Variations on a Theme by Friedman

Ali Enayat, Göteborgs Universitet

September 5, 2013

Honorary Doctorate Harvey Friedman, Universiteit Ghent

Ali Enayat, Göteborgs Universitet

Friedman's Theme

Friedman's Theme

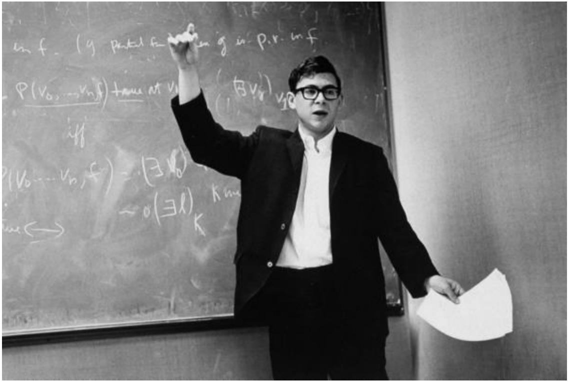
- **Friedman.** *Every countable nonstandard model of* ZF *or* PA *is isomorphic to a proper initial segment of itself*.

.

Friedman's Theme

- **Friedman.** *Every countable nonstandard model of* ZF *or* PA *is isomorphic to a proper initial segment of itself*.

  .

- 

Popular TV meets Logic

Popular TV meets Logic

- 

Jim Schmerl's Account

Jim Schmerl's Account

- Harvey was on the Flip Wilson show. It must have been in 1971 (perhaps plus/minus 1) since I was at Yale at the time and Joram Hirschfeld was just finishing his thesis then.

Jim Schmerl's Account

- Harvey was on the Flip Wilson show. It must have been in 1971 (perhaps plus/minus 1) since I was at Yale at the time and Joram Hirschfeld was just finishing his thesis then.
- He heard Harvey talk about embedding models of PA as initial segments and that gave him an idea that ended up in his thesis.

Jim Schmerl's Account

- Harvey was on the Flip Wilson show. It must have been in 1971 (perhaps plus/minus 1) since I was at Yale at the time and Joram Hirschfeld was just finishing his thesis then.
- He heard Harvey talk about embedding models of PA as initial segments and that gave him an idea that ended up in his thesis.
- Hirschfeld showed that every countable model of PA can be embedded in a nontrivial homomorphic image of the semiring R of recursive functions.

A Landmark Paper

A Landmark Paper

- H. Friedman, *Countable models of set theories*, in **Cambridge Summer School in Mathematical Logic** (Cambridge, 1971), pp. 539-573. Lecture Notes in Math., Vol. 337. Springer, Berlin, 1973.

A Landmark Paper

- H. Friedman, *Countable models of set theories*, in **Cambridge Summer School in Mathematical Logic** (Cambridge, 1971), pp. 539-573. Lecture Notes in Math., Vol. 337. Springer, Berlin, 1973.
- Introduction; pp. 539-543.

A Landmark Paper

- H. Friedman, *Countable models of set theories*, in **Cambridge Summer School in Mathematical Logic** (Cambridge, 1971), pp. 539-573. Lecture Notes in Math., Vol. 337. Springer, Berlin, 1973.
- Introduction; pp. 539-543.
- §1. Preliminaries; pp. 544-551.

A Landmark Paper

- H. Friedman, *Countable models of set theories*, in **Cambridge Summer School in Mathematical Logic** (Cambridge, 1971), pp. 539-573. Lecture Notes in Math., Vol. 337. Springer, Berlin, 1973.
- Introduction; pp. 539-543.
- §1. Preliminaries; pp. 544-551.
- §2. Standard Systems of nonstandard admissible sets; pp. 552-556.

A Landmark Paper

- H. Friedman, *Countable models of set theories*, in **Cambridge Summer School in Mathematical Logic** (Cambridge, 1971), pp. 539-573. Lecture Notes in Math., Vol. 337. Springer, Berlin, 1973.
- Introduction; pp. 539-543.
- §1. Preliminaries; pp. 544-551.
- §2. Standard Systems of nonstandard admissible sets; pp. 552-556.
- §3. The ordinals in nonstandard admissible sets; pp. 557-562.

A Landmark Paper

- H. Friedman, *Countable models of set theories*, in **Cambridge Summer School in Mathematical Logic** (Cambridge, 1971), pp. 539-573. Lecture Notes in Math., Vol. 337. Springer, Berlin, 1973.
- Introduction; pp. 539-543.
- §1. Preliminaries; pp. 544-551.
- §2. Standard Systems of nonstandard admissible sets; pp. 552-556.
- §3. The ordinals in nonstandard admissible sets; pp. 557-562.
- §4. Initial segments of nonstandard power admissible sets; pp. 563-565.

