Classical Field Theory on Lie Algebroids

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Abstract

I study the differential geometry around the following variational problem:
find the critical point of the integral of a Lagrangian function defined on a set of morphism between Lie algebroids.

It is a constrained variational problem.

It includes as particular cases the standard theory, systems with symmetry, Sigma models, Chern-Simons, ...

A generalized multisymplectic field theory is proposed.

Mechanics on Lie algebroids

(Weinstein 1996, Martínez 2001, de León *et. al.* 2004)

Lie algebroid $E \to M$.

$L \in C^\infty(E)$ or $H \in C^\infty(E^*)$

- $E = TM \to M$ Standard classical Mechanics
- $E = \mathcal{D} \subset TM \to M$ (integrable) System with holonomic constraints
- $E = TQ/G \to M = Q/G$ System with symmetry
- $E = \mathfrak{g} \to \{e\}$ System on Lie algebras
- $E = M \times \mathfrak{g} \to M$ System on a semidirect products (ej. heavy top)

Symplectic and variational

The theory is symplectic:

$$i_\Gamma \omega_L = dE_L$$

with $\omega_L = -d\theta_L$, $\theta_L = S(dL)$ and $E_L = d_\Delta L - L$.

But d is the differential on the Lie algebroid.

It is also a constrained variational theory:
Constraints: $\dot{x}^i = \rho^i_\alpha y^\alpha$ (admissible curves)
Finite admissible variations exists.
Infinitesimal variations are

$$\delta x^i = \rho^i_\alpha \sigma^\alpha$$
$$\delta y^\alpha = \dot{\sigma}^\alpha + C^\alpha_{\beta\gamma} y^\beta \sigma^\gamma$$

Time dependent systems

(Martínez, Mestdag and Sarlet 2002)

With suitable modifications one can describe time-dependent systems.

Cartan form

$$\Theta_L = S(dL) + L dt.$$

Dynamical equation

$$i_\Gamma \, d\Theta_L = 0 \qquad \text{and} \qquad \langle \Gamma, dt \rangle = 1.$$

Field theory in 1-d space-time

Example: standard case

$$
\begin{array}{ccc}
TM & \xrightarrow{T\pi} & TN \\
\downarrow & & \downarrow \\
M & \xrightarrow[\pi]{} & N
\end{array}
$$

$m \in M$ and $n = \pi(m)$

$$0 \longrightarrow \mathrm{Ver}_m \longrightarrow T_m M \longrightarrow T_n N \longrightarrow 0$$

Set of splittings: $J_m\pi = \{\, \phi\colon T_nN \to T_mM \mid T\pi \circ \phi = \mathrm{id}_{T_nN} \,\}$.

Lagrangian: $L\colon J\pi \to \mathbb{R}$

Example: principal bundle

$$\begin{array}{ccc} TQ/G & \xrightarrow{[T\pi]} & TM \\ \downarrow & & \downarrow \\ Q/G = M & \xrightarrow[\mathrm{id}]{} & M \end{array}$$

$m \in M$

$$0 \longrightarrow \mathrm{Ad}_m \longrightarrow (TQ/G)_m \longrightarrow T_m M \longrightarrow 0$$

Set of splittings: $C_m(\pi)$.

Lagrangian: $L\colon C(\pi) \to \mathbb{R}$

Other examples

- Poisson Σ-model:
  $TN \to N$ with $\dim(N) = 2$
  $(T^*M \to M, \Lambda)$ Poisson

- Chern-Simons:
  $TN \to N$ with $\dim(N) = 3$
  $TN \times \mathfrak{g} \to N$

- ...

General case

Consider

$$\begin{array}{ccc} E & \xrightarrow{\overline{\pi}} & F \\ \downarrow & & \downarrow \\ M & \xrightarrow[\underline{\pi}]{} & N \end{array}$$

with $\pi = (\overline{\pi}, \underline{\pi})$ epimorphism.

Consider the subbundle $K = \ker(\pi) \to M$.

For $m \in M$ and $n = \pi(m)$ we have

$$0 \longrightarrow K_m \longrightarrow E_m \longrightarrow F_n \longrightarrow 0$$

and we can consider the set of splittings of this sequence.

