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Plenary talks

On the Cauchy problem for p-evolution equations in Gelfand-Shilov type spaces

Marco Cappiello *(joint work with Alexandre Arias Junior and Alessia Ascanelli)*

We consider the initial value problem

$$\begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x), \\ u(0,x) = g(x), \end{cases} \tag{1}$$

for $(t,x) \in [0,T] \times \mathbb{R}$, where $P(t,x,D_t,D_x)$ is an operator of the form

$$P(t,x,D_t,D_x) = D_t u + a_p(t) D_x^p u(t,x) + \sum_{i=0}^{p-1} a_i(t,x) D_x^i u(t,x),$$

with $D = \frac{1}{i}\partial$, $p \geq 2$, $t \in [0,T], x \in \mathbb{R}$, $a_p \in C([0,T],\mathbb{R})$, $a_p(t) \neq 0$ for all $t \in [0,T]$ and $a_i(t,x) \in C([0,T], C^\infty(\mathbb{R};\mathbb{C}))$ $i = 0,\dots,p-1$. The operator P is known in literature as $p-$evolution operator. The condition that a_p is real valued means that the principal symbol (in the sense of Petrowski) of P has the real characteristic $\tau = -a_p(t)\xi^p$; by the Lax-Mizohata theorem, this condition is necessary to have a unique solution, in Sobolev spaces, of the Cauchy problem (1) for any $p \geq 1$. When the coefficients $a_i(t,x)$, $i = 0,\dots,p-1$ are real and of class $\mathcal{B}^\infty$ with respect to x (uniformly bounded together with all their x-derivatives), it is well known that the problem (1) is well-posed in $L^2(\mathbb{R})$ and in Sobolev spaces H^m, $m \in \mathbb{R}$. If some of the coefficients $a_i(t,x)$ are complex valued, then some decay conditions at infinity on the imaginary part of the coefficients a_i are needed in order to obtain well-posedness either in L^2, or in $H^\infty(\mathbb{R}) = \cap_{m\in\mathbb{R}} H^m$ or in Gevrey classes, in general with a loss of derivatives. Here we study the problem (1) taking initial data in Gelfand-Shilov type spaces and we analyze in detail the role of the exponential decay of the initial datum on the regularity and behavior of the solution for $|x| \to \infty$. To study this problem we need to introduce a suitable class of pseudo-differential operators of infinite order, that is with symbols admitting exponential growth at infinity.

Thick distributions and their Fourier transforms

Ricardo Estrada

Thick test functions and thick distributions in one variable were introduced by Estrada and Fulling in 2007 in order to explain several puzzles, apparent paradoxes in the applications of distribution theory in quantum field theory. In that study the Fourier transform of thick distributions was defined and its main properties were established.

A theory of multidimensional thick distributions was developed by Yang and Estrada in several articles, starting in 2013. The theory of thick distributions in several variables is quite different from the one dimensional theory, due to fact that removing a point from a line gives a disconnected space while doing so in spaces of higher dimension does not. Those studies did not develop a Fourier transform of thick distributions.

Very recently in a joint work by Estrada, Yang and Vindas the Fourier transform of thick distributions was constructed and its main properties were proved.

In this talk we will give a tour of these developments, from the ideas behind thick distributions, their definition and the problems they help understand, to the construction of the Fourier transform in one and several variables.

Singularity Theorems in General Relativity and Distributional Geometry

Michael Kunzinger *(joint work with M. Graf, J. Grant, R. Steinbauer, J. Vickers)*

The recent discovery of gravitational waves originating from black hole mergers has initiated a new era of experimental physics. It has also drawn a lot of attention to the mathematical aspects of singularity formation in general relativity. This field was created by Roger Penrose and Stephen Hawking about 50 years ago, when they used topological and variational techniques to show the genericity of singularity formation under mild hypotheses on spacetime curvature, resulting in their celebrated singularity theorems. It is of eminent interest, both from the point of view of physics and that of Lorentzian geometry, to extend the validity of these results to spacetime metrics of low regularity. In this talk we will highlight some of the results already established and the mathematical challenges one faces when addressing this problem. In particular we will concentrate on the interplay between geometric and analytic (in particular, distributional) methods that have been brought to bear on this question.

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Ellipticity and the Fredholm property in the Weyl-Hörmander calculus

Bojan Prangoski *(joint work with Stevan Pilipovic)*

In this talk we present results concentring the Fredholm properties and ellipticity of pseudodifferential operators with symbols in the Hörmander classes $S(M, g)$. The main result is that the Fredholm property of a ΨDO acting on Sobolev spaces in the Weyl-Hörmander calculus and the ellipticity are equivalent when the Hörmander metric is geodesically temperate and its associated Planck function vanishes at infinity. Additionally, assuming the Hörmander metric is geodesically temperate (and consequently the calculus is spectrally invariant), we prove that the inverse $\lambda \mapsto b_\lambda \in S(1, g)$ of every $\mathcal{C}^N$, $0 \leq N \leq \infty$, map $\lambda \mapsto a_\lambda \in S(1, g)$ comprised of invertible elements on L^2 is again of class $\mathcal{C}^N$.

Ultradifferentiable extension theorems

Armin Rainer *(joint work with G. Schindl)*

Whitney's classical extension theorem provides conditions for the extension of jets defined in a closed subset of $\mathbb{R}^n$ to infinitely differentiable functions on $\mathbb{R}^n$. There is a long history of research how growth constraints on the jets are preserved by their extensions. I will give an overview of the known results and focus on our recent characterisation of the validity of Whitney's extension theorem in the ultradifferentiable Roumieu setting with controlled loss of regularity. This result is based on the uniform description of ultradifferentiable function classes by weight matrices.

On a construction of self-similar solutions to nonlinear wave equations

Mitsuru Sugimoto

An attempt to construct self-similar solutions to nonlinear wave equations $\Box u = |u|^p$ is explained. The existence of self-similar solutions has been already established by Pecher (2000), Kato-Ozawa (2003), etc. based on the standard fixed point theorem. In this talk, we will discuss it by a constructive method using the theory of hypergeometric differential equations.

Analytic pseudo-differential calculus via the Bargmann transform

Joachim Toft *(joint work with Nenad Teofanov, Patrik Wahlberg)*

The Bargmann transform maps Fourier-invariant function and distribution spaces to certain spaces of formal power series expansions, which sometimes are convenient classes of analytic functions.
Pilipović spaces is a family of Fourier invariant subspaces of the Schwartz space and are defined by imposing suitable boundaries on the Hermite coefficients of the involved functions or distributions. The family of Pilipović spaces contains all Fourier invariant Gelfand-Shilov spaces as well as other spaces which are strictly smaller than any Fourier invariant non-trivial Gelfand-Shilov space. In the same way, the family of Pilipović distribution spaces contains spaces which are strictly larger than any Fourier invariant Gelfand-Shilov distribution space.
In the talk we show that the Bargmann images of Pilipović spaces and their distribution spaces are convenient classes of analytic functions or power series expansions which are suitable when investigating analytic pseudo-differential operators.
We also deduce continuity properties for such pseudo-differential operators when the symbols and target functions possess certain (weighted) Lebesgue estimates. We also show that the counter image with respect to the Bargmann transform of these results generalise some continuity results for (real) pseudo-differential operators with symbols in modulation spaces, when acting on other modulation space.

