, a') : a, a' \in \mathsf{GF}(2^{2e+1})\}$ and this is an ovoid of $\mathsf{W}(q)$, a so-called *Suzuki-Tits ovoid* or *semi-classical* ovoid. The coordinates in $\mathsf{PG}(3,q)$ are given by $\{(1,0,0,0)\}\cup\{(a'^{2^{e+1}}+a^{2^{e+1}+1}+aa', 1, a', a) : a, a' \in \mathsf{GF}(2^{2e+1})\}$.
Similarly, one can show (but we omit the proof; the calculations are a little longer than in the previous theorem, but apart from that, everything is similar):

Theorem 3.2
The classical hexagon $\mathsf{H}(q)$ is self-dual if and only if q is a power of 3. Also, $\mathsf{H}(q)$ admits a polarity if and only if $3q$ is a perfect square, i.e., $q = 3^{2e+1}$.

The corresponding ovoids in $\mathsf{H}(3^{2e+1})$ are called the *Ree-Tits ovoids* or *semi-classical* ovoids.
As for a finite generalized octagon of order (s,t), $2st$ must be a perfect square, there are no finite self-dual generalized octagons, and hence no self-polar ones either.

3.3 Ovoids and spreads of $\mathsf{H}(q)$

We now treat the classical hexagons. As already mentioned, there are three classes of orders for classical generalized hexagons: (q,q), (q,q^3) and (q^3,q). Hexagons of order (q,q^3) or (q^3,q) cannot have ovoids or spreads, so we may restrict ourselves to considering classical generalized hexagons of order (q,q), i.e., $\mathsf{H}(q)$. First we deal with spreads in $\mathsf{H}(q)$. We will construct an infinite class of spreads, one spread for each q in the next paragraph.
Let $\Gamma = \mathsf{H}(q)$ be embedded in $\Gamma' = \mathsf{H}(q^2)$ (in a natural way, i.e., if $\mathsf{H}(q^2)$ is defined over $\mathsf{GF}(q^2)$ on $\mathsf{Q}(6,q^2)$ with equation as above, then $\mathsf{H}(q)$ is obtained by restricting coordinates to $\mathsf{GF}(q)$). Let σ be the involution in $\mathsf{H}(q^2)$ defined by applying the map $x \mapsto x^q$ to every coordinate of any element of $\mathsf{H}(q^2)$. Clearly the substructure of fixed elements of $\mathsf{H}(q^2)$ is $\mathsf{H}(q)$. Now let L and M be two opposite lines of $\mathsf{H}(q)$, and let $p \mathrm{I} L$, with p a point of $\mathsf{H}(q^2)$ not belonging to $\mathsf{H}(q)$. Put $p' = \mathrm{proj}_M p^\sigma$. It is clear that the weak subhexagon $\Gamma'(p,p')$ is preserved by σ, and that $(\Gamma'^{+}(p,p'))^\sigma = \Gamma'^{+}(p',p)$. We know that $\Gamma'(p,p')$ is the double of a Desarguesian projective plane $\pi_{p,p'} \cong \mathsf{PG}(2,q^2)$ with point set $\Gamma'^{+}(p,p')$ and line set $\Gamma'^{+}(p',p)$ (see Subsection 1.4). Consequently, σ induces a polarity in $\pi_{p,p'}$. Clearly the points p and $p'^\sigma = \mathrm{proj}_M p$ are absolute points in $\pi_{p,p'}$. By the distance-3-regularity of $\mathsf{H}(q^2)$, there are exactly $q+1$ elements of $\Gamma'_3(p \bowtie p'^\sigma)\cap\Gamma'_3(p^\sigma \bowtie p')$ which belong to Γ (and hence are fixed by σ). This provides exactly $q+1$ absolute points in $\pi_{p,p'}$ incident with $p \bowtie p'^\sigma$ (which is indeed a line of $\pi_{p,p'}$).

So σ induces a unitary polarity in $\pi_{p,p'}$ and we have a set $\mathcal{S}$ of q^3+1 fixed flags, which can be seen as a set of q^3+1 fixed lines in $\mathsf{H}(q)$. As any two elements of $\mathcal{S}$ are easily seen to be opposite in $\Gamma'(p,p')$, and hence also in Γ, Theorem 2.1 implies that $\mathcal{S}$ is a spread of $\mathsf{H}(q)$. Viewed as a set of flags in $\pi_{p,p'} \cong \mathsf{PG}(2,q^2)$, it constitutes a Hermitian curve $\mathcal{H}$ union a dual one. A line of $\pi_{p,p'}$ meeting $\mathcal{H}$ in $q+1$ points corresponds to a point of $\Gamma(p,p')$ at distance 3 from $q+1$ elements of $\mathcal{S}$. We call $\mathcal{S}$ a *Hermitian* or *classical spread* of $\mathsf{H}(q)$. Note that all this implies that $\mathcal{S}$ is the intersection of the line set of $\mathsf{H}(q)$ with the line set of $\Gamma'(p,p')$. Also, we see that, if L and M are two lines of $\mathcal{S}$, and if x,y are two points at distance 3 from both L and M (in Γ), then $\Gamma_3(x)\cap\Gamma_3(y)\subseteq\mathcal{S}$. In general, we call any spread $\mathcal{S}'$ of $\Gamma=\mathsf{H}(q)$ *locally Hermitian at L* if $L\in\mathcal{S}'$ and if the above property holds for all $M\in\mathcal{S}'\setminus\{L\}$. One can easily verify that the set $\Gamma_3(x)\cap\Gamma_3(y)$ is a set of $q+1$ lines forming a regulus on the quadric $\mathsf{Q}(6,q)$.
We will see later that locally Hermitian spreads of $\mathsf{H}(q)$ give rise to spreads of $\mathsf{W}(q)$, and so do ovoids of $\mathsf{H}(q)$. So we now aim at constructing locally Hermitian spreads different from the classical one, and ovoids of $\mathsf{H}(q)$.
It is possible to slightly modify a coordinate representation of the Hermitian spread to obtain new spreads which are locally Hermitian, but not classical, in $\mathsf{H}(q)$, for all $q\equiv 1 \bmod 3$. Since the corresponding spreads of $\mathsf{W}(q)$ are the same as for the Hermitian spread, we do not give details here, but refer to Bloemen, Thas & Van Maldeghem [4].
All other known spreads and all known ovoids of $\mathsf{H}(q)$ live in the case that q is a power of 3. So let $q=3^h$. Then $\mathsf{H}(q)$ is self-dual. Hence any duality maps the Hermitian spread onto a *Hermitian* or *classical* ovoid $\mathcal{O}$. Now consider any automorphism α of $\mathsf{Q}(6,q)$. Since $\mathcal{O}$ contains q^3+1 points which are pairwise non-collinear in $\mathsf{Q}(6,q)$, the set $\mathcal{O}^\alpha$ is again a set of q^3+1 points pairwise non-collinear in $\mathsf{Q}(6,q)$, and hence pairwise opposite in $\mathsf{H}(q)$. Consequently, $\mathcal{O}^\alpha$ is an ovoid of $\mathsf{H}(q)$. In this way, many new ovoids of $\mathsf{H}(q)$ arise. The same trick can be performed with the Ree–Tits ovoid, again yielding a lot of new ovoids in $\mathsf{H}(q)$, $q=3^h$ and $h=2e+1$. Dually, this method also produces new spreads of $\mathsf{H}(q)$. In fact, there is, up to isomorphism, exactly one way to choose α such that the spread one obtains from dualizing the image under α of the dual of the Hermitian spread is locally Hermitian, but not Hermitian. This is shown by Bloemen, Thas & Van Maldeghem [4]. Let us call such a spread a *Roman spread*, for reasons that will become clear later. It is impossible to produce a locally Hermitian spread from a Ree–Tits ovoid.
There ends our present knowledge about the existence of ovoids and spreads of $\mathsf{H}(q)$. As to nonexistence, we have the following results. For q even there do not exist ovoids in $\mathsf{H}(q)$, see also Theorem 8.3. Likewise, for $q=5,7$, there do not exist ovoids in $\mathsf{H}(q)$, as proved by O'Keefe & Thas [39].

Theorem 3.3
If q is even, or if $q = 5, 7$, then no ovoid of $\mathsf{H}(q)$ exists.

3.4 From $\mathsf{H}(q)$ down to $\mathsf{W}(q)$

In this subsection, we will handle the ovoids and spreads of $\mathsf{W}(q)$. It turns out that a lot of examples of ovoids arise from $\mathsf{H}(q)$.
A *regulus* is a set of generators on a hyperbolic quadric in $\mathsf{PG}(3, q)$. We have the following theorem.

Theorem 3.4
Let $\mathcal{S}$ be a locally Hermitian spread at L of $\mathsf{H}(q)$. Let p be any point on L. Let $\mathcal{L}_p$ be the set of lines of $\mathsf{Q}(6, q)$ through p; let $\mathcal{P}_p$ be the set of planes of $\mathsf{Q}(6, q)$ through p. Then the geometry $\Gamma' = (\mathcal{P}_p, \mathcal{L}_p, \subseteq \cup \supseteq)$ is isomorphic to $\mathsf{W}(q)$. For every $M \in \mathcal{S} \setminus \{L\}$, let N_M be the line of $\mathsf{Q}(6, q)$ through p meeting every line of the regulus on $\mathsf{Q}(6, q)$ through L and M (this regulus is well defined!). Then $\mathcal{S}_p := \{L\} \cup \{N_M : M \in \mathcal{S} \setminus \{L\}\}$ forms a spread of $\mathsf{W}(q)$. If $\mathcal{S}$ is locally Hermitian at at least two lines, then $\mathcal{S}$ is locally Hermitian at every of its elements and $\mathcal{S}$ is classical. Also, if $\mathcal{O}$ is an ovoid of $\mathsf{H}(q)$ and p is a point of $\mathsf{H}(q)$ not in $\mathcal{O}$, then the set of lines $\{px : x \in \mathcal{O} \text{ and } x \in p^{\perp\!\!\perp}\}$ is a spread of $\mathsf{W}(q)$.

