Derivations of differential forms along the tangent bundle projection

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Abstract. We study derivations of the algebra of differential forms along the tangent bundle projection $\tau : TM \to M$ and of the module of vector-valued forms along τ. It is shown that a satisfactory classification and characterization of such derivations requires the extra availability of a connection on TM. The present theory completely explains and generalizes the calculus of forms associated to a given second-order vector field, which was previously introduced by one of us.

1 Introduction

The motivation for the work to be presented here to some extent originates from a paper of 1984 [19], in which a new approach was introduced to the study of geometrical features of second-order equations in general, and to the characterization of Lagrangian systems in particular. Basically, this approach relied on introducing for any given second-order vector field Γ on TM associated sets $\mathcal{X}_\Gamma$ of vector fields and $\mathcal{X}_\Gamma^*$ of 1-forms on TM. To name just a few of the interesting aspects of these sets (the definition of which can be found in section 7), recall that $\mathcal{X}_\Gamma$ contains all symmetries of the given system and that this system is Lagrangian provided $\mathcal{X}_\Gamma^*$ contains an element which is exact.

A deeper analysis of the structure of these sets led one of us [18] to a new calculus of certain differential forms on TM. First, a module structure for the sets $\mathcal{X}_\Gamma$ and $\mathcal{X}_\Gamma^*$ was identified. Then, having established their duality, the set $\mathcal{X}_\Gamma^*$ was extended to a complete algebra of scalar forms on the module $\mathcal{X}_\Gamma$ and finally, with the ordinary calculus of differential forms as a model, properly adapted, fundamental derivations of this algebra were constructed ("contraction with vector fields", "exterior derivative" and "Lie derivative"). Most of the proofs of the various statements in [18] were omitted for the sake of brevity and have not been published up to date, perhaps because the author felt his story was as yet incomplete.

The main purpose of the present paper is to introduce an alternative and more systematic approach to the subject, which will make the fundamental mechanism behind the calculus in [18] much more transparent. It will result in simpler proofs and at the same time contains the generalizations which are needed to make the theory complete.

In order to understand the origin for the present approach, observe, for example, that a vector field in $\mathcal{X}_\Gamma$ locally is of the form $X = \mu^i(\partial/\partial q^i) + \Gamma(\mu^i)(\partial/\partial v^i)$, whereas the coordinate expression of an element of $\mathcal{X}^*_\Gamma$ reads : $\alpha = \alpha_i\, dv^i + \Gamma(\alpha_i)\, dq^i$. The common feature of these expressions is that only half of the components, which are functions on TM, determine them completely. If $S = (\partial/\partial v^i) \otimes dq^i$ is the vertical endomorphism on TM, then α, for example, is completely determined by the semi-basic form $S \,\lrcorner\, \alpha = \alpha_i\, dq^i$. This suggests that we are basically looking at a calculus of semi-basic forms on TM. When one thinks of derivations of such forms, one might in the first place be led to a theory of derivations of $\bigwedge(TM)$ which preserve the subset of semi-basic forms. It turns out that such a theory is not the one which serves our purposes. Instead, we will study derivations of so-called differential forms along the map $\tau : TM \to M$ (which can be identified with semi-basic forms on TM). Perhaps another little warning is in order in this respect : in dealing with sections along τ, there is a natural way of introducing derivations which would map forms on M into forms on TM (see e.g. [16] or [22]). Such a theory again is entirely different from the kind of derivations we will study.

The plan of the paper is as follows. In the next section we recall the notion of sections along a map and fix notations for different sets which will play a key role further on. Section 3 introduces the derivations we are interested in and results in a preliminary classification of them along the lines of the pioneering work of Frölicher and Nijenhuis [10]. A much more elegant and complete classification of these derivations is obtained in Section 4, after introducing a connection on TM. Derivations of vector-valued forms along τ are considered in Section 5; among other things, they permit us to discuss the Lie algebra structure of the set of vector-valued forms. In Section 6, we establish a number of important properties of prolongations of vector fields and forms along τ; they are to a large extent the properties which explain the origin of the module and algebra structures invented in [18]. Composing the first prolongation of various objects along τ with the section of T^2M which represents a given second-order vector field, creates corresponding objects on TM. We show in Section 7 that these objects are precisely the ingredients of the calculus in [18] and that the full theory of derivations, developed in Sections 3 to 5, carries over to a new completed and generalized version of the above cited results. The paper ends with some concluding remarks.

2 Sections along a map

Let $\pi: E \to M$ be a fibre bundle and $f: N \to M$ a differentiable map. By a section of E along f we mean a section of the pull-back bundle $f^*\pi: f^*E \to N$, see [17]. Equivalently, we can consider a section σ along f as a map $\sigma: N \to E$ satisfying $\pi \circ \sigma = f$. If E is a vector bundle then the set of sections of E along f is a $C^\infty(N)$-module.

In this paper we are interested in the case that f is the projection $\tau: TM \to M$ and E is some tensorial bundle over M. Specifically, E can be one of the following vector bundles: TM, T^*M, $\Omega^k(M) = \bigcup_{q\in M} \Omega^k(T_qM)$, $\Omega(M) = \oplus\Omega^k(M)$ or $\Omega(M;TM)$, where the latter one denotes the bundle whose sections over M are the vector-valued forms on M. For these choices of E, the set of sections along τ will respectively be denoted by $\mathcal{X}(\tau)$, $\bigwedge^1(\tau)$, $\bigwedge^k(\tau)$, $\bigwedge(\tau)$, $V(\tau) = \bigwedge(\tau)\otimes\mathcal{X}(\tau)$ and we talk about vector fields, 1-forms, ..., vector-valued forms along τ.

An obvious way for obtaining forms and fields along τ is by composing forms and fields on M with the projection τ. Thus, for $X \in \mathcal{X}(M)$, $\alpha \in \bigwedge(M)$, we will introduce the notations

$$\tilde{X} = X \circ \tau \ \in \mathcal{X}(\tau), \quad \tilde{\alpha} = \alpha \circ \tau \ \in \textstyle\bigwedge(\tau).$$

We will sometimes call these basic vector fields and basic forms and denote the set of them by $\tilde{\mathcal{X}}(\tau)$, $\tilde{\bigwedge}(\tau)$. It is clear that every element of $\mathcal{X}(\tau)$ or $\bigwedge(\tau)$ can locally be written as a linear combination of basic elements, with coefficients in $C^\infty(TM) = \bigwedge^0(\tau)$. A different way for obtaining scalar and vector-valued forms along τ goes as follows. For $\alpha \in \bigwedge^{p+1}(M)$, we define $\hat{\alpha} \in \bigwedge^p(\tau)$ at each point $v \in TM$ by

$$\hat{\alpha}(v)\,(v_1,\ldots,v_p) = \alpha(\tau(v))\,(v, v_1,\ldots,v_p),$$

for all $v_1,\ldots,v_p \in T_{\tau(v)}M$. This is a generalization of the well-known association of a 1-form on M with a function on TM and we will denote the set of such $\hat{\alpha}$ (for $\alpha \in \bigwedge^{p+1}(M)$) by $\hat{\bigwedge}^p(\tau)$. A similar construction can be carried out for vector-valued forms: to each $R \in V^{p+1}(M)$ we can associate an element $\hat{R} \in \hat{V}^p(\tau) \subset V^p(\tau)$.

As in the standard theory of tensor fields on a manifold, our scalar and vector-valued forms along τ, being originally defined as sections of some appropriate pull-back bundle, can be interpreted equivalently as $C^\infty(TM)$-multilinear operators on $\mathcal{X}(\tau)$. We will often use this interpretation for constructing new objects.

At this point it is worth mentioning that there exists a natural identification between $\mathcal{X}(\tau)$ and $\mathcal{X}^V(TM)$: the set of vertical vector fields on TM. A vector field on TM is uniquely determined by its action on functions in $C^\infty(M)$ and in $\hat{\bigwedge}^0(\tau)$. Thus, we could introduce the identification of $\mathcal{X}(\tau)$ with $\mathcal{X}^V(TM)$ as follows: for $X \in \mathcal{X}(\tau)$, $X^\uparrow \in \mathcal{X}^V(TM)$ is determined by $X^\uparrow(f) = 0$ for $f \in C^\infty(M)$ and

$$X^\uparrow(\hat{\alpha}) = \tilde{\alpha}(X), \quad \text{for } \alpha \in \textstyle\bigwedge^1(M). \tag{1}$$

The above formula can serve, conversely, to define $Y^\downarrow \in \mathcal{X}(\tau)$ for each given $Y \in \mathcal{X}^V(TM)$. Incidentally, observe that vector fields along τ have a module structure over $C^\infty(TM)$, but they do not act as derivations on these scalars, which makes it perhaps a bit puzzling to define a bracket operation. However, we can make use of the above identification and the fact that the bracket (on TM) of vertical vectors is vertical to define the bracket of two elements $X, Y \in \mathcal{X}(\tau)$ via the rule

$$[X,Y] = [X^\uparrow, Y^\uparrow]^\downarrow.$$

We will recover this bracket operation in our later analysis.

3 Derivations of $\bigwedge(\tau)$

In this section we introduce the notion of a derivation of the scalar forms along τ and make a preliminary attempt towards the classification of such derivations along the lines of the standard work of Frölicher and Nijenhuis [10].

Definition 3.1 *A map $D: \bigwedge(\tau) \to \bigwedge(\tau)$ is said to be a derivation of $\bigwedge(\tau)$ of degree r if it satisfies*

1. $D(\bigwedge^p(\tau)) \subset \bigwedge^{p+r}(\tau)$,
2. $D(\alpha + \lambda\beta) = D\alpha + \lambda D\beta$, *and*
3. $D(\alpha \wedge \gamma) = D\alpha \wedge \gamma + (-1)^{pr}\alpha \wedge D\gamma$,

for all $\lambda \in \mathbf{R}$, $\alpha, \beta \in \bigwedge^p(\tau)$ and $\gamma \in \bigwedge^q(\tau)$.

As in the classical theory, one easily proves that derivations of $\bigwedge(\tau)$ are local operators.