Reference:

N. Teofanov, J. Toft *Pseudo-differential calculus in a Bargmann setting*, Ann. Acad. Sci. Fenn. Math. 45 (2020), 227–257.

Extension operators for spaces of smooth functions and Whitney jets

Jochen Wengenroth *(joint work with L. Frerick, E. Jordá)*

For a compact subset $K \subseteq \mathbb{R}^d$ we consider the spaces

$$C^\infty(K) = \{F|_K : F \in C^\infty(\mathbb{R}^d)\} \text{ and}$$
$$\mathcal{E}(K) = \{(\partial^\alpha F|_K)_{\alpha \in \mathbb{N}_0^d} : F \in C^\infty(\mathbb{R}^d)\}$$

of smooth restrictions and Whitney jets, respectively. We discuss when it is possible to construct continuous linear extension operators (or extension operators with prescribed continuity properties) for these spaces.
The ultimate goal, of course, are characterisations in terms of the geometry of K, but we also explain the functional analytic tools which are needed to obtain such results.

On some singular hyperbolic problems

Jens Wirth

The talk will focus on some special models of hyperbolic equations with singular (time-dependent) coefficients and the behaviour of their classical and generalised solutions. After giving an overview about related results, we will explain some basic ideas of an adapted phase space analysis involving additional co-variables. A diagonalisation based approach allows to extract leading order terms of representations of solutions and we will use it to obtain suitable well-posedness statements and to characterise some nonstandard behaviours related to the interaction of singularities of solutions with the singularity in the coefficient.

Contributed talks

Strong maximum principle for nonlinear cooperative elliptic systems with non-linear principal symbol

Georgi Petrov Boyadzhiev *(joint work with N. Kutev)*

In this talk we consider validity of strong maximum principle for the nonlinear system

$$G^k(x, u_k, Du^k, D^2u^k) + \sum_{i=1}^{n} c_{ki}u^i = 0 \tag{2}$$

for $x \in \Omega$ and $k = 1, ...N$. Here Ω is a bounded domain in R^n with C^1 smooth boundary $\partial\Omega$.
The principal symbols G^k are supposed uniformly elliptic ones. Furthermore, functions $G^k(x, s, p, q)$ are assumed non-decreasimg with respect to s and Lipschitz continuous with respect to p variable.
Functions $c_{ki}(x)$ are supposed continuous, $c_{ki}(x) \leq 0$ for $k \neq i$, and $\sum_{i=1}^{n} c_{ki}(x) \geq 0$.
The validity of strong interior maximum principle for the classical sub- and supersolutons of the nonlinear system (1) is shown, as well as the validity of strong boundary maximum principle for the same system.

Beurling integers with RH and large oscillation

Frederik Alexander Broucke *(joint work with G. Debruyne, J. Vindas)*

A set of Beurling generalized primes $\mathcal{P}$ is an unbounded sequence of real numbers $p_1 \leq p_2 \leq \dots$ with $p_1 > 1$. The corresponding set of generalized integers $\mathcal{N}$ is the multiplicative semigroup generated by 1 and $\mathcal{P}$. This concept was introduced by A. Beurling in 1937 to make an abstraction of the regular integers to sets with only a multiplicative structure. As in the classical case, one introduces the prime and integer counting functions, defined as

$$\pi(x) = \sum_{p \in \mathcal{P}, p \leq x} 1, \qquad N(x) = \sum_{n \in \mathcal{N}, n \leq x} 1.$$

The main goal of the theory is to investigate the relation between the asymptotic behavior of these counting functions. Usually, one assumes that one of these is close to its classical counterpart, that is, one either assumes that

$$\pi(x) \sim \mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}u}{\log u}, \quad \text{or} \quad N(x) \sim ax, \text{ for some } a > 0,$$

and investigates implications for the other counting function.
In this talk, we will discuss the existence of a prime system which satisfies the Riemann hypothesis, namely $\pi(x) = \mathrm{Li}(x) + O(\sqrt{x})$, yet the integers display extreme oscillation around the main term ax. This talk is based on joint work with Gregory Debruyne and Jasson Vindas.

Function spaces and geometric inequalities on Lie groups

Tommaso Bruno *(joint work with M. M. Peloso, M. Vallarino)*

In this talk we present a unified theory of Sobolev, Besov and Triebel–Lizorkin spaces on general Lie groups endowed with a sub-Riemannian structure. We discuss their interpolation and algebra properties, embeddings and optimal embedding constants, and related Moser–Trudinger and Poincaré inequalities.

Ternary Grassmanian algebras and generalized white noise analysis

Paula Cerejeiras

In this talk we discuss a ternary Grassmannian algebra which gives raise to a topological algebra of stochastic distributions. We develop the corresponding analogue to white noise analysis which allows the study of stochastic processes with higher order symmetries.

Applications of the Spectral theory on the S-spectrum to fractional diffusion problems

Fabrizio Colombo

The Riesz-Dunford functional calculus, and its extensions to unbounded operators, allow us to define functions of operators such as the fractional powers of a linear operator like the Laplacian. One of the reasons to replace the Laplace operator in the heat equation, by its fractional powers is that the new fractional equation has solutions with finite speed propagation of the heat. This is equivalent, in some cases, to a modification of the Fourier law in the case of homogeneous materials. In the case of non homogeneous material we have to consider more general Fourier's law defined by vector operators with suitable nonconstant coefficients that multiply the spatial derivatives. It is possible to compute directly the fractional powers of vector operators using a new functional calculus, the so called S-functional calculus, based on the notion of S-spectrum, that has been developed in more recent times. In fact, the main problem to develop a spectral theory for vector operators was to understand the natural notion of spectrum for quaternionic linear operators that contain as a particular case vector operators like the gradient operator or its generalizations. This problem was solved only in 2006 with the discovery of the S-spectrum for quaternionic linear operators and since then the quaternionic spectral theory has rapidly grown. In this talk we present the applications of the spectral theory on the S-spectrum to fractional diffusion problems. The main references for the spectral theory on the S-spectrum are listed below.

