A categorical approach to order, metric and topology

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Nothing seems to be more benign in algebra than the notion of a monoid, that is, of a set M that comes with an associative binary operation $m : M \times M \to M$ and a neutral element, written as a nullary operation $1 : 1 \to M$. In particular a monad on Set can be seen as a monoid in the monoidal category of all endofunctors on Set. A monad $\mathbb{T} = (T, m, e)$ on Set is given by a functor $T : \mathsf{Set} \to \mathsf{Set}$ and two natural transformations, the multiplication and the unit of the monad $m : TT \to T$ and $e : 1_{\mathsf{Set}} \to T$ satisfying the multiplication law and the right and left unit laws

$$m \cdot mT = m \cdot Tm, \qquad m \cdot eT = 1_T = m \cdot Te.$$

Monoids and their actions on monads provide the basic ingredients of what is called monoidal topology and provide a common setting for the description of the category of all ordered spaces, all metric spaces, all topological spaces and all approach spaces.

Inspired by the work of Manes on the role of the ultrafilter monad $\mathbb{\beta} = (\beta, m, e)$ on Set in the characterization of compact Hausdorff spaces, Barr observed that for describing convergence in an arbitrary topological space one needs a relation $a : \beta X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ between ultrafilters on X and points of X satisfying a transitivity and a reflexivity axiom

$$a\big(\overline{\beta}a(\mathfrak{X})\big) \leq a\big(m_X(\mathfrak{X})\big) \qquad \text{and} \qquad x \leq a\big(e_X(x)\big),$$

for all $\mathfrak{X} \in \beta\beta X$ and $x \in X$. These conditions make sense after extending the ultrafilter monad to the category Rel of sets with relations as morphisms. Using these ideas one obtains a characterization of topological spaces as relational algebras by means of two convergence axioms.

We will now generalize these ideas for a given quantale $\mathcal{V}$ and a monad $\mathbb{T} = (T, m, e)$ on Set, laxly extended to the category $\mathcal{V}$-Rel of sets and $\mathcal{V}$-relations. This results in the key category of interest $(\mathbb{T}, \mathcal{V})$-$\mathsf{Cat}$ whose objects, depending on the context, may be called $(\mathbb{T}, \mathcal{V})$-algebras, $(\mathbb{T}, \mathcal{V})$-spaces or $(\mathbb{T}, \mathcal{V})$-categories.

In this talk we will focus on examples of $(\mathbb{T}, \mathcal{V})$-categories describing ordered spaces, metric spaces, topological spaces and approach spaces as well as the application of these concepts in the study of topological properties.