Mechanical control systems on Lie algebroids

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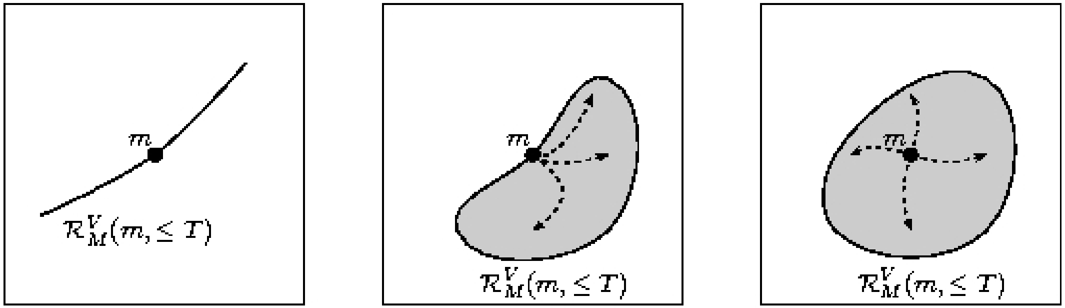
Joint work with **Eduardo Martínez**

Why control systems on Lie algebroids?

Controllability problem: find conditions that guarantee system can move locally in *any* direction.



Deciding local controllability must be addressed prior to other important control questions (e.g., motion planning, trajectory generation, etc.)

Why control systems on Lie algebroids?

Lack of unifying framework imposes separate study for various classes of systems such as

(i) simple mechanical systems (Lewis& Murray 95)

(ii) systems subject to nonholonomic constraints (Bloch et al 92, Bullo& Zefran 01, Lewis 97),

(iii) systems invariant under the action of a Lie group of symmetries (Cortés et al 02, Kelly & Murray 05, Martin & Crouch 84, Martínez & Cortés 03)

(iv) systems enjoying special homogeneity properties (Cortes et al 01, Kawski 95, Vela & Burdick 03).

(v) systems evolving on semidirect products (Shen 02)

Most works build on the rich geometric structure of these systems. Is it possible to combine both *geometric wealth* and *generality*?

Lie algebroid formalism provides an answer!

Lie algebroid notion provides framework to overcome drawback

- Underlying structure of Lie algebroid on the phase space makes possible unified treatment

- Lie algebroid formalism allows us to establish morphisms between two systems, and relate their control properties

Underlying idea: property of interest easier to decide for one system, and morphism allows us to infer property for the other one

Outline

(i) Motivation

(ii) ***Lie algebroid formalism***

(iii) Nonlinear control

(iv) Mechanical control systems on Lie algebroids

(v) Applications

Lie algebroids

Lie algebroid $\tau: E \to M$ with anchor $\rho : E \to TM$ ($\sim$ substitute of TM)

Some useful definitions:

- $\overline{\mathrm{Lie}}(\mathcal{Y})$ is distribution obtained by closing $\mathcal{Y} \subset \mathrm{Sec}(E)$ under Lie bracket
- $a : [t_0, t_1] \to E$ ***admissible*** if $\frac{d}{dt}\tau(a(t)) = \rho(a(t))$
- E ***locally transitive at*** $m \in M$ if $\rho_m : E_m \to T_mM$ is surjective (m is contained in a leaf of maximal dimension)
- $\Psi: E \to \overline{E}$ is ***morphism of Lie algebroids*** if it is admissible ($T\psi \circ \rho = \overline{\rho} \circ \Psi$) and preserves Lie algebra structure of algebroids

Linear connections

Linear E-connection on a vector bundle $\pi\colon P \longrightarrow M$ (Fernandes 02, Cantrijn & Langerock 02) is $\mathbb{R}$-bilinear map $\nabla\colon \operatorname{Sec}(E) \times \operatorname{Sec}(P) \to \operatorname{Sec}(P)$

$$\nabla_{F\sigma}\alpha = F\,\nabla_\sigma\alpha \qquad \text{and} \qquad \nabla_\sigma(F\alpha) = (\rho(\sigma)F)\alpha + F\nabla_\sigma\alpha$$

for any $F \in C^\infty(M)$, $\sigma \in \operatorname{Sec}(E)$ and $\alpha \in \operatorname{Sec}(P)$. We take $P = E$

Skew-symmetric part defines ***torsion tensor*** $T(\sigma,\eta) = \nabla_\sigma\eta - \nabla_\eta\sigma - [\sigma,\eta]$

Symmetric part determines ***symmetric product*** $\langle\sigma : \eta\rangle = \nabla_\sigma\eta + \nabla_\eta\sigma$

$\overline{\operatorname{Sym}}(\mathcal{Y})$ is distribution obtained by closing $\mathcal{Y} \subset \operatorname{Sec}(E)$ under $\langle\cdot : \cdot\rangle$

Covariant derivatives and ***geodesics of*** ∇ (admissible curves $a : \mathbb{R} \to E$ with $\nabla_{a(t)}a(t) = 0$)