References

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Sequence space representations for spaces of entire functions with rapid decay on strips

Andreas Debrouwere

In this talk we discuss sequence space representations for a class of Fréchet spaces of entire functions with rapid decay on horizontal strips. In particular, we show that the projective Gelfand-Shilov spaces Σ^1_ν and Σ^ν_1 are isomorphic to $\Lambda_\infty(n^{1/(\nu+1)})$ for $\nu > 0$. Furthermore, we present some results and open problems about sequence space representations for general Gelfand-Shilov spaces.

The saddle-point method for general partition functions

Gregory Debruyne *(joint work with G. Tenenbaum)*

We apply the saddle-point method to derive asymptotic estimates or asymptotic series for the number of partitions of a natural integer into parts chosen from a subset of the positive integers whose associated Dirichlet series satisfies certain analytic properties. This enables grouping in a single statement many cases studied in the literature, as well as a number of new ones.

Translation-Modulation Invariant Banach Spaces of Ultradistributions

Pavel Gjorgi Dimovski *(joint work with S. Pilipovic, B. Prangoski and J. Vindas)*

We define and study a new class of translation-modulation invariant Banach spaces of quasi-analytic ultradistributions. These spaces show stability under Fourier transform and tensor products; furthermore, they have a natural Banach convolution module structure over a certain associated Beurling algebra, as well as a Banach multiplication module structure over an associated Wiener-Beurling algebra.
We also investigate a new class of modulation spaces, the Banach spaces of ultradistributions $\mathcal{M}^F$ on $\mathbb{R}^d$, associated to translation-modulation invariant Banach spaces of ultradistributions F on $\mathbb{R}^{2d}$.
We define ultradistributional wave front sets with respect to translation-modulation invariant Banach spaces of ultradistributions having solid Fourier image.

Existence of strong traces for entropy solutions of degenerate parabolic equations

Marko Erceg *(joint work with D. Mitrovic)*

In this talk we study solutions to the degenerate parabolic equation

$$\partial_t u + \operatorname{div}_x f(u) = \operatorname{div}_x(a(u)\nabla u)\,,$$

subject to the initial condition $u(0,\cdot) = u_0$. Here the degeneracy appears as the matrix $a(\lambda)$ is only positive semi-definite, i.e. it can be equal to zero in some directions. Moreover, the directions can depend on λ, which is the main novelty. Equations

of this form often occur in modelling flows in porous media and sedimentation-consolidation processes.
As a consequence of the degeneracy, solutions could be singular, so one needs to justify the meaning of the initial condition. A standard way is to show that u_0 is the strong trace of a solution u at $t = 0$. The notion of strong traces proved to be very useful in showing the uniqueness of the solution to scalar conservation laws with discontinuous flux.
We prove existence of strong traces for entropy solutions to the equation above under the non-degeneracy conditions. The proof is based on the blow-up techniques, where a variant of microlocal defect functional is used and applied to the kinetic formulation of the equation above.

Smoothing and Strichartz estimates for degenerate Schrödinger-type equations

Serena Federico *(joint work with M. Ruzhansky)*

In this talk we will show that global homogeneous smoothing estimates are satisfied by some time-degenerate Schrödinger operators. Additionally, for the same class of operators, we will derive weighted Strichartz-type estimates and apply them to the local well-posedness of the corresponding semilinear Cauchy problem.

Wiener amalgams and product-convolution operators

Hans G. Feichtinger

Wiener amalgam spaces (originally called *Wiener-type spaces* have been introduced in full generality in the summer of 1980. They allow to describe the *global behaviour* of some *local norm*. The most simple cases are spaces of the form $W(L^p, \ell^q)(R^d)$, with $1 \leq p, q \leq \infty$, where the ℓ^q-sum of the local L^p-norms over the unit cubes sitting at $k \in Z^d$ is finite (and defines the norm). In the more general case one has to make use of somewhat smooth and uniform partitions of unity, such as the basis functions (shifted B-splines) for the space of cubic splines.
Product-convolution or convolution-product operators are concatenations of pointwise multiplication and convolution operators. While pointwise multiplication may increase the decay of a given function or distribution it is clear that a convolution operator will typically improve the local properties (without changing the global behaviour). Moreover, such operators are good regularizers and thus appear in the

theory of (ultra-)distributions, showing how to approximate distributions by test functions.
We will discuss a few of such situations, and if time permits we will also indicate some new results (joint work with Stevan Pilipovic and Bojan Prangovski) concerning the characterization of some new translation and modulation invariant Banach spaces of functions.

Recent results in generalized smooth functions theory

Paolo Giordano

I will review recent results in generalized smooth functions (GSF) theory. This is a minimal extension of Colombeau theory where generalized functions are directly defined as suitable set-theoretical maps between Colombeau generalized numbers so that to share a lot of properties with ordinary smooth functions. Essentially, all the classical theorems of calculus hold for GSF. In contrast to Colombeau generalized functions, GSF are freely closed with respect to composition and can also be defined on infinite numbers or on purely infinitesimal domains. I will present the new language of subpoints and the tri- and quadrichotomy law, i.e. how to bypass the lacking of total order for Colombeau generalized numbers; the notions of hypernatural numbers, hyperlimits and hyperseries; I will briefly outline multidimensional integration of GSF, with recent applications to calculus of variations; We will also talk of recent developments in well-posedness of Cauchy problems for PDE in spaces of GSF. We show e.g. that all smooth Cauchy problems are well-posed, where existence, continuity and uniqueness always hold but, in general, only in infinitesimal neighbourhoods. We finally sketch out some future developments of the theory.

Semi-Bloch-periodic Generalized functions

Maximilian F. Hasler

Recently our own work on Bloch-periodic functions and applications (Hasler *et al.* 2014) has been generalized to the notion of semi-Bloch-periodicity (Pilipovic *et al.* 2020). Here we combine this idea with our own extension to Bloch-periodic *generalized* functions (Hasler 2016). We give some new results and examples of applications, in particular, solutions to differential equations with this property, as well as new notion of asymptotically semi-periodic functions, which should be most relevant in practical applications such as physics and engineering.

Dirac states on the Weyl algebra

Günther Hörmann

The Weyl algebra $\mathcal{W}$ is the unique C^*-algebra associated with a symplectic vector space S and the corresponding canonical commutation relations in exponentiated form. Every state (normalized positive linear functional) induces a representation as operator algebra on a Hilbert space according to the GNS (Gelfand-Naimark-Segal) construction. For applications in quantum field theory (QFT) it is essential to achieve in addition an implementation of constraint equations. The latter are often expressed via some subspace L of the symplectic parameter space S (which is typically an infinite-dimensional function space), e.g. due to gauge conditions. An advanced C^*-algebraic theory for these constraints is based on finding so-called Dirac states that act trivially on the generators corresponding to L and then to define the observables in their corresponding GNS representation. States on $\mathcal{W}$ can be characterized by nonlinear functions on S and it turns out that those corresponding to Dirac states are discontinuous. We discuss some general aspects of the interplay between functions on S and states on $\mathcal{W}$, but also give a detailed analysis for an example of a non-trivial Dirac state. In the last part of the talk, we focus on the specific situation with $S = L^2(\mathbb{R}^n)$ or a test function space and illustrate relations with Colmbeau generalized functions on $\mathbb{R}^n$ or with Borel measures on spaces of test function and distributions.