A Landmark Paper

- H. Friedman, *Countable models of set theories*, in **Cambridge Summer School in Mathematical Logic** (Cambridge, 1971), pp. 539-573. Lecture Notes in Math., Vol. 337. Springer, Berlin, 1973.
- Introduction; pp. 539-543.
- §1. Preliminaries; pp. 544-551.
- §2. Standard Systems of nonstandard admissible sets; pp. 552-556.
- §3. The ordinals in nonstandard admissible sets; pp. 557-562.
- §4. Initial segments of nonstandard power admissible sets; pp. 563-565.
- §5. Submodels of Σ^1_∞-CA; pp. 566-569.

A Landmark Paper

- H. Friedman, *Countable models of set theories*, in **Cambridge Summer School in Mathematical Logic** (Cambridge, 1971), pp. 539-573. Lecture Notes in Math., Vol. 337. Springer, Berlin, 1973.
- Introduction; pp. 539-543.
- §1. Preliminaries; pp. 544-551.
- §2. Standard Systems of nonstandard admissible sets; pp. 552-556.
- §3. The ordinals in nonstandard admissible sets; pp. 557-562.
- §4. Initial segments of nonstandard power admissible sets; pp. 563-565.
- §5. Submodels of Σ^1_∞-CA; pp. 566-569.
- §6. Categoricity relative to ordinals; pp.570-572.

A Landmark Paper

- H. Friedman, *Countable models of set theories*, in **Cambridge Summer School in Mathematical Logic** (Cambridge, 1971), pp. 539-573. Lecture Notes in Math., Vol. 337. Springer, Berlin, 1973.
- Introduction; pp. 539-543.
- §1. Preliminaries; pp. 544-551.
- §2. Standard Systems of nonstandard admissible sets; pp. 552-556.
- §3. The ordinals in nonstandard admissible sets; pp. 557-562.
- §4. Initial segments of nonstandard power admissible sets; pp. 563-565.
- §5. Submodels of Σ^1_∞-CA; pp. 566-569.
- §6. Categoricity relative to ordinals; pp.570-572.
- References and Errata; p.573.

Ali Enayat, Göteborgs Universitet

Synoptic History (1)

Synoptic History (1)

- **1962.** In answer to a question of Dana Scott, Robert Vaught showed that there is a model of true arithmetic that is isomorphic to a proper initial segment of itself. This result is later included in a joint paper of Vaught and Morley.

Synoptic History (1)

- **1962.** In answer to a question of Dana Scott, Robert Vaught showed that there is a model of true arithmetic that is isomorphic to a proper initial segment of itself. This result is later included in a joint paper of Vaught and Morley.
- **1973.** Friedman's self-embedding theorem.

Synoptic History (1)

- **1962.** In answer to a question of Dana Scott, Robert Vaught showed that there is a model of true arithmetic that is isomorphic to a proper initial segment of itself. This result is later included in a joint paper of Vaught and Morley.
- **1973.** Friedman's self-embedding theorem.
- **1977.** Alex Wilkie showed the existence of *continuum-many* initial segments of every countable nonstandard model of $\mathcal{M}$ of PA that are isomorphic to $\mathcal{M}$.

Ali Enayat, Göteborgs Universitet

Synoptic History (2)

Synoptic History (2)

- **1978.** Hamid Lessan showed that a countable model $\mathcal{M}$ of Π_2^{PA} is isomorphic to a proper initial segment of itself iff $\mathcal{M}$ is 1-tall and 1-extendible.

Synoptic History (2)

- **1978.** Hamid Lessan showed that a countable model $\mathcal{M}$ of Π_2^{PA} is isomorphic to a proper initial segment of itself iff $\mathcal{M}$ is 1-tall and 1-extendible.
- Here 1-tall means that the set of Σ_1-definable elements of $\mathcal{M}$ is not cofinal in $\mathcal{M}$, and 1-extendible means that there is an end extension $\mathcal{M}^*$ of $\mathcal{M}$ that satisfies $\mathsf{I}\Delta_0$ and $\mathrm{Th}_{\Sigma_1}(\mathcal{M}) = \mathrm{Th}_{\Sigma_1}(\mathcal{M}^*)$.