Levi-Civita and constrained connections

Levi-Civita connection: For metric $\mathcal{G}\colon E \times_M E \to \mathbb{R}$, unique torsion-less connection $\nabla^{\mathcal{G}}$ on E metric with respect to $\mathcal{G}$,

$$2\mathcal{G}(\nabla_\sigma \eta, \zeta) = \rho(\sigma)\mathcal{G}(\eta,\zeta) + \rho(\eta)\mathcal{G}(\sigma,\zeta) - \rho(\zeta)\mathcal{G}(\eta,\sigma) + \mathcal{G}(\sigma,[\zeta,\eta]) + \mathcal{G}(\eta,[\zeta,\sigma])$$

- gradient of $V \in C^\infty(M)$, $\operatorname{grad}_{\mathcal{G}} V \in \operatorname{Sec}(E)$ is $\operatorname{grad}_{\mathcal{G}} V = \sharp_{\mathcal{G}}(\rho^* dV)$

Constrained connection: Let D subbundle of E, $P : E \to D$ projector, $Q = I - P$, and $D^c = \operatorname{Im}(Q)$ (note $D \oplus D^c = E$). Given ∇ connection,

$$\check{\nabla}_\sigma \eta = P(\nabla_\sigma \eta) + \nabla_\sigma (Q\eta), \quad \sigma, \eta \in \operatorname{Sec}(E)$$

Generalizes nonholonomic connection (Lewis 96, Synge 28)

- $\check{\nabla}$ restricts to D, i.e., $\check{\nabla}_\sigma \eta \in D$ for $\eta \in \operatorname{Sec}(D)$, $\sigma \in \operatorname{Sec}(E)$

Prolongation of Lie algebroid

Prolongation of E (Martinez 01) is $\tau_1 : \mathcal{T}E \longrightarrow E$ with fiber

$$\mathcal{T}_a E = \{\, (b, v) \in E_m \times T_a E \mid \rho(b) = T_a\tau(v) \,\}\,, \ a \in E_m$$

- Anchor $\rho^1 : \mathcal{T}E \longrightarrow TE$, $\rho^1(a, b, v) = v$. Also $\mathcal{T}\tau : \mathcal{T}E \longrightarrow E$, $\mathcal{T}\tau(a, b, v) = b$
- Morphisms of Lie algebroids can also be prolonged: prolongation of $\Psi : E \longrightarrow \overline{E}$ is $\mathcal{T}\Psi : \mathcal{T}E \longrightarrow \mathcal{T}\overline{E}$, $\mathcal{T}\Psi(a, b, v) = (\Psi(a), \Psi(b), T_a\Psi(v))$

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Vertical space: $\mathrm{Ver}(\mathcal{T}E) \subset \mathcal{T}E$ are elements $(a,0,v)$ with v vertical vector

Vertical lift of $\sigma \in \mathrm{Sec}(E)$ is $\sigma^V \in \mathrm{Sec}(\mathcal{T}E)$, $\sigma^V(a) = (a, 0, \sigma(m)^V_a)$

Horizontal space: $\mathrm{Hor}_m(\mathcal{T}E) \subset \mathcal{T}_{0_m}E$ along 0_M,

$$\mathrm{Hor}_m(\mathcal{T}E) = \{\,(0_m, b, v) \in \mathcal{T}_{0_m}E \mid v \in T_m M \subseteq T_{0_m}E\,\}\,, \quad m \in M$$

Along 0_M, $\mathcal{T}_{0_M}E = \mathrm{Hor}(\mathcal{T}E) \oplus \mathrm{Ver}_{0_M}(\mathcal{T}E)$

Homogeneity

Liouville section of $\mathcal{T}E$ is $\Delta(a) = (a, 0, a_a^V)$

$F \in C^\infty(E)$ is ***homogeneous of degree*** $s \in \mathbb{Z}$ if $\mathcal{L}_{\rho^1(\Delta)} F = sF$

$Z \in \mathrm{Sec}(\mathcal{T}E)$ is ***homogeneous of degree*** $s \in \mathbb{Z}$ if $[\Delta, Z] = sZ$

$\mathcal{P}_s$ is set of homogeneous sections of $\mathcal{T}E$ of degree s

Proposition: Let $r, s \in \mathbb{Z}$ and $Z \in \mathrm{Sec}(\mathcal{T}E)$. Then

(i) $[\mathcal{P}_s, \mathcal{P}_r] \subseteq \mathcal{P}_{s+r}$, and $\mathcal{P}_s = \{0\}$ if $s \leq -2$,

(ii) $Z \in \mathcal{P}_{-1}$ if and only if there exists a section σ of E such that $Z = \sigma^V$,

(iii) $Z \in \mathcal{P}_0$ if and only if Z is a projectable section,

Note that for all $Z \in \mathcal{P}_s$, $s \geq 1$, $Z(0_m) = 0_{0_m}$, $m \in M$

SODE sections

$\Gamma \in \mathrm{Sec}(\mathcal{T}E)$ is SODE section on E if $\mathcal{T}\tau \circ \Gamma = \mathrm{Id}_E$

- $\mathrm{Adm}(E)$ is set of admissible vectors $v \in T_aE$ of the form $(a, a, v) \in \mathcal{T}E$
- $\Gamma \in \mathrm{Sec}(\mathcal{T}E)$ is SODE if $\Gamma \in \mathrm{Adm}(E)$