A complete characterization of Fujita's blow-up solutions for discrete p-Laplacian paraolic equations

Jaeho Hwang *(joint work with S.-Y. Chung)*

In this talk, we discuss long-time behaviors of solutions to the discrete p-Laplacian parabolic equations with time-dependent reactions

$$u_t = \Delta_{p,\omega} u + \psi(t)|u|^{q-1}u$$

with nontrivial and nonnegative initial data, under the mixed boundary conditions, where $p \geq 2$, $q > 0$, and the function ψ is positive continuous function. The goal of this talk is to characterize completely the parameters p, q, and function ψ to see when the solutions are Fujita's blow-up solutions, general blow-up solutions, or global solutions.

Generalized white noise analysis and stochastic distributions

Uwe Kähler *(joint work with D. Alpay and P. Cerejeiras)*

In this paper we develop a framework to extend the theory of generalized stochastic processes in the Hida white noise space to more general probability spaces which include the grey noise space. To obtain a Wiener-Itô expansion we recast it as a moment problem and calculate the moments explicitly. This leads us to the importance of a family of topological algebras called strong algebras in this context. In the end we show the applicability of our approach to the study of non-Gaussian stochastic processes and stochastic distributions in the sense of Kondratiev.

Forty five years of cooperation and friendship with Stevan Pilipović

Andrzej Kaminski

Our first meeting with Stevan Pilipović had place in 1975 in Szczyrk, Poland. We participated there in a conference on generalized functions organized by Professor Jan Mikusiński. This was the beginning of our friendship and mathematical cooperation.
The aim of the talk is to present a review of results obtained during our cooperation and to reminisce some of our meetings.

(w,c)-Almost periodic generalized functions

Mohammed Taha Khalladi

The aim of this work is to introduce and study a new space of (w, c) $-$almost periodic generalized functions containing the (w, c) $-$almost periodic functions, (w, c) $-$almost periodic Schwartz distributions, the algebra of periodic generalized functions, the space of Bloch-periodic generalized functions as well as the space of (w, c) $-$periodic generalized functions.

Convolution on Weighted Spaces of Functions and Distributions

Tillmann Kleiner *(joint work with R. Hilfer)*

Weighted spaces of continuous functions on a locally compact group constitute a rich class of locally convex spaces that is convenient to construct domains for convolution operators defined by measures. The equicontinuity of weighted families of measures operating by convolution between weighted spaces can be characterized by supremal convolution inequalities [1]. This result is related to the problem of constructing joint domains for prescribed families of convolution operators that are maximal in an order theoretic sense. An application are fractional Weyl integrals on the real line. Generalizing from measures to distributions fractional derivatives and integrals of arbitrary complex order have been extended to distributional domains that are maximal with respect to convolvability of distributions [2]. The results for distribution spaces are closely related to recent work on the characterization of convolvability of distributions and topological characterizations of subsets of convolutor spaces by regularization [3]. In particular, weighted L^1-spaces of distributions as studied in [4] arise naturally from our approach.

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Selected Topics in Almost Periodicity

Marko Kostić

The class of almost periodic functions was introduced by Danish mathematician H. Bohr around 1924-1926 and later extended by numerous other authors. Any Bohr almost periodic function is continuous, which is not necessarily true in generalization concepts of V. Stepanov, A. S. Besicovitch and H. Weyl. The class of almost automorphic functions was introduced by S. Bochner in 1962. Any almost periodic function is almost automorphic, while the converse statement fails to be true in general. The Stepanov, Weyl and Besicovitch generalizations of almost automorphicity were considered in several recent research papers.

This talk aims to present some new trends in the theory of almost periodic functions and almost automorphic functions as well as their importance in the study of inhomogeneous abstract integro-differential Cauchy inclusions in Banach spaces.

Distributed order fractional wave equations with irregular coefficients

Srdan Lazendic *(joint work with Sanja Konjik and Ljubica Oparnica)*

In this talk, we derive and analyze fractional wave equations describing wave propagation in one-dimension viscoelastic media, modeled by distributed-order fractional constitutive stress-strain relation. More precisely, we consider the system of equations which consists of the equation of motion of the one-dimensional deformable-body, the constitutive equation of distributed order fractional type, which describes the mechanical properties of the linear viscoelastic body, and the strain for small local deformations. The system is equivalent to the integrodifferential wave-type equation. First, we show that that the fundamental solution to the generalized Cauchy problem for the distributed order wave equation exists and it is unique. In particular, the fundamental solutions corresponding to four thermodynamically acceptable classes of linear fractional constitutive models and power-type distributed-order models will be discussed.
Further, we are interested in obtaining similar results for the cases when viscoelastic material/media are heterogeneous. This implies that the coefficients appearing in equations become non-smooth functions depending on space (or even time) which could also be irregular, such as Dirac delta distribution. We shall concentrate on existence and uniqueness results of weak and very weak solutions. Finally, since the fractional order derivatives are difficult to handle numerically in their original form, we will motivate why the obtained mathematical model, which combines the Laplace transform together with the variational methods, is perfectly suitable for further practical applications.

The stochastic heat equation with singular potential

Tijana Levajkovic *(joint work with Snezana Gordic and Ljubica Oparnica)*

In this work we consider stochastic heat equation with space depending potential, random driving force and random initial condition. Heat equations are extensively studied in the literature and problems with random input data found applications in biology, aerodynamics, structural acoustics, financial mathematics. The heat equation with random potential, also known as the Anderson model, appears in the context of chemical kinetics and population dynamics. Singular potentials such as inverse squared potentials appear in quantum field theory. Here we allow for potential even stronger singularities like Dirac delta distribution. By applying the chaos expansion method from white noise analysis initial stochastic problem is reduced to a system of deterministic heat equations with singular potential, each of the same form. To deal with strong singularities in these deterministic equations we employ the concept of very weak solutions. Then, for the given stochastic problem we introduce notion of stochastic very weak solution. We show existence of the stochastic very weak solution and prove uniqueness and consistency in appropriate sense.

What are continuously differentiable functions on compact sets?

