Existence of strong traces for entropy solutions of degenerate parabolic equations

Marko Erceg

Department of Mathematics, Faculty of Science, University of Zagreb

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Main question

Under which conditions any \textit{solution} $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ to

$$
\partial_t u + \text{div}_x f(u) = D^2_x \cdot A(u),
$$

admits the \textbf{strong trace} at $t = 0$, i.e. does there exist $u_0 \in L^\infty(\mathbb{R}^d)$ such that

$$
\text{ess lim}_{t \to 0^+} u(t, \cdot) = u_0 \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^d).
$$

$\text{div}_x f(u) \ldots$ \textit{convective term}

$D^2_x \cdot A(u) \ldots$ \textit{diffusive term}
Main question

Under which conditions any solution \( u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R} \) to

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\]

\text{div}_x f(u) \ldots \text{convective term}

\text{diffusive term}

Let us first consider the case \( A = 0 \).
First order quasilinear equations

\[
\begin{aligned}
\partial_t u + \text{div}_x f(u) &= 0 \quad \text{in} \quad \mathbb{R}^d_+ := \mathbb{R}^+ \times \mathbb{R}^d, \\
|_{t=0} u &= u_0 \in L^\infty(\mathbb{R}^d),
\end{aligned}
\]

where \( f : \mathbb{R} \to \mathbb{R}^d \) (homogeneous) flux, \( u : \mathbb{R}^d_+ \to \mathbb{R} \) unknown.

Classical solutions are too strong (we want allow discontinuities in \( x \))

**Weak solutions:** \( u \in L^1_{\text{loc}}(\mathbb{R}^d_+) \) s.t. \( f(u) \in L^1_{\text{loc}}(\mathbb{R}^d_+; \mathbb{R}^d) \) and \( \forall \varphi \in C_\infty_c(\mathbb{R}^{1+d}) \)

\[
\int_{\mathbb{R}^d_+} u \varphi_t + f(u) \cdot \nabla_x \varphi \, dx \, dt + \int_{\mathbb{R}^d} u_0 \varphi(0, \cdot) \, dx = 0.
\]

Even for smooth f’s non-uniqueness:

\( d = 1, \ f(\lambda) = \frac{\lambda^2}{2} \) (Burgers equation), \( u_0(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \).

Both functions are a weak solution:

\( u_1(t, x) = \begin{cases} 0, & x < t/2 \\ 1, & x \geq t/2 \end{cases}, \quad u_2(x) = \begin{cases} 0, & x < 0 \\ x/t, & 0 \leq x < t \quad \text{(rarefraction wave)} \\ 1, & x \geq t \end{cases} \)
Entropy solutions

\[
\begin{aligned}
\left\{
\begin{array}{l}
\partial_t u + \text{div}_x f(u) = 0 \quad \text{in} \quad \mathbb{R}_+^d := \mathbb{R}_+^+ \times \mathbb{R}^d, \\
u|_{t=0} = u_0 \in L^\infty(\mathbb{R}^d).
\end{array}
\right.
\end{aligned}
\]

Entropy solutions: \( u \) a weak solution and s.t. \( \forall \eta \in C(\mathbb{R}) \) convex and
\( \forall \varphi \in C^\infty_c(\mathbb{R}^{1+d}), \varphi \geq 0, \)

\[
\int_{\mathbb{R}_+^d} \eta(u) \varphi_t + f^\eta(u) \cdot \nabla_x \varphi \, dx \, dt + \int_{\mathbb{R}^d} \eta(u_0) \varphi(0, \cdot) \, dx \geq 0,
\]

here \( f^\eta(\lambda) = \int_0^\lambda f' \eta' \, ds \) is an entropy-flux.

- \( \eta \) is called (mathematical) entropy \((-\eta \) corresponds to physical entropy\)
- The above inequality is due to the fact that the physical entropy has a tendency to increase in time, i.e. the mathematical entropy decreases in time
Entropy solutions: (Kružkov) \( u \in L^\infty(\mathbb{R}^d_+) \) s.t. \( \forall \lambda \in \mathbb{R} \) and \( \forall \varphi \in C_c^\infty(\mathbb{R}^{1+d}) \), \( \varphi \geq 0 \),

\[
\int_{\mathbb{R}^d_+} |u - \lambda| \varphi_t + \text{sgn}(u - \lambda)(f(u) - f(\lambda)) \cdot \nabla_x \varphi \, dx \, dt + \int_{\mathbb{R}^d} |u_0 - \lambda| \varphi(0, \cdot) \, dx \geq 0.
\]

Kružkov (1970): existence and uniqueness of entropy solutions for smooth fluxes \( f \).