Sprays are homogeneous SODE sections with degree 1

- Associated ***symmetric product:*** for $\sigma, \eta \in \mathrm{Sec}(E)$, $[\sigma^V, [\Gamma, \eta^V]]$ is homogeneous with degree -1, hence $\langle \sigma : \eta \rangle_\Gamma^V = [\sigma^V, [\Gamma, \eta^V]]$

  Symmetric product determines and *is determined* by Γ. Locally

$$[\eta^V, [\Gamma, \sigma^V]] = \left(\sigma^\gamma \rho_\gamma^k \frac{\partial \eta^\alpha}{\partial x^k} + \eta^\gamma \rho_\gamma^k \frac{\partial \sigma^\alpha}{\partial x^k} + f_{\beta\gamma}^\alpha \sigma^\beta \eta^\gamma\right) \mathcal{V}_\alpha$$

- For Γ and skew-symmetric (2,1) tensor T, $\nabla_\sigma^{\Gamma,T} \eta = \frac{1}{2}\big([\sigma, \eta] + T(\sigma, \eta)\big) + \frac{1}{2} \langle \sigma : \eta \rangle_\Gamma$ is unique connection with associated spray Γ and torsion T

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Nonlinear control

Affine nonlinear control system on manifold M,

$$\dot{m}(t) = f(m(t)) + \sum_{i=1}^{k} u_i(t) g_i(m(t)) \,,$$

where $u = (u_1, \ldots, u_k) \in U$, $0 \in U$ open set of $\mathbb{R}^k$

f is drift vector field
$g_1, \ldots, g_k$ are control vector fields

$t \mapsto u(t) = (u_1(t), \ldots, u_k(t))$ belongs to $\mathcal{U}$, set of admissible controls
(for us, piecewise constant functions with values in U)

Sample control problems

Stabilization

- Stabilize an (otherwise) unstable equilibrium
- Shape the dynamics to make a desired configuration an equilibrium

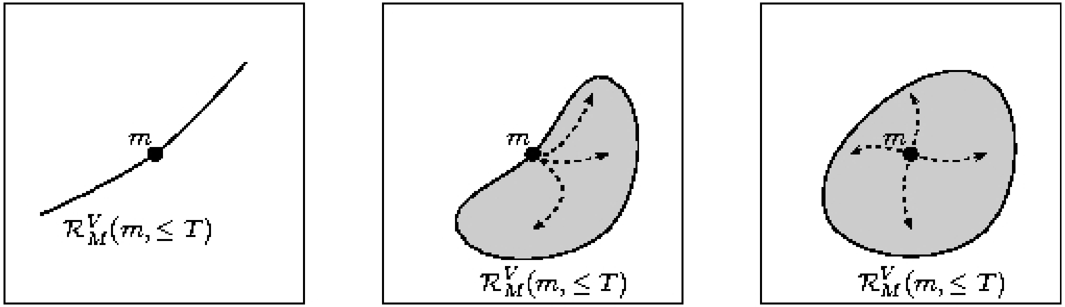
Motion planning and trajectory tracking

- Generate controls that make system go from A to B
- Generate controls that make system track a desired trajectory

Parameter uncertainty and disturbance rejection

- Design controls that cope with errors in knowledge of parameters
- Make system behavior robust to unknown disturbances

Controllability problem



$\mathcal{R}_M^V(m,T)$ is reachable set from $m \in M$ at time $T > 0$, with trajectories contained in neighborhood V of m for $t \leq T$

$$\mathcal{R}_M^V(m, \leq T) = \bigcup_{t \leq T} \mathcal{R}_M^V(m,t)$$

Locally accessible from $m \in M$: $\mathcal{R}_M^V(m, \leq T)$ contains non-empty open set of M for all neighborhoods V of m and all $T > 0$

Locally controllable from $m \in M$: $\mathcal{R}_M^V(m, \leq T)$ contains non-empty open set of M to which m belongs for all neighborhoods V of m and all $T > 0$

General control systems on Lie algebroids

$\tau : E \to M$ Lie algebroid, $\sigma, \eta_1, \ldots, \eta_k \in \mathrm{Sec}(E)$

Control problem on $E \to M$ ***with drift*** σ ***and inputs*** $\eta_1, \ldots, \eta_k$

$$\dot{m}(t) = \rho\big(\sigma(m(t)) + \sum_{i=1}^{k} u_i(t)\eta_i(m(t))\big)$$

Trajectories are admissible curves of E, and hence must lie on a leaf of E.

Only locally transitive Lie algebroids – otherwise, system cannot be locally accessible at points $m \in M$ where ρ is not surjective

With $f = \rho(\sigma)$, $g_i = \rho(\eta_i)$, standard affine nonlinear control system on M
Lie algebroid ***geometric structure*** enhances controllability analysis

Accessibility algebra and subbundle

Accessibility algebra D is smallest subalgebra in $\mathrm{Sec}(E)$ containing $\sigma, \eta_1, \ldots, \eta_k$

Elements of D are linear combinations of Lie brackets of the form

$$[\zeta_l, [\zeta_{l-1}, [\ldots, [\zeta_2, \zeta_1] \ldots]]], \quad \zeta_i \in \{\sigma, \eta_1, \ldots, \eta_k\}, \; l \in \mathbb{N}$$

Accessibility subbundle $\overline{\mathrm{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\})$ is vector subbundle of E generated by accessibility algebra D,

$$\overline{\mathrm{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\}) = \mathrm{span}\left\{\zeta(m) \mid \zeta \text{ section of } E \text{ in } \mathrm{D}\right\}, \; m \in M$$