Laurent Loosveldt *(joint work with Leonhard Frerick, Jochen Wengenroth)*

In most analysis textbooks differentiability is only treated for functions on open domains and, if needed, an ad hoc generalization for functions on compact sets is given. We propose instead to define differentiability on arbitrary compact set as the usual affine-linear approximability.
An $\mathbb{R}$-valued function f on a compact set $K \subseteq \mathbb{R}^d$ is said to belong to $C^1(K)$ if there exits a continuous function df on K with values in the linear maps from $\mathbb{R}^d$ to $\mathbb{R}$ such that, for all $x \in K$,

$$\lim_{\substack{y \to x \\ y \in K}} \frac{f(y) - f(x) - df(x)(y-x)}{|y-x|} = 0, \tag{3}$$

where $|\cdot|$ is the euclidean norm. We consider here $\mathbb{R}$-valued functions, as questions about $\mathbb{R}^n$-valued functions easily reduce to this case.
Equality (3) means that df is a continuous derivative of f on K. In general, a derivative need not be unique. For this reason, a good tool to study $C^1(K)$ is the jet space

$$\mathcal{J}^1(K) = \{(f, df) : df \text{ is a continuous derivative of } f \text{ on } K\}$$

endowed with the norm

$$\|(f, df)\|_{\mathcal{J}^1(K)} = \|f\|_K + \|df\|_K,$$

where $\|\cdot\|_K$ is the uniform norm on K and $|df(x)| = \sup\{|df(x)(v)| : |v| \le 1\}$. For the projection $\pi(f, df) = f$ we have $C^1(K) = \pi(\mathcal{J}^1(K))$, and we equip $C^1(K)$ with the quotient norm, i.e.,

$$\|f\|_{C^1(K)} = \|f\|_K + \inf\{\|df\|_K : df \text{ is a continuous derivative of } f \text{ on } K\}.$$

We start by giving a characterization of the completeness of $(C^1(K), \|\cdot\|_{C^1(K)})$ by mean of a geometric characterization of the compact set K.
It seems that the space $C^1(K)$ did not get much attention in the literature. This is in contrast to the "restriction space" $C^1(\mathbb{R}^d|K) = \{f|_K : f \in C^1(\mathbb{R}^d)\}$. Obviously, the inclusion $C^1(\mathbb{R}^d|K) \subseteq C^1(K)$ holds but it is well-known that, in general, it is strict. As Whitney showed in his famous papers from 1934, we have $C^1(\mathbb{R}^d|K) = \pi(\mathcal{E}^1(K))$ where $\mathcal{E}^1(K)$ is the spaces of jets (f, df) for which the limit (3) is uniform in $x \in K$. Moreover, $\mathcal{E}^1(K)$ endowed with the norm

$$\|(f, df)\|_{\mathcal{E}^1(K)} = \|(f, df)\|_{\mathcal{J}^1(K)} + \sup\left\{\frac{|f(y) - f(x)|}{|y - x|} : x, y \in K, y \neq x\right\}$$

is a Banach space.
In this talk, we prove that $\mathcal{E}^1(K)$ is always a dense subset of $\mathcal{J}^1(K)$. The density of $C^1(\mathbb{R}^d|K)$ in $C^1(K)$ is then an immediate consequence.
If the compact set K is topologically regular, i.e., the closure of its interior, another common way to define differentiability is the space

$$C^1_{\text{int}}(K) = \{f \in C^1(\mathring{K}) : f \text{ and } df \text{ extend continously to } K\}.$$

Equipped with the norm $\|f\|_K + \|df\|_K$, $C^1_{\text{int}}(K)$ is always a Banach space. Despite this nice aspect we will show that $C^1_{\text{int}}(K)$ has dramatic drawbacks: The chain rule fails in this setting and compositions of $C^1_{\text{int}}(K)$-functions need not be differentiable. We will present some results about equalities between $C^1_{\text{int}}(K)$, $C^1(\mathbb{R}^d|K)$ and $C^1(K)$, giving an echo to the so-called "Whitney conjecture".

Directional short-time Fourier transform of ultradistributions

Snjezana Maksimovic *(joint work with Sanja Atanasova and Stevan Pilipovic)*

We define and analyze the k-directional short-time Fourier transform and its synthesis operator over Gelfand Shilov spaces $\mathcal{S}^\alpha_\beta(\mathbb{R}^n)$ and $\mathcal{S}^\alpha_\beta(\mathbb{R}^{k+n})$ respectively, and their duals. Also, we investigate directional regular sets and their complements - directional wave fronts, for elements of $\mathcal{S}'^\alpha_\alpha(\mathbb{R}^n)$.

Multiparametric generalized algebras and application

Jean-André Marti

These algebras extend the structure of some types of generalized functions developed over the past 20 years ago by many authors. They are constructed by means of some independant parameters, a sheaf of topological algebras and a factor ring linked to the asymptotic structure adjustable to each problem to solve. We give some examples of studying and solving some differential problems with several independent singularities as non Lipschitzian nonlinearity, characteristic cases, irregular data. The singular support of the solution localizes its singularities. The singular spectrum allows a spectral analysis of these singularities and its linear, differential and nonlinear properties are studied with examples in some cases.

Semigroups of pseudo-differential operators related to basic random processes

Irina V. Melnikova *(joint work with Bovkun Vadim A.)*

The paper is devoted to semigroups of operators that are important both from the point of view of applications and of the properties of generators, which belong to the class of operators generalizing differential and integral ones. The studied semigroups describe important probabilistic characteristics of random processes, defined as functions of the transition probabilities $P(0,x;t,y)$ (see, eg, [1]).
We consider the semigroups of operators $U(t), t \geq 0$, associated with basic random processes — shift processes, Wiener and Poisson, simple and compound ones. The considered semigroups are integrally representable:

$$U(t)u(x) = \int_{\mathbb{R}} u(y)dP(0,x;t,y) = \int_{\mathbb{R}} u(y)p(0,x;t,y)dy = \langle u(\cdot), p(0,x;t,\cdot)\rangle$$

with kernels that are distributions on the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^2) \supset \mathcal{D}(\mathbb{R}^2)$. It is shown that the generators of the considered semigroups are pseudo-differential operators with symbols $p(x,\alpha)$, and the semigroups of operators themselves are families of pseudo-differential operators with symbols depending on the time parameter t. These families have similar semigroup properties, nevertheless are different in terms of the properties of the processes that generate them.
In addition, on the basis of the generalized Fourier transform techniques, or, in probabilistic terminology, the characteristic function techniques, the embedding of

the considered semigroups with pseudo-differential generators into the extended Gelfand–Shilov classification is shown. This classification is based on the classification constructed for differential operators [2, 3].

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Certain results on the existence of the convolution of Roumieu ultradistributions

Svetlana Mincheva-Kaminska

Various existence theorems on the convolution in the space of Roumieu ultradistributions are obtained. The results are connected with earlier investigations. In particular, the convolution of Roumieu ultradistributions having spiral supports is thoroughly analyzed.