Proposition 3.2 *A derivation D of $\bigwedge(\tau)$ is completely characterized by its action on $\bigwedge^0(\tau)$ and $\tilde{\bigwedge}^1(\tau)$. Moreover, the action on functions is fully determined by the action on $\tilde{\bigwedge}^0(\tau)$ and $\hat{\bigwedge}^0(\tau)$.*

Proof: If $\{\alpha^i\}_{i=1,\dots,m}$ is a local basis for $\bigwedge^1(M)$, every $\rho \in \bigwedge^p(\tau)$ can be written as

$$\rho = \rho_{i_1\cdots i_p}\,\widetilde{\alpha^{i_1}} \wedge \cdots \wedge \widetilde{\alpha^{i_p}}, \quad \rho_{i_1\cdots i_p} \in C^\infty(TM).$$

Then,

$$\begin{aligned} D\rho &= D\rho_{i_1\cdots i_p} \wedge \widetilde{\alpha^{i_1}} \wedge \cdots \wedge \widetilde{\alpha^{i_p}} \\ &\quad + \sum_{\ell=1}^{p}(-1)^{(\ell-1)r}\rho_{i_1\cdots i_\ell\cdots i_p}\widetilde{\alpha^{i_1}} \wedge \cdots \wedge D(\widetilde{\alpha^{i_\ell}}) \wedge \cdots \wedge \widetilde{\alpha^{i_p}}, \end{aligned}$$

from which the first part of the statement is obvious. For the second part, a standard but rather technical argument, involving ideals of germs of functions vanishing at a point, shows that the action of D on a function (at the point under consideration) is essentially determined by its action on the first-order Taylor expansion of that function and the result then readily follows. $\Box$

As a corollary, any map from $\tilde{\bigwedge}^0(\tau) \oplus \hat{\bigwedge}^0(\tau) \oplus \tilde{\bigwedge}^1(\tau)$ to $\bigwedge(\tau)$, satisfying the derivation axioms for $p+q \leq 1$ can be extended in a unique way to a derivation of $\bigwedge(\tau)$. It further follows, as in the usual theory, that there are no derivations of $\bigwedge(\tau)$ of degree $r < -1$ and that derivations of degree -1 vanish on $\bigwedge^0(\tau)$.

The Lie algebra structure of the set of derivations of $\bigwedge(\tau)$ is established by the following proposition, which can be proved in the same manner as in the standard theory.

Proposition 3.3 *Let D_1, D_2 and D_3 be derivations of $\bigwedge(\tau)$ of degree r_1, r_2 and r_3 respectively. Then, the graded commutator*

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1$$

is a derivation of degree $r_1 + r_2$ and we have the graded Jacobi identity

$$(-1)^{r_1 r_3}[D_1, [D_2, D_3]] + (-1)^{r_2 r_1}[D_2, [D_3, D_1]] + (-1)^{r_3 r_2}[D_3, [D_1, D_2]] = 0.$$

If one starts classifying derivations of scalar forms along the path which was set out by the work of Frölicher and Nijenhuis, one must first identify the kind of "exterior derivative" which is fundamental for the theory at hand. The natural operation which is available here is a kind of vertical derivative which can, for example, be introduced as follows: for $F \in C^\infty(TM)$, we define $d^V F \in \bigwedge^1(\tau)$ by

$$d^V F(X) = dF(X^\uparrow), \qquad X \in \mathcal{X}(\tau),$$

and for $\tilde{\alpha} \in \tilde{\bigwedge}^1(\tau)$ we set $d^V \tilde{\alpha} = 0$. It is clear that d^V has the required properties and thus extends in a unique way to a derivation of $\bigwedge(\tau)$, vanishing on $\tilde{\bigwedge}(\tau)$. In coordinates we have, for $F \in C^\infty(TM)$,

$$d^V F = (\partial F / \partial v^i)\, \widetilde{dq^i},$$

and e.g. for $\alpha = \alpha_i(q, v)\, \widetilde{dq^i} \in \bigwedge^1(\tau)$,

$$d^V \alpha = \frac{1}{2}\left(\frac{\partial \alpha_j}{\partial v^i} - \frac{\partial \alpha_i}{\partial v^j}\right) \widetilde{dq^i} \wedge \widetilde{dq^j}.$$

Proposition 3.4 *The vertical derivative on $\bigwedge(\tau)$ has the following properties:*

1. *For $\alpha \in \bigwedge^p(M)$ $(p > 0)$: $d^V \hat{\alpha} = p\, \tilde{\alpha}$.*
2. *$d^V \circ d^V = 0$.*
3. *The cohomology of d^V is trivial.*

Proof:

1. For $\alpha \in \bigwedge^1(M)$ we have

$$\forall X \in \mathcal{X}(\tau), \qquad d^V \hat{\alpha}(X) = d\hat{\alpha}(X^\uparrow) = X^\uparrow(\hat{\alpha}) = \tilde{\alpha}(X).$$

Writing a general p-form β on M as $\beta = \beta_{i_1 \cdots i_p} \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_p}$, with $\beta_{i_1 \cdots i_p} \in C^\infty(M)$ and $\alpha^{i_j} \in \bigwedge^1(M)$, it follows from the definition that $\hat{\beta} \in \bigwedge^{p-1}(\tau)$ is given by

$$\hat{\beta} = \sum_{j=1}^{p} (-1)^{j-1} \beta_{i_1 \cdots i_p}\, \widetilde{\alpha^{i_1}} \wedge \cdots \wedge \widehat{\alpha^{i_j}} \wedge \cdots \wedge \widetilde{\alpha^{i_p}}.$$

The general property now easily follows.

2. Since $d^V \circ d^V = \frac{1}{2}[d^V, d^V]$, it is a derivation, which moreover is zero on $\tilde{\bigwedge}(\tau)$. As a result of proposition 3.2, it suffices to look further at $\hat{\bigwedge}^0(\tau)$, for which we find: $d^V \circ d^V(\hat{\alpha}) = d^V\tilde{\alpha} = 0$.

3. Suppose $d^V\alpha = 0$ for $\alpha \in \bigwedge^p(\tau)$, then locally $\alpha = d^V\beta$ for some $\beta \in \bigwedge^{p-1}(\tau)$. This is clear from the local coordinate expression for $d^V\alpha$ illustrated before; in fact it essentially relies on a Poincaré type lemma with respect to the v-coordinates only. But then, since the fibres of TM are vector spaces, such a Poincaré lemma actually holds on open sets of the form $\tau^{-1}(\mathcal{U})$ with $\mathcal{U} \subset M$. The global exactness of α then follows by using a partition of unity on M. □

REMARK: Forms in $\bigwedge(\tau)$ can in a natural way be identified with semi-basic forms on TM (see later) and with this identification, d^V becomes d_S where S is the vertical endomorphism tensor field on TM. The fact that a d_S-closed semi-basic form on TM is indeed globally d_S-exact has been proved in [1].

Recall now that in the standard theory of derivations of scalar forms on a manifold, every derivation has a unique decomposition into one of type i_* and one of type d_*; the first one, moreover, necessarily is of the form i_{L_1} and the second one of the form d_{L_2}, where L_1 and L_2 are suitable vector-valued forms. We wish to investigate first to what extent a similar classification could be made for the theory of derivations of $\bigwedge(\tau)$.

Definition 3.5 *A derivation of $\bigwedge(\tau)$ is said to be of type i_* if it vanishes on $\bigwedge^0(\tau)$.*

Definition 3.6 *Let $L \in V^r(\tau)$ and $\omega \in \bigwedge^p(\tau)$ with $p > 0$, then we define $\omega\bar{\wedge}L \in \bigwedge^{p+r-1}(\tau)$ by the rule: $\forall X_i \in \mathcal{X}(\tau)$,*

$$(\omega\bar{\wedge}L)(X_1, \ldots, X_{p+r-1}) = \frac{1}{(p-1)!r!} \sum_{\sigma \in S_{p+r-1}} (\operatorname{sgn}\sigma)\, \omega\Big(L(X_{\sigma(1)}, \ldots, X_{\sigma(r)}), X_{\sigma(r+1)}, \ldots, X_{\sigma(p+r-1)}\Big).$$

For $p = 0$ we set $\omega\bar{\wedge}L = 0$.

Proposition 3.7 *For $L \in V^{r+1}(\tau)$, the map $i_L: \omega \mapsto \omega\bar{\wedge}L$ is a derivation of $\bigwedge(\tau)$ of degree r and type i_*. Conversely, every derivation D of type i_* and degree r is of the form i_L for some $L \in V^{r+1}(\tau)$*

Proof: The first part is a straightforward calculation. For the converse, let $\{\alpha^i\}$ be a local basis of $\bigwedge^1(M)$ with dual basis $\{X_i\}$. We locally construct an element $L \in V^{r+1}(\tau)$ as $L = D\widetilde{\alpha^i} \otimes \widetilde{X_i}$. This construction has the right tensorial properties with respect to a change of local basis $\{\alpha^i\}$, because the given D vanishes on functions. Furthermore, we have for all $Y_j \in \mathcal{X}(\tau)$:

$$\begin{aligned} i_L\widetilde{\alpha^i}(Y_1, \ldots, Y_{r+1}) &= \widetilde{\alpha^i}\big(L(Y_1, \ldots, Y_{r+1})\big) \\ &= D\widetilde{\alpha^i}(Y_1, \ldots, Y_{r+1}). \end{aligned}$$

Since D and i_L also coincide on $C^\infty(TM)$, we conclude that $D = i_L$. □

Theorem 3.8 *Every derivation D of $\bigwedge(\tau)$ has a unique decomposition $D = D_1 + D_2$, where D_1 is of type i_* and D_2 commutes with d^V.*

Proof: The restriction of D to $C^\infty(TM)$ can be extended to a derivation D_2 of $\bigwedge(\tau)$ by imposing that D_2 commutes with d^V. This follows from the fact that it suffices from proposition 3.2 to specify the action of D_2 on $\tilde{\bigwedge}^1(\tau)$, plus the property that each $\tilde{\alpha} \in \tilde{\bigwedge}^1(\tau)$ can be written as $\tilde{\alpha} = d^V\hat{\alpha}$. Clearly, $D_1 = D - D_2$ vanishes on functions, i.e. is of type i_*. The uniqueness of this decomposition is obvious. □

Remark: We prefer at this moment not to call the D_2-part of this decomposition a derivation of type d^V_* because, as will be shown below, its further characterization in terms of some vector-valued form is not entirely satisfactory in comparison with the elegance of the standard theory. As a matter of fact, our subsequent analysis will show that there is need for an extra tool, preferably a connection on TM, and with the aid of such a connection we will reach a much more interesting classification of the derivations of $\bigwedge(\tau)$ in the next section.

For the time being, if D is any derivation of $\bigwedge(\tau)$ of degree r, we can associate to it a map $\psi : \mathcal{X}(\tau) \times \cdots \times \mathcal{X}(\tau) \longrightarrow \mathcal{X}(TM)$, defined by

$$\psi(X_1, \ldots, X_r)\,(F) = DF\,(X_1, \ldots, X_r).$$

The map ψ can be identified (in the terminology of e.g ref. [22]) with a vector-valued r-form on TM, horizontal over M. One might conceive constructing a derivation d^V_ψ, which on functions would be defined as $d^V_\psi(F) = i_\psi dF$ and for the rest would be required to commute with d^V. This d^V_ψ would then in effect be the D_2-part of D. Such a characterization of D_2, however, is not very elegant, basically for two reasons. First of all, ψ is not an element of $V^r(\tau)$, which is the relevant set of vector-valued forms in our present approach. Secondly, d^V_ψ is not the commutator of a type i_*-derivation with d^V.

Let us then look at the commutator one would normally expect to enter in our classification.