If dimension of $\overline{\mathrm{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\})$ is constant, then $\overline{\mathrm{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\})$ is smallest Lie subalgebroid of E that has $\{\sigma, \eta_1, \ldots, \eta_k\}$ as sections

Accessibility and controllability tests

Theorem: Let E be locally transitive at $m \in M$

$\overline{\mathrm{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\})(m) + \ker \rho(m) = E_m \Rightarrow$ locally accessible from m

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Theorem: Let E be locally transitive at $m \in M$

$$\overline{\mathrm{Lie}}(\{\sigma, \eta_1, \dots, \eta_k\})(m) + \ker \rho(m) = E_m \Rightarrow \text{locally accessible from } m$$

Let B be iterated Lie bracket of elements in $\{X_0, X_1, \dots, X_k\} \subset \mathrm{Sec}(E)$

- $\delta_i(B)$ is the number of times that X_i appears in B
- $\delta(B) = \delta_0(B) + \delta_1(B) + \cdots + \delta_k(B)$, ***degree*** of B
- B ***bad*** if $\delta_0(B)$ odd and $\delta_i(B)$ even, $i \in \{1, \dots, k\}$. B ***good*** if not bad

Theorem: Assume system locally accessible from $m \in M$. If every bad Lie bracket B in $\{\sigma, \eta_1, \dots, \eta_k\}$ evaluated at m can be put as an $\mathbb{R}$-linear combination of good Lie brackets in $\{\sigma, \eta_1, \dots, \eta_k\}$ of lower degree and elements in $\ker \rho(m)$, then system is locally controllable from m

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Mechanical control systems on Lie algebroids

For Lagrangian $L : E \longrightarrow \mathbb{R}$, **Euler-Lagrange operator** $\delta L : \mathrm{Adm}(E) \longrightarrow E^*$ (Cariñena & Martinez 01, Martinez 01, Weinstein 96). Locally

$$\delta L = \left(\frac{d}{dt} \frac{\partial L}{\partial y^\alpha} + C^\gamma_{\alpha\beta} y^\beta \frac{\partial L}{\partial y^\gamma} - \rho^i_\alpha \frac{\partial L}{\partial x^i} \right) e^\alpha$$

With input forces $\{\theta_1, \ldots, \theta_k\} \subset \mathrm{Sec}(E^*)$ acting on Lagrangian system

$$\delta L = \sum_{l=1}^{k} u_l \, \theta_l$$

If system is nonholonomically constrained by subbundle D of E, with projectors $P : E \longrightarrow D$ and $Q = I - P$. Equations of motion read

$$P^*(\delta L) = \sum_{l=1}^{k} u_l P^*(\theta_l) \,, \quad Q(a) = 0$$

Connection control systems

Let ∇ connection on E, and $\{\eta, \eta_1, \ldots, \eta_k\} \subset \mathrm{Sec}(E)$

$$\nabla_{a(t)} a(t) + \eta(m(t))) = \sum_{i=1}^{k} u_i(t) \eta_i(m(t))$$

Equivalently, control system on $\mathcal{T}E \to E$

$$\dot{a}(t) = \rho^1 \Big((\Gamma_\nabla - \eta^V)(a(t)) + \sum_{i=1}^{k} u_i(t) \eta_i^V(a(t)) \Big)$$

Equations capture ***mechanical control systems***, both

- unconstrained: $L = \frac{1}{2}\mathcal{G} - V \circ \tau$, $\nabla = \nabla^{\mathcal{G}}$, $\eta = \mathrm{grad}_{\mathcal{G}} V$, $\eta_i = \sharp_{\mathcal{G}}(\theta_i) \in \mathrm{Sec}(E)$
- constrained: $L = \frac{1}{2}\mathcal{G} - V \circ \tau$, D subbundle of E, $\check{\nabla} = P(\nabla^{\mathcal{G}}_{\cdot} \cdot) + \nabla^{\mathcal{G}}_{\cdot}(Q \cdot)$, $\eta = P(\mathrm{grad}_{\mathcal{G}} V)$, $\eta_i = P(\sharp_{\mathcal{G}}(\theta_i)) \in \mathrm{Sec}(E)$

Accessibility and controllability notions

I: Full-state accessibility and controllability

W neighborhood of $0_m \in E$,

$$\mathcal{R}_E^W(0_m, \leq T) \text{ reachable points in } E \text{ from } 0_m$$

System ***locally accessible from m at zero*** if $\mathcal{R}_E^W(0_m, \leq T)$ contains non-empty open set of E for all neighb. W of 0_m in E and all $T > 0$

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$$\mathcal{R}_E^W(0_m, \leq T) \text{ reachable points in } E \text{ from } 0_m$$

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II: Base accessibility and controllability

Additional notions specialized to mechanical control systems

V neighborhood of $m \in M$,

$$\mathcal{R}_M^V(m, \leq T) = \tau(\mathcal{R}_E^{\tau^{-1}(V)}(0_m, \leq T)) \text{ reachable points in } M \text{ from } m$$

System ***locally base accessible from m*** if $\mathcal{R}_M^V(m, \leq T)$ contains non-empty open set of M for all neighborhoods V of m in M all $T > 0$