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A Fourier transform for all generalized functions

Akbarali Mukhammadiev *(joint work with P. Giordano)*

Depending on a fixed infinite number $k \in {}^{\rho}\widetilde{\mathbb{R}}$ in the ring ${}^{\rho}\widetilde{\mathbb{R}}$ of Robinson-Colombeau (here $\rho = (\rho_\varepsilon)$ generalizes the classical case $\rho_\varepsilon = \varepsilon$), we define the $n-$dimensional *hyperfinite Fourier transform* for any generalized smooth function $f \in {}^{\rho}\mathcal{GC}^{\infty}(K, \widetilde{\mathbb{C}})$, $K = [-k, k]^n$, as

$$\mathcal{F}_k\left(f\right)\left(\omega\right) := \int\limits_K f\left(x\right) e^{-ix\cdot\omega}\, dx = \int\limits_{-k}^{k} dx_1 \ldots \int\limits_{-k}^{k} f\left(x_1 \ldots x_n\right) e^{-ix\cdot\omega}\, dx_n$$

We prove that $\mathcal{F}_k : {}^{\rho}\mathcal{GC}^{\infty}(K, \widetilde{\mathbb{C}}) \longrightarrow {}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}}^n)$. We also define an appropriate notion of convolution in order to prove the convolution property of this hyperfinite Fourier transform. We then prove that the hyperfinite Fourier transform satisfies almost all the elementary properties together with the convolutional property and the corresponding inversion formula. We also consider the embedding of all Colombeau generalized functions into the space ${}^{\rho}\mathcal{GC}^{\infty}(K, \widetilde{\mathbb{C}})$. Finally, we show that this kind of Fourier transform is applicable to a larger class of spaces of generalized functions and differential problems.

The nuclearity of Gelfand-Shilov spaces and kernel theorems

Lenny Neyt *(joint work with A. Debrouwere, J. Vindas)*

Nuclear spaces play a major role in functional analysis. One of their key features is the validity of abstract Schwartz kernel theorems, which often allows for the representation and study of important classes of continuous linear mappings via kernels. Therefore, establishing whether a given function space is nuclear becomes a central question from the point of view of both applications and understanding its locally convex structure. This talk is dedicated to the characterization of nuclearity of the Gelfand-Shilov spaces of ultradifferentiable functions, both of Beurling and Roumieu type. More specific, for a Gelfand-Shilov space $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ defined by a weight sequence system $\mathfrak{M}$ and a weight function system $\mathscr{W}$, we provide concrete conditions on $\mathfrak{M}$ and $\mathscr{W}$ which are necessary and sufficient for the space $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ to be nuclear. As an application of our results, we provide kernel theorems for the spaces $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ and $\mathcal{S}'^{[\mathfrak{M}]}_{[\mathscr{W}]}$.

Colombeau solutions to stochastic hyperbolic systems

Michael Oberguggenberger *(joint work with Jelena Karakasevic, M. Schwarz)*

The talk is devoted to the stochastic Cauchy problem for linear hyperbolic systems of the form of

$$\begin{aligned} (\partial_t + \lambda(x,t)\partial_x)\, u &= f(x,t)u + g(x,t), \quad (x,t) \in \mathbb{R}^2, \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}, \end{aligned}$$

where $u = (u_1, \ldots, u_n)$, $g = (g_1, \ldots, g_n)$; λ and f are $(n\times n)$-matrix valued functions. The matrix λ is assumed to be real-valued and diagonal. The coefficient functions λ, f, g, u_0 will be random fields or stochastic processes on some probability space (Ω, Σ, P).

In many cases of applicative interest, the involved random fields or stochastic processes have paths that are too irregular as to admit classical or weak solutions. This motivates studying the system in the setting of Colombeau stochastic processes of the type $\mathcal{G}_{L^p}(\Omega, \mathbb{R}^d)$ as introduced in (Gordić, Oberguggenberger, Pilipović, Seleši, Monatshefte Math. 186(2018), 609 - 633). For $p = 0$, these processes have representatives with measurable, moderate paths, while for $p \geq 1$, they also satisfy moderate bounds on the $L^p(\Omega)$-norms.

It will be shown that, under suitable conditions on the Colombeau stochastic processes λ, f, g, u_0, the system has unique solutions in $\mathcal{G}_{L^0}(\Omega, \mathbb{R}^2)$. A new characterization of $\mathcal{G}_{L^p}(\Omega, \mathbb{R}^d)$ allows one to establish existence and uniqueness of solutions in $\mathcal{G}_{L^p}(\Omega, \mathbb{R}^2)$ for $p \geq 1$ as well.

Various applications to wave equations with random field coefficients, Klein-Gordon equations with additive or multiplicative Lévy noise as well as to stochastic transport in one space dimension will be given.

Treating strong singularities in differential equations: very weak solution concept

Ljubica Oparnica *(joint work with Michael Ruzhansky)*

The approach of very weak solution concept for treating strong (such as Delta distribution) singularities in equations was introduced in analysis of hyperbolic second order systems with non-regular time dependent coefficients. The fundamental idea is to model irregular objects in the (system of) equations by approximating nets of smooth functions with moderate asymptotics. The regularised net of problems one

can treat in a usual way and obtain net of solutions which, converging or not, but if moderate, will be called *very weak solution.*
In this talk, through several examples we will explain basic ideas and present results that confirm usefulness of the method. We consider heat equation with singular potential, wave equations with singular coefficients arising in different applications (acoustic waves, water waves, Landau Hamiltonian, fractional Zener wave equation), as well as (fractional) Schrödinger equations with singular potentials. We present different notions of moderate families which depend on the model equations under the consideration, theorems on existence and (appropriately defined) uniqueness, theorems on consistency with the classical settings, and numerical examples.
Our aim is to explain how the approach of very weak solutions deal with problems with singularities in a way that is consistent with stronger notions of solutions, and is a far-reaching research direction with possible further mathematical developments and applications in other sciences.

Global Well-posedness of a Class of Strictly Hyperbolic Cauchy Problems with Coefficients Non-Absolutely Continuous in Time

Rahul Raju Pattar *(joint work with N. Uday Kiran)*

We investigate the behavior of the solutions of a class of certain strictly hyperbolic equations defined on $[0, T] \times \mathbb{R}^n$ in relation to SG metric on the phase space. In particular, we study the global regularity and decay issues of the solution to an equation with coefficients polynomially bound in x and with their t-derivative of order $\mathrm{O}(t^{-q})$, where $q \in [1, 2)$. For this purpose, an appropriate generalized symbol class based on the metric is defined and the associated Planck function is used to define a new class of infinite order operators to perform conjugation. We demonstrate that the solution not only experiences a loss of regularity (usually observed for the case of coefficients bounded in x) but also a decay in relation to the initial datum defined in a Sobolev space tailored to the generalized symbol class.

