GEOMETRIC CHARACTERIZATION OF SEPARABLE SECOND-ORDER DIFFERENTIAL EQUATIONS

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Abstract.– We establish necessary and sufficient conditions for the separability of a system of second-order differential equations into independent one-dimensional second-order equations. The characterization of this property is given in terms of geometrical objects which are directly related to the system and relatively easy to compute. The proof of the main theorem is constructive and thus yields a practical procedure for constructing coordinates in which the system decouples.

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1. INTRODUCTION

In two recent papers [9] [10] we have developed the theory of derivations of differential forms along the tangent bundle projection $\tau: TM \to M$. The aim of the present work is to show that this theory, which may at first glance look a bit esoteric and not directly amenable to practical applications, can most certainly contribute to the solution of a problem of practical interest such as the characterization of separable systems of second-order differential equations. Our characterization of separability will lead to criteria which can be tested on given second-order equations and will show the way to construct coordinates in which the decoupling takes place.

Any result established in terms of vector fields and forms along τ has a translation in terms of geometrical objects living on TM, and vice versa. The advantage of the use of tensor fields along τ is that it allows to state results in a more economical form, as was shown in [10], [12] and [1]. This is to some extent due to the fact that a tensor field along τ can be lifted to a tensor field on TM in many diferent ways, and in this process the number of components is obviously increased. Fairly simple operations on the tensor field along τ thus may correspond to complicated operations on one of its lifts. But the main advantage of the formalism we are using is by far the existence of two derivations similar to covariant derivatives, which are in fact components of a linear connection on the pull-back bundle $\tau^*\tau: \tau^*(TM) \to TM$.

A system of second-order differential equations (SODE) defines two important operators along τ: a derivation ∇ of degree zero, called the dynamical covariant derivative, and a type (1,1) tensor field Φ, called the Jacobi endomorphism. These contain in a way most of the information about the dynamics, and the decomposition of the evolution of various objects under the flow of the SODE in terms of these two operators has been shown to be very useful.

The Jacobi endomorphism plays an important role in the study of the problem we are going to deal with. A theorem giving sufficient conditions for the separability of a SODE was proved in [3] [4] and [11]. This theorem is related to the complete integrability of the system and it is established in terms of an invariant diagonalizable tensor field on TM whose eigenspaces are involutive distributions. Such an invariant tensor, however, is associated to a tensor field U along τ which commutes with the Jacobi endomorphism. It follows that the eigenspaces of U are invariant under Φ, so that the separability of a SODE is a property which can be studied in terms of the Jacobi endomorphism.

The paper is organized as follows. In Section 2 we will present a résumé of some important results of [9] and [10]. In order to make the present paper self-contained, however, many operations will be re-defined here in a more ad hoc manner, leading more directly to the formulae which are needed for developing the present theory. In Section 3 we study distributions along τ, the natural generalization of the concept of distribution on the base manifold, and we give conditions for their integrability. Distributions defined by type (1,1) tensor fields are studied in Section 4, where we will analyze interesting properties of such tensors, such as the diagonalizability in coordinates or the separability. Section 5 contains our main theorem, characterizing separable SODEs. In Section 6 we define two tensors similar to the Nijenhuis tensor on a manifold, and we critically re-examine the separability theorem in [3]. The paper concludes with some simple examples which illustrate the theory.

2. Preliminaries

Let $\pi\colon E \to M$ be a fibre bundle and $\phi\colon N \to M$ a smooth map. A section of E along ϕ is a map $\sigma\colon N \to E$ such that $\pi \circ \sigma = \phi$. If E is a vector bundle then the set of sections along ϕ is a $C^\infty(N)$-module. The most interesting cases are $E = TM$, $(T^*M)^{\wedge p}$ or any other tensor bundle, and then a section of E along ϕ is called a vector field along ϕ, a p-form along ϕ, or a tensor field along ϕ (respectively). The set of vector fields along ϕ is denoted by $\mathcal{X}(\phi)$, and the set of p-forms along ϕ by $\bigwedge^p(\phi)$. In particular, we are interested in the case in which ϕ is the projection of the tangent bundle $\tau\colon TM \to M$. Then, a vector field along τ is a map $X\colon TM \to TM$ such that for $v \in T_qM$, $X(v)$ is a tangent vector to M at q. Similarly, a 1-form along τ is a map $\alpha\colon TM \to T^*M$ such that $\alpha(v)$ is a covector at q. The easiest example of vector field along τ is $X = Y \circ \tau$ for Y a vector field on the base manifold M. We will say that X is a basic vector field and we will not distinguish in the notation between Y and X. Similarly a basic tensor field is a tensor field on M thought of as a tensor field along τ via composition with τ. In natural coordinates (x^i, v^i) in TM the coordinate expresion of a vector field X along τ and a 1-form α along τ is

$$X = X^i(x,v)\frac{\partial}{\partial x^i} \qquad\qquad \alpha = \alpha_i(x,v)\mathrm{d}x^i \tag{2.1}$$

and the coordinate expresion of a (1,1) type tensor field U along τ is

$$U = U^i_j(x,v)\frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j. \tag{2.2}$$

There exists a canonical vector field along τ denoted by $\mathbf{T}$ and defined by the identity map on TM thought of as a section of TM along τ. Its coordinate expression is

$$\mathbf{T} = v^i\frac{\partial}{\partial x^i}. \tag{2.3}$$

In two previous papers [9] [10] we have studied the algebra of derivations of scalar and vector valued forms along τ. Special emphasis has been put in what we called self-dual derivations, that is, derivations D of degree zero satisfying the rule

$$D\langle X,\alpha\rangle = \langle DX,\alpha\rangle + \langle X, D\alpha\rangle. \tag{2.4}$$

They have the property that they extend to derivations of tensor fields along τ by means of the relation

$$\begin{aligned}[DW](\alpha_1,\ldots,\alpha_q,X_1,\ldots,X_p) &= D[W(\alpha_1,\ldots,\alpha_q,X_1,\ldots,X_p)]\\ &\quad-\sum_{i=1}^{q} W(\alpha_1,\ldots,D\alpha_i,\ldots,\alpha_q,X_1,\ldots,X_p)\\ &\quad-\sum_{j=1}^{p} W(\alpha_1,\ldots,\alpha_q,X_1,\ldots,DX_j,\ldots,X_p)\end{aligned} \tag{2.5}$$

where W is a p-covariant q-contravariant tensor field along τ, $X_j \in \mathcal{X}(\tau)$ and $\alpha_i \in \bigwedge^1(\tau)$. In coordinates, a self-dual derivation is determined by its action on $C^\infty(TM)$ and by functions D^i_j, called the coefficients of D, defined by $D(\partial/\partial x^j) = D^i_j\,\partial/\partial x^i$. Then equation (2.4) implies that $D(\mathrm{d}x^i) = -D^i_j\,\mathrm{d}x^j$. One way of constructing self-dual derivations is by extension of derivations of $\mathcal{X}(\tau)$, i.e. $\mathbb{R}$-linear maps from $\mathcal{X}(\tau)$ into itself such that $D(FX) = (DF)X + F(DX)$ and $D(FG) = (DF)G + F(DG)$ for F, $G \in C^\infty(TM)$ and $X \in \mathcal{X}(\tau)$. If D is a derivation of $\mathcal{X}(\tau)$, then equation (2.4) implicitly defines $D\alpha$. We will use this procedure in order to define the fundamental derivations that we are going to use. In fact we will define them as derivations of $\mathcal{X}(\tau)$, and the extension to self-dual derivations should be understood.