Definition 3.9 *For $L \in V^r(\tau)$ we define the derivation d^V_L as*

$$d^V_L = [i_L, d^V] = i_L \circ d^V - (-1)^{r-1} d^V \circ i_L.$$

Such derivations are said to be of type d^V_*.

Note that $d^V = d^V_{I\circ\tau}$, where I denotes the identity type (1,1) tensor on M.

Proposition 3.10 *A derivation D is of type d^V_* if and only if it commutes with d^V and vanishes on basic functions.*

Proof: Any d^V_L obviously commutes with d^V and vanishes on functions on M. For the converse, let D be a derivation of degree r with the above two properties and consider the derivation $\overline{D}$ of degree $r-1$ which is determined by the following action on $\bigwedge^0(\tau) \oplus \tilde{\bigwedge}^1(\tau)$:

$$\overline{D}F = 0, \qquad \overline{D}\tilde{\alpha} = D\hat{\alpha}.$$

$\overline{D}$ is of type i_* and therefore of the form i_L for some $L \in V^r(\tau)$. The claim is that $D = d^V_L$. Now, as observed in the proof of theorem 3.8, derivations which commute with d^V are completely determined by their action on functions. Since D and d^V_L certainly coincide on basic functions, we merely have to look at functions in $\hat{\bigwedge}^0(\tau)$. We find that

$$d^V_L\hat{\alpha} = i_L d^V\hat{\alpha} = i_L\tilde{\alpha} = D\hat{\alpha},$$

which confirms the claim. □

Theorem 3.11 *A derivation D of $\bigwedge(\tau)$ of degree r, which vanishes on basic functions has a unique decomposition in the form*

$$D = i_{L_1} + d^V_{L_2}$$

for some $L_1 \in V^{r+1}(\tau)$ and $L_2 \in V^r(\tau)$.

Proof: In the unique decomposition guaranteed by theorem 3.8, we know that D_1 is of the form i_{L_1} by proposition 3.7 and, in view of the extra assumption, D_2 is of the form $d^V_{L_2}$ by proposition 3.10. □

We have now clearly identified the problem concerning an elegant classification of all derivations D: the D_2-part of such a derivation need not vanish on basic functions. In order to characterize that part of a derivation which contributes to its action on functions on M, we need some way of extending derivations of scalar forms on M to a corresponding action on $\bigwedge(\tau)$. It will be sufficient for that purpose to extend the fundamental derivation, i.e. the exterior derivative of $\bigwedge(M)$ and the most natural way for achieving this is with the aid of a (generalized) connection on TM.

4 Connections and complete classification

Let us now suppose we are given a connection on the bundle $\tau\colon TM \to M$ (for general references see e.g. [7] and [13]), i.e. we have a splitting of the exact sequence

$$0 \longrightarrow V(TM) \xrightarrow{i} T(TM) \xrightarrow{j} \tau^*(TM) \longrightarrow 0$$

where j is the map $j(u) = (v, \tau_* u)$, for $v \in TM$ and $u \in T_v(TM)$, and i is the inclusion map of the vertical subbundle $V(TM)$ into $T(TM)$. In other words, considering sections of $T(TM)$ and $\tau^*(TM)$ over TM, we have a module homomorphism

$$\xi^H\colon \mathcal{X}(\tau) \to \mathcal{X}(TM)$$

satisfying $\tau_* \circ \xi^H = \mathrm{id}_{\mathcal{X}(\tau)}$.

Associated to the connection we can define a derivation d^H of degree 1 by means of its action on $\bigwedge^0(\tau)$ and $\tilde{\bigwedge}^1(\tau)$:

1. $d^H F(v)(w) = [\xi^H_v(w)]F$ for $F \in \bigwedge^0(\tau)$ and $v,\, w \in TM$

2. $d^H\tilde{\alpha} = \widetilde{d\alpha}$ for $\alpha \in \bigwedge^1(M)$.

Note that $\xi^H_v : T_{\tau(v)}M \to T_v(TM)$ is related to $\xi^H : \mathcal{X}(\tau) \to \mathcal{X}(TM)$ by the rule $\xi^H(X)(v) = \xi^H_v(X(v))$, $\forall X \in \mathcal{X}(\tau)$ and that $d^H F$ alternatively can be defined by $d^H F(X) = \xi^H(X)(F)$. In coordinates, denoting the connection coefficients by Γ^i_j, we have

$$H_i = \xi^H\left(\widetilde{\frac{\partial}{\partial q^i}}\right) = \frac{\partial}{\partial q^i} - \Gamma^j_i \frac{\partial}{\partial v^j}$$

and the action on d^H on $\bigwedge^0(\tau) \oplus \tilde{\bigwedge}^1(\tau)$ is given by

$$d^H F = H_i(F)\widetilde{dq^i}, \quad d^H\alpha = \frac{1}{2}(H_i\alpha_j - H_j\alpha_i)\,\widetilde{dq^i} \wedge \widetilde{dq^j}.$$

It is easy to see that in particular for basic functions we have $d^H\tau^* f = \widetilde{df}$.

Definition 4.1 *Let L be a vector valued r-form along τ. We define the derivation d^H_L by*

$$d^H_L = [i_L, d^H] = i_L \circ d^H - (-1)^{r-1} d^H \circ i_L.$$

A derivation of this type is said to be a type d^H_* derivation.

Theorem 4.2 *Given a connection on TM, every derivation D of $\bigwedge(\tau)$ of degree r has a unique decomposition of the form*

$$D = i_{L_1} + d^V_{L_2} + d^H_{L_3},$$

for some $L_2, L_3 \in V^r(\tau)$ and $L_1 \in V^{r+1}(\tau)$.

Proof: For $X_1, \ldots, X_r \in \mathcal{X}(\tau)$, consider the map $\varphi_{X_1,\ldots,X_r}$ from $C^\infty(M)$ to $C^\infty(TM)$ given by

$$\varphi_{X_1,\ldots,X_r} f = [D(f \circ \tau)](X_1, \ldots, X_r).$$

Then $\varphi_{X_1,\ldots,X_r}$ is a vector field along τ (see e.g. [22]). Since the dependence of $\varphi_{X_1,\ldots,X_r}$ on $X_1, \ldots, X_r$ is multilinear and skew-symmetric, it actually defines some $L_3 \in V^r(\tau)$: $\varphi_{X_1,\ldots,X_r} = L_3(X_1, \ldots, X_r)$. It follows that D and $d^H_{L_3}$ coincide on basic functions, so that proposition 3.11 implies the existence of $L_1 \in V^{r+1}(\tau)$ and $L_2 \in V^r(\tau)$, such that $D - d^H_{L_3} = i_{L_1} + d^V_{L_2}$. □

For the sake of clarity, we should emphasize that $d^H_{L_3}$ in general need not commute with d^V, so that i_{L_1} differs from the D_1 in the decomposition of theorem 3.8. The rest of this section is devoted to further considerations and properties of d^H, in a way which shows close ressemblance to the theory developed in [7].

Due to the fact that d^V vanishes on basic forms, the commutator $[d^H, d^V]$ will vanish on basic functions. Moreover, the Jacobi identity shows that $[d^H, d^V]$ commutes with d^V. According to proposition 3.10, we therefore conclude that

$$[d^H, d^V] = d^V_T, \tag{2}$$

for some $T \in V^2(\tau)$, which we call the torsion of the given connection. In local coordinates, T is given by

$$T = \frac{1}{2!}\left(\frac{\partial \Gamma^k_i}{\partial v^j} - \frac{\partial \Gamma^k_j}{\partial v^i}\right) \widetilde{dq^i} \wedge \widetilde{dq^j} \otimes \widetilde{\frac{\partial}{\partial q^k}}.$$

Similarly, the commutator $[d^H, d^H]$ vanishes on basic functions:

$$[d^H, d^H]\tau^* f = 2\, d^H(d^H \tau^* f) = 2\, d^H(\widetilde{df}) = 2\, \widetilde{d^2 f} = 0.$$

Thus, there exist $P \in V^3(\tau)$ and $R \in V^2(\tau)$ such that

$$\frac{1}{2}[d^H, d^H] = i_P + d^V_R \tag{3}$$

The vector-valued 2-form R can rightly be called the curvature of the connection and has the local expression

$$R = \frac{1}{2!} R^i_{jk}\, \widetilde{dq^j} \wedge \widetilde{dq^k} \otimes \widetilde{\frac{\partial}{\partial q^i}}, \tag{4}$$

where

$$R^i_{jk} = \frac{\partial \Gamma^i_j}{\partial q^k} - \frac{\partial \Gamma^i_k}{\partial q^j} + \frac{\partial \Gamma^i_k}{\partial v^l}\Gamma^l_j - \frac{\partial \Gamma^i_j}{\partial v^l}\Gamma^l_k.$$

The coordinate expression of P reads

$$P = \frac{1}{3!} P^i_{jkl}\, \widetilde{dq^j} \wedge \widetilde{dq^k} \wedge \widetilde{dq^l} \otimes \widetilde{\frac{\partial}{\partial q^i}},$$

where

$$P^i_{jkl} = \frac{\partial R^i_{jl}}{\partial v^k} + \frac{\partial R^i_{lk}}{\partial v^j} + \frac{\partial R^i_{kj}}{\partial v^l}.$$

The geometrical meaning of P is reflected by the following properties, which are corresponding generalizations of the Bianchi identities.

Proposition 4.3 *The following relations hold*

$$d^V_P = [d^H, d^V_T], \tag{5}$$
$$d^H_P = [d^H, d^V_R]. \tag{6}$$

Proof: Applying the Jacobi identity to d^V, d^H and d^H we obtain

$$[d^V, [d^H, d^H]] + 2\,[d^H, [d^V, d^H]] = 0$$

and taking into account (3) we have

$$-2\, d^V_P + 2\,[d^V, d^V_R] + 2\,[d^H, d^V_T] = 0.$$

Since d^V_R commutes with d^V, equation (5) follows. The property (6) follows in the same way from $[d^H, [d^H, d^H]] = 0$. □

A very interesting particular case (see e.g. [11], [12], [14], [9]) concerns the connection defined by a second-order differential equation field Γ on M (henceforth called SODE). If the local expression of Γ is

$$\Gamma = v^i \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial v^i},$$

then, the coefficients of this connection are

$$\Gamma^i_j = -\frac{1}{2}\frac{\partial f^i}{\partial v^j}.$$

It follows that its torsion vanishes. Conversely, if a connection is torsionless there exists a SODE defining it and proposition 4.3 further implies:

$$P = 0 \quad \text{and} \quad [d^{\scriptscriptstyle H}, d^{\scriptscriptstyle V}_R] = 0.$$

5 Derivations of vector-valued forms along τ

The set of vector-valued differential forms along τ can be endowed with a graded module structure over the graded ring $\bigwedge(\tau)$, with the definitions

$$\begin{aligned}
&(L_1 + L_2)(X_1, \ldots, X_p) = L_1(X_1, \ldots, X_p) + L_2(X_1, \ldots, X_p) \\
&(\omega \wedge L)(X_1, \ldots, X_{p+q}) = \\
&\qquad \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} (\operatorname{sgn} \sigma)\, \omega(X_{\sigma(1)}, \ldots, X_{\sigma(q)}) L(X_{\sigma(q+1)}, \ldots, X_{\sigma(p+q)})
\end{aligned} \tag{7}$$

where $\omega \in \bigwedge^q(\tau)$ and L, L_1, $L_2 \in V^p(\tau)$. For the sake of completeness, we now study derivations of this module. The reader can consult some of the work of Michor (see e.g. [15]) for comparable ideas.