We define the sets

$$\mathcal{L}_m\pi = \{\, w\colon F_n \to E_m \;\mid w \text{ is linear}\,\}$$
$$\mathcal{J}_m\pi = \{\, \phi \in \mathcal{L}_m\pi \;\mid \overline{\pi} \circ \phi = \mathrm{id}_{F_n}\,\}$$
$$\mathcal{V}_m\pi = \{\, \psi \in \mathcal{L}_m\pi \;\mid \overline{\pi} \circ \psi = 0\,\}\,.$$

Projections

$\underline{\tilde{\pi}_{10}}\colon \mathcal{L}\pi \to M$	vector bundle
$\underline{\pi_{10}}\colon \mathcal{J}\pi \to M$	affine subbundle
$\underline{\boldsymbol{\pi}_{10}}\colon \mathcal{V}\pi \to M$	vector subbundle

Local expressions

Take $\{e_a, e_\alpha\}$ adapted basis of $\mathrm{Sec}(E)$, i.e. $\{\overline{\pi}(e_a) = \bar{e}_a\}$ is a basis of $\mathrm{Sec}(F)$ and $\{e_\alpha\}$ basis of $\mathrm{Sec}(K)$. Also take adapted coordinates (x^i, u^A) to the bundle $\underline{\pi}\colon M \to N$.

An element of $\mathcal{L}\pi$ is of the form

$$w = (y^b_a e_b + y^\alpha_a e_\alpha) \otimes e^a$$

Thus we have coordinates $(x^i, u^A, y^b_a, y^\alpha_a)$ on $\mathcal{L}\pi$.

An element of $\mathcal{J}\pi$ is of the form

$$\phi = (e_a + y^\alpha_a e_\alpha) \otimes e^a$$

Thus we have coordinates (x^i, u^A, y^α_a) on $\mathcal{J}\pi$.

Anchor and bracket

We will assume that F and E are Lie algebroids and π is a morphism of Lie algebroids. The anchors are

$$\rho(\bar{e}_a) = \rho_a^i \frac{\partial}{\partial x^i} \qquad \begin{cases} \rho(e_a) = \rho_a^i \dfrac{\partial}{\partial x^i} + \rho_a^A \dfrac{\partial}{\partial u^A} \\ \rho(e_\alpha) = \rho_\alpha^A \dfrac{\partial}{\partial u^A} \end{cases}$$

and the brackets are

$$[\bar{e}_a, \bar{e}_b] = C_{bc}^a \bar{e}_a \qquad \begin{cases} [e_a, e_b] = C_{ab}^\gamma e_\gamma + C_{bc}^a e_a \\ [e_a, e_\beta] = C_{a\beta}^\gamma e_\gamma \\ [e_\alpha, e_\beta] = C_{\alpha\beta}^\gamma e_\gamma \end{cases}$$

Affine functions

Given a section θ of $(\mathcal{L}\pi)^*$, we define the affine function $\hat{\theta} \in C^\infty(\mathcal{J}\pi)$ by

$$\hat{\theta}(\phi) = \mathrm{tr}(\theta_m \circ \phi),$$

where $m = \underline{\pi_{10}}(\phi)$.

In coordinates, if

$$\theta = (\theta^a_b e^b + \theta^a_\alpha e^\alpha) \otimes \bar{e}_a$$

then

$$\hat{\theta} = \theta^a_a + \theta^a_\alpha y^\alpha_a.$$

Total derivative of f with respect to a section $\eta \in \mathrm{Sec}(F)$

$$\widehat{df \otimes \eta} = \acute{f}_{|a} \eta^a.$$

where

$$\acute{f}_{|a} = \rho^i_a \frac{\partial f}{\partial x^i} + (\rho^A_a + \rho^A_\alpha y^\alpha_a) \frac{\partial f}{\partial u^A}.$$

Affine structure functions:

$$Z^\alpha_{a\gamma} = (d_{e_\gamma} \widehat{e^\alpha) \otimes} \bar{e}_a = C^\alpha_{a\gamma} + C^\alpha_{\beta\gamma} y^\beta_a$$

$$Z^\alpha_{ac} = (d_{e_c} \widehat{e^\alpha) \otimes} \bar{e}_a = C^\alpha_{ac} + C^\alpha_{\beta c} y^\beta_a$$

$$Z^b_{a\gamma} = (d_{e_\gamma} \widehat{e^b) \otimes} \bar{e}_a = 0$$

$$Z^b_{ac} = (d_{e_c} \widehat{e^b) \otimes} \bar{e}_a = C^b_{ac}$$

Variational Problem

Only for $F = TN$.

Let ω be a fixed volume form on N.

Variational problem: Given a function $L \in C^\infty(\mathcal{J}\pi)$ find those morphisms $\Phi\colon F \to E$ of Lie algebroids such that $\pi \circ \Phi = \mathrm{id}_F$ and are critical points of the action

$$\mathcal{S}(\Phi) = \int_N L(\Phi)\,\omega$$

It is a constrained variational problem since Φ must be a morphism.

In coordinates,

$$\underline{\Phi}(x) = (x, u^A(x)) \qquad \text{and} \qquad \overline{\Phi} = (e_a + y_a^\alpha(x) e_\alpha) \otimes \bar{e}^a$$

and $\omega = dx^1 \wedge \cdots \wedge dx^r$.