Proof. The first part of the theorem is a special case of a well known general property of polar spaces. For the second part of the theorem, we remark that $\mathcal{S}_p$ has q^2+1 elements, hence it suffices to show that two distinct elements are opposite in $\mathsf{W}(q)$. Suppose that N_M and $N_{M'}$ are not opposite, i.e., that they are coplanar. If we denote by w the intersection of $N_{M'}$ and M', then this implies that w is, in $\mathsf{H}(q)$ at distance at most 4 from every point of N_M. By Subsection 1.4, N_M is the intersection of $u^\perp$ with $v^{\perp\!\!\perp}$, for certain points u, v of $\mathsf{H}(q)$. It readily follows that either $w = u$ or w lies at distance exactly four from every point of N_M. In both cases the line M' of $\mathsf{H}(q)$ lies at distance ≤ 3 from some point of N_M, and hence M' is not opposite the element of $\mathcal{S}$ through that point, a contradiction.
We will not prove here the third assertion of the previous theorem (see Bloemen, Thas & Van Maldeghem [4] for a proof.
The fourth assertion is easy: If px and py, $x, y \in \mathcal{O}$ are coplanar, then x and y are not opposite in $\mathsf{H}(q)$. ∎

Let us now apply the methods of this theorem to the known examples. If q is not a power of 3, then there are only two classes of spreads of $\mathsf{H}(q)$ known, and they are both locally Hermitian, but they both give rise to the same class of spreads in $\mathsf{W}(q)$ by the method explained in the previous theorem. This spread of $\mathsf{W}(q)$ is in fact also called a *classical spread* and can be obtained by considering the lines of $\mathsf{W}(q)$ concurrent with a fixed line outside $\mathsf{W}(q)$ in $\mathsf{H}(3, q^2)$, when $\mathsf{W}(q)$ is seen as a subquadrangle of $\mathsf{H}(3, q^2)$ (see the dual

of Theorem 2.5). A third way to obtain it is by intersecting $\mathsf{Q}(4,q)$ with a hyperplane intersecting $\mathsf{Q}(4,q)$ in an elliptic quadric, and then dualizing.
If we apply the previous method to the locally Hermitian Roman spread, then we obtain a new spread of $\mathsf{W}(q)$, which can also be constructed using the method of Theorem 2.5 with the so-called *Roman* quadrangles; see the chapter on generalized quadrangles. Here, 3 divides q.
A third class of spreads of $\mathsf{W}(q)$ is obtained from the Ree–Tits ovoids in $\mathsf{H}(q)$ by the fourth assertion of Theorem 3.4. There, with respect to a Ree–Tits ovoid $\mathcal{O}$ in $\mathsf{H}(q)$, two inequivalent choices of points outside $\mathcal{O}$: those on an absolute line with respect to the corresponding polarity, and the other points. This gives rise to two inequivalent spreads of $\mathsf{W}(q)$. Here, q is an odd power of 3.
A fourth class of spreads of $\mathsf{W}(q)$ is given by the dual of the Suzuki–Tits ovoids, see above. Here, q is an odd power of 2.
Finally, there is at present only one other infinite class of spreads of $\mathsf{W}(q)$ known, and it is due to Kantor. In fact, it arises from a coordinatization representation of the classical spread by a little distortion. We omit the details. Here, q is odd and not a prime (otherwise the classical spread is obtained).
There is one sporadic spread, for $q = 3^5$, see Penttila & Williams [43].
Concerning ovoids of $\mathsf{W}(q)$, one can, for q even, dualize the classical spreads. For $q = 2^{2e+1}$, one also has the Suzuki–Tits ovoids. These are the only known classes of ovoids of $\mathsf{W}(q)$ and it has been conjectured that there are no other ovoids.
If q is odd, then there are no ovoids in $\mathsf{W}(q)$:

Theorem 3.5
$\mathsf{W}(q)$ has no ovoids if and only if q is odd.

Proof. We only have to show that for q odd no ovoid exists in $\mathsf{W}(q)$. So let, by way of contradiction, $\mathcal{O}$ be an ovoid of $\mathsf{W}(q)$, q odd, and let $\mathsf{W}(q)$ arise from the symplectic polarity ρ in $\mathsf{PG}(3,q)$. Let p be any point of $\mathsf{W}(q)$ not in $\mathcal{O}$. Every line of $\mathsf{W}(q)$ through p contains a point of $\mathcal{O}$. Suppose three such points x, y, z are collinear in $\mathsf{PG}(3,q)$. Let L be the line of $\mathsf{PG}(3,q)$ incident with x, y, z. Suppose that $|L \cap \mathcal{O}| = \ell \geq 3$. Every point $w \in \mathcal{O}$ not on L is incident with a unique line L_w of $\mathsf{W}(q)$ meeting L (and incident with the intersection of L with w^ρ). Counting the number of such pairs (w, L_w), we obtain $q^2 + 1 - \ell = (q + 1 - \ell)(q + 1)$, hence $\ell = 2$, a contradiction. So the the plane p^ρ meets $\mathcal{O}$ in an oval. But all lines through p in p^ρ are tangents, and so we obtain a $(q + 2)$-arc, contradicting the fact that q is odd. ■

3.5 The other classical quadrangles

Let us now have a look at the quadrangle $\mathsf{H}(3,q^2)$. By intersecting with a non-tangent plane in $\mathsf{PG}(3,q^2)$, one obtains the point set of a Hermitian curve, which is an ovoid of $\mathsf{H}(3,q^2)$. Indeed, it is clear that we obtain q^3+1 points, and that no two of them are collinear (the intersection does not contain lines). Furthermore, the order of $\mathsf{H}(3,q^2)$ is (q^2,q), and hence by the dual of Theorem 2.6 there are no spreads in $\mathsf{H}(3,q^2)$. Dually, there exists a spread in $\mathsf{Q}(5,q)$, but no ovoids. There are plenty of other ovoids in $\mathsf{H}(3,q^2)$. One can for instance replace a line of the Hermitian curve above by the set of points collinear with all points of that line.

Remark

We note that the Hermitian unital in $\mathsf{PG}(2,q^2)$ "is" a spread of both $\mathsf{Q}^-(5,q)$ and $\mathsf{H}(q)$. A way of seeing this explicitly is given by the following construction of Thas [53]: we intersect $\mathsf{Q}(6,q)$ with a hyperplane such that the intersection is a $\mathsf{Q}^-(5,q)$. The lines of $\mathsf{Q}(6,q)$ which are lines of both $\mathsf{H}(q)$ and $\mathsf{Q}^-(5,q)$ form a Hermitian spread in both $\mathsf{H}(q)$ and $\mathsf{Q}^-(5,q)$.
Now we consider the classical quadrangle $\mathsf{H}(4,q^2)$. It is not known whether this has spreads; only a computer result of Brouwer (unpublished) is available saying that $\mathsf{H}(4,4)$ has no spread.
Finally, one has the following result:

Theorem 3.6
$\mathsf{H}(4,q^2)$ does not have any ovoid.

Proof. An ovoid $\mathcal{O}$ of $\mathsf{H}(4,q^2)$ would contain q^5+1 points. Intersecting $\mathsf{H}(4,q^2)$ with a hyperplane of $\mathsf{PG}(4,q^2)$, we see that every hyperplane of $\mathsf{PG}(4,q^2)$ contains either 1 (the hyperplane is tangent to $\mathsf{H}(4,q^2)$ at a point of $\mathcal{O}$) or q^3+1 (the hyperplane is tangent to $\mathsf{H}(4,q^2)$ at a point off $\mathcal{O}$, or it intersects $\mathsf{H}(4,q^2)$ in an $\mathsf{H}(3,q^2)$, where $\mathcal{O}$ induces an ovoid of size q^3+1) points. Let L be a line of $\mathsf{PG}(4,q^2)$ having at least two points in common with $\mathcal{O}$. Let π be any plane through L, and let X be the number of points of $\mathcal{O}$ in π. Clearly, there are $(q^3+1-X)(q^2+1)+X=q^5+q^3+q^2+1-Xq^2$ points in $\mathcal{O}$. Hence $X=q+1$. Let Y be the number of points of $\mathcal{O}$ on L. Then similarly counting the number of points of $\mathcal{O}$ in some fixed hyperplane H through L yields $(q+1-Y)(q^2+1)+Y=q^3+1$, hence $Y=(q+1)/q$, a contradiction. ∎

4 Relation with ovoids and spreads of $\mathsf{PG}(3,q)$ and $\mathsf{Q}(6,q)$

4.1 Ovoids and plane spreads of $\mathsf{Q}(6,q)$

Theorem 4.1
Let $\mathcal{O}$ be a set of points of $\mathsf{Q}(6,q)$, and let $\mathsf{H}(q)$ be a classical hexagon on $\mathsf{Q}(6,q)$, as before. Then $\mathcal{O}$ is an ovoid of $\mathsf{Q}(6,q)$ if and only if it is an ovoid

of $\mathsf{H}(q)$*. Also, if* $\mathcal{O}$ *is an ovoid of* $\mathsf{H}(q)$*, then the set of planes* $\{p^\perp : p \in \mathcal{O}\}$ *is a plane spread of* $\mathsf{Q}(6,q)$*.*

Proof. This follows immediately from the fact that two points of $\mathsf{H}(q)$ are opposite if and only if they are not collinear in $\mathsf{Q}(6,q)$. The second assertion follows from the fact that every point of $\mathsf{H}(q)$, and hence of $\mathsf{Q}(6,q)$, is collinear in $\mathsf{H}(q)$ with a unique point of $\mathcal{O}$. ∎

Remark

Let $\mathsf{Q}^+(7,q)$ be a non-degenerate hyperbolic quadric in $\mathsf{PG}(7,q)$. Define the following geometry Ω. The 1-points of Ω are the points of $\mathsf{Q}^+(7,q)$; the 2-points are the members of one class of generators of $\mathsf{Q}^+(7,q)$; the 3-points are the members of the other class of generators; the lines are the lines of $\mathsf{Q}^+(7,q)$; incidence is containment, reverse containment, or meeting in a plane of $\mathsf{Q}^+(7,q)$. Then we have a rank 4 geometry which is symmetric in the i-points, i.e., interchanging the i-points and j-points, $i,j \in \{1,2,3\}$ does not change the isomorphism type of the geometry. In fact, there always exists a isomorphism of Ω of order 3 mapping the i-points to the $(i+1)$-points, $i+1$ modulo 3. Such a map is called a *triality*. The classical generalized hexagons now arise from a triality by taking the absolute 1-points and the fixed lines (an i-point is *absolute* if it is incident with its image). Now let $\mathcal{O}$ be an ovoid of $\mathsf{Q}(6,q)$. Embed $\mathsf{Q}(6,q)$ into $\mathsf{Q}^+(7,q)$. Then obviously, $\mathcal{O}$ is an ovoid of $\mathsf{Q}^+(7,q)$. Applying a triality, we obtain a spread of $\mathsf{Q}^+(7,q)$. Intersecting each element of that spread with $\mathsf{Q}(6,q)$, we obtain a plane spread of $\mathsf{Q}(6,q)$. This spread is equivalent with the spread obtained from the previous theorem.