Definition 5.1 *A map $D: V(\tau) \to V(\tau)$ is said to be a derivation of $V(\tau)$ of degree r if it satisfies*

1. $D(V^p(\tau)) \subset V^{p+r}(\tau)$,
2. $D(L_1 + \lambda L_2) = DL_1 + \lambda DL_2, \qquad \lambda \in \mathbf{R}$,
3. *There exists a map $\overline{D}: \bigwedge(\tau) \to \bigwedge(\tau)$ such that for all $\omega \in \bigwedge^q(\tau)$ and $L \in V^p(\tau)$*

$$D(\omega \wedge L) = (\overline{D}\omega) \wedge L + (-1)^{qr} \omega \wedge DL. \tag{8}$$

It can be easily shown that the following properties hold true.

- $\overline{D}$ is a derivation of scalar forms along τ.
- D is a local operator and is determined by its action on $V^0(\tau) = \mathcal{X}(\tau)$ and $V^1(\tau)$. There are no derivations of degree $r < -1$, and those of degree -1 vanish on $V^0(\tau)$.

- Alternatively, D is determined by its associated derivation $\overline{D}$ and its action on $V^0(\tau)$, whereby it suffices for the latter part to specify the action on $\tilde{\mathcal{X}}(\tau)$.

- The set of derivations of $V(\tau)$ is a graded Lie algebra with the bracket defined by

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1.$$

In order to classify the derivations of $V(\tau)$, knowing that they are in part determined by their $\overline{D}$, let us start from the known classification of such derivations and search for a simple way for extending their action to $V(\tau)$. The only feature which must be added in such an extension is a consistent action on $\tilde{\mathcal{X}}(\tau)$. If D_0 is any derivation of $\bigwedge(\tau)$ and we look for a covering D such that $\overline{D} = D_0$, the most simple attempt would be to define the restriction of D to $\tilde{\mathcal{X}}(\tau)$ as being zero. For this to make sense, however, thinking of the product of a basic vector field $\tilde{X}$ with a basic function f and the general rule (8), we must have

$$0 = D(f\,\tilde{X}) = D_0 f \otimes \tilde{X} + f\,D\tilde{X} = D_0 f \otimes \tilde{X},$$

which implies that D_0 should vanish on basic functions. Hence, we can use this simple direct construction to extend derivations of type i_* or of type d^{V}_* and we keep the same notation i_{L_1} or $d^{V}_{L_2}$ for the corresponding covering derivation of $V(\tau)$. The following property of such an extension will be useful later on.

Lemma 5.2 *Let D be a derivation of $V(\tau)$ of degree r, which vanishes on basic vector fields. Then, for any $L \in V^p(\tau)$, the action of $DL \in V^{p+r}(\tau)$ on basic 1-forms $\tilde{\alpha} \in \tilde{\bigwedge}^1(\tau)$ is given by*

$$DL(\tilde{\alpha}) = \overline{D}(L(\tilde{\alpha})). \tag{9}$$

Proof: We have, in all generality, for $X_1, \ldots, X_{p+r} \in \mathcal{X}(\tau)$

$$DL(\tilde{\alpha})\,(X_1, \ldots, X_{p+r}) = \langle DL(X_1, \ldots, X_{p+r}), \tilde{\alpha} \rangle.$$

Looking at this relation term by term, we can regard L to be of the form $L = \omega \otimes \tilde{X}$, with $\omega \in \bigwedge^p(\tau)$ and $\tilde{X} \in \tilde{\mathcal{X}}(\tau)$. Then, $DL = \overline{D}\omega \otimes \tilde{X}$ so that the right-hand side becomes $\overline{D}\omega(X_1, \ldots, X_{p+r})\,\langle \tilde{X}, \tilde{\alpha} \rangle$ or $\overline{D}(L(\tilde{\alpha}))\,(X_1, \ldots, X_{p+r})$, from which the result follows. $\square$

We need a more careful argumentation to find a natural extension of a type d^{H}_* derivation. Again, of course, it suffices for a start to look at $\tilde{\mathcal{X}}(\tau)$. So, we want to define, for $\tilde{X} \in \tilde{\mathcal{X}}(\tau)$, $d^{H}\tilde{X}$ as a vector-valued 1-form along τ. This we do, in agreement with general observations of section 2, by defining the action of $d^{H}\tilde{X}$ on basic vector fields, together with an imposed $C^\infty(TM)$-linearity. Thus, for $\tilde{Z} \in \tilde{\mathcal{X}}(\tau)$, we put

$$d^{H}\tilde{X}(\tilde{Z}) = [\xi^{H}(\tilde{Z}), \tilde{X}^{\uparrow}]^{\downarrow}.$$

To verify that this makes sense, note that the right-hand side starts with computing, essentially, the bracket of a horizontal and vertical lift of vector fields on M. It is well known [6] that such a bracket is vertical indeed, that it behaves linearly with respect to multiplication of Z by functions on M and has the right derivation property concerning multiplication of X by such functions. As argued above, this d^H is subsequently extended to a derivation of the whole $V(\tau)$ by requiring that its associated derivation of scalar forms is the d^H we already know. In this way, for example, the coordinate expression for $d^H X$, when X is a general element of $\mathcal{X}(\tau)$ (not necessarily basic), is found to be:

$$d^H X = \left(H_i(X^k) + X^j \frac{\partial \Gamma_i^k}{\partial v^j} \right) \widetilde{dq^i} \otimes \widetilde{\frac{\partial}{\partial q^k}}.$$

One can further verify that a direct definition of $d^H X$, for all $X \in \mathcal{X}(\tau)$, can be given pointwise as follows: $\forall Z \in \mathcal{X}(\tau)$ and arbitrary $v \in TM$,

$$(d^H X)(Z)\,(v) = [\xi^H(\tilde{Z}_0), X^{\uparrow}]^{\downarrow}\,(v),$$

where Z_0 is any vector field on M which at the point $\tau(v)$ coincides with $Z(v)$. An extension of $d^H_{L_3}$, for which we again maintain the same notation, now is readily obtained as the commutator of the extensions for i_{L_3} and d^H.

Let now D be an arbitrary derivation of $V(\tau)$ of degree r and consider the above proposed extension $D_1 = i_{L_1} + d^V_{L_2} + d^H_{L_3}$ of its associated $\overline{D}$. Then, $D_2 = D - D_1$ is a derivation of algebraic type, by which we mean that $\overline{D_2} = 0$. For $X_1, \ldots, X_r \in \mathcal{X}(\tau)$, we consider the map $T_{X_1,\ldots,X_r} : \mathcal{X}(\tau) \to \mathcal{X}(\tau)$, defined by $X \longmapsto D_2 X\,(X_1, \ldots, X_r)$. As a result of $\overline{D_2} = 0$, this map is $C^\infty(TM)$-linear and therefore determines a vector-valued 1-form along τ. Moreover, $T_{X_1,\ldots,X_r}$ depends multilinearly and skew-symmetrically on $X_1, \ldots, X_r$. This implies that there exists a tensor field $Q \in \bigwedge^r(\tau) \otimes V^1(\tau)$ such that $Q(X_1, \ldots, X_r) = T_{X_1,\ldots,X_r}$. This tensor field completely characterizes the derivation D_2, so it is convenient to introduce an appropriate notation reflecting the interference of Q. We define the map $a_Q : \mathcal{X}(\tau) \to V^r(\tau)$ by

$$a_Q X(X_1, \ldots, X_r) = Q(X_1, \ldots, X_r)\,(X),$$

and extend its action to $V(\tau)$ by putting $\overline{a_Q} = 0$. Then, clearly, $a_Q X = D_2 X$ for $X \in \mathcal{X}(\tau)$ and the fact that also $\overline{D_2} = 0$ implies that $D_2 = a_Q$.

It is of some interest to derive an explicit defining relation for the action of this new type of derivation on a general vector-valued form along τ. Consider an element $L \in V^p(\tau)$ which is of the form $\omega \otimes X$, with $\omega \in \bigwedge^p(\tau)$ and $X \in \mathcal{X}(\tau)$. Then we have $a_Q L = (-1)^{pr} \omega \wedge a_Q X$, so that the definition (7) yields

$$\begin{aligned}
&a_Q L(X_1, \ldots, X_{p+r}) \\
&\quad = (-1)^{pr} \frac{1}{p!r!} \sum_{\sigma \in S_{p+r}} (\operatorname{sgn} \sigma)\, \omega(X_{\sigma(1)}, \ldots, X_{\sigma(p)})\, Q(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+r)})\,(X) \\
&\quad = (-1)^{pr} \frac{1}{p!r!} \sum_{\sigma \in S_{p+r}} (\operatorname{sgn} \sigma)\, Q(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+r)}) \Big(\omega(X_{\sigma(1)}, \ldots, X_{\sigma(p)})\, X \Big)
\end{aligned}$$

from which it follows that for a general $L \in V^p(\tau)$:

$$\begin{aligned}
&a_Q L(X_1, \ldots, X_{p+r}) \\
&\qquad = \frac{1}{p!r!} \sum_{\sigma \in S_{p+r}} (\operatorname{sgn} \sigma)\, Q(X_{\sigma(1)}, \ldots, X_{\sigma(r)}) \Big(L(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+p)}) \Big).
\end{aligned}$$

Summarizing the results of the preceding analysis, we come to the following statement about the characterization of derivations of $V(\tau)$.