The variational problem is: find the critical points of

$$\int_N L(x^i, u^A, y_a^\alpha)\, dx^1 \wedge \cdots \wedge dx^r$$

subject to the constraints

$$\frac{\partial u^A}{\partial x^a} = \rho_a^A + \rho_\alpha^A y_a^\alpha$$
$$\frac{\partial y_c^\alpha}{\partial x^b} - \frac{\partial y_b^\alpha}{\partial x^c} + C_{b\gamma}^\alpha y_c^\gamma - C_{c\gamma}^\alpha y_b^\gamma + C_{\beta\gamma}^\alpha y_b^\beta y_c^\gamma + C_{bc}^\alpha = 0$$

Variations

Let σ be a section of E projectable over a section η of F.

Let Ψ_s the flow of σ and Φ_s the flow of η,

$$\begin{array}{ccc} E & \xrightarrow{\Psi_s} & E \\ {\scriptstyle \overline{\pi}}\downarrow & & \downarrow{\scriptstyle \overline{\pi}} \\ F & \xrightarrow[\Phi_s]{} & F \end{array}$$

Define the map $\mathcal{J}\Psi_s \colon \mathcal{J}\pi \to \mathcal{J}\pi$ by

$$\mathcal{J}\Psi_s(\phi) = \Psi_s \circ \phi \circ \Phi_{-s}.$$

for $\phi \in \mathcal{J}\pi$.

- $\mathcal{J}\Psi_s$ is an affine bundle map (over $\psi_s \colon M \to M$, the flow of $\rho(\sigma)$).
- If $\Phi \colon F \to E$ is a morphism then so is $\mathcal{J}\Psi_s(\Phi) = \Psi_s \circ \Phi \circ \Phi_{-s}$.
- $\mathcal{J}\Psi_s$ is a local flow. The vector field it defines is to be called the complete lift $X_\sigma^{(1)} \in \mathfrak{X}(\mathcal{J}\pi)$ of σ.

  If σ projects to the zero section $\eta = 0$ then

$$X_\sigma^{(1)} = \rho^A_\alpha \sigma^\alpha \frac{\partial}{\partial u^A} + \left(\acute{\sigma}^\alpha_{|a} dx^a + Z^\alpha_{a\beta}\sigma^\beta \right) \frac{\partial}{\partial y^\alpha_a}$$

with $Z^\alpha_{a\beta} = C^\alpha_{a\beta} + C^\alpha_{\gamma\beta} y^\gamma_a$

Euler-Lagrange equations

Infinitesimal admissible variations are

$$\delta u^A = \rho^A_\alpha \sigma^\alpha$$
$$\delta y^\alpha_a = \frac{d\sigma^\alpha}{dx^a} + Z^\alpha_{a\beta}\sigma^\beta.$$

Integrating by parts we get the Euler-Lagrange equations

$$\frac{d}{dx^a}\left(\frac{\partial L}{\partial y^\alpha_a}\right) = \frac{\partial L}{\partial y^\gamma_a} Z^\gamma_{a\alpha} + \frac{\partial L}{\partial u^A}\rho^A_\alpha,$$

$$u^A_{,a} = \rho^A_a + \rho^A_\alpha y^\alpha_a$$
$$\left(y^\alpha_{a,b} + C^\alpha_{b\gamma} y^\gamma_a\right) - \left(y^\alpha_{b,a} + C^\alpha_{a\gamma} y^\gamma_b\right) + C^\alpha_{\beta\gamma} y^\beta_b y^\gamma_a + C^\alpha_{ba} = 0.$$

Euler-Lagrange equations: autonomous case

If $M = N \times Q$ and $E = F \times G$, then

$$\rho^a_\alpha = 0 \qquad \text{and} \qquad C^\alpha_{a\beta} = 0.$$

Thus

$$\frac{d}{dx^a}\left(\frac{\partial L}{\partial y^\alpha_a}\right) = C^\gamma_{\beta\alpha} y^\beta_a \frac{\partial L}{\partial y^\gamma_a} + \frac{\partial L}{\partial u^A}\rho^A_\alpha,$$

$$u^A_{,a} = \rho^A_\alpha y^\alpha_a$$
$$y^\alpha_{a,b} - y^\alpha_{b,a} + C^\alpha_{\beta\gamma} y^\beta_b y^\gamma_a = 0$$

Repeated jets

From now on F is again an arbitrary Lie algebroid.

E-tangent to $\mathcal{J}\pi$.