On a nonlinear stochastic fractional heat equation

Danijela Rajter-Ciric *(joint work with Milos Japundzic)*

We consider a stochastic fractional heat equation with variable thermal conductivity, in infinite domain, with both a deterministic and a stochastic source and a stochastic initial data. First we consider a space-fractional case with the regularized Riesz derivative of order α, $1 < \alpha < 2$. Then we consider a case with the Caputo time-fractional derivative of order β, $0 < \beta < 1$, and with a regularized (integer or fractional) space differential operator. In both cases we prove the existence and uniqueness of solutions within certain spaces of generalized stochastic processes. In our solving procedure, we use the theory of generalized uniformly continuous semigroups of operators and the theory of generalized uniformly continuous solution operators. We justify our procedure by comparing, under certain conditions, non-regularized and the corresponding regularized problems.

Wavelet expansions in Gelfand-Shilov spaces

Dusan Zoran Rakic *(joint work with S. Pilipovic, N. Teofanov, J. Vindas)*

We study wavelet expansions of Gelfand-Shilov functions and ultradistributions in the multidimensional case. First, it is given important example for orthonormal wavelets in Gelfand-Shilov spaces mainly motivated by Dziubański-Hernández construction of band-limited wavelets of Lemarié-Meyer type with subexponential decay. In order to obtain convergence result for wavelet series expansion necessary growth estimates for wavelet coefficients of Gelfand-Shilov functions and ultradistributions are studied. Again, it is noted loss of regularity phenomenon established in the previous work on continuity properties of the wavelet transform in Gelfand-Shilov spaces.

Paley Wiener Schwartz type theorem for the wavelet transform

Abhishek Singh *(joint work with R. S. Pathak)*

In the present paper, we discuss about extension of the wavelet transform on distribution space of compact support and develop the Paley Wiener Schwartz type theorem for the wavelet transform on the same. Also, Paley Wiener Schwartz type theorem for the wavelet transform is also established using the relation between the wavelet transform and double Fourier transform. Furthermore, we obtain a generalization of Paley-Wiener-Schwartz type theorem for the distributional wavelet transform. An example is given by interpreting the Mexican hat wavelet transform in the Fourier space.

Exact Solution of Boussinesq Equations

Sudhir Singh *(joint work with K. Sakkarvarthi, L. Kaur, R. Sakthivel, K. Murugesan)*

In this work, we have obtained the rogue wave solutions of a new (2+1)-dimensional integrable Boussinesq model governing the evolution of high and steep gravity water waves. The evolution dynamics of obtained rogue waves along with the identification of their type, bright or dark type localized structures, and manipulation of their amplitude, depth, and width is being discussed. The objective of this study is achieved by employing Hirotas bilinear operator and a generalized polynomial recursive test function. In particularly one, two and three order rogue wave solutions are obtained, and their control and dynamics are discussed categorically. Also, the non local Boussinesq equation is being studies. The obtained results help to demonstrate complete dynamics of rogue waves in higher dimension integrable systems and its various application over-controlling mechanism of rogue waves in optics, atomic condensates, and deep water oceanic waves.

Fréchet frames and generalized functions

Diana Todorova Stoeva *(joint work with S. Pilipović)*

This talk is dedicated to the 70th birthday of Professor Stevan Pilipović and it is based on our joint work with him throughout the years. We will give a brief overview of the theory of Fréchet frames we have developed. A focus will be given on possibilities for frame expansions of the elements in projective and inductive limits of Banach spaces, in particular in the spaces of tempered distributions and ultradistributions.

Certain aspects of bilinear operators

Nenad Teofanov *(joint work with A. Abdeljawad, P. Balazs, F. Bastianoni, S. Coriasco)*

In this lecture we discuss two types of results related to bilinear extensions of some classes of operators.
We first consider bilinear version of pseudodifferential operators with Gevrey-Hörmander symbols. Such symbols enjoy Gevrey type regularity and may have a subexponential growth at infinity.

Our second example are bilinear localization operators which can be identified with particular pseudodifferential operators.
Both types of operators are studied by using the techniques of time-frequency analysis. In particular, we use the short-time Fourier transform and modulation spaces in our analysis.
We present continuity properties of those operators on modulation spaces and, consequently, on Gelfand-Shilov spaces and their distribution spaces. Finally, we give an interpretation of bilinear localization operators as bilinear continuous frame multipliers.
The lecture is dedicated to Professor Stevan Pilipović on the occasion of his 70th birthday.

Hyper-power series and analytic generalized smooth functions

Diksha Tiwari *(joint work with P.Giordano)*

Since the ring ${}^{\rho}\widetilde{\mathbb{R}}$ of Robinson-Colombeau (here $\rho = (\rho_\varepsilon)$ generalizes the classical case $\rho_\varepsilon = \varepsilon$) is non-Archimedean, a classical series $\sum_{n=0}^{\infty} a_n$ of generalized numbers $a_n \in {}^{\rho}\widetilde{\mathbb{R}}$ is convergent if and only if $a_n \to 0$ in the sharp topology. Therefore, this property does not permit us to generalize several classical results, mainly in the study of analytic generalized functions. We define the notion of hyperseries, i.e. series ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ where the sum is extended to the set ${}^{\sigma}\widetilde{\mathbb{N}} \subseteq {}^{\sigma}\widetilde{\mathbb{R}}$ of hyperfinite natural numbers. A real hyper-power series is hence of the form ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-c)^n$, and for this notion we recover classical and new examples such as ${}^{\rho}\sum_{n\in{}^{\rho}\widetilde{\mathbb{N}}} \frac{x^n}{n!} = e^x$ for $x \in {}^{\rho}\widetilde{\mathbb{R}}$ finite and ${}^{\rho}\sum_{n\in{}^{\rho}\widetilde{\mathbb{N}}} \frac{\delta_n(0)}{n!} x^n = \delta(x)$, $\forall x \in {}^{\rho}\widetilde{\mathbb{R}}$. We can also prove classical results such as: algebraic operations and composition of hyper-power series, every hyper-power series defines a generalized smooth function, derivation and integration of hyper-power series, principle of permanence, characterization of real hyper-analytic functions by uniform majoration of derivatives on functionally compact sets, embedding of distributions as analytic generalized smooth functions. The final aim of this research is to generalize the Cauchy-Kowalevski theorem to the wider class of analytic generalized functions.

Tsunami propagation for singular topographies

Niyaz Tokmagambetov *(joint work with A. Altybay, M. Ruzhansky, M. Sebih)*

We consider a tsunami wave equation with singular coefficients and prove that it has a very weak solution. Moreover, we show the uniqueness results and consistency theorem of the very weak solution with the classical one in some appropriate sense. Numerical experiments are done for the families of regularised problems in one- and two-dimensional cases. In particular, the appearance of a substantial second wave is observed, travelling in the opposite direction from the point/line of singularity. Its structure and strength are analysed numerically. In addition, for the two-dimensional tsunami wave equation, we develop GPU computing algorithms to reduce the computational cost.

