Classical field theories with nonholonomic constraints

Some general remarks

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Overview of this talk

Aim: *to show that geometric methods from nonholonomic mechanics carry over quite naturally to classical field theory, and to make a start on some examples.*

Overview of this talk

1. Geometric formalism;

2. Derivation of the constrained field equations;

3. The nonholonomic projector;

4. Example: ideal incompressible fluids;

5. A tentative example: the nonholonomic Cosserat rod.

6. Conclusions.

Not in this talk: Cauchy formalism, aspects of symmetry, nonholonomic momentum map & Noether theorem; linear constraints . . .

References

Based on work together with F. Cantrijn, M. de León & D. Martín de Diego. In particular, see:

1. F. Cantrijn, M. de León, D. Martín de Diego, J. Vankerschaver: *Geometric aspects of nonholonomic field theories*. To appear in *Rep. Math. Phys.*
2. J. Vankerschaver: *The momentum map for nonholonomic field theories with symmetry*. To appear in *Int. J. Geom. Meth. Mod. Phys.*

All proofs omitted in this talk (as well as lots more) can be found in these articles.

Classical field theories: overview

- Fields: sections of a bundle $\pi : Y \to X$.
  $\dim X = n + 1,$ coordinate system (x^i)
  $\dim Y = n + 1 + m,$ coordinate system (x^i, u^a)
  We take X to be oriented with vol. form μ.

- First-order jet bundle $J^1\pi$, coordinate system $(x^i, u^a; u^a_i)$. *Projections:*
  source $\pi_1 : J^1\pi \to X$ $\pi_1(j^1_x\phi) = x$
  target $\pi_{1,0} : J^1\pi \to Y$ $\pi(j^1_x\phi) = \phi(x)$

Classical field theories: overview

- **First-order field theory:** characterised by a Lagrangian $L : J^1\pi \to \mathbb{R}$. Regularity: $\frac{\partial^2 L}{\partial u_i^a \partial u_j^b}$ is invertible.
- **Euler-Lagrange equations:**

$$\frac{\partial L}{\partial u^a}(j^1\phi) - \frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\partial L}{\partial u_i^a}(j^1\phi) = 0.$$

- **Associated multisymplectic form:** $\Omega_L = -\mathrm{d}\Theta_L \in \Omega^{n+2}(J^1\pi)$, with

$$\Theta_L = \frac{\partial L}{\partial u_i^a}(\mathrm{d}u^a - u_j^a \mathrm{d}x^j) \wedge \mathrm{d}^n x_i + L\mathrm{d}^{n+1}x.$$

- **De Donder-Weyl equations:** look for a connection in π_1 with horizontal projector $\mathbf{h}$ satisfying

$$i_{\mathbf{h}}\Omega_L - n\Omega_L = 0.$$

Geometric formalism: ingredients

We start from a fibre bundle $\pi : Y \longrightarrow X$ and a first-order Lagrangian $L : J^1\pi \longrightarrow \mathbb{R}$.

Constraints: modelled by

1. a *constraint submanifold* $\mathcal{C} \hookrightarrow J^1\pi$, (locally) given as the zero set of k independent functions φ^α, $\alpha = 1, \ldots, k$:

$$\mathcal{C} = \left\{ \gamma \in J^1\pi : \varphi^\alpha(\gamma) = 0 \right\}.$$

We assume that $(\pi_{1,0})_{|\mathcal{C}} : \mathcal{C} \rightarrow Y$ is a fibre bundle.

2. a *bundle of constraint forms* $F \subset \wedge^{n+1}(J^1\pi)$ along $\mathcal{C}$, locally generated by forms

$$\Phi^\alpha = (C^\alpha)^i_a(\mathrm{d}u^a - u^a_j \mathrm{d}x^j) \wedge \mathrm{d}^n x_i.$$

Geometric formalism: Chetaev principle

A priori, $\mathcal{C}$ and F are completely unrelated!

Chetaev principle: assumes that F is linked to $\mathcal{C}$, by defining

$$\begin{aligned} \Phi^\alpha &= S_\mu^*(\mathrm{d}\varphi^\alpha) \\ &= \frac{\partial \varphi^\alpha}{\partial u_i^a}(\mathrm{d}u^a - u_j^a \mathrm{d}x^j) \wedge \mathrm{d}^n x_i, \end{aligned}$$

and putting $F_\gamma = \langle \Phi^\alpha(\gamma) \rangle$ for $\gamma \in \mathcal{C}$.

Field equations

We take variations of the action and integrate by parts

$$\delta S = \int_U \left(\frac{\partial L}{\partial u^a} - \frac{\mathrm{d}}{\mathrm{d}x^i} \frac{\partial L}{\partial u^a_i} \right) \xi^a \mathrm{d}^{n+1} x,$$

where $\xi = \xi^a \frac{\partial}{\partial u^a}$ is an infinitesimal variation.