Consider $\tau^E_{\mathcal{J}\pi}\colon \mathcal{T}^E\mathcal{J}\pi \to \mathcal{J}\pi$

$$\mathcal{T}^E\mathcal{J}\pi = \left\{\, (\phi, a, V) \in \mathcal{J}\pi \times E \times T\mathcal{J}\pi \;\middle|\; T_\phi\underline{\pi_{10}}(V) = \rho(a) \,\right\}$$

and the projection $\pi_1 = \pi \circ \pi_{10} = (\overline{\pi} \circ \overline{\pi_{10}}, \underline{\pi} \circ \underline{\pi_{10}})$

$$\begin{array}{ccc} \mathcal{T}^E\mathcal{J}\pi & \xrightarrow{\overline{\pi_1}} & F \\ \downarrow & & \downarrow \\ \mathcal{J}\pi & \xrightarrow[\underline{\pi_1}]{} & N \end{array}$$

A repeated jet $\psi \in \mathcal{J}\pi_1$ at the point $\phi \in \mathcal{J}\pi$ is a map $\psi\colon F_n \to \mathcal{T}^E_\phi \mathcal{J}\pi$ such that $\overline{\pi_1} \circ \psi = \mathrm{id}_{F_n}$.

Explicitly ψ is of the form $\Psi = (\phi, \zeta, V)$ with

- $\underline{\pi_{10}}(\phi) = \underline{\pi_{10}}(\zeta)$,
- $V\colon F_n \to T_\phi \mathcal{J}\pi$ satisfying

$$T\underline{\pi_{10}} \circ V = \rho \circ \zeta.$$

Locally

$$\psi = (\mathcal{X}_a + \Psi^\alpha_a \mathcal{X}_\alpha + \Psi^\alpha_{ab} \mathcal{V}^b_\alpha) \otimes \bar{e}^a.$$

Contact forms

An element $(\phi, a, V) \in \mathcal{T}^E \mathcal{J}\pi$ is horizontal if $v_\phi(a) = 0$;

$$Z = a^b(\mathcal{X}_b + y_b^\beta \mathcal{X}_\beta) + V_b^\beta \mathcal{V}_\beta^b.$$

An element $\mu \in \mathcal{T}^{*E}\mathcal{J}\pi$ is vertical if it vanishes on horizontal elements.

A **contact 1-form** is a section of $\mathcal{T}^{*E}\mathcal{J}\pi$ which is vertical at every point. They are spanned by

$$\theta^\alpha = \mathcal{X}^\alpha - y_a^\alpha \mathcal{X}^a.$$

The $\bigwedge \mathcal{T}^E \mathcal{J}\pi$-module generated by contact 1-forms is the **contact module** $\mathcal{M}^c$

$$\mathcal{M}^c = \langle \theta^\alpha \rangle.$$

The differential ideal generated by contact 1-forms is the **contact ideal** $\mathcal{I}^c$.

$$\mathcal{I}^c = \langle \theta^\alpha, d\theta^\alpha \rangle$$

Second order jets

- A jet $\psi \in \mathcal{J}_\phi \pi_1$ is **semiholonomic** if $\psi^\star \theta = 0$ for every θ in $\mathcal{M}^c$.

  The jet $\psi = (\phi, \zeta, V)$ is semiholonomic if and only if $\phi = \zeta$.

- A jet $\psi \in \mathcal{J}_\phi \pi_1$ is **holonomic** if $\psi^\star \theta = 0$ for every θ in $\mathcal{I}^c$.

  The jet $\psi = (\phi, \zeta, V)$ is holonomic if and only if $\phi = \zeta$ and $\mathcal{M}^\gamma_{ab} = 0$, where

  $$\mathcal{M}^\gamma_{ab} = y^\gamma_{ab} - y^\gamma_{ba} + C^\gamma_{b\alpha} y^\alpha_a - C^\gamma_{a\beta} y^\beta_b - C^\gamma_{\alpha\beta} y^\alpha_a y^\beta_b + y^\gamma_c C^c_{ab} + C^\gamma_{ba}.$$

  The set of holonomic jets will be denoted $\mathcal{J}^2\pi$.

Jet prolongation of sections

A bundle map $\Phi = (\overline{\Phi}, \underline{\Phi})$ section of π is equivalent to a bundle map $\check{\Phi} = (\check{\overline{\Phi}}, \underline{\Phi})$ from N to $\mathcal{J}\pi \to M$ section of π_1

$$\check{\overline{\Phi}}(n) = \overline{\Phi}\Big|_{F_n}.$$

The jet prolongation of Φ is the section $\Phi^{(1)} \equiv \mathcal{T}^{\Phi}\check{\Phi}$ of π_1.

In coordinates

$$\Phi^{(1)} = (\mathcal{X}_a + \Phi^{\alpha}_{a}\mathcal{X}_{\alpha} + \acute{\Phi}^{\alpha}_{b|a}\mathcal{V}^{b}_{\alpha}) \otimes \bar{e}^{a}.$$

Theorem: Let $\Psi \in \mathrm{Sec}(\pi_1)$ be such that the associated map $\check{\Psi}$ is a semiholonomic section and let $\check{\Phi}$ be the section of $\underline{\pi_1}$ to which it projects. Then

1. The bundle map Ψ is admissible if and only if Φ is admissible and $\Psi = \Phi^{(1)}$.
2. The bundle map Ψ is a morphism of Lie algebroids if and only if $\Psi = \Phi^{(1)}$ and Φ is a morphism of Lie algebroids.