In order to give a complete classification of the set of derivations of forms along τ we need an Ehresmann connection in the tangent bundle, i.e. a horizontal subbundle of $T(TM)$. A connection defines an isomorphism between the modules of vector fields along τ and horizontal vector fields on TM, which is called the horizontal lift. Locally a connection is given by n^2 functions $\Gamma^i_j \in C^\infty(TM)$, called the coefficients of the connection, which define the horizontal lift $H_i = \partial/\partial x^i - \Gamma^j_i\partial/\partial v^j$ of the coordinate vector field $\partial/\partial x^i$. In the presence of a connection, any vector field $Z \in \mathcal{X}(TM)$ can be decomposed into a sum $Z = X_1^H + X_2^V$ for X_1, $X_2 \in \mathcal{X}(\tau)$, where the indices H and V refer to horizontal and vertical lift, respectively. In coordinates, if Z has the expression $Z = X^i\partial/\partial x^i + Y^i\partial/\partial v^i$ then X_1 and X_2 are $X_1 = X^i\partial/\partial x^i$ and $X_2 = (Y^i + \Gamma^i_j X^j)\partial/\partial x^i$.

The necessity of a connection does not introduce any extra assumption, since in the problems we are going to deal with a SODE will always be present, and every SODE induces a connection on the tangent bundle. In coordinates, if f^i are the forces defined by a SODE Γ, $f^i = \Gamma(v^i)$, then the coefficients of the connection are $\Gamma^i_j = -(1/2)\partial f^i/\partial v^j$. It follows that the functions $\Gamma^i_{jk} =$

$\partial\Gamma^i_j/\partial v^k$ are symmetric in j, k, that is, the torsion of the connection vanishes. In fact, the vanishing of the torsion is the necessary and sufficient condition for the connection to be defined by a SODE. In what follows we will assume that the connection is the one associated to a SODE.

With the above decomposition in mind, we define two fundamental derivations D^H_X and D^V_X for every $X \in \mathcal{X}(\tau)$, called respectively the horizontal and vertical covariant derivative, by means of the equation

$$[X^H, Y^V] = \{\mathrm{D}^H_X Y\}^V - \{\mathrm{D}^V_Y X\}^H. \tag{2.6}$$

The word 'covariant' means that they are $C^\infty(TM)$-linear in the argument noted as a subscript. It can be shown that a self-dual derivation D can be uniquely decomposed as a sum

$$D = \mathrm{D}^H_{X_1} + \mathrm{D}^V_{X_2} + \mu_U$$

for X_1, $X_2 \in \mathcal{X}(\tau)$ and U a (1,1) tensor field along τ, and where the derivation μ_U is defined by $\mu_U(X) = UX$, for $X \in \mathcal{X}(\tau)$. If X, Y are vector fields along τ with local expression $X = X^i\partial/\partial x^i$ and $Y = Y^i\partial/\partial x^i$, then the coordinate expressions of $\mathrm{D}^H_X Y$ and $\mathrm{D}^V_X Y$ are

$$\begin{aligned} \mathrm{D}^H_X Y &= X^i(H_i Y^j + Y^k\Gamma^j_{ik})\frac{\partial}{\partial x^j} \\ \mathrm{D}^V_X Y &= X^i(V_i Y^j)\frac{\partial}{\partial x^j}, \end{aligned} \tag{2.7}$$

where $V_i = \partial/\partial v^i$ and $\Gamma^i_{jk} = V_k(\Gamma^i_j)$.

From the $C^\infty(TM)$-linearity of the covariant derivatives, it follows that we can define operations D^H and D^V on tensor fields by

$$\begin{aligned} [\mathrm{D}^H W](\alpha_1, \dots, \alpha_q, X_1, \dots, X_p) &= [\mathrm{D}^H_{X_1} W](\alpha_1, \dots, \alpha_q, X_2, \dots, X_p) \\ [\mathrm{D}^V W](\alpha_1, \dots, \alpha_q, X_1, \dots, X_p) &= [\mathrm{D}^V_{X_1} W](\alpha_1, \dots, \alpha_q, X_2, \dots, X_p). \end{aligned}$$

It can be easily shown that a vector field X along τ is basic if and only if $\mathrm{D}^V X$ vanishes. Similarly, a tensor field W along τ is basic iff $\mathrm{D}^V W = 0$.

The commutator of two horizontal lifts and two vertical lifts is given by

$$\begin{aligned} [X^V, Y^V] &= \{[X, Y]_V\}^V \\ [X^H, Y^H] &= \{[X, Y]_H\}^H + \{R(X, Y)\}^V \end{aligned} \tag{2.8}$$

where R is the curvature of the connection and where the horizontal and vertical commutators, $[\cdot,\cdot]_H$ and $[\cdot,\cdot]_V$, are given by

$$[X, Y]_H = \mathrm{D}^H_X Y - \mathrm{D}^H_Y X \tag{2.9}$$

$$[X, Y]_V = \mathrm{D}^V_X Y - \mathrm{D}^V_Y X. \tag{2.10}$$

It is easy to see (e.g. in coordinates) that the horizontal bracket of two basic vector fields coincides with the Lie bracket of such vector fields on M.