Synoptic History (2)

- **1978.** Hamid Lessan showed that a countable model $\mathcal{M}$ of Π_2^{PA} is isomorphic to a proper initial segment of itself iff $\mathcal{M}$ is 1-tall and 1-extendible.
- Here 1-tall means that the set of Σ_1-definable elements of $\mathcal{M}$ is not cofinal in $\mathcal{M}$, and 1-extendible means that there is an end extension $\mathcal{M}^*$ of $\mathcal{M}$ that satisfies $\mathsf{I}\Delta_0$ and $\mathrm{Th}_{\Sigma_1}(\mathcal{M}) = \mathrm{Th}_{\Sigma_1}(\mathcal{M}^*)$.
- **1978.** Craig Smorynski's influential lectures and expositions systematized and extended Friedman-style embedding theorems around the key concept of (partial) recursive saturation.

Ali Enayat, Göteborgs Universitet

Synoptic History (3)

Synoptic History (3)

- **1979.** Leonard Lipshitz showed that a countable nonstandard model of PA is Diophantine correct iff it can be embedded into arbitrarily low nonstandard initial segments of itself (the result was suggested by Stanley Tennenbaum).

Synoptic History (3)

- **1979.** Leonard Lipshitz showed that a countable nonstandard model of PA is Diophantine correct iff it can be embedded into arbitrarily low nonstandard initial segments of itself (the result was suggested by Stanley Tennenbaum).
- **1981.** Jeff Paris noted that an unpublished construction of Robert Solovay shows that every countable recursively saturated model of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$ is isomorphic to a proper initial segment of itself.

Synoptic History (4)

Synoptic History (4)

- **1985.** Costas Dimitracopoulos showed that every countable nonstandard model of $\mathsf{I}\Delta_0 + \mathrm{B}\Sigma_2$ is isomorphic to a proper initial segment of itself.

Synoptic History (4)

- **1985.** Costas Dimitracopoulos showed that every countable nonstandard model of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_2$ is isomorphic to a proper initial segment of itself.
- **1987.** Jean-Pierre Ressayre proved an optimal result: for every countable nonstandard model $\mathcal{M}$ of $\mathrm{I}\Sigma_1$ and for every $a \in \mathcal{M}$ there is an embedding j of $\mathcal{M}$ onto a proper initial segment of itself such that $j(x) = x$ for all $x \leq a$; moreover, this property characterizes countable models of $\mathrm{I}\Sigma_1$ among countable models of $\mathrm{I}\Delta_0$.

Synoptic History (4)

- **1985.** Costas Dimitracopoulos showed that every countable nonstandard model of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_2$ is isomorphic to a proper initial segment of itself.
- **1987.** Jean-Pierre Ressayre proved an optimal result: for every countable nonstandard model $\mathcal{M}$ of $\mathrm{I}\Sigma_1$ and for every $a \in \mathcal{M}$ there is an embedding j of $\mathcal{M}$ onto a proper initial segment of itself such that $j(x) = x$ for all $x \leq a$; moreover, this property characterizes countable models of $\mathrm{I}\Sigma_1$ among countable models of $\mathrm{I}\Delta_0$.
- **1988.** Independently of Ressayre, Dimitracopoulos and Paris showed that every countable nonstandard model of $\mathrm{I}\Sigma_1$ is isomorphic to a proper initial segment of itself.

Synoptic History (4)

- **1985.** Costas Dimitracopoulos showed that every countable nonstandard model of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_2$ is isomorphic to a proper initial segment of itself.
- **1987.** Jean-Pierre Ressayre proved an optimal result: for every countable nonstandard model $\mathcal{M}$ of $\mathrm{I}\Sigma_1$ and for every $a \in \mathcal{M}$ there is an embedding j of $\mathcal{M}$ onto a proper initial segment of itself such that $j(x) = x$ for all $x \leq a$; moreover, this property characterizes countable models of $\mathrm{I}\Sigma_1$ among countable models of $\mathrm{I}\Delta_0$.
- **1988.** Independently of Ressayre, Dimitracopoulos and Paris showed that every countable nonstandard model of $\mathrm{I}\Sigma_1$ is isomorphic to a proper initial segment of itself.
- Dimitracopoulos and Paris also generalized Lessan's aforementioned result by weakening Π_2^{PA} to $\mathrm{I}\Delta_0 + \exp + \mathrm{B}\Sigma_1$.

Ali Enayat, Göteborgs Universitet

Synoptic History (5)

Synoptic History (5)

- **1991.** Richard Kaye's text gave a systematic presentation of various Friedman-style embedding theorems.