Corollary: Let Φ an admissible map and a section of π. Then Φ is a morphism if and only if $\Phi^{(1)}$ is holonomic.

Lagrangian formalism

$L \in C^\infty(\mathcal{J}\pi)$ Lagrangian, $\omega \in \bigwedge^r F$ 'volume' form.

Canonical form.

For every $\phi \in \mathcal{J}_m\pi$

$$h_\phi(a) = \phi(\overline{\pi}(a)) \qquad \text{and} \qquad v_\phi(a) = a - \phi(\overline{\pi}(a))$$

They define the map $\vartheta\colon \underline{\pi_{10}}^*E \to \underline{\pi_{10}}^*E$ by

$$\vartheta(\phi, a) = v_\phi(a).$$

Vertical lifting.

As in any affine bundle

$$\psi_\phi^V f = \frac{d}{dt} f(\phi + t\psi)\Big|_{t=0}, \qquad \psi \in \mathcal{V}_m\pi, \quad \phi \in \mathcal{J}_m\pi.$$

Thus we have a map $\xi^V : \underline{\pi_{10}}^*(\mathcal{L}\pi) \to \mathcal{T}^E\mathcal{J}\pi$

$$\xi^V(\phi, \varphi) = (\phi, (v_\phi \circ \varphi)_\phi^V).$$

Vertical endomorphism.

Every $\nu \in \mathrm{Sec}(F^*)$ defines $S_\nu \colon \mathcal{T}^E\mathcal{J}\pi \to \mathcal{T}^E\mathcal{J}\pi$

$$S_\nu(\phi, a, V) = \xi^V(\phi, a \otimes \nu) = (\phi, 0, v_\phi(a) \otimes \nu).$$

In coordinates

$$S = \theta^\alpha \otimes \bar{e}_a \otimes \mathcal{V}^a_\alpha.$$

Finally

$$S_\omega = \theta^\alpha \wedge \omega_a \otimes \mathcal{V}^a_\alpha.$$

Cartan forms.

$$\Theta_L = S_\omega(dL) + L\omega$$

$$\Omega_L = -d\Theta_L$$

In coordinates

$$\Theta_L = \frac{\partial L}{\partial y^\alpha_a}\theta^\alpha \wedge \omega_a + L\omega$$

Euler-Lagrange equations.

A solution of the field equations is a morphism $\Phi \in \mathrm{Sec}(\pi)$ such that

$$\Phi^{(1)\star}(i_X \Omega_L) = 0$$

for all π_1-vertical section $X \in \mathrm{Sec}(\mathcal{T}^E \mathcal{J}\pi)$.

More generally one can consider the **De Donder** equations

$$\Psi^\star(i_X \Omega_L) = 0.$$

If L is regular then $\Psi = \Phi^{(1)}$.

In coordinates we get the Euler-Lagrange partial differential equations

$$\acute{u}^A_{|a} = \rho^A_a + \rho^A_\alpha y^\alpha_a$$

$$y^\gamma_{a|b} - y^\gamma_{b|a} + C^\gamma_{b\alpha} y^\alpha_a - C^\gamma_{a\beta} y^\beta_b - C^\gamma_{\alpha\beta} y^\alpha_a y^\beta_b + y^\gamma_c C^c_{ab} + C^\gamma_{ab} = 0$$

$$\left(\frac{\partial L}{\partial y^\alpha_a}\right)'_{|a} + \frac{\partial L}{\partial y^\alpha_a} C^b_{ba} - \frac{\partial L}{\partial y^\gamma_a} Z^\gamma_{a\alpha} - \frac{\partial L}{\partial u^A} \rho^A_\alpha = 0,$$

Hamiltonian formalism

Consider the affine dual of $\mathcal{J}\pi$ considered as the bundle $\underline{\pi_{10}}^\dagger\colon \mathcal{J}^\dagger\pi \to M$ with fibre over m

$$\mathcal{J}^\dagger\pi = \{\, \lambda \in (E_m^*)^{\wedge r} \mid i_{k_1} i_{k_2}\lambda = 0 \text{ for all } k_1, k_2 \in K_m \,\}$$

We have a canonical form Θ in $\mathcal{T}^E\mathcal{J}^\dagger\pi$, given by

$$\Theta_\lambda = (\pi_{10}^\dagger)^\star\lambda.$$

Explicitly

$$\Theta_\lambda(Z_1, Z_2, \ldots, Z_r) = \lambda(a_1, a_2, \ldots, a_r),$$

for $Z_i = (\lambda, a_i, V_i)$.