The vertical commutator defines a Lie algebra structure on $\mathcal{X}(\tau)$, while the horizontal one does not except if the curvature vanishes, as follows from the equations

$$\begin{aligned}
&[\mathrm{D}^V_X, \mathrm{D}^V_Y] = \mathrm{D}^V_{[X,Y]_V} \\
&[\mathrm{D}^V_X, \mathrm{D}^H_Y] = \mathrm{D}^H_{\mathrm{D}^V_X Y} - \mathrm{D}^V_{\mathrm{D}^H_Y X} + \mu_{\theta(X,Y)} \\
&[\mathrm{D}^H_X, \mathrm{D}^H_Y] = \mathrm{D}^H_{[X,Y]_H} + \mathrm{D}^V_{R(X,Y)} + \mu_{\mathrm{Rie}(X,Y)}.
\end{aligned} \tag{2.11}$$

These expressions define the tensor fields θ and Rie that should be considered as 2-covariant tensor fields taking values in the set of (1,1) tensor fields. Their coordinate expresions are

$$\begin{aligned}
\theta &= \Gamma^k_{jml} \mathrm{d}x^l \otimes \mathrm{d}x^j \otimes \left(\mathrm{d}x^m \otimes \frac{\partial}{\partial x^k}\right) \\
\mathrm{Rie} &= \frac{1}{2}\big(H_k(\Gamma^i_{lj}) - H_l(\Gamma^i_{kj}) + \Gamma^i_{kr}\Gamma^r_{lj} - \Gamma^i_{lr}\Gamma^r_{kj}\big)\mathrm{d}x^k \wedge \mathrm{d}x^l \otimes \left(\mathrm{d}x^j \otimes \frac{\partial}{\partial x^i}\right)
\end{aligned} \tag{2.12}$$

Notice that Rie is a (1,1) tensor-valued 2-form while θ is completely symmetric. From the coordinate expression of Rie it follows that $\mathrm{Rie} = -\mathrm{D}^V R$, i.e. if X, Y, Z are vector fields along τ, then $\mathrm{Rie}(X,Y)Z = -[\mathrm{D}^V_Z R](X,Y)$.

Fundamental derivations of degree 1, d^H and d^V, can be defined as the skew-symmetric part of D^H and D^V, respectively, and they are called the horizontal and vertical exterior differentials. For instance, if $X, Y \in \mathcal{X}(\tau)$, $\alpha \in \bigwedge^1(\tau)$ and U is a (1,1) tensor field along τ then

$$\begin{aligned}
(\mathrm{d}^V X)(Y) &= \mathrm{D}^V_Y X, \quad \text{i.e. } \mathrm{d}^V X = \mathrm{D}^V X \\
(\mathrm{d}^V \alpha)(X,Y) &= \mathrm{D}^V\alpha(X,Y) - \mathrm{D}^V\alpha(Y,X) = [\mathrm{D}^V_X\alpha](Y) - [\mathrm{D}^V_Y\alpha](X) \\
(\mathrm{d}^V U)(X,Y) &= \mathrm{D}^V U(X,Y) - \mathrm{D}^V U(Y,X) = [\mathrm{D}^V_X U](Y) - [\mathrm{D}^V_Y U](X),
\end{aligned} \tag{2.13}$$

and similar relations hold for d^H. The vertical differential is nilpotent, $\mathrm{d}^V \circ \mathrm{d}^V = 0$, and commutes with the horizontal differential, i.e. $\mathrm{d}^V \circ \mathrm{d}^H + \mathrm{d}^H \circ \mathrm{d}^V = 0$. The horizontal differential is nilpotent iff the curvature vanishes.

Another important tensor field associated to the connection is the tension. It is the (1,1) tensor field $\mathbf{t} = -\mathrm{D}^H \mathbf{T}$. Its coordinate expression is

$$\mathbf{t} = (\Gamma^i_j - \Gamma^i_{jk} v^k)\, \mathrm{d}x^j \otimes \frac{\partial}{\partial x^i}. \tag{2.14}$$

As a consequence of the vanishing of the torsion we have $\mathrm{d}^V \mathbf{t} = 0$. Notice that $\mathbf{t} = 0$ iff the connection is linear, i.e. the Γ^i_{jk} do not depend on v^l. In this case the tensor Rie coincides with the Riemann curvature tensor of the linear connection. Notice also that $\mathrm{D}^V\mathbf{T} = I$.

More directly related to the SODE, there is a self-dual derivation ∇ called the dynamical covariant derivative and a (1,1) tensor field Φ called the Jacobi endomorphism. They can be simultaneously defined by the equation

$$[\Gamma, X^H] = \{\nabla X\}^H + \{\Phi(X)\}^V. \tag{2.15}$$

∇ and Φ somehow contain most of the information about the SODE. In fact, on functions ∇ coincides with $\mathcal{L}_\Gamma$. We further have

$$[\Gamma, X^V] = -X^H + \{\nabla X\}^V. \tag{2.16}$$

In coordinates, the coefficients of ∇ are equal to the coefficients of the connection:

$$\nabla\left(\frac{\partial}{\partial x^j}\right) = \Gamma^i_j \frac{\partial}{\partial x^i}, \tag{2.17}$$

so that the dynamical covariant derivative of $X = X^i\partial/\partial x^i$ has the expression

$$\nabla X = (\Gamma X^i + \Gamma^i_j X^j)\frac{\partial}{\partial x^i}. \tag{2.18}$$

The coordinate expression of the Jacobi endomorphism is

$$\Phi = \left(-\frac{\partial f^i}{\partial x^j} - \Gamma^i_k\Gamma^k_j - \Gamma(\Gamma^i_j)\right) \mathrm{d}x^j \otimes \frac{\partial}{\partial x^i}, \tag{2.19}$$

and Φ satisfies the following relation

$$\Phi(X) = R(\mathbf{T}, X) - \mathrm{D}^H_X \nabla\mathbf{T}. \tag{2.20}$$

The exterior differentials of Φ are related to the curvature by means of

$$\mathrm{d}^V\Phi = 3R \qquad\qquad \mathrm{d}^H\Phi = \nabla R. \tag{2.21}$$

We further have the following commutators

$$[\nabla, \mathrm{D}^V_X]Y = \mathrm{D}^V_{\nabla X}Y - \mathrm{D}^H_X Y \tag{2.22}$$

$$[\nabla, \mathrm{D}^H_X]Y = \mathrm{D}^H_{\nabla X}Y + \mathrm{D}^V_{\Phi(X)}Y - [\mathrm{D}^V_Y\Phi](X) - R(X,Y) \tag{2.23}$$

for X and Y vector fields along τ.

Finally, we recall that a SODE Γ is Lagrangian if there exists a function L on TM such that Γ represents the Euler-Lagrange equations defined by L. The conditions for a SODE to be (locally) Lagrangian are known as Helmholtz conditions and can be formulated within the present framework as follows: a SODE Γ is (locally) Lagrangian iff there exists a symmetric 2-covariant tensor field g alond τ satisfying

(1) $\nabla g = 0$
(2) $\mathrm{D}^V g$ is symmetric
(3) Φ is symmetric with respect to g.

The Lagrangian L and the tensor g are related by $g = \mathrm{D}^V\mathrm{D}^V L$.