4.2 Ovoids and line spreads of $\mathsf{PG}(3,q)$

Theorem 4.2
Let $\mathsf{W}(q)$ *be embedded in* $\mathsf{PG}(3,q)$ *in a natural way. Every spread of* $\mathsf{W}(q)$ *is a line spread of* $\mathsf{PG}(3,q)$*. Every ovoid of* $\mathsf{W}(q)$ *is an ovoid of* $\mathsf{PG}(3,q)$*. Every ovoid of* $\mathsf{PG}(3,q)$*,* q *even, is projectively equivalent with an ovoid of* $\mathsf{W}(q)$*.*

Proof. The first assertion follows from the fact that a spread in $\mathsf{W}(q)$ is just a partition of the point set in lines, and the same is true in $\mathsf{PG}(3,q)$. The second assertion follows from the proof of Theorem 3.5, where we proved that no line of $\mathsf{PG}(3,q)$ can meet an ovoid of $\mathsf{W}(q)$ in more than 2 points (and this part was independent of q). We now show the third assertion.
Let $\mathcal{O}$ be an ovoid in $\mathsf{PG}(3,q)$. We will use the fact that at every point of $\mathcal{O}$, there is a unique tangent plane. From this, one deduces that a non-tangent plane π meets $\mathcal{O}$ in an oval. Let x_π be the nucleus of that oval. Suppose that x_π is also the nucleus of the oval $\mathcal{O} \cap \pi'$. Let L be any line of $\mathsf{PG}(3,q)$

through x_π. If L is contained in π or π', then it is tangent to $\mathcal{O}$. If L is not contained in π and in π', then let π'' be any plane through L. The intersection $\pi \cap \pi''$ and $\pi' \cap \pi''$ are tangent lines to $\mathcal{O}$, hence π'' meets $\mathcal{O}$ in an oval. Consequently L is a tangent line. So every line through x_π is a tangent line, contradicting the fact that there are q^2+q+1 such lines, but only q^2+1 points of $\mathcal{O}$. So the mapping $\pi \mapsto x_\pi$ is injective. Since there are q^3+q non-tangent planes, and equally many points off $\mathcal{O}$, every point off $\mathcal{O}$ is the nucleus of exactly one plane intersection oval. Now we define the following map ρ from the set of points of $\mathsf{PG}(3,q)$ to the set of planes of $\mathsf{PG}(3,q)$. A point $x \in \mathcal{O}$ is mapped onto the tangent plane of $\mathcal{O}$ at x. A point x off $\mathcal{O}$ is mapped by ρ onto the plane π for which $x = x_\pi$.
We remark that the only tangent lines to $\mathcal{O}$ through a point x are contained in x^ρ. Indeed, for $x \in \mathcal{O}$, this is obvious. Now let $x \notin \mathcal{O}$. If there were a tangent line L through x not in x^ρ, then, as above, any plane through L would meet $\mathcal{O}$ in an oval with nucleus x, a contradiction.
Let x, y, z be three collinear points. If they are contained in a tangent line L, then the images x^ρ, y^ρ and z^ρ contain the line L, hence they are collinear. Suppose now that they are contained in a non-tangent line L. We remark that the mapping ρ is bijective. Indeed, it is bijective when restricted to the the points off $\mathcal{O}$ and the non-tangent planes to $\mathcal{O}$. It is now clear that ρ is bijective. Let π be any plane through L and let $w^\rho = \pi$. Since all tangent lines to $\mathcal{O}$ through w are contained in π, the lines wx, wy and wz are tangent lines, hence x^ρ, y^ρ and z^ρ contain w. But as there are $q+1$ choices for w, those planes share the same $q+1$ points, which must necessarily be collinear. Hence those planes are collinear themselves. So ρ can be uniquely extended to a correlation. Now, for any point x, it is clear that the image under ρ of any point y in x^ρ contains x (because such an image contains the tangent line xy). Hence ρ has order 2. But every point is incident with its image. Consequently ρ is a symplectic polarity, and, obviously, the fixed lines are the tangent lines to $\mathcal{O}$, implying that $\mathcal{O}$ is an ovoid in the corresponding symplectic quadrangle. ■

Part II: Ovoids and spreads of polar spaces

5 Polar spaces: definition and properties

Polar spaces describe the geometry of vector spaces carrying a reflexive sesquilinear form or a quadratic form in much the same way as projective spaces describe the geometry of vector spaces. In the sequel we will always assume that everything is finite. There are three types of forms with an associated polar space. The space associated with the alternating bilinear form is called the *symplectic* polar space; the one with the Hermitian form is called the *unitary or Hermitian* polar space; finally with the quadratic forms is associated the *orthogonal* polar space. These polar spaces are commonly called *classical polar spaces*. Although we assume here some preknowledge on the theory of polar spaces, we will give some of the basic definitions and properties. The reader is referred to the literature for more information, e.g. [8], [10], [28].

Let $V(n+1, q)$ be a vector space carrying a reflexive sesquilinear form σ of one of the three types. Recall that the sesquilinear form defines uniquely the polarity in the associated projective space $\mathsf{PG}(n, q)$, unless q is even and σ is orthogonal, in which case one has to use the *quadratic form* which we will also denote by σ. A subspace W of V is called *totally isotropic* if the sesquilinear form σ vanishes identically on W, i. e. if $W \subseteq W^\sigma$. In case of an orthogonal polarity, a subspace on which the quadratic form σ vanishes is called a *totally singular subspace*. We shall often call the maximal totally isotropic subspaces or maximal totally singular subspaces of a polar space $\mathcal{S}$, the *generators* of $\mathcal{S}$.

One can prove that the classical polar space, regarded as a geometry on the points of the projective space $\mathsf{PG}(n, q)$, whose flats are the totally isotropic or totally singular subspaces (called subspaces for short) having the following properties (see for instance [10] for a proof).

- **[P1]** Each subspace is isomorphic to a projective space of dimension at most $r-1$.
- **[P2]** The intersection of any family of subspaces is a subspace.
- **[P3]** If W is a subspace of dimension $r-1$, and p a point not in W, then the set of points p' such that the line pp' is totally isotropic or totally singular, is a hyperplane in W, and the union of those lines pp' is a subspace of dimension $r-1$.
- **[P4]** There exist two disjoint subspaces of dimension $r-1$.

A geometry consisting of a set of points with a collection of distinguished subsets called subspaces, satisfying the axioms **[P1]**-**[P4]** is called an *abstract polar space of rank* r.

An important simplification of the axioms **[P1]**-**[P4]** was obtained by Buekenhout and Shult [9]. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be an incidence structure of points and lines (or a point-line geometry). A *subspace* of $\mathcal{S}$ will be a point set which contains any line through two of its points; it is *singular* provided any two of its points are collinear.

Theorem 5.1 ([9])
Suppose that an incidence structure $\mathcal{S}$ has the following properties:

(i) if p is a point not on a line L, then p is collinear with one or all points of L;

(ii) any line contains at least three points;

(iii) no point is collinear with all others;

(iv) any chain (w.r.t. inclusion) of singular subspaces is finite.

Then the singular subspaces constitute an abstract polar space.

Remark

It is clear that the generalized quadrangles are the polar spaces of rank 2, and these are far of being completely classified. We might expect however a classification to be possible for sufficiently large rank. This has been proved indeed by Tits, building on work of Veldkamp [64].

Theorem 5.2 ([61])
All finite polar spaces of rank at least 3 are classical.

We shall use the following notation:

> $\mathsf{W}_n(q)$: the polar space arising from a symplectic polarity of $\mathsf{PG}(n, q)$, n odd and $n \geq 3$: here $r = (n+1)/2$;
>
> $\mathsf{Q}(2n, q)$: the polar space arising from a non-singular quadric in $\mathsf{PG}(2n, q)$, $n \geq 2$: here $r = n$;
>
> $\mathsf{Q}^+(2n+1, q)$: the polar space arising from a non-singular hyperbolic quadric in $\mathsf{PG}(2n+1, q)$, $n \geq 1$: here $r = n+1$;
>
> $\mathsf{Q}^-(2n+1, q)$: the polar space arising from a non-singular elliptic quadric in $\mathsf{PG}(2n+1, q)$, $n \geq 2$: here $r = n$;
>
> $\mathsf{H}(n, q^2)$: the polar space arising from a non-singular Hermitian variety H in $\mathsf{PG}(n, q^2)$, $n \geq 3$: for n odd $r = (n+1)/2$, for n even $r = n/2$.

Theorem 5.3
For q even, the polar space $\mathsf{Q}(2n, q)$ is isomorphic to the polar space $\mathsf{W}_{2n-1}(q)$.

Proof. The theorem is easily proved by projection of $\mathsf{Q}(2n,q)$ from the nucleus onto a $\mathsf{PG}(2n-1,q) \subset \mathsf{PG}(2n,q)$ not containing the nucleus. ∎

Let $|\mathcal{S}|$ denote the number of points of the polar space $\mathcal{S}$, and let $\Sigma(\mathcal{S})$ be the set of generators of $\mathcal{S}$; all elements of $\Sigma(\mathcal{S})$ have dimension $r-1$. For a proof of the following theorems we refer e.g. to [28] or [55].