Theorem 5.3 *If D is a derivation of $V(\tau)$ of degree r, there exist $L_1 \in V^{r+1}(\tau)$, $L_2, L_3 \in V^r(\tau)$ and $Q \in \bigwedge^r(\tau) \otimes V^1(\tau)$, such that*

$$D = i_{L_1} + d^V_{L_2} + d^H_{L_3} + a_Q.$$

Note that the coordinate expression of $d^V L$ and $d^H L$, if $L \in V(\tau)$ is of the form $L = L^i \otimes \widetilde{\frac{\partial}{\partial q^i}}$, is given by

$$d^V L = d^V L^i \otimes \widetilde{\frac{\partial}{\partial q^i}} \tag{10}$$

and

$$d^H L = \left(d^H L^i + \frac{\partial \Gamma^i_k}{\partial v^j} \widetilde{dq^k} \wedge L^j \right) \otimes \widetilde{\frac{\partial}{\partial q^i}}. \tag{11}$$

To finish this section, we will have a closer look at the way one can define a graded Lie algebra structure on the module $V(\tau)$. For any L_1, $L_2 \in V(\tau)$, the corresponding derivations $d^V_{L_1}$, $d^V_{L_2}$ of $\bigwedge(\tau)$ vanish on basic functions and, therefore, so does their commutator. Moreover, it follows from the Jacobi identity that $[d^V_{L_1}, d^V_{L_2}]$ commutes with d^V so that proposition 3.10 implies the existence of a vector valued form along τ, denoted by $[L_1, L_2]$, such that

$$[d^V_{L_1}, d^V_{L_2}] = d^V_{[L_1, L_2]} \tag{12}$$

In addition, the graded Jacobi identity satisfied by d^V_* derivations directly translates to a graded Jacobi identity for this bracket on $V(\tau)$.

Proposition 5.4 *The bracket $[L, M]$ can be expressed as:*

$$[L, M] = d^V_L M - (-1)^{\ell m} d^V_M L \tag{13}$$

where ℓ and m are the degrees of L and M, respectively.

Proof: We will prove this equality by proving that their corresponding type d^V_* derivations coincide, for which in turn it suffices that the action on $\hat{\bigwedge}^0(\tau)$ is the same. For $\alpha \in \bigwedge^1(M)$, we have $d^V \hat{\alpha} = \tilde{\alpha} \in \tilde{\bigwedge}^1(\tau)$, and making use of the property (9) we find

$$\begin{aligned}
\left\{ d^V_{d^V_L M} - (-1)^{\ell m} d^V_{d^V_M L} \right\} \hat{\alpha} &= i_{d^V_L M}\, d^V \hat{\alpha} - (-1)^{\ell m} i_{d^V_M L}\, d^V \hat{\alpha} \\
&= d^V_L\, i_M\, d^V \hat{\alpha} - (-1)^{\ell m} d^V_M\, i_L\, d^V \hat{\alpha} \\
&= d^V_L\, d^V_M\, \hat{\alpha} - (-1)^{\ell m} d^V_M\, d^V_L\, \hat{\alpha} \\
&= d^V_{[L,M]} \hat{\alpha},
\end{aligned}$$

which indeed implies (13). □

A particular case of special interest concerns the bracket of two elements of $V^0(\tau)$, i.e. vector fields along τ. The formula (13) then reduces to $[X,Y] = d^V_X Y - d^V_Y X$, which in coordinates becomes

$$[X,Y] = \left(X^k \frac{\partial Y^i}{\partial v^k} - Y^k \frac{\partial X^i}{\partial v^k} \right) \widetilde{\frac{\partial}{\partial q^i}}$$

and is seen to be exactly the bracket we mentioned in section 2. Note further that, in view of $d^V_X = [i_X, d^V]$, it can equally be written in the form

$$[X,Y] = d^V Y(X) - d^V X(Y), \tag{14}$$

where a type (1,1) tensor field along τ, such as $d^V X$, in accordance with (10) has the coordinate expression

$$d^V X = \frac{\partial X^j}{\partial v^i} \widetilde{dq^i} \otimes \widetilde{\frac{\partial}{\partial q^j}}. \tag{15}$$

Remark: The basic obstruction for d^H_* derivations to define a Lie algebra structure on $V(\tau)$ is the fact that, in general, $[d^H, d^H] \neq 0$. Only in the case of a flat connection we can endow $V(\tau)$ with another Lie algebra structure.

The decomposition of most commutators of derivations of $\bigwedge(\tau)$ can be expressed in terms of d^H and d^V acting on $V(\tau)$. It is, for example, an instructive exercise to verify the following relation:

$$[d^H, d^V_L] = (-1)^\ell i_{d^V d^H L} + d^V_{d^H L} - d^H_{d^V L}\,. \tag{16}$$

The procedure for obtaining such a decomposition explicitly by a coordinate calculation involves three steps (in agreement with the theory developed in section 4). First, the action of the derivation in question on functions on M easily leads to the identification of the d^H_* part. Next, subtracting this part from the given derivation, one obtains the d^V_* part by calculating the action on functions in $\hat{\bigwedge}^0(\tau)$. Finally, subtracting this result again, the i_* part is found by computing the action on the basis $\widetilde{dq^i}$ of $\tilde{\bigwedge}^1(\tau)$.

Using (16), it is worthwhile re-expressing the basic commutator (2), together with the generalized Bianchi identities (5) and (6). One readily finds that these are equivalent to

$$\begin{aligned} d^V(I \circ \tau) &= 0, & d^H(I \circ \tau) &= T, \\ d^V T &= 0, & d^H T &= P, \\ d^V R &= -P, & d^H R &= 0. \end{aligned}$$

Some of these relations are trivially satisfied, others amount to defining relations; the only non-trivial equalities are the two Bianchi identities which now take the form

$$d^H R = 0, \qquad d^H T + d^V R = 0.$$

6 Prolongations of vector fields and forms along τ

As is well known [8], every function $f \in C^\infty(M)$ can be lifted to a function $f^1 \in C^\infty(TM)$ by means of

$$f^1(v) = \frac{d}{dt}\{f \circ \rho\}\Big|_{t=0}$$

where $v \in TM$ and $\rho: \mathbf{R} \to M$ is any representative curve of v (notice that $f^1 = \widehat{df}$). Similarly, for $F \in \bigwedge^0(\tau)$, if $j_0^2\rho$ is any point of T^2M, we define $F^1 \in C^\infty(T^2M)$ by

$$F^1(j_0^2\rho) = \frac{d}{dt}\left\{F \circ j^1\rho\right\}\Big|_{t=0}.$$

Observe here that for $f \in C^\infty(M)$: $(f \circ \tau)^1 = f^1 \circ \tau_{21}$. We can now use these facts to lift other objects along τ to corresponding objects along $\tau_{21} : T^2M \to TM$ as follows. First, for $X \in \mathcal{X}(\tau)$ we define $X^1 \in \mathcal{X}(\tau_{21})$ by

$$X^1 f^1 = (Xf)^1$$

for all $f \in C^\infty(M)$. Next, for $\alpha \in \bigwedge^p(\tau)$ and $L \in V^p(\tau)$, we define the prolongations $\alpha^1 \in \bigwedge^p(\tau_{21})$ and $L^1 \in V^p(\tau_{21})$ by requiring that for all $X_1, \ldots, X_p \in \mathcal{X}(\tau)$:

$$\alpha^1(X_1^1, \ldots X_p^1) = [\alpha(X_1, \ldots, X_p)]^1, \quad L^1(X_1^1, \ldots X_p^1) = [L(X_1, \ldots, X_p)]^1.$$

That such requirements completely determine X^1, α^1 and L^1 can be argued along the same lines as in [23] for the complete lifts of various objects on a manifold to its higher-order tangent bundles. In the present situation, for example, we can choose at every $a \in T^2M$ which is not in the zero section over M, a basis for $T_{\tau_{21}(a)}TM$, consisting of $2n$ vectors of the form $X_k^1(a)$ with $X_k \in \mathcal{X}(\tau)$. The argument then is completed by continuity. In fact, the same kind of reasoning applies just as well for the determination of any element of $\bigwedge(\tau_{21})$ or $V(\tau_{21})$ (not just prolongations) and will indeed tacitly be used further on.

From the definition of F^1 it follows that

$$(FG)^1 = (\tau_{21}^* F)G^1 + F^1(\tau_{21}^* G) \tag{17}$$

for $F, G \in C^\infty(TM)$.

It is easy to see that the map $\alpha \mapsto \alpha^1$ is a τ_{21}^*-derivation (in the sense of Pidello and Tulczyjew [16]) and (with some obvious identifications) is but the total time derivative associated to the canonical vector field $\mathbf{T}$ along τ. It can be shown [2] that, given $X \in \mathcal{X}(\tau)$, X^1 is the unique vector field along τ_{21} for which $\tau_* \circ X^1 = X \circ \tau_{21}$ and $\mathbf{T}^1 \circ X = X^1 \circ \mathbf{T}$ (as operators on $C^\infty(M)$).

If S is the vertical endomorphism on TM, we can extend its action to vector fields along τ_{21} as follows: $\forall Y \in \mathcal{X}(\tau_{21})$, $S(Y) \in \mathcal{X}(\tau_{21})$ is defined pointwise by $S(Y)(a) = S_{\tau_{21}(a)}(Y(a))$. In particular, for the prolongation of $X \in \mathcal{X}(\tau)$, it is easy to see that $S(X^1) = X^\uparrow \circ \tau_{21}$.

Proposition 6.1 *If G is a function on TM and X is a vector field along τ then*

$$(GX)^1 = (\tau_{21}^* G)X^1 + G^1 S(X^1). \tag{18}$$

Proof: We first prove that $\tau_{21}^*(Xf) = S(X^1)f^1$. Indeed, if $a \in T^2M$ and $v = \tau_{21}(a)$, we have

$$\begin{aligned} S(X^1)(f^1)(a) &= X^\uparrow(v)(f^1) \;=\; X^\uparrow(f^1)(v) \\ &= X^\uparrow(\widehat{df})(v) \;=\; \widetilde{df}(X)(v) \\ &= X(f)(v), \end{aligned}$$

where we made use of property (1). Next, we find

$$\begin{aligned} (GX)^1 f^1 = (G(Xf))^1 &= G^1 \tau_{21}^*(Xf) + \tau_{21}^* G (Xf)^1 \\ &= G^1 S(X^1) f^1 + (\tau_{21}^* G) X^1 f^1, \end{aligned}$$

from which the result follows. □

Proposition 6.2 *For $\alpha \in \bigwedge^p(\tau)$ and $X_1, \ldots, X_p \in \mathcal{X}(\tau)$*

$$\alpha^1(X_1^1, \ldots, S(X_i^1), \ldots, X_p^1) = \alpha(X_1, \ldots, X_i, \ldots, X_p) \circ \tau_{21}. \tag{19}$$

Proof: If F is a function on TM then using (18) we have

$$\begin{aligned} \alpha^1(X_1^1, \ldots, (FX_i)^1, \ldots, X_p^1) &= F^1 \, \alpha^1(X_1^1, \ldots, S(X_i^1), \ldots, X_p^1) \\ &\quad + \tau_{21}^* F \, [\alpha(X_1, \ldots, X_p)]^1. \end{aligned}$$

On the other hand, from the definition of α^1 and (17) we obtain

$$\begin{aligned} &\alpha^1(X_1^1, \ldots, (FX_i)^1, \ldots, X_p^1) = [F \, \alpha(X_1, \ldots, X_p)]^1 \\ &\quad = F^1 \, \tau_{21}^*[\alpha(X_1, \ldots, X_p)] + \tau_{21}^* F \, [\alpha(X_1, \ldots, X_p)]^1. \end{aligned}$$

Comparison of both expressions yields the desired result. □

An interesting consequence of this proposition is that the covariant tensor field along τ_{21}, formally denoted by $S \lrcorner \alpha^1$ and defined, for all $Y_k \in \mathcal{X}(\tau_{21})$, by

$$S \lrcorner \alpha^1 \, (Y_1, \ldots, Y_p) = \alpha^1(S(Y_1), \ldots, Y_p),$$

is actually skew-symmetric and thus is a proper p-form along τ_{21}, for which we have the property

$$S \lrcorner \alpha^1 = \imath_0 \alpha \circ \tau_{21}, \tag{20}$$

where $\imath_0$ stands for the natural identification of the graded algebra $\bigwedge(\tau)$ and the graded algebra $\bigwedge_0(TM)$ of semi-basic forms on TM. The isomorphism $\imath_0 \colon \bigwedge(\tau) \to \bigwedge_0(TM)$ is defined by

$$(\imath_0 \alpha)(Y_1, \ldots, Y_p) = \alpha(\tau_* \circ Y_1, \ldots, \tau_* \circ Y_p), \tag{21}$$

for $\alpha \in \bigwedge^p(\tau)$ and $Y_1, \ldots, Y_p \in \mathcal{X}(TM)$.