Synoptic History (5)

- **1991.** Richard Kaye's text gave a systematic presentation of various Friedman-style embedding theorems.
- **Theorem.** (Fine-tuned Friedman Theorem) *Suppose M is a countable nonstandard model of $\mathrm{I}\Sigma_1$ and $\{a, b\} \subseteq N$ with $a < b$. The following statements are equivalent*:

Synoptic History (5)

- **1991.** Richard Kaye's text gave a systematic presentation of various Friedman-style embedding theorems.
- **Theorem.** (Fine-tuned Friedman Theorem) *Suppose M is a countable nonstandard model of $\mathrm{I}\Sigma_1$ and $\{a, b\} \subseteq N$ with $a < b$. The following statements are equivalent*:
- **(1)** *For every Σ_n-formula $\sigma(x, y)$ we have*:

$$\mathcal{M} \models \exists y\ \sigma(a, y) \longrightarrow \exists y < b\ \sigma(a, y).$$

Synoptic History (5)

- **1991.** Richard Kaye's text gave a systematic presentation of various Friedman-style embedding theorems.
- **Theorem.** (Fine-tuned Friedman Theorem) *Suppose M is a countable nonstandard model of* $I\Sigma_1$ *and* $\{a, b\} \subseteq N$ *with* $a < b$. *The following statements are equivalent*:
- **(1)** *For every* Σ_n*-formula* $\sigma(x, y)$ *we have*:

$$\mathcal{M} \models \exists y\ \sigma(a, y) \longrightarrow \exists y < b\ \sigma(a, y).$$

- **(2)** *There is a* Σ_n-elementary-*initial embedding* $j : \mathcal{M} \to \mathcal{M}$ *with* $j(a) = a$ *and* $a < j(M) < b$

Synoptic History (6)

Synoptic History (6)

- **1997.** Kazuyuki Tanaka extended Ressayre's aforementioned result by showing that every countable nonstandard model of WKL_0 has a nontrivial self-embedding in the following sense:
  Given $(\mathcal{M}, \mathcal{A}) \models \mathsf{WKL}_0$ there is a proper initial segment I of $\mathcal{M}$ such that:

$$(\mathcal{M}, \mathcal{A}) \cong (I, \mathcal{A} \restriction I),$$

  where

$$\mathcal{A} \restriction I := \{A \cap I : A \in \mathcal{A}\}.$$

Ali Enayat, Göteborgs Universitet

Some Recent Results (1)

Some Recent Results (1)

- **Definition.** A (total) function f from M to M is a *total* $\mathcal{M}$*-recursive function* if the graph of f is definable in $\mathcal{M}$ by a parameter-free Σ_1-formula.

Some Recent Results (1)

- **Definition.** A (total) function f from M to M is a *total $\mathcal{M}$-recursive function* if the graph of f is definable in $\mathcal{M}$ by a parameter-free Σ_1-formula.

- **Theorem.** (V. Shavrukov, 2013) Suppose M is a countable nonstandard model of $\mathrm{I}\Sigma_1$, and $\{a, b\} \subseteq M$ *with* $a < b$. *The following statements are equivalent:*

Some Recent Results (1)

- **Definition.** A (total) function f from M to M is a *total* $\mathcal{M}$*-recursive function* if the graph of f is definable in $\mathcal{M}$ by a parameter-free Σ_1-formula.
- **Theorem.** (V. Shavrukov, 2013) Suppose M is a countable nonstandard model of $\mathrm{I}\Sigma_1$, and $\{a, b\} \subseteq M$ *with* $a < b$. *The following statements are equivalent:*
- **(1)** *There is an initial embedding* $j : \mathcal{M} \to \mathcal{M}$ *with* $a < j(M) < b$.

Ali Enayat, Göteborgs Universitet

Some Recent Results (1)

- **Definition.** A (total) function f from M to M is a *total $\mathcal{M}$-recursive function* if the graph of f is definable in $\mathcal{M}$ by a parameter-free Σ_1-formula.

- **Theorem.** (V. Shavrukov, 2013) Suppose M is a countable nonstandard model of $I\Sigma_1$, and $\{a, b\} \subseteq M$ *with $a < b$. The following statements are equivalent:*

- **(1)** *There is an initial embedding $j : \mathcal{M} \to \mathcal{M}$ with $a < j(M) < b$.*

- **(2)** *$f(a) < b$ for all $\mathcal{M}$-total recursive functions f.*

.

Some Recent Results (2)

Some Recent Results (2)

- **Theorem.** (T. Wong, to appear). *Suppose* $(\mathcal{M}, \mathcal{A})$ *is a countable nonstandard recursively saturated model of* RCA_0^*. *The following are equivalent*: .

Some Recent Results (2)

- **Theorem.** (T. Wong, to appear). *Suppose* $(\mathcal{M}, \mathcal{A})$ *is a countable nonstandard recursively saturated model of* RCA_0^*. *The following are equivalent*: .
- **(1)** *There is a self-embedding of* $(\mathcal{M}, \mathcal{A})$ *onto a proper initial segment of itself*.