Ultradistributions as boundary values of analytic functions

Filip Tomić *(joint work with S. Pilipović, N. Teofanov)*

We investigate new spaces of ultradistributions as dual spaces of test functions whose derivatives are controlled by the two parameter sequences of the form $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $\tau > 0$, $\sigma > 1$. We prove that they can be represented as boundary values of analytic functions in corresponding infinitesimal wedge domain. Essential condition for this purpose, condition $(M.2)$, in the classical ultradistribution theory, is changed by a new one, $\widetilde{(M.2)}$, which involves a new technique. Finally, we analyze corresponding wave front sets by using boundary value representations and specific functions with logarithmic behavior.

Dispersion, spreading and sparsity of Gabor wave packets

S. Ivan Trapasso *(joint work with Elena Cordero and Fabio Nicola)*

Sparsity properties for phase-space representations of several types of operators have been extensively studied in recent articles, including pseudodifferential, metaplectic and Fourier integral operators, with applications to the analysis of dispersive evolution equations on phase space. It has been proved that such operators are approximately diagonalized by Gabor wave packets. While the latter are expected to undergo some spreading phenomenon, there is no record of this issue in the aforementioned results. We recently proved refined estimates for the Gabor matrix of metaplectic operators, also of generalized type, where sparsity, spreading and dispersive properties are all noticeable. We also provide applications to the propagation of singularities for the Schrödinger equation; in this connection, our results can be regarded as a microlocal refinement of known estimates.

Boundedness of the Hilbert transform on Lorentz spaces and applications

Kanat Tulenov *(joint work with F. Sukochev and D. Zanin)*

In this work, it is investigated the optimal range of the classical Hilbert transform and its noncommutative counterparts including the triangular truncation operator on Lorentz spaces of functions and Schatten-Lorentz ideals respectively. Some applications of obtained results to operator Lipschitz functions and commutator estimates in Schatten-Lorentz ideals of compact operators are shown.

Some generalizations of the almost periodic functions

Daniel Velinov *(joint work with M. Kostić, S. Pilipović, B. Chaouchi, M. T. Khalladi, A. Rahmani)*

This talk is devoted to the semi-Bloch (p,k)-periodic functions and the almost (ω,c)-pseudo periodic functions as a generalization of many of the previously developed concepts in theory of almost periodic functions. A various types of almost (ω,c)-pseudo periodic composition results are given. Also, a qualitative analysis in this context of a semilinear (fractional) Cauchy inclusions is provided.

Convolution with the kernel $e^{s\langle x\rangle^q}, q \geq 1$ within ultradistribution spaces

Dorde Vucković *(joint work with S. Pilipovic, B. Prangoski)*

We consider the existence of convolution of Roumieu type ultradistribution with the kernel $e^{s(1+|x|^2)^{q/2}}, q \geq 1$, $s \in \mathbb{R}\backslash\{0\}$.
This talk is based on joint work with Stevan Pilipović and Bojan Prangoski.

Reconstruction of the One-dimensional Thick Distribution Theory

Yunyun Yang

The theory of thick distributions (both in dimension 1 and in higher dimensions) was constructed in recent years by Estrada, Fulling and Yang. However this theory of distributions with one thick point in dimension one is very different from that in higher dimensions. In this talk we uses the language of asymptotic analysis to reconstruct the 1-dimensional thick distribution theory and to incorporate it into the framework of the higher-dimensional thick distribution theory. Some new concepts and interesting results appear from viewing singular functions in a different way.

Very weak solutions to wave equations on graded groups

Nurgissa Yessirkegenov *(joint work with Prof. M. Ruzhanksy)*

In this talk we discuss the well-posedness of the Cauchy problem for the Rockland wave equations on graded groups (which include the cases of $\mathbb{R}^n$, Heisenberg, and general stratified Lie groups) when the time-dependent propagation speed $a(t)$ is Hölder and distributional. In the case $a(t)$ is a distribution, we introduce the notion of "very weak solutions" to the Cauchy problem, and prove its existence and uniqueness in an appropriate sense.

Invertibility of matrix type operators of infinite order with exponential off-diagonal decay

Milica Zigic *(joint work with S. Pilipovic, B. Prangoski)*

Recently, a series of papers were related to the matrix type characterisation of various classes of pseudo-differential operators and Fourier integral operators through the matrix representation related to the Gabor wave packets and the almost diagonalisation of ΨDOs and FIOs. Matrix classes with polynomial or sub-exponential off-diagonal decay are known to be spectrally invariant. Here we analyse various exponential off-diagonal decay rates of the elements of infinite matrices and their inverses. This decay rate of the elements of an infinite matrix does not imply inverse–closedness, i.e. the inverse, if exists, does not have the same order of decay. We discuss some consequences and extensions of this result.

List of participants

Jelena Aleksic (Serbia)
Sanja Atanasova (North Macedonia)
Antonio de Oliveira Nginamau Barbosa (China)
Georgi Boyadzhiev (Bulgaria)
Fred Brackx (Belgium)
Frederik Broucke (Belgium)
Tommaso Bruno (Belgium)
Marco Cappiello (Italy)
Paula Cerejeiras (Portugal)
Fabrizio Colombo (Italy)
Sandro Coriasco (Italy)
Andreas Debrouwere (Belgium)
Gregory Debruyne (Belgium)
Pavel Dimovski (North Macedonia)
Marko Erceg (Croatia)
Celine Esser (Belgium)
Ricardo Estrada (USA)
Serena Federico (Belgium)
Hans Feichtinger (Austria)
Paolo Giordano (Austria)
Maximilian Hasler (France)
Rudolf Hilfer (Germany)
Günther Hörmann (Austria)
Jaeho Hwang (Korea, Republic of)
Uwe Kähler (Portugal)
Andrzej Kaminski (Poland)
Tillmann Kleiner (Germany)
Marko Kostic (Serbia)
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Laurent Loosveldt (Belgium)
Frederick Maes (Belgium)
Snjezana Maksimovic (Bosnia and Herzegowina)
Jean-Andre Marti (France)
Irina Melnikova (Russian Federation)
Svetlana Mincheva-Kaminska (Poland)
Arman Molla (Belgium)
Akbarali Mukhammadiev (Austria)

Marko Nedeljkov (Serbia)
Lenny Neyt (Belgium)
Michael Oberguggenberger (Austria)
Ljubica Oparnica (Belgium)
Rahul Pattar (India)
Stevan Pilipovic (Serbia)
Bojan Prangoski (North Macedonia)
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Danijela Rajter-Ciric (Serbia)
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Abhishek Singh (India)
Christian Spreitzer (Austria)
Roland Steinbauer (Austria)
Diana Stoeva (Austria)
Mitsuru Sugimoto (Japan)
Nenad Teofanov (Serbia)
Diksha Tiwari (Austria)
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