Using (17) and (19), it is now easy to derive the following further properties for $\alpha \in \bigwedge^p(\tau)$, $\beta \in \bigwedge^q(\tau)$, $G \in C^\infty(TM)$,

$$(G\alpha)^1 = (\tau_{21}^* G) \, \alpha^1 + G^1 (S \lrcorner \alpha^1) \tag{22}$$

$$(\alpha \wedge \beta)^1 = \alpha^1 \wedge (S \lrcorner \beta^1) + (S \lrcorner \alpha^1) \wedge \beta^1. \tag{23}$$

Similar relations hold true for the prolongation of vector-valued forms. Following the pattern of the proof of (19), it is easy to establish the next result.

Proposition 6.3 *For $L \in V^p(\tau)$ and $X_1, \dots, X_p \in \mathcal{X}(\tau)$*

$$L^1(X_1^1, \dots, S(X_i^1) \dots, X_p^1) = S\Big([L(X_1, \dots, X_i \dots, X_p)]^1\Big). \tag{24}$$

Again it follows that $S \lrcorner L^1$ is a vector-valued p-form along τ_{21} and we propose to write the above property formally as

$$S \lrcorner L^1 = S \circ L^1. \tag{25}$$

Still along the lines of the above considerations for scalar forms, it is simple to show, using (18) and (25) that for $L \in V^p(\tau)$, $\alpha \in \bigwedge^q(\tau)$, $G \in C^\infty(TM)$,

$$(GL)^1 = (\tau_{21}^* G)\, L^1 + G^1(S \circ L^1) \tag{26}$$

$$(\alpha \wedge L)^1 = \alpha^1 \wedge (S \circ L^1) + (S \lrcorner \alpha^1) \wedge L^1. \tag{27}$$

Finally, as a direct consequence of the defining relations for prolongations, we mention that for $\alpha \in \bigwedge^1(\tau)$, $L \in V^p(\tau)$ one has the obvious property

$$(\alpha \circ L)^1 = \alpha^1 \circ L^1. \tag{28}$$

7 Differential calculus relative to a second-order differential equation

Let Γ be a SODE on M, i.e. $\Gamma \in \mathcal{X}(TM)$ is a vector field such that $\tau_* \circ \Gamma = \mathbf{T}$. Associated to Γ, we recall from [18] the definition of the following sets:

$$\begin{aligned}
&\mathcal{X}_\Gamma = \{\, X \in \mathcal{X}(TM) \,|\, S(\mathcal{L}_\Gamma X) = 0 \,\}, \\
&\textstyle\bigwedge_\Gamma^p = \{\, \omega \in \bigwedge^p(TM) \,|\, S \lrcorner \omega \text{ is a } p\text{-form and} \mathcal{L}_\Gamma(S \lrcorner \omega) = \omega \,\}, \\
&T_\Gamma^{(1,1)} = \{\, R \in T^{(1,1)}(TM) \,|\, S \circ R = R \circ S \text{ and } S \circ \mathcal{L}_\Gamma R = 0 \,\},
\end{aligned}$$

which were endowed with a $C^\infty(TM)$-module structure by the product rules

$$\begin{aligned}
F * X &= FX + (\Gamma F) S(X), \\
F * \omega &= F\omega + (\Gamma F) S \lrcorner \omega, \\
F * R &= FR + (\Gamma F) S \circ R.
\end{aligned}$$

Moreover, setting $\bigwedge_\Gamma^0 = C^\infty(TM)$, the module $\bigwedge_\Gamma = \oplus_p \bigwedge_\Gamma^p$ is endowed with a graded algebra structure if we define

$$\alpha \overset{\circ}{\wedge} \omega = (S \lrcorner \alpha) \wedge \omega + \alpha \wedge (S \lrcorner \omega),$$

for forms of degree different from zero and for $F \in \bigwedge_\Gamma^0$,

$$F \overset{\circ}{\wedge} \omega = F * \omega.$$

In order to complete this picture, we now consider vector-valued forms, mapping elements of $\mathcal{X}_\Gamma$ to elements of $\mathcal{X}_\Gamma$.

Proposition 7.1 *A vector-valued form $L \in V^p(TM)$ preserves the set $\mathcal{X}_\Gamma$ if and only if*

1. $S \lrcorner L = S \circ L$, *and*
2. $S \circ \mathcal{L}_\Gamma L = 0$.

Proof: Let L be a vector valued p-form preserving $\mathcal{X}_\Gamma$. Then for $X_1, \ldots, X_p \in \mathcal{X}_\Gamma$ and $F \in C^\infty(TM)$,

$$F * \Big(L(X_1, \ldots, X_i, \ldots, X_p)\Big) - L(X_1, \ldots, F * X_i, \ldots, X_p) = \\ (\Gamma F)\{S \circ L(X_1, \ldots, X_i, \ldots, X_p) - L(X_1, \ldots, S(X_i), \ldots, X_p)\}$$

is an element of $\mathcal{X}_\Gamma$ for all functions F. But this is impossible except if

$$S \circ L(X_1, \ldots, X_i, \ldots, X_p) = L(X_1, \ldots, S(X_i), \ldots, X_p),$$

for all $X_1, \ldots, X_p \in \mathcal{X}_\Gamma$. Now, away from the zero section of TM, it is easy to see (e.g. by straightening out Γ) that a local basis for $\mathcal{X}(TM)$ can be constructed out of elements of $\mathcal{X}_\Gamma$. We thus conclude that $S \circ L = S \lrcorner L$ outside the zero section, and by continuity also on the whole of TM.

Secondly, we have

$$\begin{aligned}
&(S \circ \mathcal{L}_\Gamma L)(X_1, \ldots, X_p) \\
&\quad = S[\mathcal{L}_\Gamma(L(X_1, \ldots, X_p))] - \sum_{i=1}^{p} S \circ L(X_1, \ldots, \mathcal{L}_\Gamma X_i, \ldots, X_p) \\
&\quad = S[\mathcal{L}_\Gamma(L(X_1, \ldots, X_p))] - \sum_{i=1}^{p} L(X_1, \ldots, S(\mathcal{L}_\Gamma X_i), \ldots, X_p) = 0,
\end{aligned}$$

for all $X_1, \ldots, X_p \in \mathcal{X}_\Gamma$. By the same reasoning as above, this implies $S \circ \mathcal{L}_\Gamma L = 0$. The converse is a matter of re-arranging arguments. □

The set of vector-valued forms preserving $\mathcal{X}_\Gamma$ will be denoted by V_Γ^p, and for $p = 0$ we set $V_\Gamma^0 = \mathcal{X}_\Gamma$. Clearly $V_\Gamma^1 = T_\Gamma^{(1,1)}$. Defining the product by functions again as

$$F * L = FL + (\Gamma F) S \circ L$$

the set V_Γ^p is endowed with a $C^\infty(TM)$-module structure. Finally, we define a "wedge" product by scalar forms in a way similar to the product on $\bigwedge_\Gamma$:

$$\omega \overset{\circ}{\wedge} L = (S \lrcorner \omega) \wedge L + \omega \wedge (S \lrcorner L)$$

for $\omega \in \bigwedge_\Gamma$ and $L \in V_\Gamma$. Showing that $\omega \overset{\circ}{\wedge} L$ is indeed an element of V_Γ is an easy matter with the help of the properties $S \circ L = S \lrcorner L$, $S \circ \mathcal{L}_\Gamma L = 0$ and $S \circ \mathcal{L}_\Gamma S = -S$.

The rest of this section is devoted to re-obtaining and generalizing, in a much simpler fashion, the results of [18], using the theory developed in the previous sections. This will be achieved in two steps: first we will consider the SODE as a section of $\tau_{21}: T^2M \to TM$, and subsequently we will establish, via the results of section 6, an isomorphism between $\bigwedge(\tau)$, $V(\tau)$ and $\bigwedge_\Gamma$, V_Γ, respectively.

A SODE can be considered as a section $\gamma: TM \to T^2M$ of the projection $\tau_{21}: T^2M \to TM$, see e.g. [8]. The relation between Γ and γ is given by $\Gamma = \mathbf{T}^1 \circ \gamma$, or equivalently, $\mathcal{L}_\Gamma = \gamma^* \circ d_{\mathbf{T}^1}$ as operators on $\bigwedge(TM)$, see [2]. In local coordinates, if Γ has the form

$$\Gamma = v^i \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial v^i},$$

then the expression for γ reads

$$\gamma(q^i, v^i) = (q^i, v^i, f^i(q^j, v^j)).$$

Integral curves of Γ are curves σ in M whose second extension σ^2 lies in the image of γ. Using this alternative description, we will associate to every element of $\bigwedge(\tau)$ an element of $\bigwedge_\Gamma$ by restriction of its first prolongation to the image of γ. To be precise, consider the map $I_\Gamma: \bigwedge^p(\tau) \to \bigwedge^p(TM)$ given by $I_\Gamma: \alpha \mapsto \alpha^1 \circ \gamma$ for $p > 0$. For $p = 0$ we define I_Γ as the identity on $C^\infty(TM)$, for reasons which will become clear later on. Similarly, we consider the map $J_\Gamma: V^p(\tau) \to V^p(TM)$ given by $J_\Gamma: L \mapsto L^1 \circ \gamma$. In [2] it was shown that the image of a vector field X along τ by J_Γ is a vector field in $\mathcal{X}_\Gamma$, and every vector field in $\mathcal{X}_\Gamma$ can be obtained in this way. Here, we will prove more generally that the image of I_Γ is $\bigwedge_\Gamma$ and the image of J_Γ is V_Γ. This could be done in a direct way, but it is easier to make use of the isomorphism $\imath_0$ between the graded algebra $\bigwedge(\tau)$ of forms along τ and the graded algebra $\bigwedge_0(TM)$ of semi-basic forms on TM, as introduced before. For a function $F \in C^\infty(TM)$, $\imath_0 F$ is defined to be F. Then, with a similar identification between forms along τ_{21} and forms on T^2M which are semi-basic with respect to τ_{21}, it is clear that for any $\alpha \in \bigwedge(\tau)$, the prolongation α^1 is essentially $d_{\mathbf{T}^1}\imath_0\alpha$, from which it follows that