Some Recent Results (2)

- **Theorem.** (T. Wong, to appear). *Suppose* $(\mathcal{M}, \mathcal{A})$ *is a countable nonstandard recursively saturated model of* RCA_0^*. *The following are equivalent*: .
- **(1)** *There is a self-embedding of* $(\mathcal{M}, \mathcal{A})$ *onto a proper initial segment of itself*.
- **(2)** $(\mathcal{M}, \mathcal{A}) \models \mathsf{WKL}_0^*$.

.

Some Recent Results (2)

- **Theorem.** (T. Wong, to appear). *Suppose* $(\mathcal{M}, \mathcal{A})$ *is a countable nonstandard recursively saturated model of* RCA_0^*. *The following are equivalent*: .
- **(1)** *There is a self-embedding of* $(\mathcal{M}, \mathcal{A})$ *onto a proper initial segment of itself*.
- **(2)** $(\mathcal{M}, \mathcal{A}) \models \mathrm{WKL}_0^*$.

.

- RCA_0^* is formulated in the language of second-order arithmetic, and consists of basic recursive axioms for addition, multiplication, and exponentiation; augmented with Δ_0^1-Comprehension and Δ_0^0-Induction.

Some Recent Results (2)

- **Theorem.** (T. Wong, to appear). *Suppose* $(\mathcal{M}, \mathcal{A})$ *is a countable nonstandard recursively saturated model of* RCA_0^*. *The following are equivalent*: .
- **(1)** *There is a self-embedding of* $(\mathcal{M}, \mathcal{A})$ *onto a proper initial segment of itself.*
- **(2)** $(\mathcal{M}, \mathcal{A}) \models \mathsf{WKL}_0^*$.

.

- RCA_0^* is formulated in the language of second-order arithmetic, and consists of basic recursive axioms for addition, multiplication, and exponentiation; augmented with Δ_0^1-Comprehension and Δ_0^0-Induction.
-
$$\frac{\mathsf{WKL}_0^*}{\mathsf{I}\Delta_0 + \mathsf{Exp} + \mathsf{B}\Sigma_1} = \frac{\mathsf{WKL}_0}{\mathsf{I}\Sigma_1}$$

Some Recent Results (3)

Some Recent Results (3)

- **Theorem.** (K. Yokoyama, to appear). *Suppose* $(\mathcal{M}, \mathcal{A})$ *is a countable nonstandard model of* RCA_0. *The following are equivalent*:

  .

Some Recent Results (3)

- **Theorem.** (K. Yokoyama, to appear). *Suppose* $(\mathcal{M}, \mathcal{A})$ *is a countable nonstandard model of* RCA_0. *The following are equivalent*:

.

- **(1)** $(\mathcal{M}, \mathcal{A}) \models \Pi^1_1\text{-CA}$.

.

Some Recent Results (3)

- **Theorem.** (K. Yokoyama, to appear). *Suppose* $(\mathcal{M}, \mathcal{A})$ *is a countable nonstandard model of* RCA_0. *The following are equivalent*:

.

- **(1)** $(\mathcal{M}, \mathcal{A}) \models \Pi^1_1$-CA.

.

- **(2)** *There is a self-embedding of* $(\mathcal{M}, \mathcal{A})$ *onto a proper initial segment of itself such that* $j(M)$ *is a "Ramsey cut" in* $\mathcal{M}$.

Ali Enayat, Göteborgs Universitet

Some Recent Results (4)

Some Recent Results (4)

- **Theorem.** (A.E., to appear) *The following conditions are equivalent for a countable nonstandard model of* PA:

.

Some Recent Results (4)

- **Theorem.** (A.E., to appear) *The following conditions are equivalent for a countable nonstandard model of* PA:

.

- **(1)** $\mathcal{M}$ *has a self-embedding onto a proper initial segment of itself such that* $\mathrm{Fix}(j) = K^1(\mathcal{M})$.

.

Ali Enayat, Göteborgs Universitet

Some Recent Results (4)

- **Theorem.** (A.E., to appear) *The following conditions are equivalent for a countable nonstandard model of* PA:

.

- **(1)** $\mathcal{M}$ *has a self-embedding onto a proper initial segment of itself such that* $\mathrm{Fix}(j) = K^1(\mathcal{M})$.

.

- **(2)** $\mathbb{N}$ *is a strong cut of* $\mathcal{M}$.

Envoi

Envoi

- Thank you, and Congratulations Harvey! .

Envoi

- Thank you, and Congratulations Harvey! .