3. Distributions along τ and integrability

One of the more important tools in Differential Geometry is the notion of distribution (see e.g. [14]). The central result of the theory of distributions is Frobenius theorem, which states that a distribution is integrable iff it is involutive. In such a case there exist coordinates (x^α, y^i) such that the distribution is locally generated by the vector fields $\partial/\partial x^\alpha$, i.e. the integral submanifolds of the distribution are locally given by slices y^i = constant. The concept of distribution along τ generalizes the concept of distribution in the same way as the concept of vector field along τ generalizes the concept of vector field on the base manifold. In this section we will define that notion and we will study conditions for the integrability of such distributions.

Definition 3.1. *A d-*dimensional distribution $\mathcal{D}$ along τ *is a smooth choice of a d-dimensional subspace $\mathcal{D}(v)$ of $T_{\tau(v)}M$ for every $v \in TM$. We say that a vector field X along τ belongs to $\mathcal{D}$, $X \in \mathcal{D}$, if $X(v) \in \mathcal{D}(v)$ for each $v \in TM$.*

By "smooth choice" we mean that in a neighbourhood of each v in TM there exist d independent vector fields along τ which belong to $\mathcal{D}$. Alternatively, we can define a distribution along τ as a vector subbundle of the pull-back bundle $\tau^*\tau: \tau^*(TM) \to TM$. A vector field belongs to $\mathcal{D}$ if it is a section of such subbundle.

Definition 3.2. *A distribution $\mathcal{D}$ along τ is said to be* basic *if there exists a distribution $\mathcal{E}$ on the base manifold M such that $\mathcal{D}(v) = \mathcal{E}(\tau(v))$ for each $v \in TM$. We will say that $\mathcal{D}$ is involutive if it is basic and the distribution $\mathcal{E}$ in the base is involutive. An integral submanifold of $\mathcal{E}$ is said to be an integral submanifold of $\mathcal{D}$.*

It is clear that $\mathcal{D}$ is basic if and only if it is locally generated by basic vector fields along τ with coefficients in $C^\infty(TM)$. In this case, $\mathcal{D}$ is D^V-invariant, i.e. if Z belongs to the distribution and X is any vector field along τ then $\mathrm{D}^V_X Z$ also belongs to $\mathcal{D}$. Indeed, if $Z_1, \ldots, Z_d$ are basic vector fields that span $\mathcal{D}$ then every vector field Z along τ in $\mathcal{D}$ can be written as a linear combination $Z = \rho^\alpha Z_\alpha$, where the coefficients ρ^α are functions on TM. Thus, $\mathrm{D}^V_X Z = (\mathrm{D}^V_X \rho^\alpha) Z_\alpha$ belongs to $\mathcal{D}$. That the converse also holds is shown by the following result.

Proposition 3.3. *A distribution along τ is basic if and only if it is D^V-invariant.*

Proof: Let $\mathcal{D}$ be a D^V-invariant d-dimensional distribution along τ and choose a local basis $\{Z_\alpha\}_{\alpha=1,\ldots,d}$ for $\mathcal{D}$. If $\{X_i\}_{i=1,\ldots,n}$ is a local basis of $\mathcal{X}(M)$, there exist functions Λ^i_α on TM such that $Z_\alpha = \Lambda^i_\alpha X_i$. Since the vector fields Z_α are linearly independent, the matrix Λ^i_α contains a regular $d \times d$ submatrix. We can assume that such submatrix is the upper left corner one Λ^β_α. Denoting by

θ^α_β the inverse of Λ^β_α, $\{Y_\alpha = \theta^\beta_\alpha Z_\beta\}$ is also a local basis for $\mathcal{D}$. We claim that the vector fields Y_α are basic. Indeed, the expression of Y_α in the basis $\{X_i\}$ is $Y_\alpha = X_\alpha + \sum_{i=d+1}^n \sigma^i_\alpha X_i$, with $\sigma^i_\alpha = \theta^\beta_\alpha \Lambda^i_\beta$. For every $X \in \mathcal{X}(\tau)$ we have that $\mathrm{D}^V_X Y_\alpha = \sum_{i=d+1}^n (\mathrm{D}^V_X \sigma^i_\alpha) X_i$ belongs to $\mathcal{D}$ and therefore should be spanned by the Y_α. But this is impossible except if $\mathrm{D}^V_X \sigma^i_\alpha = 0$, that is $\mathrm{D}^V_X Y_\alpha = 0$. □

From a distribution $\mathcal{D}$ along τ we can find distributions on TM by lifting every vector field of $\mathcal{D}$ to TM. Since there are many different lifts we will get different lifted distributions, for instance with the vertical and the horizontal lift. A very interesting lift associated to a SODE is the following. Given a local basis $\{Z_\alpha\}$ for $\mathcal{D}$ we let $J_\Gamma \mathcal{D}$ be the distribution spanned by the vector fields $\{Z^V_\alpha, J_\Gamma Z_\alpha\}$, where $J_\Gamma Z_\alpha = \{Z_\alpha\}^H + \{\nabla Z_\alpha\}^V$. Such kind of distributions has been studied in [7]. As a matter of fact, in [7], an integrable distribution on TM is said to be S-regular if it is regular and locally generated by vertical and complete lifts of vector fields on the base. Since the complete lift of a basic vector field X is just $J_\Gamma X$, it follows that S-regularity of $J_\Gamma \mathcal{D}$ is just a matter of $\mathcal{D}$ being basic. Hence, D^V-invariance of $\mathcal{D}$ is the criterion for $J_\Gamma \mathcal{D}$ to be S-regular and every S-regular distribution on TM can be interpreted this way.

From the fact that the horizontal bracket of two basic vector fields is equal to their Lie bracket we deduce that a basic distribution is involutive if and only if it is closed under the horizontal bracket. It follows that a distribution along τ is involutive if and only if it is D^V-invariant and closed under the horizontal bracket. Notice that this condition is independent of the SODE connection. The dual statement reads as follows: a co-distribution along τ, i.e. a vector subbundle of $\tau^*(T^*M)$, is basic iff it is D^V-invariant, and it is integrable iff it is D^V-invariant and d^H-closed (meaning that the ideal generated by 1-forms in the codistribution is d^H-closed). This easily follows from the general formula $\mathrm{d}^H\alpha(X,Y) = \mathrm{D}^H_X(\alpha(Y)) - \mathrm{D}^H_Y(\alpha(X)) - \alpha([X,Y]_H)$.

When dealing with integrable complementary distributions along τ, the following result ensures the existence of systems of local coordinates for M simultaneously adapted to every distribution.