The differential of Θ is a multisymplectic form

$$\Omega = -d\Theta.$$

For a section h of the projection $\mathcal{J}^\dagger\pi \to \mathcal{V}^*\pi$ we consider the Liouville-Cartan forms

$$\Theta_h = (\mathcal{T}h)^\star\Theta \qquad \text{and} \qquad \Omega_h = (\mathcal{T}h)^\star\Omega$$

We set the Hamilton equations

$$\Lambda^\star(i_X\Omega_h) = 0,$$

for a morphism Λ.

In coordinates we get the Hamiltonian field PDEs

$$\acute{u}^A_{|a} = \rho^A_a + \rho^A_\alpha \frac{\partial H}{\partial \mu^a_\alpha}$$

$$\left(\frac{\partial H}{\partial \mu^a_\alpha}\right)'_{|b} - \left(\frac{\partial H}{\partial \mu^b_\alpha}\right)'_{|a} + C^\alpha_{\beta\gamma}\frac{\partial H}{\partial \mu^b_\beta}\frac{\partial H}{\partial \mu^a_\gamma} + C^\alpha_{b\gamma}\frac{\partial H}{\partial \mu^a_\gamma} - C^\alpha_{a\gamma}\frac{\partial H}{\partial \mu^b_\gamma} = C^\alpha_{ab}$$

$$\acute{\mu}^c_{\alpha|c} x^i + \mu^b_\alpha C^c_{bc} = -\rho^A_\alpha \frac{\partial H}{\partial u^A} + \mu^c_\gamma\left(C^\gamma_{c\alpha} + C^\gamma_{\beta\alpha}\frac{\partial H}{\partial \mu^c_\beta}\right).$$

Legendre transformation

There is a Legendre transformation $\widehat{\mathfrak{F}}_{\mathfrak{L}} : \mathcal{J}\pi \to \mathcal{J}^\dagger\pi$ defined by affine approximation of the Lagrangian as in the standard case. We have similar results:

- $\Theta_L = (\mathcal{T}\widehat{\mathfrak{F}}_{\mathfrak{L}})^\star \Theta$
- $\Omega_L = (\mathcal{T}\widehat{\mathfrak{F}}_{\mathfrak{L}})^\star \Omega$
- For hyperregular Lagrangian L: if Φ is a solution of the Euler-Lagrange equations then $\Lambda = \mathcal{T}\mathfrak{F}_{\mathfrak{L}} \circ \Phi^{(1)}$ is a solution of the Hamiltonian field equations. Conversely if Λ is a solution of the Hamiltonian field equations, then there exists a solution Φ of the Euler-Lagrange equations such that $\Lambda = \mathcal{T}\mathfrak{F}_{\mathfrak{L}} \circ \Phi^{(1)}$.

Example: Standard case

$E = TM$ and $F = TN$. In pseudocordinates

$$\bar{e}_i = \frac{\partial}{\partial x^i}, \qquad \text{and} \qquad e_i = \frac{\partial}{\partial x^i} + \Gamma_i^A \frac{\partial}{\partial u^A}, \qquad e_A = \frac{\partial}{\partial u^A}.$$

We have the brackets

$$[e_i, e_j] = -R_{ij}^A e_A, \qquad [e_i, e_B] = \Gamma_{iB}^A e_A \qquad \text{and} \qquad [e_A, e_B] = 0,$$

where we have written $\Gamma_{iA}^B = \partial \Gamma_i^B / \partial u^A$.

The components of the anchor maps are $\rho_j^i = \delta_j^i$, $\rho_i^A = \Gamma_i^A$ and $\rho_B^A = \delta_B^A$.

The Euler-Lagrange equations are

$$\begin{aligned}
&\frac{\partial u^A}{\partial x^i} = \Gamma_i^A + y_i^A \\
&\frac{\partial y_i^A}{\partial x^j} - \frac{\partial y_j^A}{\partial x^i} + \Gamma_{jB}^A y_i^B - \Gamma_{iB}^A y_j^B = R_{ij}^A \\
&\frac{d}{dx^i}\left(\frac{\partial L}{\partial y_i^A}\right) - \Gamma_{iA}^B \frac{\partial L}{\partial y_i^B} = \frac{\partial L}{\partial u^A}.
\end{aligned}$$

Example: Chern-Simons

We consider a Lie algebra $\mathfrak{g}$ with an ad-invariant metric so that the structure constants $C_{\alpha\beta\gamma}$ are skewsymmetric.

Let N be a 3-dimensional manifold, $E = TN \times \mathfrak{g} \to N$ and $F = TN \to N$.

A section $\Phi\colon F \to E$ is of the form $\Phi(v) = (v, A^\alpha(v)\epsilon_\alpha)$ for some 1-forms A^α on N.