$$\alpha^1 \circ \gamma = \mathcal{L}_\Gamma(\imath_0\alpha). \tag{29}$$

Let us now further extend the action of $\imath_0$ to vector-valued forms along τ. First, for $\alpha \in \bigwedge^1(\tau)$, $X \in \mathcal{X}(\tau)$, the relation

$$\imath_0\Big(\alpha(X)\Big) = \imath_0\alpha(\imath_0 X)$$

defines an identification between $\mathcal{X}(\tau)$ and the quotient module $\mathcal{X}(TM)/\mathcal{X}^V(TM)$. If Y is a representative of an element of $\mathcal{X}(TM)/\mathcal{X}^V(TM)$, the inverse correspondence is simply given by $\imath_0^{-1}[Y] = \tau_* \circ Y \in \mathcal{X}(\tau)$. Combining the action of $\imath_0$ on $\bigwedge(\tau)$ and $\mathcal{X}(\tau)$, we obtain the extension to $V(\tau)$: $L \mapsto \imath_0 L$, where $\imath_0 L$ is an equivalence class of certain vector-valued forms on TM. In order to make this correspondence clear, consider the subset of vector-valued forms N on TM, with the property that $S \circ N$ vanishes when one of its arguments is vertical. In coordinates, if we write N as $N = \lambda^i \otimes (\partial/\partial q^i) + \mu^i \otimes (\partial/\partial v^i)$, with λ^i, $\mu^i \in \bigwedge(TM)$, this requirement simply expresses the fact that the λ^i are semi-basic. Then, for each such N, it makes sense to define $\tau_* \circ N \in V(\tau)$ via the requirement: $\forall X_1, \ldots, X_p \in \mathcal{X}(\tau)$ ($p > 0$ being the degree of N),

$$\tau_* \circ N(X_1, \ldots, X_p) = \tau_* \circ \Big(N(\overline{X_1}, \ldots, \overline{X_p})\Big), \tag{30}$$

where the $\overline{X_i} \in \mathcal{X}(TM)$ are such that $\tau_* \circ \overline{X_i} = X_i$. Within this subset, we can obviously pass to the quotient, modulo forms with values in $\mathcal{X}^V(TM)$. Then, for $L \in V(\tau)$, $\imath_0 L$ is one of these classes, for which a representative $\overline{L}$ is determined by: $\tau_* \circ \overline{L} = L$.

Under the isomorphism $\imath_0$, derivations of $\bigwedge(\tau)$ can be regarded as derivations $\tilde{D}$ of $\bigwedge_0(TM)$, the correspondence being given by $\tilde{D} = \imath_0 \circ D \circ \imath_0^{-1}$. It is clear that $[\widetilde{D_1}, \widetilde{D_2}] = \widetilde{[D_1, D_2]}$. For instance, to i_L corresponds $i_{\overline{L}}$, where $\overline{L}$ is any representative of $\imath_0 L$; to d^V corresponds d_S, and to d^H corresponds d_{P_H}, where P_H is the horizontal projector associated to the connection and defined by $P_H(X)(v) = \xi^H_v\Big(\tau_*(X(v))\Big)$, for all $X \in \mathcal{X}(TM)$. At this point, we want to repeat that the theory of derivations of $\bigwedge_0(TM)$ we are constructing this way is not equivalent to the theory of derivations of $\bigwedge(TM)$ which preserve semi-basic forms.

In order to prove now that the image of I_Γ is $\bigwedge_\Gamma$ we recall property (20), from which it follows that

$$S \,\lrcorner\, I_\Gamma \alpha = S \,\lrcorner (\alpha^1 \circ \gamma) = \imath_0 \alpha \circ \tau_{21} \circ \gamma = \imath_0 \alpha.$$

This expresses that $S \,\lrcorner\, I_\Gamma \alpha$ is a form and we further have (using (29))

$$\mathcal{L}_\Gamma(S \,\lrcorner\, I_\Gamma \alpha) = \mathcal{L}_\Gamma(\imath_0 \alpha) = I_\Gamma \alpha,$$

so that $I_\Gamma \alpha \in \bigwedge_\Gamma$. In addition, for any $\omega \in \bigwedge_\Gamma$, $S \,\lrcorner\, \omega$ is a semi-basic form and we have

$$I_\Gamma\Big(\imath_0^{-1}(S \,\lrcorner\, \omega)\Big) = \mathcal{L}_\Gamma(S \,\lrcorner\, \omega) = \omega.$$

This shows that $I_\Gamma\Big(\bigwedge(\tau)\Big) = \bigwedge_\Gamma$ and that the inverse $I_\Gamma^{-1} : \bigwedge_\Gamma \to \bigwedge(\tau)$ is given by $\omega \mapsto \imath_0^{-1}(S \,\lrcorner\, \omega)$ (injectivity of I_Γ is obvious from the definition).

For the map J_Γ on $\mathcal{X}(\tau)$ and more generally on $V(\tau)$, we will proceed by duality and make repeated use of the fact that, roughly speaking (see the argumentation in the proof of proposition 7.1), a local basis of vector fields on TM can be constructed out of elements of $\mathcal{X}_\Gamma$ and similarly, a local basis of 1-forms can be built with elements of $\bigwedge^1_\Gamma$.

So let $X \in \mathcal{X}(\tau)$ and note first that for all $\alpha \in \bigwedge^1(\tau)$,

$$\langle X^1 \circ \gamma, \alpha^1 \circ \gamma \rangle = \Big(\alpha(X)\Big)^1 \circ \gamma = \mathcal{L}_\Gamma \langle \overline{X}, \imath_0 \alpha \rangle$$

with $\tau_* \circ \overline{X} = X$. It follows that

$$\begin{aligned} \langle S \circ \mathcal{L}_\Gamma(X^1 \circ \gamma), \alpha^1 \circ \gamma \rangle &= \langle \mathcal{L}_\Gamma(X^1 \circ \gamma), \imath_0 \alpha \rangle \\ &= \mathcal{L}_\Gamma \langle X^1 \circ \gamma, \imath_0 \alpha \rangle - \langle X^1 \circ \gamma, \alpha^1 \circ \gamma \rangle \\ &= \mathcal{L}_\Gamma \langle X^1 \circ \gamma - \overline{X}, \imath_0 \alpha \rangle \;=\; 0, \end{aligned}$$

since $X^1 \circ \gamma - \overline{X}$ is vertical. This implies $S \circ \mathcal{L}_\Gamma(X^1 \circ \gamma) = 0$ and thus $J_\Gamma(X) = X^1 \circ \gamma \in \mathcal{X}_\Gamma$. To show that the map is onto, let now X be an element of $\mathcal{X}_\Gamma$ and consider $\tau_* \circ X \in \mathcal{X}(\tau)$. Then, for all $\alpha \in \bigwedge^1(\tau)$:

$$\begin{aligned} \langle (\tau_* \circ X)^1 \circ \gamma, \alpha^1 \circ \gamma \rangle &= \Big(\alpha(\tau_* \circ X)\Big)^1 \circ \gamma \\ &= \mathcal{L}_\Gamma\Big(\alpha(\tau_* \circ X)\Big) \;=\; \mathcal{L}_\Gamma \langle X, \imath_0 \alpha \rangle \\ &= \langle [\Gamma, X], \imath_0 \alpha \rangle + \langle X, \alpha^1 \circ \gamma \rangle. \end{aligned}$$

The first term of the last line vanishes because $[\Gamma, X]$ is vertical. Since every 1-form in $\bigwedge^1_\Gamma$ can be written as $\alpha^1 \circ \gamma$ for some α, we conclude that $(\tau_* \circ X)^1 \circ \gamma = X$. This says that $J_\Gamma(\mathcal{X}(\tau)) = \mathcal{X}_\Gamma$ and $J_\Gamma^{-1}(X) = \tau_* \circ X$.

Let us finally turn to $V^p(\tau)$ for $p > 0$ and note for a start that the property (25) directly implies $S \lrcorner (J_\Gamma L) = S \circ (J_\Gamma L)$. Next we have $L^1 \circ \gamma(X_1^1 \circ \gamma, \ldots, X_p^1 \circ \gamma) = [L(X_1, \ldots, X_p)]^1 \circ \gamma$, which says that $J_\Gamma L(J_\Gamma X_1, \ldots, J_\Gamma X_p) = J_\Gamma\Big(L(X_1, \ldots, X_p)\Big) \in \mathcal{X}_\Gamma$. Taking the Lie derivative with respect to Γ of this relation and composing with S, the right-hand side will vanish. Moreover, the property first mentioned allows us, for the term involving $S \circ J_\Gamma L$, to move S over to any of the arguments we like. The result can be written as

$$S \circ \mathcal{L}_\Gamma J_\Gamma L(J_\Gamma X_1, \ldots, J_\Gamma X_p) + \sum_{i=1}^{p} J_\Gamma L\Big(J_\Gamma X_1, \ldots, S(\mathcal{L}_\Gamma J_\Gamma X_i), \ldots, J_\Gamma X_p\Big) = 0$$

and ultimately reduces to the first term being zero. Knowing already that every element of $\mathcal{X}_\Gamma$ can be regarded as $J_\Gamma X$ for some X, this implies $S \circ \mathcal{L}_\Gamma J_\Gamma L = 0$ and thus $J_\Gamma(V^p(\tau)) \subset V_\Gamma^p$. For the converse, if $R \in V_\Gamma^p$, the property $S \lrcorner R = S \circ R$ implies that $S \circ R$ vanishes when one of its arguments is vertical. Therefore, the definition (30) applies and gives $\tau_* \circ R \in V(\tau)$. We wish to prove that $J_\Gamma(\tau_* \circ R) = R$. Now,

$$\begin{aligned} \Big((\tau_* \circ R)^1 \circ \gamma\Big)(X_1^1 \circ \gamma, \ldots, X_p^1 \circ \gamma) &= \Big(\tau_* \circ R(X_1, \ldots, X_p)\Big)^1 \circ \gamma \\ &= \Big(\tau_* \circ (R(J_\Gamma X_1, \ldots, J_\Gamma X_p))\Big)^1 \circ \gamma\,, \end{aligned}$$

where we have used (30) and the fact that $\tau_* \circ J_\Gamma X_i = X_i$. By the result on vector fields, the right-hand side is just $R(J_\Gamma X_1, \ldots, J_\Gamma X_p)$, which shows that J_Γ indeed is onto again and $J_\Gamma^{-1}(R) = \tau_* \circ R$.