Theorem 5.4
The numbers of points of the finite classical polar spaces are given by the following formulae:

$$\begin{aligned}
|\mathsf{W}_n(q)| &= (q^{n+1}-1)/(q-1),\\
|\mathsf{Q}(2n,q)| &= (q^{2n}-1)/(q-1),\\
|\mathsf{Q}^+(2n+1,q)| &= (q^n+1)(q^{n+1}-1)/(q-1),\\
|\mathsf{Q}^-(2n+1,q)| &= (q^n-1)(q^{n+1}+1)/(q-1),\\
|\mathsf{H}(n,q^2)| &= (q^{n+1}+(-1)^n)(q^n-(-1)^n)/(q^2-1).
\end{aligned}$$

Theorem 5.5
The numbers of generators of the finite classical polar spaces are given by the following formulae:

$$\begin{aligned}
|\Sigma(\mathsf{W}_n(q))| &= (q+1)(q^2+1)\ldots(q^{(n+1)/2}+1),\\
|\Sigma(\mathsf{Q}(2n,q))| &= (q+1)(q^2+1)\ldots(q^n+1),\\
|\Sigma(\mathsf{Q}^+(2n+1,q))| &= 2(q+1)(q^2+1)\ldots(q^n+1),\\
|\Sigma(\mathsf{Q}^-(2n+1,q))| &= (q^2+1)(q^3+1)\ldots(q^{n+1}+1),\\
|\Sigma(\mathsf{H}(2n,q^2))| &= (q^3+1)(q^5+1)\ldots(q^{2n+1}+1),\\
|\Sigma(\mathsf{H}(2n+1,q^2))| &= (q+1)(q^3+1)\ldots(q^{2n+1}+1).
\end{aligned}$$

6 m-systems, ovoids and spreads of polar spaces

Let $\mathcal{S}$ be a finite classical polar space of rank r, with $r \geq 2$. A *partial m-system* [46] of $\mathcal{S}$, with $0 \leq m \leq r-1$, is any set $\{\pi_1, \pi_2, \ldots, \pi_k\}$ of $k(\neq 0)$ totally singular m-spaces of $\mathcal{S}$ such that no generator containing π_i has a point in common with $(\pi_1 \cup \pi_2 \cup \ldots \cup \pi_k) \setminus \pi_i$, $i = 1, 2, \ldots, k$. Note that a partial 0-system is a k-cap, while a partial $(r-1)$-system, a set of generators that are pairwise disjoint, is called a *partial spread*.

Theorem 6.1 ([46])
Let M be a partial m-system of the finite classical polar space $\mathcal{S}$, then

$$\begin{array}{lcll}
\text{for } \mathcal{S} & = & \mathsf{W}_n(q), & |M| \leq q^{(n+1)/2}+1, \\
\text{for } \mathcal{S} & = & \mathsf{Q}(2n,q), & |M| \leq q^n+1, \\
\text{for } \mathcal{S} & = & \mathsf{Q}^+(2n+1,q), & |M| \leq q^n+1, \\
\text{for } \mathcal{S} & = & \mathsf{Q}^-(2n+1,q), & |M| \leq q^{n+1}+1, \\
\text{for } \mathcal{S} & = & \mathsf{H}(2n,q^2), & |M| \leq q^{2n+1}+1, \\
\text{for } \mathcal{S} & = & \mathsf{H}(2n+1,q^2), & |M| \leq q^{2n+1}+1.
\end{array}$$

If for $|M|$ the upper bound is reached, which is by the way independent of m, then M is called an *m-system* of $\mathcal{S}$. A 0-system is called *an ovoid* of $\mathcal{S}$, while an $(r-1)$-system is called a *spread.*

Example

Every spread of the generalized hexagon $\mathsf{H}(q)$ is a 1-system of the quadric $\mathsf{Q}(6,q)$.

We will restrict for the sequel to constructions of ovoids and spreads, for more details on m-systems we refer to [46] and [47].

7 Spreads of polar spaces

7.1 Spreads of symplectic polar spaces

Definition

A *t-spread of* $\mathsf{PG}(n,q)$ with $1 \leq t < n$ is a set of t-dimensional subspaces of $\mathsf{PG}(n,q)$ that partition the point set of $\mathsf{PG}(n,q)$.

Theorem 7.1
A spread of $\mathsf{W}_n(q)$ *is a t-spread of* $\mathsf{PG}(n,q)$, $n = 2t+1$.

Proof. A spread of $\mathsf{W}_n(q)$ is a set of $\frac{n-1}{2}$-dimensional totally isotropic subspaces, which indeed partition the point set of $\mathsf{PG}(n,q)$.

Definition

Assume that $\pi_0, \pi_1, \ldots, \pi_t$ are $(\frac{n-t}{t+1})$-dimensional spaces of $\mathsf{PG}(n,q^{t+1})$ generating the whole space and such that they are conjugated w.r.t. the extension $\mathsf{GF}(q^{t+1})$ of $\mathsf{GF}(q)$. Assume that x_0 is a point of π_0 and that $x_1, x_2, \ldots, x_t$ are the conjugate points of x_0. These $t+1$ points generate a t-dimensional space. The intersection of $\mathsf{PG}(t,q^{t+1})$ and $\mathsf{PG}(n,q)$ is a t-dimensional space $\mathsf{PG}(t,q)$. The set $\{\pi_0, \pi_1, \ldots, \pi_t\}$ defines a set Φ of $\frac{q^{n+1}-1}{q^{t+1}-1}$ t-dimensional subspaces $\mathsf{PG}(t,q)$ of $\mathsf{PG}(n,q)$, partitioning the point set of $\mathsf{PG}(n,q)$, hence Φ is a spread and is called a *regular t-spread* of $\mathsf{PG}(n,q)$.

A *t-regulus* in $\mathsf{PG}(2t+1,q)$ is a set R of $q+1$ mutually skew t-dimensional subspaces satisfying the following property.

> *Every line L of $\mathsf{PG}(2t+1,q)$ intersecting 3 elements of R, intersects all elements of R.*

Theorem 7.2 ([7])
If $q>2$, then a t-spread Φ is regular if it satisfies the following property.

> *As soon as three elements of a t-regulus are elements of Φ, then all elements of the t-regulus are elements of Φ.*

Remark

For an overview of existence of t-spreads in $\mathsf{PG}(2t+1,q)$, we refer to e.g. [59]. For a proof of the following theorem we refer for instance to [52].

Theorem 7.3
The polar space $\mathsf{W}_n(q)$ contains always a spread which is a regular $\frac{n-1}{2}$-spread of $\mathsf{PG}(n,q)$.

Proof. Let $t=\frac{n-1}{2}$ and let $C=\{\pi_1,\pi_2,\pi_3,\ldots,\pi_{q^{\frac{n+1}{2}}+1}\}$ denote a regular spread of t-dimensional spaces of $\mathsf{PG}(n,q)$. As the spread is regular, there are exactly $(t+1)$ lines l_i, $1\leq i\leq(t+1)$ of $\mathsf{PG}(n,q^{t+1})$ that are generating the space, are conjugate w.r.t. the $(t+1)$-extension $\mathsf{GF}(q^{t+1})$ of $\mathsf{GF}(q)$, and such that each line intersects each element of C. Hence there exists a collineation τ of order $t+1$ of $\mathsf{PG}(n,q^{t+1})$ fixing each point of $\mathsf{PG}(n,q)$ and such that $l_i^\tau=l_{i+1}$, $1\leq i\leq t$ and $l_{t+1}^\tau=l_1$. Denote by σ the symplectic polarity of $\mathsf{PG}(n,q)$ such that π_1,π_2,π_3,π_4 are absolute $(t+1)$-dimensional spaces. The extension of σ w.r.t. $\mathsf{PG}(n,q^{t+1})$ will also be denoted by σ. Let $\pi_i\cap l_1=x_i$, $\quad 1\leq i\leq 4$. Hence l_1 contains the four points x_1,x_2,x_3,x_4. From $x_1\in\pi_1=\pi_1^\sigma$ it follows that $\pi_1\subset x_1^\sigma$. As $l_1^\sigma=x_1^\sigma\cap x_2^\sigma$, the space $\pi_1\cap l_1^\sigma=\pi_1\cap x_1^\sigma\cap x_2^\sigma=\pi_1\cap x_2^\sigma$ is a line K_1 of π_1. Using the same argument for π_2,π_3 and π_4 we can conclude that l_1^σ is an $(n-2)$-dimensional space that intersects each t-dimensional space π_i in a line K_i, $1\leq i\leq 4$. Note that $K_1=l_1^\sigma\cap\pi_1=(l_1\pi_1)^\sigma$ because $\pi_1^\sigma=\pi_1$. The space $K_1^\sigma=l_1\pi_1$, hence $(l_1\cap\pi_1)$ is an element of K_1^σ. Assume that $l_i\cap\pi_1$ is not an element of K_1, hence K_1^σ is not in $(l_i\cap\pi_1)^\sigma$. Then K_1 is a subspace of K_1^σ not lying in $(l_i\cap\pi_1)^\sigma$, which is a contradiction. Hence $(l_i\cap\pi_1)$ is an element of K_1. In the same way one can prove that $(l_i\cap\pi_j)$ is an element of K_j, $\quad 1\leq i\leq t+1$ and $1\leq j\leq 4$. Hence $\langle l_2,l_3,\ldots,l_{t+1}\rangle\subseteq l_1^\sigma$ from which it follows that $l_1^\sigma=\langle l_2,l_3,\ldots,l_{t+1}\rangle$. As $(l_1^\tau)^\sigma=\langle l_2^\tau,l_3^\tau,\ldots,l_{t+1}^\tau\rangle$ one may conclude that $l_{i_1}^\sigma=\langle l_{i_2},\ldots l_{i_{t+1}}\rangle$ with $\{i_1,i_2,\ldots,i_{t+1}\}=\{1,2,\ldots,t+1\}$. Let $l_j\cap\pi_i=x_{ij}$, $1\leq j\leq t+1$, $5\leq i\leq q^{\frac{n+1}{2}}+1$. Then $x_{ij}^\sigma=l_j^\sigma\cup\pi_i^\sigma=\langle l_1,l_2,\ldots,l_{j-1},l_{j+1},\pi_i\rangle$. So x_{il} is a point of x_{ij}^σ with $l\neq j$ and $1\leq l\leq t+1$. As σ is symplectic, $x_{il}\in x_{il}^\sigma$, hence

$x_{il} \in x^{\sigma}_{ij}$, $1 \leq l, j \leq t+1$. We can conclude that $(x_{i_1} \cup x_{i_2} \cup \ldots \cup x_{i_{t+1}})$ is a subset of $(x^{\sigma}_{i_1} \cup x^{\sigma}_{i_2} \cup \ldots \cup x^{\sigma}_{i_{t+1}})$ hence that $\pi_i \subset \pi_i^{\sigma}$ from which it follows that π_i , $1 \leq i \leq q^{\frac{n+1}{2}+1}$, is absolute w.r.t. σ. ∎

Remark

Many other examples of spreads of $\mathsf{W}_n(q)$ are known. We refer for instance to [1, 22, 33, 38, 49].