Lemma 3.4. *Let $\mathcal{D}_A$, $A = 1, \ldots, r$, be complementary distributions along τ, i.e. $T_{\tau(v)}M = \oplus_{A=1}^r \mathcal{D}_A(v)$ $\forall v \in TM$. Assume further that every $\mathcal{D}_A$ is D^V- and $[\cdot,\cdot]_H$-closed and that also every sum of $\mathcal{D}_A$ is $[\cdot,\cdot]_H$-closed. Then for each point $q \in M$ there exists an open neighbourhood $\mathcal{U} \subset M$ of q such that $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_r$, where each $\mathcal{U}_A$ is an open neighbourhood of q in the maximal integral submanifold of $\mathcal{D}_A$ through q.*

Proof: Let $\mathcal{D}_A^\perp$ be the annihilator of $\mathcal{D}_A$ and $\mathcal{D}^*_A = \cap_{B\neq A} \mathcal{D}^\perp_B$. Then $\mathcal{D}^*_A$ are complementary codistributions along τ. Since each $\mathcal{D}_A$ (and each sum of $\mathcal{D}_A$'s) is D^V-invariant and $[\cdot,\cdot]_H$-closed, we have that each $\mathcal{D}^*$ is D^V-invariant and d^H-closed. It follows that $\mathcal{D}^*_A$ is basic and that there exists a codistribution $\mathcal{E}^*_A$

on M such that $\mathcal{D}^*_A(v) = \mathcal{E}^*_A(\tau(v))$. Since $\mathcal{D}^*_A$ is d^H-closed, we have that $\mathcal{E}^*_A$ is d-closed (d being the exterior derivative on M). Frobenius theorem in such a situation yields coordinates which integrate all distributions simultaneously. To see this, it suffices to consider the case $r = 2$, i.e. to look at the case of two complementary codistributions $\mathcal{E}^*_A$ and $\mathcal{E}^*_B$ on M (the more general case following easily by induction). In coordinates $(y^i) = (y^{A\alpha}, y^{B\beta})$ adapted to the codistribution $\mathcal{E}^*_A$, the complementarity ensures that $\mathcal{E}^*_B$ is spanned by 1-forms of the form $\{\mathrm{d}y^{B\beta} + \Lambda^{B\beta}_{A\alpha}\mathrm{d}y^{A\alpha}\}_{\beta=1,\dots,\dim\mathcal{E}^*_B}$. Integrability of $\mathcal{E}^*_B$ subsequently implies the existence of functions $x^{B\beta}$ such that $\mathrm{d}x^{B\beta}$ generates the same codistribution. Putting further $x^{A\alpha} = y^{A\alpha}$ for $\alpha = 1,\dots,\dim\mathcal{E}^*_A$ completes the construction of coordinates with the desired feature. In this way, for our general situation, there exist functions $\{x^{A\alpha}\}_{A=1,\dots,r;\alpha=1,\dots,\dim\mathcal{D}_A}$ on a neighbourhood $\mathcal{U}$ of each point in M such that $\mathcal{D}^*_A$ is generated by $\{\mathrm{d}x^{A\alpha}\}_{\alpha=1,\dots,\dim\mathcal{D}_A}$. The slices $x^{B\beta} =$ constant for $B \neq A$ are the open sets $\mathcal{U}_A$, which are of course integral submanifolds of $\mathcal{D}_A$. $\square$

It is worth to emphasize that a distribution along τ which is closed under the vertical bracket is not necessarily basic. In fact, it can be shown that in such a case the distribution is locally generated by vector fields which pairwise commute with respect to the vertical bracket. This in turns implies that there exist d^V-exact 1-forms along τ generating the annihilator of the distribution.

4. Distributions defined by type (1,1) tensor fields

In the same way as distributions often appear as defined by invariant subspaces of type (1,1) tensor fields on a manifold, distributions along τ appear as defined by invariant subspaces of type (1,1) tensor fields along τ. In particular we are interested in diagonalizable tensor fields and the distributions defined by their eigenspaces.

A tensor field U along τ is said to be diagonalizable if for each $v \in TM$ the endomorphism $U(v)\colon T_{\tau(v)}M \to T_{\tau(v)}M$ is diagonalizable, there exist (locally) smooth functions μ_A (called eigenfunctions) such that $\mu_A(v)$ is an eigenvalue of $U(v)$ and the rank of $\mu_A - U$ is constant. In this case, the eigenspaces of U define distributions along τ, called the eigendistributions of U and denoted by $\mathcal{D}_A$, i.e. $\mathcal{D}_A = \ker(\mu_A - U)$. An integral submanifold of an eigendistribution is said to be an eigenmanifold of U. In what follows, U will denote a diagonalizable tensor field along τ and the dimension of $\mathcal{D}_A$ will be denoted d_A.

Proposition 4.1. *Let D be a self-dual derivation. The eigendistributions of a diagonalizable U are invariant under D if and only if $[DU, U] = 0$. In such a case DU is diagonalizable and its eigenvalues are $D\mu_A$.*

Proof: Since D is self-dual, if $X \in \mathcal{D}_A$ we have

$$DU(X) = D(UX) - U(DX) = (\mu_A - U)DX + (D\mu_A)X, \tag{4.1}$$

and thus

$$[DU, U](X) = DU(\mu_A X) - U(DU(X)) = (\mu_A - U)DU(X) = (\mu_A - U)^2 DX.$$

It follows that DX belongs to $\mathcal{D}_A$ if and only if $[DU, U](X) = 0$. Since U is diagonalizable, a local basis of $\mathcal{X}(\tau)$ can be made up of eigenvectors of U. Equation (4.1) then reduces to $DU(X) = (D\mu_A)X$ and the proposition easily follows. □

The above result enables us to analyse whether the eigendistributions of a tensor field U along τ are $\mathrm{D}^{\scriptscriptstyle V}$-invariant, and hence basic. This will happen iff $[\mathrm{D}^{\scriptscriptstyle V}_X U, U] = 0$ for each $X \in \mathcal{X}(\tau)$. As a consequence of the linearity of the vertical covariant derivative, the commutator $[\mathrm{D}^{\scriptscriptstyle V}_X U, U]$ defines a type (1,2) tensor field $\mathrm{C}^{\scriptscriptstyle V}_U$ along τ:

$$\mathrm{C}^{\scriptscriptstyle V}_U(X, Y) = [\mathrm{D}^{\scriptscriptstyle V}_X U, U](Y). \tag{4.2}$$

An equivalent definition of this tensor is

$$\mathrm{C}^{\scriptscriptstyle V}_U(X, Y) = \mathrm{D}^{\scriptscriptstyle V} U(X, UY) - U\mathrm{D}^{\scriptscriptstyle V} U(X, Y). \tag{4.3}$$