Principle of d'Alembert for field theories: restrict to variations ξ satisfying

$$\xi^a \frac{\partial \varphi^\alpha}{\partial u^a_i} = 0 \quad \text{for } \alpha = 1, \ldots, k.$$

Affine constraints: $\varphi^\alpha_i = A^\alpha_a u^a_i + B^\alpha_i$. Hence we obtain that $\xi^a A^\alpha_a = 0$.

Constrained Euler-Lagrange equations

By use of the principle of d'Alembert, we conclude that $\phi \in \mathrm{Sec}(\pi)$ is a solution of the *constrained field equations* if

$$\boxed{\frac{\partial L}{\partial u^a}(j^1\phi) - \frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\partial L}{\partial u^a_i}(j^1\phi) = \lambda_{\alpha i}\frac{\partial \varphi^\alpha}{\partial u^a_i}}$$

together with the constraint equations $\varphi^\alpha \circ j^1\phi = 0$.

$$\boxed{\frac{\partial L}{\partial u^a}(j^1\phi) - \frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\partial L}{\partial u^a_i}(j^1\phi) = \lambda_{\alpha i}\frac{\partial \varphi^\alpha}{\partial u^a_i}}$$

together with the constraint equations $\varphi^\alpha \circ j^1\phi = 0$.

Warning: $\lambda_{\alpha i}$ are Lagrange multipliers. Have to be determined from the constraint equations. *This is not possible in general*!

Solution:

- dependent on modelling;
- sometimes not all multipliers are needed

Constrained De Donder-Weyl equation

In De Donder-Weyl form, we look for connections $\mathbf{h}$ in π_1 satisfying

$$\boxed{i_{\mathbf{h}}\Omega_L - n\Omega_L \in \mathcal{I}(F), \quad \text{and} \quad \operatorname{Im}\mathbf{h} \subset T\mathcal{C}.}$$

If integrable, integral sections of $\mathbf{h}$ solve constrained Euler-Lagrange equations.

Our aim: to turn a connection $\mathbf{h}$ solving the (free) De Donder-Weyl equation

$$i_{\mathbf{h}}\Omega_L = n\Omega_L,$$

into a solution $\mathbf{h}'$ of the constrained field equations.

Geometric treatment: the bundle D

- Construct a "complement" D to F, in the sense that

$$X \in D \Leftrightarrow i_X \Omega_L \in F.$$

  Straightforward for symplectic manifolds ($D = F^{\perp}$), in general *impossible* for generic multisymplectic manifolds!

- Possible here because of special form of $\Phi^\alpha = S^*_\mu(\mathrm{d}\varphi^\alpha)$: there exist X_α such that $i_{X_\alpha}\Omega_L = \Phi^\alpha$, with

$$X_\alpha = (X_\alpha)^A_i \frac{\partial}{\partial u^A_i} \quad \text{where } (X_\alpha)^A_i \frac{\partial^2 L}{\partial u^A_i \partial u^B_j} = \frac{\partial \varphi_\alpha}{\partial u^B_j}.$$

- *Compatibility*: we demand that, for each $\gamma \in \mathcal{C}$, $D(\gamma) \cap T_\gamma \mathcal{C} = 0$. This gives rise to a decomposition along $\mathcal{C}$

$$T_\gamma J^1 \pi = D(\gamma) \oplus T_\gamma \mathcal{C}$$

Geometric treatment: nonholonomic projector

We recall the decomposition

$$T_\gamma J^1\pi = D(\gamma) \oplus T_\gamma \mathcal{C} \quad \text{along } \mathcal{C}$$

and consider the projector $\mathcal{P} : TJ^1\pi \longrightarrow T\mathcal{C}$.

Claim: if $\mathbf{h}$ solves free DDW, then $\mathcal{P} \circ \mathbf{h}$ is a solution of the constrained field equations.

Proof:
1. By definition, $\operatorname{Im} \mathcal{P} \circ \mathbf{h} \subset T\mathcal{C}$.
2. On the other hand,

$$\begin{aligned} i_{\mathcal{P}\circ\mathbf{h}}\Omega_L - n\Omega_L &= (i_{\mathbf{h}}\Omega_L - n\Omega_L) - i_{\mathcal{Q}\circ\mathbf{h}}\Omega_L \\ &= \lambda_{\alpha i} \mathrm{d}x^i \wedge \Phi^\alpha . \end{aligned}$$

(we omit the proof that $\mathcal{P} \circ \mathbf{h}$ is a connection).

Example: ideal incompressible fluids

- **Setting:** a fluid filling Euclidian space $\mathbb{R}^3$.

$$X = \mathbb{R} \times \mathbb{R}^3, \quad \text{coordinates } (t, X^I)$$
$$Y = X \times \mathbb{R}^3, \quad \text{coordinates } (t, X^I; x^i)$$

  Jet bundle $J^1\pi$: coordinates $(t, X^I; x^i; v^i, F^i_I)$.

- **Barotropic fluid:** Lagrangian density

$$L = \frac{1}{2}\rho(X)\,\|v\|^2\,\mathrm{d}^4x - \rho(X)W(J)\mathrm{d}^4x,$$

  where $J = \det F^i_I$.