Strong traces

\[
\begin{cases}
\partial_t u + \text{div}_x f(u) = 0 & \text{in } \mathbb{R}^d_+ := \mathbb{R}^+ \times \mathbb{R}^d, \\
 u|_{t=0} = u_0 \in L^\infty(\mathbb{R}^d).
\end{cases}
\]

\[
\forall \lambda \in \mathbb{R} \text{ and } \forall \varphi \in C^\infty_c(\mathbb{R}^{1+d}), \varphi \geq 0:
\int_{\mathbb{R}^d_+} |u - \lambda| \varphi_t + \text{sgn}(u - \lambda)(f(u) - f(\lambda)) \cdot \nabla_x \varphi \, dx \, dt + \int_{\mathbb{R}^d} |u_0 - \lambda| \varphi(0, \cdot) \, dx \geq 0.
\]
Strong traces

\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t u + \text{div}_x f(u) = 0 \quad \text{in} \quad \mathbb{R}^d_+ := \mathbb{R}^+ \times \mathbb{R}^d, \\
u|_{t=0} = u_0 \in L^\infty(\mathbb{R}^d) .
\end{array} \right.
\end{aligned}
\]

\[\forall \lambda \in \mathbb{R} \quad \text{and} \quad \forall \varphi \in C^\infty_c(\mathbb{R}^{1+d}), \varphi \geq 0: \]
\[
\int_{\mathbb{R}^d_+} |u - \lambda| \varphi_t + \text{sgn}(u - \lambda)(f(u) - f(\lambda)) \cdot \nabla x \varphi \, dx \, dt + \int_{\mathbb{R}^d} |u_0 - \lambda| \varphi(0, \cdot) \, dx \geq 0 .
\]

\[\iff \quad (\text{a.e. } \lambda \in \mathbb{R}) \quad \partial_t |u - \lambda| + \text{div}_x \left( \text{sgn}(u - \lambda)(f(u) - f(\lambda)) \right) \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d_+),
\]
\[\text{ess lim}_{t \to 0^+} u(t, \cdot) = u_0 \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^d) .
\]
Strong traces

\[
\left\{ \begin{array}{l}
\partial_t u + \text{div}_x f(u) = 0 \quad \text{in} \quad \mathbb{R}^d_+ := \mathbb{R}^+ \times \mathbb{R}^d, \\
u|_{t=0} = u_0 \in L^\infty(\mathbb{R}^d).
\end{array} \right.
\]

(a.e. \( \lambda \in \mathbb{R} \))
\[
\partial_t |u - \lambda| + \text{div}_x \left( \text{sgn}(u - \lambda)(f(u) - f(\lambda)) \right) \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d_+),
\]
\[
\text{ess lim}_{t \to 0^+} u(t, \cdot) = u_0 \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^d). \quad \text{strong trace}
\]

Vasseur (2001): existence of strong traces for entropy solutions for smooth fluxes \( f \) and with a non-degeneracy condition

Panov (2005): existence of strong traces for entropy solutions (without non-degeneracy conditions)

Neves, Panov, Silva (2018): existence of strong traces for entropy solutions for heterogeneous fluxes \( f \) and with a non-degeneracy condition
Degenerate parabolic equation

\begin{equation}
\begin{aligned}
\begin{cases}
\partial_t u + \text{div}_x f(u) &= D_x^2 \cdot A(u) \quad \text{in} \quad \mathbb{R}^d_+,
\end{cases}
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
_{t=0} u &= u_0 \in L^\infty(\mathbb{R}^d),
\end{aligned}
\end{equation}

where \( f : \mathbb{R} \to \mathbb{R}^d, A : \mathbb{R} \to \mathbb{R}^{d \times d}_{\text{sym}}, \) and \( u : \mathbb{R}^d_+ \to \mathbb{R} \) unknown.