7.2 Spreads of quadrics

7.2.1 Relations between spreads of quadrics

Theorem 7.4

1. *The quadric $\mathsf{Q}^+(4n+1, q)$ has no spread.*
2. *The existence of spreads of $\mathsf{Q}^+(4n+3, q)$, $n \geq 1$, is equivalent with the existence of spreads of $\mathsf{Q}(4n+2, q)$.*
3. *If $\mathsf{Q}(2n, q)$, $n \geq 3$, has a spread, then $\mathsf{Q}^-(2n-1, q)$ has a spread.*

Proof. It is commonly known that the quadric $\mathsf{Q}^+(2m+1, q)$ has two families Σ_1 and Σ_2 of maximal singular subspaces. Two maximal singular subspaces π and π' of $\mathsf{Q}^+(2m+1, q)$, will be of the same family iff $\dim(\pi \cap \pi')$ has the same parity of m.

1. Assuming $m = 2n$, two maximal subspaces are disjoint iff they are in different families. This implies that no spread of the quadric $\mathsf{Q}^+(4n+1, q)$ can exist.
2. Let Φ^+ be a spread of $\mathsf{Q}^+(4n+3, q)$, $n \geq 1$, and interect $\mathsf{Q}^+(4n+3, q)$ with a hyperplane $\mathsf{PG}(4n+2, q)$ of $\mathsf{PG}(4n+3, q)$ which is not tangent to $\mathsf{Q}^+(4n+3, q)$. Clearly the intersection of Φ^+ with $\mathsf{PG}(4n+2, q)$ is a spread Φ of $\mathsf{Q}(4n+2, q) = \mathsf{Q}^+(4n+3, q) \cap \mathsf{PG}(4n+2, q)$. Conversely let Φ be a spread of $\mathsf{Q}(4n+2, q)$, $n \geq 1$, and let $\mathsf{Q}(4n+2, q)$ be embedded in the non-singular hyperbolic quadric $\mathsf{Q}^+(4n+3, q)$. The two families of maximal singular subspaces of $\mathsf{Q}^+(4n+3, q)$ are denoted by Σ_1 and Σ_2. One easily shows that the elements of Σ_i containing the respective elements of Φ constitute a spread Φ^+ of $\mathsf{Q}^+(4n+3, q)$.
3. Let Φ be a spread of $\mathsf{Q}(2n, q)$. Let $\pi = \mathsf{PG}(2n-1, q)$ be a hyperplane intersecting the quadric $\mathsf{Q}(2n, q)$ in an elliptic quadric $\mathsf{Q}^-(2n-1, q)$. Then $\Phi_\pi = \Phi \cap \pi$ is a spread of $\mathsf{Q}^-(2n-1, q)$.

∎

Corollary

If q is even then $\mathsf{Q}(2n,q)$ is isomorphic to $\mathsf{W}_{2n-1}(q)$ and hence has spreads, hence by the above theorem, $\mathsf{Q}^-(2n-1,q)$, q even, and $Q^+(4m+3,q)$, $(n=2m)$ q even has spreads.

Remark

The construction used in the above proof for constructing spreads of $\mathsf{Q}(4n+2,q)$ from spreads of $\mathsf{Q}^+(4n+3,q)$, might yield non-isomorphic spreads, see 9.3, for more information.

Open problem

The existence of spreads of $\mathsf{Q}^-(2m+1,q)$, q odd, is still open. See section 7.2.3 for more information on spreads of $Q^+(4m+3,q)$, q odd.

7.2.2 Spreads of $\mathsf{Q}(2n,q)$.

By theorem 5.3 we may restrict to the case q is odd.

Theorem 7.5 ([58])
The polar space $\mathsf{Q}(4n,q)$, q odd and $n \geq 1$ has no spread.

Proof. Let Φ be a spread of $\mathsf{Q}(4n,q)$, q odd and $n \geq 1$. Let $\pi = \mathsf{PG}(4n-1,q)$ be an intersecting hyperplane of $\mathsf{PG}(4n,q)$, such that $\mathsf{PG}(4n-1,q) \cap \mathsf{Q}(4n,q)$ is a non-singular hyperbolic quadric $\mathsf{Q}^+(4n-1,q)$. Assume that Φ contains r maximal singular subspaces of $\mathsf{Q}^+(4n-1,q)$. We count in two different ways, the ordered pairs of the set $V = \{(x,\pi) \| x \in \mathsf{Q}^+(4n-1,q),\ x \in \pi,\ \pi \in \Phi\}$. Given a point x_0 of $\mathsf{Q}^+(4n-1,q)$, there is a unique ordered pair $(x_0,\pi) \in V$, hence $|V| = |\mathsf{Q}^+(4n-1,q)| = \frac{(q^{2n-1}+1)(q^{2n}-1)}{q-1}$. On the other hand, either π is one of the r maximal singular subspaces of $\mathsf{Q}^+(4n-1,q)$, in which case it counts for $\frac{q^{2n}-1}{q-1}$ points x, or is one of the $q^{2n}+1-r$ elements of Φ intersecting $\mathsf{Q}^+(4n-1,q)$ in the $\frac{q^{2n-1}-1}{q-1}$ elements of a $(2n-2)$-dimensional subspace. Hence

$$
\begin{aligned}
|V| &= r\Big(\frac{q^{2n}-1}{q-1}\Big) + (q^{2n}+1-r)\frac{q^{2n-1}-1}{q-1} \\
&= \frac{(q^{2n-1}+1)(q^{2n}-1)}{q-1}.
\end{aligned}
$$

From which it follows that $r=2$.
The two spaces π_1 en π_2 are elements of the same family of maximal singular subspaces of $\mathsf{Q}^+(4n-1,q)$. Assume that π_0 is an element of Φ and that

ξ is a maximal singular subspace of $\mathsf{Q}(4n,q)$ intersecting π_0 in a $(2n-2)$-dimensional space. The elements of $\Phi \backslash \{\pi_0\}$ having a non-empty intersection with ξ, have exactly one point in common with ξ. The number of points in $\xi \setminus (\xi \cap \pi_0)$ is $\frac{q^{2n}-q^{2n-1}}{q-1} = q^{2n-1}$. We denote the elements of Φ intersecting ξ in one point with $\pi_1, \ldots, \pi_{q^{2n-1}}$. Let η be a third projective $(2n-1)$-dimensional space of $\mathsf{Q}(4n,q)$ containing $\pi_0 \cap \xi$. In the polar space $\mathsf{Q}(4n,q)$ there is no point of $\xi \setminus \pi_0$ collinear with a point of $\eta \setminus \pi_0$. Hence $\pi_i \cap \eta = \emptyset$, $i = 1, 2, \ldots, q^{2n-1}$. If we denote by γ_i the $(4n-1)$-dimensional space generated by π_i and η, then $\gamma_i \cap \mathsf{Q}(4n,q)$ is a non-singular hyperbolic quadric $\mathsf{Q}_i^+(4n-1,q)$, $i = 1, 2, \ldots, q^{2n-1}$. In the first part of the proof we have shown that $\mathsf{Q}_i^+(4n-1,q)$ contains two elements of Φ, one of them being π_i. Call the other one ρ_i. Note that π_i and ρ_i are members of the same family Σ of maximal singular subspaces of $\mathsf{Q}_i^+(4n-1,q)$. Clearly ξ is contained in Q_i^+ and as $(\xi \cap \pi_i)$ is one point, one can conclude that $\xi \notin \Sigma$. Hence $\xi \cap \rho_i \neq \emptyset$, $i = 1, 2, \ldots, q^{2n-1}$. Consequently $\rho_i \in \{\pi_0, \pi_1, \ldots, \pi_{q^{2n-1}}\} \backslash \{\pi_i\}$. Since $\xi \cap \eta$ is contained in exactly two $(2n-1)$-dimensional subspaces of $\mathsf{Q}_i^+(4n-1,q)$, namely ξ and η, the space π_0 is not contained in $\mathsf{Q}_i^+(4n-1,q)$. Hence $\rho_i \neq \pi_0$. Now it is clear that the mapping $\theta : \pi_i \mapsto \rho_i$, $i = 1, 2, \ldots, q^{2n-1}$ is an involution without fixed elements in the set $\{\pi_1, \ldots, \pi_{q^{2n-1}}\}$. This contradicts q being odd. ∎

The special case $\mathsf{Q}(6,q)$

From theorem 5.3, it follows that $\mathsf{Q}(6,q)$, q even has spreads. On the other hand, from 3.3 and 4.1, we know that for every ovoid $\mathcal{O}$ of the generalized hexagon $\mathsf{H}(q)$ (being also an ovoid of $\mathsf{Q}(6,q)$) the set of planes $\{p^\perp \| p \in \mathcal{O}\}$ is a spread of $\mathsf{Q}(6,q)$. As $\mathsf{H}(q)$, has ovoids in the case $q = 3^h$, the quadric $\mathsf{Q}(6,q)$, has spreads for $q = 3^h$. As an ovoid of $\mathsf{Q}^+(7,q)$ yields by triality (see 4.1), a spread of $\mathsf{Q}^+(7,q)$, and hence by theorem 7.4 (2) a spread of $\mathsf{Q}(6,q)$, theorem 7.7 (1) yields all values for which spreads of $\mathsf{Q}(6,q)$ exist. We summarize the existence in the following theorem.

Theorem 7.6
The polar space $\mathsf{Q}(6,q)$ has a spread if q is even, if q is an odd prime, or if $q \equiv 0$ or $2 \pmod 3$.

Open problems

The existence of spreads of $\mathsf{Q}(4n+2,q)$, $n > 1$ and q odd, is still open. So far there are no spreads known of the quadric $\mathsf{Q}(6,q)$, q odd and $q \equiv 1 \pmod 3$.