The Lagrangian density for Chern-Simons theory is

$$L\, dx^1 \wedge dx^2 \wedge dx^3 = \frac{1}{3!}\, C_{\alpha\beta\gamma}\, A^\alpha \wedge A^\beta \wedge A^\gamma.$$

There is no admissibility condition in this case, since there are no coordinates u^A.

The morphism conditions can be written conveniently in terms of the 1-forms A^α as

$$dA^\alpha + \frac{1}{2} C^\alpha_{\beta\gamma} A^\beta \wedge A^\gamma = 0.$$

The Euler-Lagrange equations reduce to a linear combination of the morphism condition, and thus vanish identically.

The conventional Lagrangian density for the Chern-Simons theory is

$$L'\omega = k_{\alpha\beta}\left(A^\alpha \wedge dA^\beta + \frac{1}{3}C^\beta_{\mu\nu}A^\alpha \wedge A^\mu \wedge A^\nu\right)$$

The difference between L' and L is a multiple of the morphism condition

$$L'\omega - L\omega = k_{\alpha\mu}A^\mu\left[dA^\alpha + \frac{1}{2}C^\alpha_{\beta\gamma}A^\beta \wedge A^\gamma\right].$$

Therefore both Lagrangians coincide on the set of morphisms, which is the set where the action is defined.

Example: Poisson sigma-model

Is an example of autonomous theory.

Let N be a 2-dimensional and $F = TN$.

Let (Q, Λ) Poisson manifold and $G = T^*Q$ with the associated Lie algebroid structure.

The Lagrangian density $\mathcal{L}(\phi) = -\phi^\star\Lambda$. For a morphism Φ, we write $A_K = \Phi^\star(\partial/\partial u^K) = y_{Ki}dx^i$, so that the Lagrangian density reads

$$\mathcal{L} = -\frac{1}{2}\Lambda^{JK}A_J \wedge A_K$$

The Euler-Lagrange equations reduce to a linear combination of the morphism condition.

Thus the field equations are just

$$
\begin{aligned}
&d\phi^J + \Lambda^{JK} A_K = 0 \\
&dA_J + \frac{1}{2}\Lambda^{KL}_{,J} A_K \wedge A_L = 0.
\end{aligned}
$$

The conventional Lagrangian density for the Poisson Sigma model is $\mathcal{L}' = \mathrm{tr}(\overline{\Phi} \wedge T\underline{\Phi}) + \frac{1}{2}\Phi^\star\Lambda$. The difference between $\mathcal{L}'$ and $\mathcal{L}$ is a multiple of the admissibility condition $d\phi^J + \Lambda^{JK}A_K$;

$$\mathcal{L}' - \mathcal{L} = A_J \wedge (d\phi^J + \Lambda^{JK}A_K).$$

Therefore both Lagrangians coincide on admissible maps, and hence on morphisms.

Example: Time-dependent Mechanics

Consider the case of a Lie algebroid $E \to M$ and $F = T\mathbb{R} \to N = \mathbb{R}$. Define the affine space

$$A = \left\{ a \in E \;\middle|\; \overline{\pi}(a) = \frac{\partial}{\partial t} \right\},$$

modeled on the kernel of π. Then $(A^\dagger)^* = E$ and the Lie algebroid on E restricts to a Lie algebroid structure on the affine bundle A (a Lie Afffffgebroid structure).

Conversely, given a structure of Lie algebroid on an affine bundle $A \to M$, with $\underline{\pi}\colon M \to \mathbb{R}$ fiber bundle, we have that $E = (A^\dagger)^*$ has a Lie algebroid structure. If $\tilde{\rho}$ is the anchor on E then $\overline{\pi}(z) = T\underline{\pi}(\tilde{\rho}(z))$ is a morphism form E to $T\mathbb{R}$ and we have a canonical identification $I\colon A \to \mathcal{J}\pi$ given by

$$I(a) = a\,dt$$

Thus we recover the time dependent formulation.

Many more examples

- Variational problems for holomorphic maps.
- Systems with symmetry.
- Other sigma-models.
- ...

The End

Appendices

Flow of a derivation

Let $\tau \colon E \to M$ be a vector bundle.

A derivation of the $C^\infty(M)$-module $\mathrm{Sec}(E)$ is a $\mathbb{R}$-linear map $D \colon \mathrm{Sec}(E) \to \mathrm{Sec}(E)$ for which there exists a vector field D^M on M such that

$$D(f\sigma) = (D^M f)\sigma + f D\sigma.$$

The action of D can be extended to $\mathrm{Sec}(E^*)$ by duality: if α is a section of E^* then $D\alpha$ is defined by the equation

$$D\langle \alpha, \sigma \rangle = \langle D\alpha, \sigma \rangle + \langle \alpha, D\sigma \rangle.$$

- There exists a linear local flow $\phi_s \colon E \to E$ projecting to the flow of the vector field D^M such that

$$D\sigma = \frac{d}{ds}\phi_s^*\sigma\Big|_{s=0} \qquad \text{and} \qquad D\alpha = \frac{d}{ds}\phi_s^*\alpha\Big|_{s=0}.$$

- There exists a vector field D^E on E such that

$$D^E\hat{\sigma} = \widehat{D\sigma},$$

  for every section σ of E. The vector field D^E projects to D^M and its flow is precisely ϕ_s.