Accessibility and controllability notions

III: Accessibility and controllability with regards to a manifold

Let $\psi : M \to N$ be open. System is ***locally base accessible from*** m ***with regards to*** N if $\psi(\mathcal{R}_M^V(m, \leq T))$ contains non-empty open set of N for all neighborhoods V of m and all $T > 0$

Analogous definitions for ***controllability***

Base accessibility and controllability with regards to M with $\mathrm{Id}_M : M \to M$ corresponds to base accessibility and controllability

System base accessible (respec. controllability) $\Rightarrow$ system base accessible (respec. controllability) with regards to N

Structure of control algebra

Objective: analyze $\overline{\mathrm{Lie}}(\{\Gamma - \eta^V, \eta_1^V, \ldots, \eta_k^V\})$ at points of the form 0_m, $m \in M$

Analysis

- extends Lewis & Murray 95 to mechanical control systems defined on Lie algebroids
- relies on ***homogeneity*** and geometry of $\mathcal{T}E$ along zero-section

Strategy: $\overline{\mathrm{Lie}}(\{\Gamma - \eta^V, \eta_1^V, \ldots, \eta_k^V\}) \subset \overline{\mathrm{Lie}}(\{\Gamma, \eta_1^V, \ldots, \eta_k^V, \eta^V\})$

Theorem: Let $m \in M$. Then,

$$\overline{\mathrm{Lie}}(\{\Gamma, \eta_1^V, \ldots, \eta_k^V, \eta^V\}) \cap \mathrm{Ver}_{0_m}(\mathcal{T}E) = \overline{\mathrm{Sym}}(\{\eta, \eta_1, \ldots, \eta_k\})(m)^V$$
$$\overline{\mathrm{Lie}}(\{\Gamma, \eta_1^V, \ldots, \eta_k^V, \eta^V\}) \cap \mathrm{Hor}_m(\mathcal{T}E) = \overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\{\eta, \eta_1, \ldots, \eta_k\}))(m)$$

Vertical and horizontal distributions

Let $\mathcal{X}' = \{\Gamma - \eta^V, \eta_1^V, \ldots, \eta_k^V\}$ and $\mathcal{X} = \{\Gamma, \eta_1^V, \ldots, \eta_k^V, \eta^V\}$

For B' in $\mathrm{Br}(\mathcal{X}')$, let $S(B') \subset \mathrm{Br}(\mathcal{X})$ contain $B \in \mathrm{Br}(\mathcal{X})$ obtained by replacing any $\Gamma - \eta^V$ in B' by either Γ or η^V (denote $\delta_{k+1}(B)$ number of occurrences of η^V in B)

$$B' = \sum_{B \in S(B')} (-1)^{\delta_{k+1}(B)} B$$

Reciprocally, for $B \in \mathrm{Br}(\mathcal{X})$, determine B' such that $B \in S(B')$ by substituting occurrence of Γ or η^V in B by $\Gamma - \eta^V$ ($\mathrm{pseudoinv}(B) = B'$)

Define $\sigma \in \mathcal{C}^{(k)}_{\mathrm{ver}}(\eta; \eta_1, \ldots, \eta_k) \subset \mathrm{Sec}(E)$ iff

$$\sigma^V = B'', B'' = \sum_{\substack{\tilde{B} \in S(\mathrm{pseudoinv}(B)) \\ \cap \mathrm{Br}_{-1}(\mathcal{X}) \cap \mathrm{Br}_0(\mathcal{X})}} (-1)^{\delta_{k+1}(\tilde{B})} \tilde{B}\,, \; B \in \mathrm{Br}^{2k-1}(\mathcal{X}) \text{ primitive}$$

and $\sigma \in \mathcal{C}^{(k)}_{\mathrm{hor}}(\eta; \eta_1, \ldots, \eta_k) \subset \mathrm{Sec}(E)$ iff

$$\sigma = \sigma_{B''}, B'' = \sum_{\substack{\tilde{B} \in S(\mathrm{pseudoinv}(B)) \\ \cap \mathrm{Br}_{-1}(\mathcal{X}) \cap \mathrm{Br}_0(\mathcal{X})}} (-1)^{\delta_{k+1}(\tilde{B})} \tilde{B}\,, \; B \in \mathrm{Br}^{2k}(\mathcal{X}) \text{ primitive}$$

The accessibility subbundle

$$\mathcal{C}_{\mathrm{ver}}(\eta;\eta_1,\ldots,\eta_k) = \cup\mathcal{C}^{(k)}_{\mathrm{ver}}(\eta;\eta_1,\ldots,\eta_k)$$
$$\mathcal{C}_{\mathrm{hor}}(\eta;\eta_1,\ldots,\eta_k) = \cup\mathcal{C}^{(k)}_{\mathrm{hor}}(\eta;\eta_1,\ldots,\eta_k)$$

Subbundles of E are $C_{\mathrm{ver}}(\eta;\eta_1,\ldots,\eta_k)$ and $C_{\mathrm{hor}}(\eta;\eta_1,\ldots,\eta_k)$