Similar comments and expressions apply to the horizontal counterpart. The coordinates expressions of these tensors are

$$\begin{aligned}
\mathrm{C}^{\scriptscriptstyle V}_U &= \big\{(V_i U^k_l)U^l_j - (V_i U^l_j)U^k_l\big\}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \frac{\partial}{\partial x^k}\\
\mathrm{C}^{\scriptscriptstyle H}_U &= \big\{(H_i U^k_l + U^r_l \Gamma^k_{ir} - U^k_r \Gamma^r_{il})U^l_j\\
&\qquad - (H_i U^l_j + U^r_j \Gamma^l_{ir} - U^l_r \Gamma^r_{ij})U^k_l\big\}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \frac{\partial}{\partial x^k}
\end{aligned} \tag{4.4}$$

We also define tensor fields $\mathrm{H}^{\scriptscriptstyle V}_U$ and $\mathrm{H}^{\scriptscriptstyle H}_U$ along τ, whose skew-symmetric parts bear resemblance to the Haantjes tensor in [5],

$$\begin{aligned}
\mathrm{H}^{\scriptscriptstyle V}_U(X, Y) &= \mathrm{C}^{\scriptscriptstyle V}_U(U^2X, Y) - 2U\mathrm{C}^{\scriptscriptstyle V}_U(UX, Y) + U^2\mathrm{C}^{\scriptscriptstyle V}_U(X, Y)\\
\mathrm{H}^{\scriptscriptstyle H}_U(X, Y) &= \mathrm{C}^{\scriptscriptstyle H}_U(U^2X, Y) - 2U\mathrm{C}^{\scriptscriptstyle H}_U(UX, Y) + U^2\mathrm{C}^{\scriptscriptstyle H}_U(X, Y).
\end{aligned} \tag{4.5}$$

They are important for the characterization of diagonalizable tensor fields which can be put into diagonal form via a coordinate transformation. We will say that a tensor field U along τ is diagonalizable in coordinates if for each q in M there exist local coordinates in an open neighbourhood $\mathcal{U} \subset M$ of q such that U is diagonal in these coordinates.

Theorem 4.2. *A diagonalizable tensor field U along τ is diagonalizable in coordinates if and only if*

(1) $\mathrm{C}^{\scriptscriptstyle V}_U = 0$, *and*
(2) $\mathrm{H}^{\scriptscriptstyle H}_U$ *is symmetric.*

Proof: The first condition implies that each eigendistribution of U is $\mathrm{D}^{\scriptscriptstyle V}$-invariant and hence basic. On the other hand, if $X \in \mathcal{D}_A$ and $Y \in \mathcal{D}_B$, then

$$\mathrm{H}^{\scriptscriptstyle H}_U(X,Y) = (\mu_A - U)^2 \mathrm{C}^{\scriptscriptstyle H}_U(X,Y) = (\mu_A - U)^2(\mu_B - U)^2 \mathrm{D}^{\scriptscriptstyle H}_X Y,$$

so that $\mathrm{H}^{\scriptscriptstyle H}_U$ is symmetric iff $\mathrm{D}^{\scriptscriptstyle H}_X Y - \mathrm{D}^{\scriptscriptstyle H}_Y X = [X,Y]_{\scriptscriptstyle H} \in \mathcal{D}_A + \mathcal{D}_B$. It follows from Lemma 3.4 that the eigendistributions are simultaneously integrable. Thus, in a neighbourhood of each point there exist coordinates $(x^{A\alpha})_{A=1,\dots,r;\alpha=1,\dots,d_A}$ such that $\{\partial/\partial x^{A\alpha}\}_{\alpha=1,\dots,d_A}$ span the eigendistribution $\mathcal{D}_A$. Hence, $U(\partial/\partial x^{A\alpha}) = \mu_A\, \partial/\partial x^{A\alpha}$ and U is diagonal in these coordinates. The proof of the converse is a straighforward calculation. $\square$

Notice that the above characterization is independent of the choice of the connection, provided that, as assumed throughout this paper, it is torsionless (i.e. induced by a SODE).

As a consequence of the relation (2.22) we have the following sufficient conditions for the diagonalizability of a tensor field in coordinates.

Proposition 4.3. *If a diagonalizable tensor field U satisfies*

(1) $\mathrm{C}^{\scriptscriptstyle V}_U = 0$ *and*
(2) $[\nabla U, U] = 0$,

then U is diagonalizable in coordinates. Moreover, in coordinates which diagonalize U the coefficients of the connection $\Gamma^{A\alpha}_{B\beta}$ vanish for $A \neq B$.

Proof: Under the conditions of the Proposition, the eigendistributions of U are $\mathrm{D}^{\scriptscriptstyle V}$- and ∇-invariant. From equation (2.22) we then see that they are also $\mathrm{D}^{\scriptscriptstyle H}$-invariant. It follows that $[X,Y]_{\scriptscriptstyle H}$ belongs to $\mathcal{D}_A + \mathcal{D}_B$ for $X \in \mathcal{D}_A$, $Y \in \mathcal{D}_B$. In view of Lemma 3.4, there exist coordinates $(x^{A\alpha})_{A=1,\dots,r;\alpha=1,\dots,d_A}$ in which U is diagonal. Moreover, since the eigendistributions are ∇-invariant, we have $\nabla(\partial/\partial x^{A\alpha}) = \Gamma^{B\beta}_{A\alpha}\, \partial/\partial x^{B\beta} \in \mathcal{D}_A$, from which we deduce that $\Gamma^{A\alpha}_{B\beta} = 0$ for $A \neq B$. $\square$

Incidentally, one can prove the general relation

$$[U, \mathrm{D}^{\scriptscriptstyle H}_X U] = \nabla[\mathrm{D}^{\scriptscriptstyle V}_X U, U] + \mathrm{D}^{\scriptscriptstyle V}_X[U, \nabla U] - [\mathrm{D}^{\scriptscriptstyle V}_{\nabla X} U, U] - 2[\mathrm{D}^{\scriptscriptstyle V}_X U, \nabla U]$$

from which it again follows that if U satisfies the hypotheses of the above proposition then $[U, \mathrm{D}^{\scriptscriptstyle H}_X U]$ also vanishes.

We now study whether a tensor field U along τ is separable, meaning that U is diagonalizable in coordinates and that each eigenvalue μ_A depends only on the coordinates $(x^{A\alpha}, v^{A\alpha})_{\alpha=1,\dots,d_A}$ corresponding to the eigendistribution $\mathcal{D}_A$. Such a U projects onto every eigenmanifold and is the direct sum of these projections.