- **Incompressibility constraint:** $J = \det F^i_I = 1$. This is really a divergence:

$$J = \frac{\mathrm{d}}{\mathrm{d}X^I}\left(\frac{1}{3}Jx^i(F^{-1})^I_i - X^I\right).$$

Example: ideal incompressible fluids

- **Field equations:**

$$\rho(X)\delta_{ij}\frac{\mathrm{d}v^j}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}X^I}\left(\rho(X)W'J(F^{-1})^I_i\right) = \lambda_I J(F^{-1})^I_i,$$

  supplemented with $J \equiv 1$.

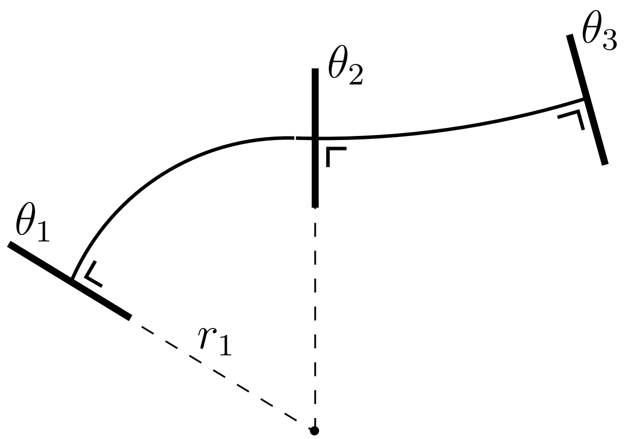
- Comparison with vakonomic approach (Marsden et al.) shows that there exist a multiplier p ("pressure") such that

$$\lambda_I = \frac{\mathrm{d}p}{\mathrm{d}X^I}.$$

- In agreement with usual treatment of incompressibility. Not so surprising, given the divergence property…

A tentative example

Setting: imagine N wheels interconnected by flexible beams, being able to twist and to bend. Beams counteract twisting & binding.



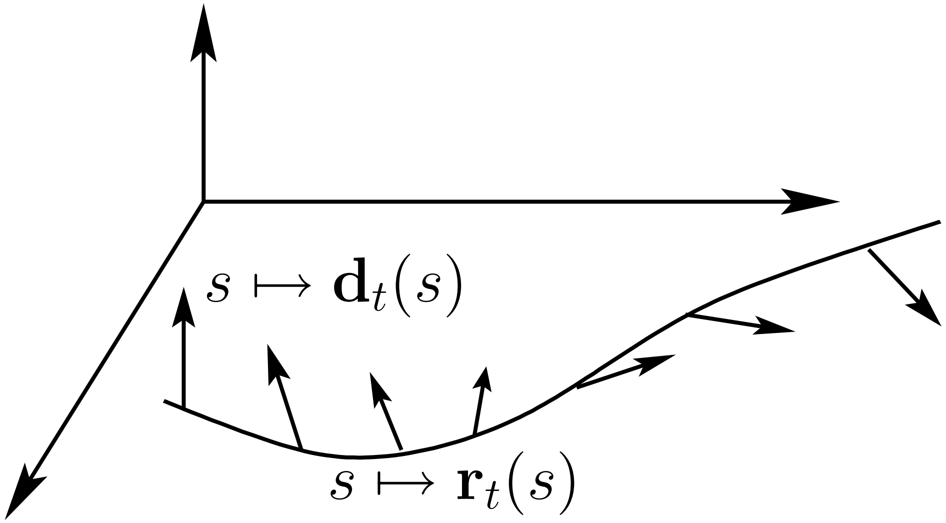
Lagrangian:

$$L = \sum_{i=1}^{N} L_{\text{r.d.}}(x_i, y_i, \phi_i, \theta_i) - \frac{A}{2} \sum_{i=1}^{N-1} (\theta_{i+1} - \theta_i)^2 - \frac{B}{2} \sum_{i=1}^{N-1} \frac{1}{r_i^2},$$

where $L_{\text{r.d.}} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I}{2}\dot{\phi}^2 + \frac{J}{2}\dot{\theta}^2$.
Constraint: each wheel rolls without sliding.

The Cosserat rod

- In the limit $N \to +\infty$, we obtain something called a *Cosserat rod*…



- Lagrangian density for such a model:

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\pi}{4}\dot{\theta}^2 - \frac{\pi\mu}{4}(\rho - \theta')^2.$$

- Constraint (rolling without sliding) survives in the continuum limit as well.
- **Problem:** equations of motion very hard to make sense of!

Conclusions

- Mathematical formulation carries along nicely;
- Examples are a another question:
  1. Vakonomic equations are much more prominent;
  2. Field equations computationally very difficult!
- Future work:
  1. Examples, esp. computer simulations;
  2. Linear constraints: many interesting mathematical results;
  3. Classification of constraints: vakonomic *vs.* nonholonomic.

The End (for now)

Thank you for listening!