- rough flux \( f \) \( (L^p, p > 1) \)
- \( A' \geq 0 \) (degenerate parabolicity)

Motivation: flow in porous media \( (CO_2 \text{ sequestration}) \)
- heterogeneous layers \( \longrightarrow \) discontinuous flux and a lack of diffusion in some directions
Definition of solutions (kinetic formulation)

\[ \left\{ \begin{array}{l} \partial_t u + \text{div}_x f(u) = D_x^2 \cdot A(u) \quad \text{in} \quad \mathbb{R}^d_+, \\ u|_{t=0} = u_0 \in L^\infty(\mathbb{R}^d). \end{array} \right. \]

A measurable function \( u \) defined on \( \mathbb{R}^d_+ \) is called a quasi-solution to \( (\text{DP}_1) \) if \( f_k(u), A_{kj}(u) \in L^1_{\text{loc}}(\mathbb{R}^d_+) \), \( k, j = 1, \ldots, d \), and for a.e. \( \lambda \in \mathbb{R} \)

\[
\partial_t |u - \lambda| + \text{div}_x \left( \text{sgn}(u - \lambda) (f(u) - f(\lambda)) \right) \\
- D_x^2 \cdot [\text{sgn}(u - \lambda)(A(u) - A(\lambda))] = -\gamma(t, x, \lambda),
\]

holds in \( \mathcal{D}'(\mathbb{R}^d_+) \), where \( \gamma \in C(\mathbb{R}_\lambda; \mathcal{M}_+(\mathbb{R}^d_+)) \).

For \( A = 0 \) coincides with the previous definition of entropy solutions.
Definition of solutions (kinetic formulation)

(DP)
\[
\begin{aligned}
\partial_t u + \text{div}_x f(u) &= D_x^2 \cdot A(u) \quad \text{in} \quad \mathbb{R}^d_+, \\
|t=0 &= u_0 \in L^\infty(\mathbb{R}^d).
\end{aligned}
\]

Theorem

If function $u$ is a quasi-solution to ($DP_1$), then the function

\[
h(t, x, \lambda) := \text{sgn}(u(t, x) - \lambda) = -\partial_\lambda |u(t, x) - \lambda|
\]

is a weak solution to the following linear equation (entropy solution):

\[
\partial_t h + \text{div}_x (f'(h) - D_x^2 \cdot [A'(\lambda)h]) = \partial_\lambda \gamma(t, x, \lambda).
\]

Lions, Perthame, Tadmor (1994)
Existence of strong traces for \((\text{DP}_1)\)

\[
(\text{DP}_1) \quad \partial_t u + \text{div}_x f(u) = D_x^2 \cdot A(u) \quad \text{in} \quad \mathbb{R}_+^d.
\]

\[
\text{ess lim}_{t \to 0^+} u(t, \cdot) = u_0 \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^d).
\]

Kwon (2009): scalar diffusion matrices \(A(u) = a(u)I\) without non-degeneracy conditions

Aleksić, Mitrović (2014): traceable fluxes \(f\) and ultra-parabolic \(A\) (i.e. \(A = B \oplus 0\) where \(B > 0\)) without non-degeneracy conditions

“Fully degenerate” matrices \(A\) not covered, e.g.

\[
a(\lambda) = \left( \frac{1}{\sqrt{\lambda^2 + 1}} \begin{bmatrix} \lambda & 1 \\ 1 & -\lambda \end{bmatrix} \right) \begin{bmatrix} 0 & 0 \\ 0 & \lambda^2 + 1 \end{bmatrix} \left( \frac{1}{\sqrt{\lambda^2 + 1}} \begin{bmatrix} \lambda & 1 \\ 1 & -\lambda \end{bmatrix} \right) = \begin{bmatrix} 1 & -\lambda \\ -\lambda & \lambda^2 \end{bmatrix}
\]
Existence of strong traces for \((\text{DP}_1)\)

\[(\text{DP}_1) \quad \partial_t u + \text{div}_x f(u) = D^2_x \cdot A(u) \quad \text{in} \quad \mathbb{R}^d_+ .\]

\[\text{ess lim}_{t \to 0^+} u(t, \cdot) = u_0 \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^d) \quad ???\]

**Theorem (E., Mitrović)**

Let \(f \in C^1(\mathbb{R}; \mathbb{R}^d)\) and let \(A \in C^{1,1}(\mathbb{R}; \mathbb{R}^{d \times d})\) be such that

i) \((\exists \sigma \in C^{0,1}(\mathbb{R}; \mathbb{R}^{d \times d}))\) \(A'(\lambda) = \sigma(\lambda)^T \sigma(\lambda)\);

ii) \(\sup_{(\tau,\xi) \in \mathbb{S}^d} \text{meas}\left\{\lambda \in \mathbb{R} : \tau + \langle f'(\lambda), \xi \rangle = \langle A'(\lambda)\xi, \xi \rangle = 0\right\} = 0\)

(non-degeneracy condition).