Theorem 4.4. *Let U be a diagonalizable tensor field along τ satisfying*

(1) $\mathrm{C}^{V}_U = 0$,
(2) $[\nabla U, U] = 0$,
(3) $\mathrm{d}^V U(UX, Y) = \mathrm{d}^V U(X, UY)$,
(4) $\mathrm{d}^H U(UX, Y) = \mathrm{d}^H U(X, UY)$,

for every X, $Y \in \mathcal{X}(\tau)$. Then U is separable. Conversely, if U is a separable tensor field, then there exist SODE*s such that (1)–(4) hold.*

Proof: From (1) and (2) we have that U is diagonalizable in coordinates. Moreover, if $X \in \mathcal{D}_A$ and $Y \in \mathcal{D}_B$, the D^V-invariance of the eigenspaces yields,

$$\begin{aligned} \mathrm{d}^V U(X, Y) &= (\mathrm{D}^V_X U)Y - (\mathrm{D}^V_Y U)X \\ &= (\mathrm{D}^V_X \mu_B)Y - (\mathrm{D}^V_Y \mu_A)X, \end{aligned} \tag{4.6}$$

where we have used Proposition 4.1. Thus, from (3) we have

$$(\mu_A - \mu_B)\big\{(\mathrm{D}^V_X \mu_B)Y - (\mathrm{D}^V_Y \mu_A)X\big\} = 0. \tag{4.7}$$

Hence, if $A \neq B$ then X and Y are linearly independent, so that $\mathrm{D}^V_X \mu_B = 0$ for every $X \in \mathcal{D}_A$ with $A \neq B$. It follows that μ_B depends only on the velocities $(v^{B\beta})_{\beta=1,\dots,d_B}$.

Similarly, from condition (4) we have that $\mathrm{D}^H_X \mu_B = 0$ for every $X \in \mathcal{D}_A$ with $A \neq B$. Since $[\nabla U, U] = 0$ implies that the coefficients $\Gamma^{A\alpha}_{B\beta}$ of the connection vanish for $A \neq B$, it follows that μ_B ultimately depends on the coordinates $(x^{B\beta}, v^{B\beta})_{\beta=1,\dots,d_B}$ only.

Conversely, if U is separable, for each point $q \in M$ there exists an open neighbourhood $\mathcal{U} \subset M$ of q such that $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_r$, where $\mathcal{U}_A$ is an open neighbourhood of q in the maximal integral submanifold of $\mathcal{D}_A$ through q. Choosing a SODE $\Gamma_A \in \mathcal{X}(T\mathcal{U}_A)$ on every $\mathcal{U}_A$ and defining the SODE $\Gamma \in \mathcal{X}(T\mathcal{U})$ by $\Gamma = \sum_{A=1}^{r} \Gamma_A$ we have that U satisfies the above conditions with respect to the connection associated to Γ. The global result follows by using a partition of unity on the base manifold M. □

It is not difficult to prove that conditions (1) and (3) in this theorem are equivalent to

$$\mathrm{D}^V U(UX, Y) = \mathrm{D}^V U(X, UY) = U\mathrm{D}^V U(X, Y),$$

provided, as we are assuming, that U is diagonalizable.

A particular case of a separable tensor field is the case in which the degenerate eigenvalues are constant.

Theorem 4.5. *Let U be a diagonalizable tensor field along τ satisfying*

(1) $\mathrm{C}^{V}_{U} = 0$,
(2) $[\nabla U, U] = 0$,
(3) $\mathrm{d}^{V} U = 0$,
(4) $\mathrm{d}^{H} U = 0$.

Then U is separable and the degenerate eigenvalues are constant.

Proof: From Theorem 4.4 we know that U is separable. If μ_B is a degenerate eigenvalue, we have that for every pair of linearly independent vector fields X, Y in $\mathcal{D}_B$, equation (4.6) for $A = B$ implies also that $\mathrm{D}^{V}_{X}\mu_B = 0$ for every $X \in \mathcal{D}_B$, and hence $\mathrm{D}^{V}_{X}\mu_B = 0\ \forall X \in \mathcal{X}(\tau)$. Similarly, from condition $\mathrm{d}^{H} U = 0$ we have that $\mathrm{D}^{H}_{X}\mu_B = 0\ \forall X \in \mathcal{X}(\tau)$. It follows that μ_B is constant. □

As we will see in the next section the case in which the degenerate eigenvalues are constant is important for our purposes.

5. Separability of second-order differential equations

In this section we will solve the following problem: Given a system of second-order differential equations

$$\ddot{x}^i = f^i(x^1, \ldots, x^n, \dot{x}^1, \ldots, \dot{x}^n), \qquad i = 1, \ldots, n \tag{5.1}$$

when does there exist a coordinate transformation

$$\bar{x}^i = \phi^i(x^1, \ldots x^n), \qquad i = 1, \ldots, n \tag{5.2}$$

such that in the coordinates $\bar{x}^i$ the system (5.1) decouples into n independent 1-dimensional second-order equations

$$\begin{cases} \ddot{\bar{x}}^1 = \bar{f}^1(\bar{x}^1, \dot{\bar{x}}^1) \\ \qquad \vdots \\ \ddot{\bar{x}}^n = \bar{f}^n(\bar{x}^n, \dot{\bar{x}}^n). \end{cases} \tag{5.3}$$

If this is the case, we will say that the system of second-order equations is separable. In more geometric terms:

Definition 5.1. *A* SODE Γ *is separable if for each point $q \in M$ there exists an open neighbourhood $\mathcal{U} \subset M$ of q of the form $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_n$ and n* SODEs $\Gamma_i \in \mathcal{X}(T\mathcal{U}_i)$ *such that $\Gamma|_{T\mathcal{U}} = \Gamma_1 + \cdots + \Gamma_n$. We say that Γ is globally separable if $\mathcal{U} = M$.*

We will characterize such system in terms of geometric objects directly related to the SODE, namely the induced connection and the Jacobi endomorphism defined by Γ. First we prove that separability is a property depending only on the (torsionless) connection and Φ, and not on the particular SODE.

Proposition 5.2. *Let Γ_1, Γ_2 be two* SODEs *on TM and X the vector field along τ defined by $\Gamma_2 - \Gamma_1 = X^V$. Then*

(1) *Γ_1 and Γ_2 define the same connection and the same Jacobi endomorphism if and only if $\mathrm{D}^V X = 0$ and $\mathrm{D}^H X = 0$.*
(2) *Under these circumstances, Γ_1 is separable if and only if Γ_2 is separable.*

Proof: The connections defined by Γ_1 and Γ_2 coincide iff X is a basic vector field (this is clear in coordinates). In this case the difference between the Jacobi endomorphisms associated to Γ_1 and Γ_2 is

$$\begin{aligned}(\Phi_2 - \Phi_1)(Y) &= R(\mathbf{T}, Y) - \mathrm{D}^H_Y \nabla_2 \mathbf{T} - R(\mathbf{T}, Y) + \mathrm{D}^H_Y \nabla_1 \mathbf{T} \\ &= -\mathrm{D}^H_Y (\nabla_2 - \nabla_1)\mathbf{T} \\ &= -\mathrm{D}^H_Y X.\end{aligned}$$

Thus $\Phi_1 = \Phi_2$ iff $\mathrm{D}^H X = 0$.