Theorem: Let $m \in M$. Then,

$$\overline{\mathrm{Lie}}(\{\Gamma - \eta^V, \eta_1^V, \ldots, \eta_k^V\}) \cap \mathrm{Ver}_{0_m}(\mathcal{T}E) = C_{\mathrm{ver}}(\eta;\eta_1,\ldots,\eta_k)(m)^V$$
$$\overline{\mathrm{Lie}}(\{\Gamma - \eta^V, \eta_1^V, \ldots, \eta_k^V\}) \cap \mathrm{Hor}_m(\mathcal{T}E) = C_{\mathrm{hor}}(\eta;\eta_1,\ldots,\eta_k)(m)$$

When $\eta = 0$ (no potential)

$$C_{\mathrm{ver}}(0;\eta_1,\ldots,\eta_k) = \overline{\mathrm{Sym}}(\{\eta_1,\ldots,\eta_k\})$$
$$C_{\mathrm{hor}}(0;\eta_1,\ldots,\eta_k) = \overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\{\eta_1,\ldots,\eta_k\}))$$

Accessibility tests

Theorem: Let $m \in M$ and assume E is locally transitive at m. Then, system

- locally base accessible from m if $C_{\mathrm{hor}}(\eta; \eta_1, \dots, \eta_k)(m) + \ker\rho = E_m$
- locally accessible from m at zero if $C_{\mathrm{hor}}(\eta; \eta_1, \dots, \eta_k)(m) + \ker\rho = E_m$ and $C_{\mathrm{ver}}(\eta; \eta_1, \dots, \eta_k)(m) = E_m$

When $\eta = 0$ (no potential), if $\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\{\eta_1, \dots, \eta_k\}))(m) + \ker\rho \neq E_m$, let N denote maximal integral manifold of $\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\{\eta_1, \dots, \eta_k\}))(m)$ through m

For each neighborhood V of m in M and each T sufficiently small, $\mathcal{R}_M^V(m, \leq T) \subset N$ contains a non-empty open subset of N

Controllability tests

Let P be symmetric product in $\{\eta, \eta_1, \ldots, \eta_k\}$

P is *bad* if the number of occurrences of each η_i in P is even

P is *good* otherwise

Accordingly, $\langle \eta_i : \eta_i \rangle$ is bad and $\langle \langle \eta : \eta_j \rangle : \langle \eta_i : \eta_i \rangle \rangle$ is good

Theorem: Let $m \in M$. System is locally base controllable from m if locally base accessible from m and every bad symmetric product in $\{\eta, \eta_1, \ldots, \eta_k\}$ evaluated at m can be put as an $\mathbb{R}$-linear combination of good symmetric products of lower degree and elements of $\ker \rho$

Similar tests for

- locally controllable at zero
- base accessibility/controllability with regards to manifold

Morphism-related mechanical systems (I)

Let $\Psi : E \to \overline{E}$ be morphism of Lie algebroids and weakly $\mathcal{T}\Psi$-related systems $(\Gamma - \eta^V, \{\eta_1^V, \dots, \eta_k^V\})$ on E, $(\overline{\Gamma} - \overline{\eta}^V, \{\overline{\eta}_1^V, \dots, \overline{\eta}_k^V\})$ on $\overline{E}$

Using homogeneity, one can deduce that

- associated connections are also Ψ-related
- $\mathcal{T}\Psi$-relation among vertical lifts of potential terms and input sections translates into a Ψ-relation of potential terms and input sections

Theorem: Under above conditions, with ψ open,

if system on E is locally base accessible (respectively locally base controllable) from $m \Rightarrow$ system on $\overline{E}$ is locally base accessible (respectively locally base controllable) from $\psi(m)$

Morphism-related mechanical systems (II)

If Ψ isomorphism between fibers of Lie algebroids, then conditions are either simultaneously satisfied on E and $\overline{E}$ or simultaneously not satisfied

Theorem: Let $\Psi : E \to \overline{E}$ be a morphism of Lie algebroids which is an isomorphism on each fiber. Consider two mechanical control systems on E and $\overline{E}$, with $k \geq \overline{k}$, that are Ψ-related. Let $m \in M$. Then

(i) $C_{\text{ver}}(\eta; \eta_1, \dots, \eta_k)(m) = E_m$ if and only if $C_{\text{ver}}(\overline{\eta}; \overline{\eta}_1, \dots, \overline{\eta}_{\overline{k}})(\psi(m)) = \overline{E}_{\psi(m)}$,

(ii) $C_{\text{hor}}(\eta; \eta_1, \dots, \eta_k)(m) + \ker \rho = E_m$ if and only if $C_{\text{hor}}(\overline{\eta}; \overline{\eta}_1, \dots, \overline{\eta}_{\overline{k}})(\psi(m)) + \ker \overline{\rho} = \overline{E}_{\psi(m)}$,

(iii) Every bad symmetric product in $\{\eta, \eta_1, \dots, \eta_k\}$ evaluated at m can be put as an $\mathbb{R}$-linear combination of good symmetric products of lower degree if and only if every bad symmetric product in $\{\overline{\eta}, \overline{\eta}_1, \dots, \overline{\eta}_{\overline{k}}\}$ evaluated at $\psi(m)$ can be put as an $\mathbb{R}$-linear combination of good symmetric products of lower degree.