We have finally reached the point where we can easily transport all the structures on $\bigwedge(\tau)$ and $V(\tau)$ to $\bigwedge_\Gamma$ and V_Γ, respectively. It turns out that the maps I_Γ and J_Γ are graded algebra homomorphisms and since the ring $C^\infty(TM)$ plays the role of the set of scalars for both algebra's, it is now clear why we defined I_Γ to be the identity on functions.

Proposition 7.2 *For $F \in C^\infty(TM)$, α, $\omega \in \bigwedge(\tau)$ and $L \in V(\tau)$ the following relations hold*

1. $I_\Gamma(F\alpha) = F * I_\Gamma \alpha$
2. $J_\Gamma(FL) = F * J_\Gamma L$
3. $I_\Gamma(\omega \wedge \alpha) = (I_\Gamma \omega) \overset{\circ}{\wedge} (I_\Gamma \alpha)$
4. $J_\Gamma(\omega \wedge L) = (I_\Gamma \omega) \overset{\circ}{\wedge} (J_\Gamma L)$

Proof: They follow immediately from (22), (26), (23) and (27). □

As indicated in the introduction, most of the effort in [18] went into discovering what could be called the basic derivations of the algebra $\bigwedge_\Gamma$: they were denoted respectively by $\hat\imath_X$, $\hat d$ and $\hat{\mathcal{L}}_X = [\hat\imath_X, \hat d]$, with $X \in \mathcal{X}_\Gamma$. It is obvious now that these derivations must correspond to derivations of $\bigwedge(\tau)$. If D is a derivation of $\bigwedge(\tau)$ then $\hat D = I_\Gamma \circ D \circ I_\Gamma^{-1}$ is a derivation of $\bigwedge_\Gamma$ and we have $I_\Gamma \circ [D_1, D_2] \circ I_\Gamma^{-1} = [\widehat{D_1}, \widehat{D_2}]$. We wish to prove that the derivations discussed in [18] come from the derivations i_X and $d^{\scriptscriptstyle V}$ of $\bigwedge(\tau)$ and their commutator. In addition, we will of course obtain this time a much more complete picture of the derivations of $\bigwedge_\Gamma$.

Note that, when $I_\Gamma \circ D \circ I_\Gamma^{-1}$ does not act on functions and the result is not a function, the following relation holds:

$$I_\Gamma \circ D \circ I_\Gamma^{-1} = I_\Gamma \circ \imath_0^{-1} \circ \imath_0 \circ D \circ \imath_0^{-1} \circ \imath_0 \circ I_\Gamma^{-1} = \mathcal{L}_\Gamma \circ D_0 \circ (S \lrcorner\,),$$

where $D_0 = \imath_0 \circ D \circ \imath_0^{-1}$. This expression will simplify some of the calculations which follow.

Proposition 7.3 *If X is any vector field along τ, we have*

$$\widehat{i_X} = I_\Gamma \circ i_X \circ I_\Gamma^{-1} = \hat\imath_{J_\Gamma X}.$$

Proof: We have to consider separately the cases where i_X acts on functions, 1-forms or p-forms with $p > 1$, because the meaning of I_Γ and I_Γ^{-1} is different in each of these cases. For a function $F \in C^\infty(TM)$ it is obvious that $\widehat{i_X} F = 0$. For a 1-form $\alpha \in \bigwedge_\Gamma^1$ we find

$$\widehat{i_X}\,\alpha = i_X\, \imath_0^{-1}(S \lrcorner \alpha) = i_{J_\Gamma X}(S \lrcorner \alpha) = \langle S(J_\Gamma X), \alpha\rangle.$$

For $\omega \in \bigwedge_\Gamma^p$, $p > 1$,

$$\begin{aligned}
\widehat{i_X}\omega &= \mathcal{L}_\Gamma\, i_{J_\Gamma X}(S \lrcorner \omega) \\
&= i_{[\Gamma, J_\Gamma X]}(S \lrcorner \omega) + i_{J_\Gamma X}\, \mathcal{L}_\Gamma(S \lrcorner \omega) \\
&= i_{J_\Gamma X}\omega
\end{aligned}$$

because $S \lrcorner \omega$ is semibasic, $[\Gamma, J_\Gamma X]$ is vertical and $\mathcal{L}_\Gamma(S \lrcorner \omega) = \omega$. These results precisely correspond to the definition of $\hat\imath_{J_\Gamma X}$ in [18]. □

Obviously, we now know of a more general type i_* derivation of $\bigwedge_\Gamma$: for any $L \in V(\tau)$ we can consider $\widehat{i_L} = I_\Gamma \circ i_L \circ I_\Gamma^{-1}$. It turns out that $\widehat{i_L}$ coincides with $i_{J_\Gamma L}$. The proof is similar to the case of vector fields.

Proposition 7.4 $\widehat{d^{\scriptscriptstyle V}} = I_\Gamma \circ d^{\scriptscriptstyle V} \circ I_\Gamma^{-1} = \hat d$

Proof: For a function F on TM

$$(I_\Gamma \circ d^{\scriptscriptstyle V} \circ I_\Gamma^{-1})\, F = \mathcal{L}_\Gamma(d_S F) = \mathcal{L}_\Gamma(S \lrcorner dF),$$

and for a p-form ω, $p > 0$,

$$\begin{aligned}(I_\Gamma \circ d^V \circ I_\Gamma^{-1})\,\omega &= \mathcal{L}_\Gamma(d_S(S \,\lrcorner\, \omega)) \\ &= d_{\mathcal{L}_\Gamma S}(S \,\lrcorner\, \omega) + d_S \mathcal{L}_\Gamma(S \,\lrcorner\, \omega) \\ &= d_{\mathcal{L}_\Gamma S}(S \,\lrcorner\, \omega) + d_S \omega\,.\end{aligned}$$

Again, this is in agreement with the definition of $\hat{d}$ in [18]. □

An interesting completion of the theory which now trivially follows from proposition 3.4 is that the cohomology of $\hat{d}$ is trivial.

Note also that, since $d^V_X = [i_X, d^V]$, we have

$$\widehat{d^V_X} = I_\Gamma \circ d^V_X \circ I_\Gamma^{-1} = \hat{\mathcal{L}}_{J_\Gamma X}$$

and this further implies that the bracket $[X, Y]$ defined by (14) must transfer to the bracket $\{J_\Gamma X, J_\Gamma Y\}_\Gamma$ defined in [18]. One of the defining relations for the latter was of the form

$$\{J_\Gamma X, J_\Gamma Y\}_\Gamma = R_{J_\Gamma Y}(J_\Gamma X) - R_{J_\Gamma X}(J_\Gamma Y),$$

with $R_{J_\Gamma X} \in V^1_\Gamma$. It is clear that this relation now directly follows from (14) and that in fact $R_{J_\Gamma X} = J_\Gamma(d^V X)$ (compare also the coordinate formula (15) for $d^V X$ with the formula for $R_{J_\Gamma X}$ in [18]).

Again, apart from rediscovering the operations introduced in [18], our present analysis immediately produces certain generalizations. For example, we could consider for $L \in V^p_\Gamma$ the derivation $\hat{d}_L = [i_L, \hat{d}]$, which reduces to the $\hat{\mathcal{L}}$ derivative for the case $p = 0$. Also, knowing now that the tensors of type R_X, for $X \in \mathcal{X}_\Gamma$ (as discussed first in [19]) essentially come form $d^V(\tau_* \circ X)$, we could in the same way, for $L \in V^p_\Gamma$ define $R_L \in V^{p+1}_\Gamma$ by

$$R_L = J_\Gamma\Big(d^V(\tau_* \circ L)\Big).$$

It then follows that $R_L = 0$ if and only if there exist a vector-valued form $M \in V_\Gamma$, such that $L = R_M$.

Finally, the most important new feature which is added now to the theory developed in [18] is the constatation that the picture is not complete without considering a connection. Moreover, there is no need here to introduce something extra, as a SODE naturally comes with an associated connection, whose horizontal projector is given by $P_H = \frac{1}{2}(I - \mathcal{L}_\Gamma S)$. The meaning of the corresponding derivation $\widehat{d^H}$ on $\bigwedge_\Gamma$ then is obtained from the following relations

$$(I_\Gamma \circ d^H \circ I_\Gamma^{-1})\,F = \mathcal{L}_\Gamma(P_H \,\lrcorner\, dF)$$

$$(I_\Gamma \circ d^H \circ I_\Gamma^{-1})\,\omega = d_{\mathcal{L}_\Gamma P_H}(S \,\lrcorner\, \omega) + d_{P_H}\omega$$

for $F \in C^\infty(TM)$ and $\omega \in \bigwedge^p_\Gamma$, $p > 0$. They can be proved in a way similar to the proof of proposition 7.4.

8 Concluding remarks

It is clear that the calculus of forms along τ which we just developed calls for some applications. We can safely say, however, that a few applications are already available. Indeed, the whole construction was motivated at the start by the interest of the sets $\mathcal{X}_\Gamma$ and $\bigwedge^1_\Gamma$ and a nice application in which these sets play a key role concerns the interplay between symmetries and adjoint symmetries of a second-order equation (see [20] and time-dependent generalizations in [5] and [21]). Another application, concerning the inverse problem of Lagrangian mechanics can be found in [3].

The nice thing about calculations involving elements of $\mathcal{X}_\Gamma$, $\bigwedge_\Gamma$ or V_Γ is that all the objects one is dealing with live on the same space TM. At the same time, however, one feels the need for economizing the calculations because, roughly speaking, only half of the components of such objects are important. The answer to this need lies exactly in the calculus of forms along τ. Note, by the way, that many of the relevant objects in mechanics are indeed sections along τ. This is true for example for the Cartan 1-form θ_L or the Euler-Lagrange form δL (which are semi-basic forms). Sections along τ were recently shown to be of interest also for a better understanding of Noether's theorem (see [2], as well as a generalization to higher-order mechanics in [4]).

Now that we have a more economical machinary at our disposal, it is our belief that further and perhaps more deep applications will follow. For that we will undoubtedly have to supplement the calculus of sections along τ with operations which incorporate more of the dynamics of the given system. We are thinking here in the first place of a covariant derivative operator ∇, associated to the connection which comes with the given second-order system Γ. This operator could for example, as a derivation of $\bigwedge(\tau)$, be defined as follows :

$$\nabla F = \Gamma(F), \qquad \text{for } F \in C^\infty(TM)$$

$$\nabla \alpha = \imath_0^{-1}(P_H \,\lrcorner\, I_\Gamma \alpha), \qquad \text{for } \alpha \in \bigwedge{}^1(\tau)$$

the covariant derivative of vector fields along τ then following by duality. With this operation one can arrive at very elegant formulae for the equations for symmetries (a generalization of the Jacobi equation) and for adjoint symmetries, but it would take us too far to enter into this subject at the moment.

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