7.2.3 Spreads of $\mathsf{Q}^+(4n+3,q)$.

The quadric $\mathsf{Q}^+(7,q)$ is of course a special case here. As a consequence of the discussion on $\mathsf{Q}(6,q)$ given in the former section, we can summarize the results as given in part (1) of the next theorem, see also theorem 8.8 where references are given.

Theorem 7.7

(1) The quadric $\mathsf{Q}^+(7,q)$, has a spread if q is even, or q is an odd prime, or $q \equiv 0$ or $2 \pmod 3$.

(2) The quadric $\mathsf{Q}^+(4n+3,q)$ has a spread if and only if $\mathsf{Q}(4n+2,q)$ has a spread. Hence $\mathsf{Q}^+(4n+3,q)$, q even, has always spreads.

Open problems

The existence of spreads of $\mathsf{Q}^+(4n+3,q)$, q odd and $n > 1$ as well as the existence of ovoids of $\mathsf{Q}^+(7,q)$, with q odd, not a prime and $q \equiv 1 \pmod 3$ are still open.

7.3 Spreads of Hermitian polar spaces

In 3.5 it has been remarked that $\mathsf{H}(3,q^2)$ and $\mathsf{H}(4,4)$ have no spreads, while the existence is still open for $\mathsf{H}(4,q^2)$, $q > 2$. Although it is quite interesting to find spreads of Hermitian polar spaces, as can be seen in the following theorems, no spreads are known.

Theorem 7.8
If $\mathsf{H}(2n+1,q^2)$, $n \geq 2$, has a spread, then also $\mathsf{H}(2n,q^2)$ has a spread.

Proof. Let Φ be a spread of $\mathsf{H}(2n+1,q^2)$, $n \geq 2$, and intersect $\mathsf{H}(2n+1,q^2)$ with a hyperplane $\mathsf{PG}(2n,q^2)$ of $\mathsf{PG}(2n+1,q^2)$ which is not tangent to $\mathsf{H}(2n+1,q^2)$. Then the intersection of Φ with $\pi = \mathsf{PG}(2n,q^2)$ is a spread Φ_π of $\mathsf{H}(2n,q^2) = \pi \cap \mathsf{H}(2n+1,q^2)$. ■

Theorem 7.9 ([58])

1. *If $\mathsf{H}(2n,q^2)$ $n \geq 2$, has a spread then also $\mathsf{Q}^-(4n+1,q)$ has a spread.*

2. *If $\mathsf{H}(2n+1,q^2)$, $n \geq 1$, has a spread then also $\mathsf{Q}^+(4n+3,q)$ has a spread.*

Proof.

1. Consider the polar space $\mathsf{Q}^-(4n+1,q)$, $n \geq 1$. In the extension $\mathsf{PG}(4n+1,q^2)$ of $\mathsf{PG}(4n+1,q)$, the polar space $\mathsf{Q}^-(4n+1,q)$ extends to the polar space $\mathsf{Q}^+(4n+1,q^2)$. On $\mathsf{Q}^+(4n+1,q^2)$ it is possible to choose a projective $2n$-space π with $\pi \cap \overline{\pi} = \emptyset$, where $\overline{\pi}$ is conjugate

to π w.r.t. the quadratic extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$. The lines of $\mathsf{Q}^-(4n+1,q)$ whose extensions intersect π and $\overline{\pi}$ form a line spread T^- of $\mathsf{Q}^-(4n+1,q)$. It can be shown that the points common to π and the extensions of the lines of T^- form a Hermitian variety $\mathsf{H}(2n,q^2)$ of π. Consider a spread Φ of $\mathsf{H}(2n,q^2)$, $n \geq 2$. If $\xi \in \Phi$, then the union of the lines of T^- whose extensions contain a point of ξ is a projective $(2n-1)$-space, i.e. a generator on $\mathsf{Q}^-(4n+1,q)$. It is clear that any two such generators are skew. Since $|\Phi| = q^{2n+1}+1$, the spaces ξ form a spread of $\mathsf{Q}^-(4n+1,q)$.

2. Consider the polar space $\mathsf{Q}^+(4n+3,q)$, $n \geq 0$. In the extension $\mathsf{PG}(4n+3,q^2)$ of $\mathsf{PG}(4n+3,q)$, the polar space $\mathsf{Q}^+(4n+3,q)$ extends to the polar space $\mathsf{Q}^+(4n+3,q^2)$. It can be shown that the polar space $\mathsf{Q}^+(4n+3,q^2)$ contains a maximal singular subspace π with $\pi \cap \overline{\pi} = \emptyset$, where $\overline{\pi}$ is conjugate to π w.r.t. the quadratic extension $\mathsf{GF}(q^2)$ of $\mathsf{GF}(q)$. The lines of $\mathsf{Q}^+(4n+3,q)$ whose extensions intersect π and $\overline{\pi}$ form a line spread T^+ of $\mathsf{Q}^+(4n+3,q)$. It can be shown that the points common to π and the extensions of the lines of T^+ form a Hermitian variety $\mathsf{H}(2n+1,q^2)$ of π. The rest of the proof is similar to the proof of the first part.

■

An important non-existence theorem however is the following theorem.

Theorem 7.10 ([58])
The polar space $\mathsf{H}(2n+1,q^2)$ *has no spread.*

Open problem

The existence of spreads in $\mathsf{H}(2n,q^2)$, $n \geq 2$, and $(n,q) \neq (2,2)$ is still open. Note that especially the case q is odd is tempting, as the existence of a spread of $\mathsf{H}(2n,q^2)$, q odd, would imply a solution (by theorem 7.9) of the open problem on the existence of spreads of $\mathsf{Q}^-(4n+1,q)$, q odd.

Remark

P. Kleidman classified in an unpublished paper [36] all spreads of finite polar spaces admitting a 2-transitive group.

8 Ovoids of polar spaces

8.1 Ovoids of the symplectic polar space

The following lemma will be used in the proof of theorem 8.2.

Lemma 8.1 ([50])
If $S_{m,q}$ is a point set of $\mathsf{PG}(m,q)$ such that each hyperplane intersects $S_{m,q}$ in 1 or n, $n > 1$ points , and such that there is at least one hyperplane intersecting in 1 point, then $S_{m,q}$ is either a line of $\mathsf{PG}(m,q)$ or $m = 3$ and $S_{3,q}$ is an ovoid of $\mathsf{PG}(3,q)$.

Theorem 8.2 ([55])
The polar space $\mathsf{W}_n(q)$, $n = 2t+1$ and $t > 1$ has no ovoid.

Proof. Assume that $\mathcal{O}$ is an ovoid of $\mathsf{W}_n(q)$, $n > 3$. We investigate the intersection of $\mathcal{O}$ with a hyperplane $\Pi = \mathsf{PG}(n-1,q)$ of $\mathsf{PG}(n,q)$. Let σ be the symplectic polarity defining $\mathsf{W}_n(q)$, and let $x = \Pi^\sigma$.
Suppose that $x \in \mathcal{O}$, and that $y \in \mathcal{O} \cap \Pi$, $y \neq x$. A maximal totally isotropic subspace through x and y should intersect $\mathcal{O}$ in a unique point, a contradiction, hence $|\Pi \cap \mathcal{O}| = 1$.
Suppose that $x \notin \mathcal{O}$ and let y be a point of $\mathcal{O} \cap \Pi$. A maximal isotropic subspace through x intersects $\mathcal{O}$ in a unique point, which is a point of x^σ. We count in two ways the ordered pairs of the set $\{(x,y) \| \pi$ is a maximal totally isotropic subspace in x^σ containing x and y, $y \in \pi \cap \mathcal{O}\}$. Fixing y, a maximal totally isotropic subspace through x and y is contained in $(xy)^\sigma$, being an $(n-2)$-dimensional subspace intersecting $\mathsf{W}_n(q)$ in a cone with vertex the line xy projecting a polar space $\mathsf{W}_{n-4}(q)$. Each maximal totally isotropic subspace of $\mathsf{W}_{n-4}(q)$ union xy, is a maximal totally isotropic subspace of $\mathsf{W}_n(q)$. Hence the number of maximal totally isotropic subspaces through xy equals $|\Sigma(\mathsf{W}_{n-4}(q))| = (q+1)(q^2+1)\dots(q^{\frac{n-3}{2}}+1)$. We now fix π, $x \in \pi$. The space π defines exactly one point y of $\mathcal{O}$. The hyperplane x^σ intersects $\mathsf{W}_n(q)$ in a cone with vertex x projecting $\mathsf{W}_{n-2}(q)$. The maximal isotropic subspaces of $\mathsf{W}_{n-2}(q)$ together with x define the maximal isotropic subspaces of $\mathsf{W}_n(q)$. Hence through x there are $|\Sigma(\mathsf{W}_{n-2}(q))| = (1+q)(1+q^2)\dots(q^{\frac{n-1}{2}}+1)$ maximal totally isotropic subspaces.
Hence
$$|\Sigma(\mathsf{W}_{n-2}(q))| = |\Pi \cap \mathcal{O}|.|\Sigma(\mathsf{W}_{n-4}(q))|,$$
or
$$|\Pi \cap \mathcal{O}| = q^{\frac{n-1}{2}} + 1.$$
We may conclude that $|\Pi \cap \mathcal{O}| \in \{1, q^{\frac{n-1}{2}}+1\}$. By lemma 8.1 $n = 3$, a contradiction. ∎

8.2 Ovoids of quadrics

In 4.1 it has been proved that every ovoid of the classical generalized hexagon $\mathsf{H}(q)$ is also an ovoid of $\mathsf{Q}(6,q)$ (and conversely) and we have seen that ovoids of $\mathsf{H}(q)$ exist for $q = 3^h$. For general odd values of q the following theorem is known.

Theorem 8.3 ([39])
If every ovoid of $\mathsf{Q}(4,q)$, $q \neq 3$ *and* q *odd, is an elliptic quadric, then* $\mathsf{Q}(6,q)$ *has no ovoids.*

Corollary

The polar spaces $\mathsf{Q}(6,5)$ and $\mathsf{Q}(6,7)$ have no ovoids.

Open problem

If q is odd and $q \neq 5, 7, 3^h$ then no ovoids of $\mathsf{Q}(6,q)$ are known to exist.