Outline

(i) Motivation

(ii) Lie algebroid formalism

(iii) Nonlinear control

(iv) Mechanical control systems on Lie algebroids

(v) *Applications*

Applications: Simple Mechanical Control Systems (SMCS)

A ***simple mechanical control system*** $(Q, \mathcal{G}, V, \mathcal{F})$,

- Q is the manifold of configurations of the system
- $\mathcal{G}$ is a Riemannian metric on Q (kinetic energy metric of the system)
- $V \in C^\infty(Q)$ is the potential function
- $\mathcal{F} = \{F^1, \ldots, F^k\}$ is a set of k linearly independent 1-forms on Q

The dynamics of simple mechanical control systems is classically described by the forced Euler-Lagrange's equations

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \sum_{i=1}^{k} u_i(t) F^i$$

where $L : TQ \to \mathbb{R}$, $L(q, \dot{q}) = \frac{1}{2}\mathcal{G}(\dot{q}, \dot{q}) - V(q)$

Applications: *Simple Mechanical Control Systems (SMCS)*

Lie algebroid $\tau = \tau_Q : E_1 = TQ \to M_1 = Q$, with $\rho = \mathrm{Id}_{TQ} : TQ \to TQ$
Forces $\mathcal{F}$ correspond to sections of the dual bundle $E_1^* = T^*Q$

Dynamics is intrinsically written as

$$\dot{a}(t) = \rho^1\Big(\Gamma(a(t)) - (\mathrm{grad}_{\mathcal{G}} V)^V(a(t)) + \sum_{i=1}^{k} u_i(t) Y_i^V(a(t))\Big)$$

where Γ is SODE associated with $\nabla^{\mathcal{G}}$ and $\rho^1 = \mathrm{Id}_{TTM}$

(i) Base accessibility (resp. accessibility at zero) in $M = Q$ ***corresponds to*** configuration accessibility (resp. accessibility at zero velocity) in Q (Lewis & Murray 95)

(ii) Tests on Lie algebroid render previously known tests for accessibility (Lewis & Murray 95)

Analogous situation with ***controllability***

Applications: SMCS with symmetry

SMCS $(Q, \mathcal{G}, V, \mathcal{F})$ invariant under free and proper action Φ of Lie group G

Then $Q(Q/G, G, \pi)$ principal fiber bundle with bundle space Q, base space Q/G, structure group G and projection π

Φ induces lifted (free and proper) action of G on TQ, $\hat{\Phi} : G \times TQ \to TQ$, $\hat{\Phi}_g = T\Phi_g$, with $p : TQ \to TQ/G$, $p(v_q) = [v_q]$, surjective submersion

Lie algebroid $E_2 = TQ/G \to M_2 = Q/G$, with

$$\tau_2([v_q]) = [q] \qquad \rho_2([v_q]) = T\pi(v_q)$$

SMCS induces mechanical control system on E_2:

- $\mathcal{Y}$ induce sections $\mathcal{B} = \{B_i : Q/G \to TQ/G\}_{i=1}^k$ such that $p \circ Y_i = B_i \circ \pi$;
- V and $\mathcal{G}$ induce $\overline{\operatorname{grad}_{\mathcal{G}} V}$ such that $p \circ \operatorname{grad}_{\mathcal{G}} V = \overline{\operatorname{grad}_{\mathcal{G}} V} \circ \pi$

Applications: SMCS with symmetry – tests

Accessibility notions:

(i) Base accessibility on E_2 ***corresponds to*** configuration accessibility in Q/G

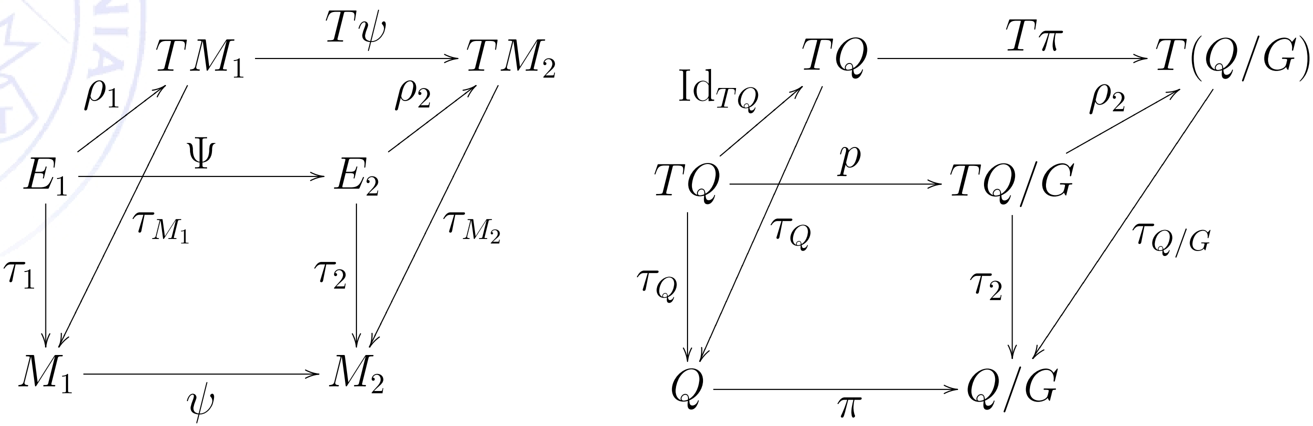
(ii) Accessibility at zero in E_2 (reachable sets in TQ/G) ***stronger than*** accessibility at zero velocity in Q/G (accessible sets in $T(Q/G)$)

(iii) If bundle $Q = G \times Q/G$ trivial, consider $\tau : G \times Q/G \to G$. Then, base accessibility with regards to G ***corresponds to*** fiber configuration accessibility (Cortes et al 02)

Same deal with ***controllability***

What about accessibility/controllability tests in Q which make use of the SMCS symmetry?