- There exists a vector field D^{E^*} on E^* such that

$$D^{E^*}\hat{\alpha} = \widehat{D\alpha},$$

  for every section α of E^*. The vector field $X_D^{E^*}$ projects to D^M and its flow is ϕ_{-s}^*. (i.e. $\phi_{-s}^*(\mu) = \mu \circ \phi_{-s}$.)

Examples

- **Lie derivative.**

  $E = TM$ and $D = \mathfrak{L}_X$, so that $D^M = X$. Then $D^{TM} = X^c$ the complete lift of X. The flow is $\phi_s = T\varphi_s$.

- **Lie derivatives on a Lie algebroid.**

  More generally, if E is a Lie algebroid and $D = d_\sigma$ then $D^{TM} = \rho^1(\sigma^c)$ the vector field associated to the complete lift of σ.

Covariant derivative.

On a vector bundle with a linear connection, take $D = \nabla_X$ then $D^E = X^h$ the horizontal lift of X to E. The flow is parallel transport along the integral curves of X.

ρ-Covariant derivative.

More generally, if we have a ρ-covariant derivative on a vector bundle E, take $D = \nabla_\sigma$ then $D^E = \rho^1(\sigma^h)$ the horizontal lift of σ to E. The flow is parallel transport along the integral curves of σ.

Lie Algebroids

A Lie algebroid structure on the vector bundle $\tau\colon E \to M$ is given by

- a Lie algebra structure $(\mathrm{Sec}(E), [\,,\,])$ on the set of sections of E, and
- a morphism of vector bundles $\rho\colon E \to TM$ over the identity, such that
  - $\rho([\sigma, \eta]) = [\rho(\sigma), \rho(\eta)]$
  - $[\sigma, f\eta] = f[\sigma, \eta] + (\rho(\sigma)f)\,\eta$,

  where $\rho(\sigma)(m) = \rho(\sigma(m))$.

The first condition is actually a consequence of the second and the Jacobi identity.

Examples-

- **Tangent bundle.**

  $E = TM$,

  $\rho = \mathrm{id}$,

  $[\,,\,] =$ bracket of vector fields.

- **Integrable subbundle.**

  $E \subset TM$, integrable distribution

  $\rho = i$, canonical inclusion

  $[\,,\,] =$ restriction of the bracket to vector fields in E.

Lie algebra.

$E = \mathfrak{g} \to M = \{e\}$, Lie algebra (fiber bundle over a point)
$\rho = 0$, trivial map (since $TM = \{0_e\}$)
$[\,,\,] =$ the bracket in the Lie algebra.

Atiyah algebroid.

Let $\pi\colon Q \to M$ a principal G-bundle.
$E = TQ/G \to M$, (Sections are equivariant vector fields)
$\rho([v]) = T\pi(v)$ induced projection map
$[\,,\,] =$ bracket of equivariant vectorfields (is equivariant).

Transformation Lie algebroid.

Let $\Phi \colon \mathfrak{g} \to \mathfrak{X}(M)$ be an action of a Lie algebra $\mathfrak{g}$ on M.

$E = M \times \mathfrak{g} \to M$,

$\rho(m, \xi) = \Phi(\xi)(m)$ value of the fundamental vectorfield

$[\,,\,] =$ induced by the bracket on $\mathfrak{g}$.

Exterior differential

On 0-forms

$$df(\sigma) = \rho(\sigma)f$$

On p-forms ($p > 0$)

$$d\omega(\sigma_1, \ldots, \sigma_{p+1}) = \\ = \sum_{i=1}^{p+1} (-1)^{i+1} \rho(\sigma_i)\omega(\sigma_1, \ldots, \widehat{\sigma_i}, \ldots, \sigma_{p+1}) \\ - \sum_{i<j} (-1)^{i+j} \omega([\sigma_i, \sigma_j], \sigma_1, \ldots, \widehat{\sigma_i}, \ldots, \widehat{\sigma_j}, \ldots, \sigma_{p+1}).$$

Admissible maps and Morphisms

A bundle map $\Phi = (\overline{\Phi}, \underline{\Phi})$ between E and E' is said to be admissible map if

$$\Phi^\star df = d\Phi^\star f.$$

A bundle map $\Phi = (\overline{\Phi}, \underline{\Phi})$ between E and E' is said to be a morphism of Lie algebroids if

$$\Phi^\star d\theta = d\Phi^\star \theta.$$

Obviously every morphism is an admissible map.