Theorem 8.4 ([58])

1. *If the polar space* $\mathsf{Q}(2n,q)$, $n \geq 3$, *has ovoids then the polar space* $\mathsf{Q}(2n-2,q)$ *has also ovoids.*
2. *If the polar space* $\mathsf{Q}^-(2n+1,q)$, $n \geq 3$, *has ovoids then* $\mathsf{Q}^-(2n-1,q)$ *has also ovoids.*
3. *If the polar space* $\mathsf{Q}^+(2n+1,q)$, $n \geq 2$, *has ovoids then* $\mathsf{Q}^+(2n-1,q)$ *has also ovoids.*

Proof. We only prove here the first assertion. Assume that $\mathcal{O}$ is an ovoid of $\mathsf{Q}(2n,q)$, $n \geq 3$. Let x be a point of $\mathsf{Q}(2n,q) \setminus \mathcal{O}$. Let σ be the polarity defining the quadric. The cone $x^\sigma \cap \mathsf{Q}(2n,q)$ with vertex x projects a non-singular quadric $\mathsf{Q}(2n-2,q)$. The projection $\mathcal{O}'$ of $\mathcal{O} \cap x^\sigma$ from x on $\mathsf{Q}(2n-2,q)$ defines an ovoid of $\mathsf{Q}(2n-2,q)$. By construction no two points on $\mathcal{O}'$ are collinear. A maximal totally singular subspace of $\mathsf{Q}(2n-2,q)$ together with x defines a maximal totally singular subspace of $\mathsf{Q}(2n,q)$, which contains exactly one point of $\mathcal{O}$. Hence the projection contains exactly one point of $\mathcal{O}'$.
The 2 other assertions are proved in the same way. ∎

Theorem 8.5 ([24])
If $n \geq 4$ *then the polar space* $\mathsf{Q}(2n,q)$, q *odd has no ovoid.*

Theorem 8.6 ([55])
The polar space $\mathsf{Q}^-(2n+1,q)$ *with* $n > 1$ *has no ovoid.*

Theorem 8.7
The polar space $\mathsf{Q}^+(5,q)$ *has ovoids.*

Proof. Let $\mathcal{S}$ be a spread of $\mathsf{PG}(3,q)$ and let π be a plane of $\mathsf{PG}(3,q)$. It is clear that π contains at most one line of $\mathcal{S}$. As the number of planes containing a line of $\mathcal{S}$ equals $|\mathcal{S}|.(q+1) = q^3+q^2+q+1$, which is the total number of planes, we can conclude that π contains exactly one line of $\mathcal{S}$. Let $\mathcal{O}$ be the subset of points of $\mathsf{Q}^+(5,q)$, being the elements under the

Klein correspondence of elements of $\mathcal{S}$. One easily proves that each plane of $\mathsf{Q}^+(5,q)$ intersects $\mathcal{O}$ in exactly one point, hence that $\mathcal{O}$ is an ovoid of $\mathsf{Q}^+(5,q)$. ■

Theorem 8.8
The polar space $\mathsf{Q}^+(7,q)$ has an ovoid if q is even, or q is an odd prime, or $q \equiv 0$ or $2 \pmod 3$.

Remark

Because of triality, this theorem is the same as theorem 7.7 (1). Up to isomorphism there is exactly one ovoid (and hence one spread) of $\mathsf{Q}^+(7,2)$ [33] and $\mathsf{Q}^+(7,3)$ [41]. A sporadic ovoid of $\mathsf{Q}^+(7,5)$ has been constructed in [15], while one of $\mathsf{Q}^+(7,7)$ was constructed in [44]. For constructions of ovoids for the other values of q we refer to [33, 32, 31, 53]. If p is an odd prime, the existence of ovoids in $\mathsf{Q}^+(7,p)$ was proved in [14].

Theorem 8.9

1. *The polar space $\mathsf{Q}^+(2n+1,2)$, $n \geq 4$, has no ovoid* [33].
2. *The polar space $\mathsf{Q}^+(2n+1,3)$, $n \geq 4$, has no ovoid* [45].
3. *The polar space $\mathsf{Q}^+(2n+1,q)$, $q = p^h$, has no ovoid if $p^n > \binom{2n+p}{p-1}$* [6].

Corollary

If $n \geq 4$ then $\mathsf{Q}^+(2n+1,q)$, $q = p^h$, has no ovoid if $p = 2$ or 3. If $n \geq 5$ then $\mathsf{Q}^+(2n+1,q)$, $q = p^h$, has no ovoid if $p = 5$ or 7.

8.3 Ovoids of Hermitian polar spaces

Theorem 8.10 ([55])
The polar space $\mathsf{H}(2n,q^2)$ $n \geq 2$, has no ovoid.

Theorem 8.11 ([58])

1. *If the polar space $\mathsf{H}(2n+1,q^2)$, $n \geq 2$, has an ovoid then $\mathsf{H}(2n-1,q^2)$ has an ovoid.*
2. *If the polar space $\mathsf{Q}^-(4n+1,q)$, $n \geq 2$ has an ovoid then $\mathsf{H}(2n,q^2)$ has an ovoid.*
3. *If the polar space $\mathsf{Q}^+(4n+3,q)$, $n \geq 1$ has an ovoid then $\mathsf{H}(2n+1,q^2)$ has an ovoid.*

Open problem

The existence of ovoids in $\mathsf{H}(2n+1,q^2)$ for $n > 1$ is still open.

Remark

In [35], Kleidman classifies all ovoids of polar spaces admitting a 2-transitive group.

9 Applications

9.1 Designs arriving from ovoids and spreads

Let Φ be a spread of the polar space $\mathsf{Q}(2n,q)$. The line set of Q is denoted by $\mathcal{L}$, and the set of all elements of $\mathcal{L}$ belonging to an element of Φ is denoted by $\mathcal{L}^*$. Then the incidence structure $\mathcal{D} = (\mathcal{P},\mathcal{B},\mathrm{I})$ is defined as follows: $\mathcal{P} = \Phi$, $\mathcal{B} = \mathcal{L}\setminus\mathcal{L}^*$; if $\mathsf{PG}(n-1,q) \in \Phi$ and $L \in \mathcal{L}\setminus\mathcal{L}^*$, then $\mathsf{PG}(n-1,q)\ \mathrm{I}\ L$ iff $\mathsf{PG}(n-1,q)\cap L \neq \emptyset$. One easily proves that $\mathcal{D}$ is a $3-(q^n+1, q+1, \frac{q^{n-1}-1}{q-1})$ design.
If $n=3$ then $\mathcal{D}$ is a $3-(q^3+1,q+1,q+1)$ design. In this case another design can be constructed as follows. Let us denote the set of all totally singular planes by Ω^*. Then the incidence structure $\mathcal{D}' = (\mathcal{P}',\mathcal{B}',\mathrm{I}')$ is defined as follows: $\mathcal{P}' = \Phi$, $\mathcal{B} = \Omega^*\setminus\Phi$. If $\pi\in\Phi$ and $\pi'\in\Omega^*\setminus\Phi$, then $\pi\ \mathrm{I}'\ \pi'$ iff $\pi\cap\pi'\neq\emptyset$. One easily proves that $\mathcal{D}'$ is a $3-(q^3+1, q^2+1, (q+1)(q^2-1))$ design.
Let $\mathsf{Q}^+(4n+3,q)$ be a non-singular hyperbolic quadric of $\mathsf{PG}(4n+3,q)$ and let Φ be a spread of $\mathsf{Q}^+(4n+3,q)$. The set of all lines on $\mathsf{Q}^+(4n+3,q)$ is denoted by $\mathcal{L}$, and the set of all elements of $\mathcal{L}$ belonging to an element of Φ is denoted by $\mathcal{L}^*$. Then $\mathcal{D} = (\mathcal{P},\mathcal{B},\mathrm{I})$ is defined as follows: $\mathcal{P}=\Phi$, $\mathcal{B} = \mathcal{L}\setminus\mathcal{L}^*$; if $\mathsf{PG}(2n+1,q)\in\Phi$ and $L\in\mathcal{L}\setminus\mathcal{L}^*$, then $\mathsf{PG}(2n+1,q)\ \mathrm{I}\ L$ iff $\mathsf{PG}(2n+1,q)\cap L\neq\emptyset$. One easily proves that $\mathcal{D}$ is a $3-(q^{2n+1}+1, q+1, \frac{q^{2n+2}-1}{q-1})$ design.
Let $\mathcal{O}$ be an ovoid of $\mathsf{Q}^+(4n+3,q)$. Define $\mathcal{D} = (\mathcal{P},\mathcal{B},\mathrm{I})$ as follows: $\mathcal{P} = \mathcal{O}$, $\mathcal{B} = \mathsf{Q}^+(4n+3,q)\setminus\mathcal{O}$; if $x\in\mathcal{O}$ and $y\in\mathsf{Q}^+(4n+3,q)\setminus\mathcal{O}$, then $x\ \mathrm{I}\ y$ iff xy is a line of $\mathsf{Q}^+(4n+3,q)$. One easily proves that $\mathcal{D}$ is a $3-(q^{2n+1}+1, q^{2n}+1, \frac{q^{4n}-1}{q-1})$ design.

9.2 Codes and spreads

A (simple) graph Γ is called *distance regular* iff the number of vertices at distance i from x and distance j from y is only dependent of the distance $\mathsf{d}(x,y)$ of x and y. For a positive integer e, a perfect e code is a nonempty subset C of the vertex set with the property that any vertex lies at distance at most e from a unique vertex in C.
In [12], [52] and [54] the following infinite classes of perfect 1-codes in distance regular graphs were constructed.
Consider the polar space $\mathsf{W}_5(q)$. Define a graph $\Gamma(\mathsf{W}_5(q))$ with vertex set, the set of all maximal totally isotropic subspaces, and $\pi\sim\pi'$ iff $\pi\cap\pi'$ is a line. One can prove that $\Gamma(\mathsf{W}_5(q))$ is a distance regular graph and a perfect

1-code is the same thing as a spread of $\mathsf{W}_5(q)$. Hence for any q, q a prime power, there arises perfect 1-codes.
Next, consider the polar space $\mathsf{Q}(6,q)$. Define a graph $\Gamma(\mathsf{Q}(6,q))$ with vertex set the set of all maximal totally singular spaces on $\mathsf{Q}(6,q)$, and $\pi \sim \pi'$ iff $\pi \cap \pi'$ is a line. One can prove that $\Gamma(\mathsf{Q}(6,q))$ is a distance regular graph and a perfect 1-code is the same thing as a spread of $\mathsf{Q}(6,q)$, hence for any prime power q, $q \not\equiv 1 \pmod 3$, there arises a perfect code. Note that $\Gamma(\mathsf{W}_5(q))$ and $\Gamma(\mathsf{Q}(6,q))$ have the same parameters, but are isomorphic iff q is even. Finally remark that the point graph $\Gamma(\mathsf{H}(q))$ of the generalized hexagon $\mathsf{H}(q)$ is a distance regular subgraph of the graph of $\Gamma(\mathsf{Q}(6,q))$ and that an ovoid of $\mathsf{H}(q)$ yields a perfect 1-code of $\Gamma(\mathsf{H}(q))$ as well as of $\Gamma(\mathsf{Q}(6,q))$.