Then any bounded quasi-solution \(u\) to \((\text{DP}_1)\) admits the strong trace at \(t = 0\).
Strategy of the proof

1. kinetic formulation
2. existence of a weak trace
3. rescaling (blow-up)
4. equation in new variables
5. localisation principle and non-degeneracy condition
Strategy of the proof

1. kinetic formulation

\[ h = \text{sgn}(u - \lambda) = -\partial_\lambda |u - \lambda| \text{ satisfies } \]
\[ \partial_t h + \text{div}_x (f' h) - D^2 \cdot [A'(\lambda)h] = \partial_\lambda \gamma \]

2. existence of a weak trace

3. rescaling (blow-up)

4. equation in new variables

5. localisation principle and non-degeneracy condition
Strategy of the proof

1. kinetic formulation

2. existence of a weak trace

There exists $h_0 \in L^\infty(\mathbb{R}^{d+1})$, such that

$$h(t, \cdot, \cdot) \rightharpoonup h_0, \text{ weakly-}* \text{ in } L^\infty(\mathbb{R}^{d+1}), \text{ as } t \to 0, \ t \in E,$$

where $E$ is a set of full measure

3. rescaling (blow-up)

4. equation in new variables

5. localisation principle and non-degeneracy condition
Strategy of the proof

1. kinetic formulation

2. existence of a weak trace

3. rescaling (blow-up)

If $(\exists \alpha > 0)(\forall \rho \in C^1_c(\mathbb{R}))$

$$
\int_{\mathbb{R}} h\left(\frac{\hat{t}}{m^\alpha}, \frac{\hat{x}}{m^\alpha} + y, \lambda\right) \rho(\lambda) \, d\lambda \to \int_{\mathbb{R}} h_0(y, \lambda) \rho(\lambda) \, d\lambda \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}_+^d \times \mathbb{R}^d),
$$

then $u$ admits the strong trace at $t = 0$ which is equal to $\frac{1}{2} \int_{\mathbb{R}} h_0(\cdot, \lambda) \, d\lambda$

4. equation in new variables

5. localisation principle and non-degeneracy condition
Strategy of the proof

1. kinetic formulation

2. existence of a weak trace

3. rescaling (blow-up)

4. equation in new variables

\[ h^m(\hat{t}, \hat{x}, y, \lambda) := h(\frac{\hat{t}}{m}, \frac{\hat{x}}{m} + y, \lambda) \] satisfies

\[ \frac{1}{m} \left( \partial_i h^m + \text{div}_x (f' h^m) \right) - D_{\hat{x}}^2 \cdot [A'(\lambda) h^m] = \frac{1}{m^2} \partial_\lambda \gamma^m \]

We test by a suitable function and take \( m \to \infty \)

5. localisation principle and non-degeneracy condition
Strategy of the proof

1. kinetic formulation
2. existence of a weak trace
3. rescaling (blow-up)
4. equation in new variables
5. localisation principle and non-degeneracy condition

On the limit we get (μ is a suitable variant of microlocal defect object):

\[(\forall \phi) \langle \mu, F \phi \rangle = 0 \quad F \text{ non-degenerate} \quad \mu \equiv 0 \quad \Rightarrow \quad \text{strong convergence of (3)}\]
Adaptive H-measures

- $(u_n)$ bounded in $L^2(\mathbb{R}^{2d+2})$ and $(v_n)$ bounded in $L^2(\mathbb{R}^{2d+1})$
- $\psi$ continuous

\[
\langle \mu, \psi \rangle = \lim_{n} \int_{\mathbb{R}^{2d+2}} \psi(y, \lambda, \frac{(\tau, \xi)}{\pi_n(\lambda, \tau, \xi)}, \frac{n\langle A'(\lambda)\xi | \xi \rangle}{\pi_n(\lambda, \tau, \xi)}) \hat{u}_n(\tau, \xi, y, \lambda) \hat{v}_n(\tau, \xi, y) d\tau d\xi dy d\lambda ,
\]

where

\[
\pi_n(\lambda, \tau, \xi) = 1 + |(\tau, \xi)| + n\langle A'(\lambda)\xi | \xi \rangle
\]
...thank you for your attention :)