Applications: SMCS with symmetry – geometry



- Both algebroids have fibers of the same dimension $n = \dim Q$
- Ψ is surjective

Applications: SMCS with symmetry – tests

Theorem (accessibility):

$$C_{\text{hor}}(\text{grad}_{\mathcal{G}} V; \mathcal{Y}) = TQ \iff C_{\text{hor}}(\overline{\text{grad}_{\mathcal{G}} V}; \mathcal{B}) = TQ/G$$
$$C_{\text{ver}}(\text{grad}_{\mathcal{G}} V; \mathcal{Y}) = TQ \iff C_{\text{ver}}(\overline{\text{grad}_{\mathcal{G}} V}; \mathcal{B}) = TQ/G$$

(resp. accessibility at zero velocity)

- reduced representation (space of smaller dimension)
- extends results in Cortes et al 02 to nontrivial potential terms.

Theorem (controllability): Enough to check bad symmetric products in $\{\overline{\text{grad}_{\mathcal{G}} V}, B_1, \ldots, B_k\}$ are $\mathbb{R}$-linear combinations of good ones in TQ/G (plus accessibility)

Furthermore, if reduced system is not base accessible (resp. controllable), then original system is not base accessible (resp. controllable)

Applications: semidirect products – geometry

Let $\mathfrak{g} \to \mathfrak{X}(M)$, $\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$, surjective Lie algebra homomorphism

Lie algebroid $\tau : E = M \times \mathfrak{g} \to M$, with $\rho(m, \xi) = \xi_M(m)$

With $TE \equiv TM \times T\mathfrak{g} \equiv TM \times \mathfrak{g} \times \mathfrak{g}$ (left multiplication),

$(a, b, v) \in \mathcal{T}E$ is $\big((m, \xi), (m, \eta), (v_m, \xi, \zeta)\big)$, with $v_m = \eta_M(m)$

Therefore, $\mathcal{T}E \equiv M \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$, with

$$\tau_1(m, \xi, \eta, \zeta) = (m, \xi), \quad \mathcal{T}\tau(m, \xi, \eta, \zeta) = (m, \eta), \quad \rho^1(m, \xi, \eta, \zeta) = (\eta_M(m), \xi, \zeta)$$

Let $(\mathcal{G}, V, \{\theta_1, \dots, \theta_k\})$ be mechanical control system on E
Assume $\mathcal{G}$ comes from inner product on $\mathfrak{g}$, $\mathcal{G}((m, \xi_1), (m, \xi_2)) = \mathcal{G}(\xi_1, \xi_2)$
For $\xi \in \mathfrak{g}$, define $\mathrm{ad}^\dagger_\xi : \mathfrak{g} \to \mathfrak{g}$ by $\mathcal{G}(\mathrm{ad}^\dagger_\xi \eta_1, \eta_2) = \mathcal{G}(\eta_1, [\xi, \eta_2]_\mathfrak{g})$

Applications: *semidirect products – dynamics*

SODE reads $\Gamma_{\nabla^{\mathcal{G}}}(m,\xi) = (m,\xi,\xi,\mathrm{ad}^\dagger_\xi \xi)$, and controlled equations

$$\dot a - \mathrm{ad}^\dagger_a a = -\,\mathrm{grad}_{\mathcal{G}} V(m) + \sum_{i=1}^{k} u_i \eta_i(m)$$

For constant sections $\sigma_i(m) = (m,\xi_i)$, $i = 1, 2$,

$$\nabla^{\mathcal{G}}_{\sigma_1}\sigma_2(m) = \left(m, \frac{1}{2}[\xi_1,\xi_2]_{\mathfrak{g}} - \frac{1}{2}(\mathrm{ad}^\dagger_{\xi_1}\xi_2 + \mathrm{ad}^\dagger_{\xi_2}\xi_1)\right)$$
$$\langle \sigma_1 : \sigma_2\rangle(m) = \left(m, -(\mathrm{ad}^\dagger_{\xi_1}\xi_2 + \mathrm{ad}^\dagger_{\xi_2}\xi_1)\right)$$

(i) Tests can be applied to these problems to determine base accessibility (resp. controllability) – generalizes Shen 02

(ii) Systems appear frequently as mechanical systems defined on homogeneous spaces for a given group action

Conclusions

- Investigated controllability properties of systems on Lie algebroids. Established controllability results for nonlinear affine control systems
- Introduced mechanical control system on Lie algebroid. Defined controllability notions and investigated sufficient tests. Applications to systems related by morphism of Lie algebroids
- Illustrated results with the classes of simple mechanical control systems and of systems evolving on semidirect products

Future work

- investigation of controllability tests along relative equilibria of mechanical control systems on Lie algebroids
- treatment of models that include gyroscopic forces and dissipation

Thanks for your attention!

Check out

http://www.ams.ucsc.edu/~jcortes !