9.3 Ovoids, spreads and translation planes

Let Φ be a spread of $\mathsf{W}_{2m-1}(q)$, then Φ is also a spread of $\mathsf{PG}(2m-1,q)$ (see theorem 7.1). Hence by the Bruck-Bose construction (see page 1 in the chapter on unitals for the case $m=2$), there corresponds a translation plane $\Pi(\Phi)$ of order q^m with Φ. Now, let q be even, $q=2^h$, and consider a spread Φ' of $\mathsf{Q}(2m,q)$. If we project Φ' from the nucleus of Q onto a $\mathsf{PG}(2m-1,q)$ not containing x, then there arises a spread Φ' of some $\mathsf{W}_{2m-1}(q)$. Moreover, every spread of $\mathsf{W}_{2m-1}(q)$, q even, can be obtained in such a way.
Let Φ be a spread of $\mathsf{W}_{4n+1}(q)$, q even, and let Φ' be the corresponding spread of $\mathsf{Q}(4n+2,q)$. Embed Q in a hyperbolic quadric $\mathsf{Q}^+(4n+3,q)$ of $\mathsf{PG}(4n+3,q)$, and consider all elements of one family of maximal singular subspaces, which are $(2n+1)$-spaces, of $\mathsf{Q}^+(4n+3,q)$ which contain an element of Φ'. Then there arises a spread Φ'' of $\mathsf{Q}^+(4n+3,q)$. Now we intersect Φ'' with a hyperplane $\pi = \mathsf{PG}(4n+2,q)$ of $\mathsf{PG}(4n+3,q)$, which yields a spread Φ'_π of $\mathsf{Q}_\pi(4n+2,q) = \mathsf{Q}^+(4n+3,q)\cap\pi$. By projection of Φ'_π from the nucleus of $\mathsf{Q}_\pi(4n+2,q)$, there arises a spread Φ_π of the polar space $\mathsf{W}_{4n+1}(q)$. It is proved by Kantor [34] that Φ_π depends on the choice of π and that Φ_π is not necessarily isomorphic to Φ. Hence, one spread Φ of $\mathsf{W}_{4n+1}(q)$, yields a lot of spreads Φ_π of $\mathsf{W}_{4n+1}(q)$. The spreads Φ_π as well as the planes $\Pi(\Phi_\pi)$ are called by Kantor, the *cousins* of Φ and $\Pi(\Phi)$. If Φ is the regular $2n$-spread of $\mathsf{PG}(4n+1,q)$, q even and $(n,q)\neq(1,2)$ then $\Pi(\Phi)$ is the Desarguesian affine plane $\mathsf{AG}(2,q^{2n+1})$. It is proved by Kantor [34] that all cousins of Φ are not regular, moreover he proves that this construction yields at least $\frac{q-1}{h}+1$, pairwise non-isomorphic non-Desarguesian cousins of $\mathsf{AG}(2,q^{2n+1})$.
An ovoid of $\mathsf{Q}^+(7,q)$ yields also an ovoid of the Klein quadric $\mathsf{Q}^+(5,q)$ (see theorem 8.4), and hence a line spread of $\mathsf{PG}(3,q)$, which in turn yields by the Bruck-Bose construction a translation plane of order q^2. By the triality an ovoid of $\mathsf{Q}^+(7,q)$ yields also a spread, and hence when q is even, it yields by the above construction, translation planes of order q^3. For a discussion on translation planes and ovoids of quadrics we refer to [30].

9.4 Kerdock sets and spreads

Let G be the group of all non-singular skew symmetric $2n \times 2n$ matrices with zero diagonal over $\mathsf{GF}(q)$. A *Kerdock* set is a subset K of G consisting of q^{2n-1} matrices such that the difference of any two of them is non-singular. Kerdock sets are used to construct non-linear error-correcting codes. Kantor [32], [31] proves that the existence of a spread of $\mathsf{Q}^+(4n-1,q)$ is equivalent to the existence of a Kerdock set. Actually the uniqueness of the spread of $\mathsf{Q}^+(7,3)$ was proved by Patterson [41] using Kerdock sets.

9.5 Spreads and maximal arcs

In a finite projective plane π of order q (not necessarily Desarguesian) each set $\mathcal{K}(\neq \emptyset)$ of k points can be seen as a $\{k;d\}$-arc, where $d(\neq 0)$ is the maximum number of points of $\mathcal{K}$ on a line of the plane. Given q and d, then $k \leq (q+1)(d-1)+1$. If equality holds the set $\mathcal{K}$ is called a *maximal arc of degree* d. Equivalently such an arc $\mathcal{K}$ can be defined as a non-empty set of points in the plane, such that any line intersects $\mathcal{K}$ in 0 or d points. If $d=1$, then of course $\mathcal{K}$ is a point, if $d=q$ then $\mathcal{K}$ is an affine plane of order q obtained by deleting a line L of π. If $\mathcal{K}$ is a maximal arc of degree d, $d \leq q$, then one easily shows that $\overline{\mathcal{K}} = \{$lines L of π such that $L \cap \mathcal{K} = \emptyset\}$ is a $\{q(q-d+1)/d; q/d\}$-arc, i.e. a maximal arc of degree q/d, in the dual projective plane $\overline{\pi}$. Hence, if π is Desarguesian then a maximal arc of degree d, $d \leq q$, yields one of degree q/d. In [3] (see also [2]) it is proved that a Desarguesian plane of odd order q cannot contain a maximal arc of degree $1 < d < q$. This result generalizes an old result of Cossu [16] who proved that $\mathsf{PG}(2,9)$ has no $\{21;3\}$-arc, and of Thas [51] who proved the nonexistence of maximal arcs of degree 3 in Desarguesian projective planes of order $q = 3^h (h > 1)$. By computer Denniston has proved that none of the 3 non-Desarguesian planes of order 9 has a maximal arc of degree 3. Desarguesian planes of order $q = 2^h$ do have maximal arcs of degree d for any divisor d of q. One of the constructions goes back to Denniston [21].
The following construction using spreads is taken from [53]. Let $\mathsf{Q}^-(2n-1,q)$, $n \geq 2$ be an elliptic quadric, and let Φ^- be a spread of $\mathsf{Q}^-(2n-1,q)$, hence it consists of q^n+1 $(n-2)$-dimensional spaces which partition the point set of $\mathsf{Q}^-(2n-1,q)$. Let $\Phi = \{\pi_1, \ldots, \pi_{q^n+1}\}$ be an $(n-1)$-spread of $\mathsf{PG}(2n-1,q)$, such that $\{\pi_1 \cap \mathsf{Q}^-(2n-1,q), \pi_2 \cap \mathsf{Q}^-(2n-1,q), \pi_{q^n+1} \cap \mathsf{Q}^-(2n-1,q)\} = \Phi^-$. Embed $\mathsf{PG}(2n-1,q)$ as a hyperplane Π_∞ in $\mathsf{PG}(2n,q) = \Pi$ and consider a point x of $\Pi \setminus \Pi_\infty$. Then it is proved that the set $\mathcal{K} = \{\cap xy \| y \in \mathsf{Q}^-(2n-1,q)\} \setminus \mathsf{Q}^-(2n-1,q)$ is a maximal arc of degree q^{n-1} in the projective plane of order q^n defined by Φ. The construction is also valid if $n=2$ and replacing Φ^- by the points of an ovoid. If q is odd, then a pair $\{\Phi^-, \Phi\}$ with the above property does not exist [5]. If q is even, a spread Φ^* of the quadric $\mathsf{Q}(2n,q)$ will yield a pair $\{\Phi^-, \Phi\}$ with the above

property. The spread Φ^- is constructed as explained in theorem 7.4, while the spread Φ is the symplectic spread obtained by projecting down from the nucleus. We remark that this construction has recently been generalized [26], by using m-systems.

9.6 Partial and semipartial geometries and spreads

A (finite) *partial geometry* $\mathsf{pg}(s,t,\alpha)$ is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order (s,t) such that for all anti-flags (x,L) the incidence number $\alpha(x,L)$ is a constant $\alpha(\neq 0)$. The numbers s, t and α are called the *parameters* of $\mathcal{S}$.

Partial geometries arising from maximal arcs can be constructed. Moreover, spreads of the quadric $\mathsf{Q}^+(4n+3,q)$ for $q = 2,3$ yield partial geometries [18], [56], [17]. We refer to [19] for more details.

A *semipartial geometry* $\mathsf{spg}(s,t,\alpha,\mu)$ with parameters s,t,α,μ is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order (s,t), such that for each anti-flag (x,L), the incidence number $\alpha(x,L)$ equals 0 or a constant $\alpha(>0)$ and such that for any two points which are not collinear, there are μ ($\mu > 0$) points collinear with both (μ-condition). Semipartial geometries were introduced in [20]. For an overview of theory and examples we refer to [19].

In [57] a construction method for semipartial geometries based on spreads of polar spaces is introduced. A spread Φ of $\mathsf{Q}^-(2m+3,q)$ $(m \geq 0)$ yields a $\mathsf{spg}(q^{m+1}-1, q^{m+2}, q^m, q^{m+1}(q^{m+1}-1))$. If $m > 1$, and q is even, then the quadric $\mathsf{Q}^-(2m+3,q)$ has spreads, hence this yields semipartial geometries. A spread of $\mathsf{H}(n,q^2)$ would yield a semipartial geometry $\mathsf{spg}(q^n-1, q^{n+1}, q^{n-1}, q^n(q^n-1))$.

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