Assume that Γ_1 is separable in the coordinates $(x^i)_{i=1,\ldots,n}$. Then the coefficients of the connection $\Gamma^i_j = -(1/2)\partial f^i/\partial v^j$ vanish for $i \neq j$, and the functions Γ^i_{jk} also vanish except for $i = j = k$. Thus $0 = \left\langle \mathrm{d}x^i\, , \mathrm{D}^H_{\partial/\partial x^j} X \right\rangle = H_j(X^i) = \partial X^i/\partial x^j$ for $i \neq j$ and X^i therefore depends on x^i only, which implies that Γ_2 is also separable. □

The procedure which we will follow in the study of separability is to construct coordinates in which the Jacobi endomorphism and the matrix of the coefficients of the connection are diagonal. We will first consider two simple cases, to which the general case will subsequently be reduced.

As a preliminary remark, observe first that for affine connections, whose coefficients are affine functions of the velocities, the tension is a basic tensor field. Conversely, if the tension is basic and the connection coefficients are assumed to be smooth on the zero section, they must be of the form

$$\Gamma^i_j(x, v) = \mathbf{t}^i_j(x) + \Gamma^i_{jk}(x)v^k. \tag{5.4}$$

The functions Γ^i_{jk} then define a linear connection (with coefficients $\Gamma^i_{jk}(x)v^k$), whose Riemann curvature is equal to the Rie tensor of the affine connection. It follows that if the curvature R of the affine connection vanishes then the curvature of the linear connection also vanishes.

Proposition 5.3. *Let Γ be a* SODE *such that $\Phi = \mu I$ and $\mathbf{t} = \beta I$, with $\mu, \beta \in \mathbf{R}$. Then Γ is separable.*

Proof: Since the tension is a basic tensor field the connection is affine. In view of the relation $\mathrm{d}^V \Phi = 3R$ we have that the curvature vanishes, and the associated linear connection is flat. It follows that there exist coordinates in which the coefficients of the linear connection vanish. In these coordinates the matrix of the tension is $\mathbf{t}^i_j = \Gamma^i_j$, and from $\mathbf{t} = \beta I$ we find $\Gamma^i_j = \beta\delta^i_j$, which implies $\partial f^i/\partial v^j = 0$ for $i \neq j$. Furthermore, since Φ is diagonal we have for $i \neq j$,

$$0 = \Phi^i_j = -\frac{\partial f^i}{\partial x^j} - \Gamma^i_k\Gamma^k_j - \Gamma(\Gamma^i_j) = -\frac{\partial f^i}{\partial x^j}.$$

We conclude that f^i is a function of the corresponding x^i and v^i, and thus Γ is separable. □

Note that in the adapted coordinates of the proof, $\Phi = \mu I$ and $\mathbf{t} = \beta I$ imply that the forces are of the form

$$f^i = -(\mu + \beta^2)x^i - 2\beta v^i + \gamma^i, \qquad \gamma^i \in \mathbf{R} \tag{5.5}$$

which physically corresponds to an isotropic harmonic oscillator submerged in an isotropic fluid.

Remark: Smoothness on the zero section of the SODE Γ apparently plays a key role in the above argumentation (and therefore also in the main theorem which is to follow). In a global setting, this is of course not a supplementary restriction. If, for a local application of the criteria of Theorem 5.6, Γ is not required to be smooth on the zero section, the fact that the tension is basic in principle would allow the connection coefficients to contain a part which is merely homogeneous of degree 1 in the velocities (and not linear). In such a case, however, we can never separate the system. Indeed, in coordinates which would separate the equations, non-vanishing connection coefficients can at most depend on one velocity coordinate and as such necessarily are linear. But then they would also be linear in any other chart, contradicting the original assumption.

We next consider the case $\Phi = \mu I$ and study the conditions which must be imposed on the tension for ensuring separability of the SODE.

Lemma 5.4. *If $\Phi = \mu I$, $\mu \in \mathrm{R}$, then*

(1) $\nabla \mathbf{t} = 0$
(2) $\mathrm{d}^H \mathbf{t} = 0$.

Proof: Since the curvature vanishes we have $\Phi = -\mathrm{d}^H\nabla\mathbf{T}$ and equation (2.23) yields, taking into account that $\mathrm{D}^V\mathbf{T} = I$ and $\mathrm{D}^V\Phi = 0$,

$$\begin{aligned}\Phi(X) &= -\mathrm{D}^H_X\nabla\mathbf{T}\\ &= -\nabla\mathrm{D}^H_X\mathbf{T} + \mathrm{D}^H_{\nabla X}\mathbf{T} + \mathrm{D}^V_{\Phi(X)}\mathbf{T}\\ &= \nabla\mathbf{t}(X) - \mathbf{t}(\nabla X) + \Phi(X)\\ &= (\nabla\mathbf{t})(X) + \Phi(X).\end{aligned}$$

It follows that $\nabla \mathbf{t} = 0$. The property $\mathrm{d}^H \mathbf{t} = 0$ follows from the fact that $\mathrm{d}^H \circ \mathrm{d}^H = 0$ as a result of of the vanishing of the curvature. □

Proposition 5.5. *Let* Γ *be a* SODE *such that* $\Phi = \mu I$ *with* $\mu \in \mathrm{R}$. *Then* Γ *is separable if and only if* $\mathbf{t}$ *is diagonalizable and* $\mathrm{C}^V_{\mathbf{t}} = 0$.

Proof: From the preceding lemma, recalling further that $\mathrm{d}^V \mathbf{t} = 0$, it follows that the tension satisfies all hypotheses of Theorem 4.5. As observed in Theorem 4.4, in the coordinates which diagonalize $\mathbf{t}$, $\Gamma^{A\alpha}_{B\beta} = 0$ for $A \neq B$. Moreover, since Φ is diagonal we find for $A \neq B$ that $0 = \Phi^{A\alpha}_{B\beta} = -\partial f^{A\alpha}/\partial x^{B\beta}$. Thus $f^{A\alpha}$ depends only on $(x^{A\gamma}, v^{A\gamma})_{\gamma=1,\ldots,d_A}$ and Γ projects onto every eigenmanifold. If d_A is greater than 1, we further know that the corresponding degenerate eigenvalue of $\mathbf{t}$ is constant. Thus, for such a subsystem, Proposition 5.3 applies and ensures the total separation of Γ. The proof of the converse is straightforward. □

Theorem 5.6. *A* SODE Γ *is separable if and only if the following conditions hold*

(1) Φ *is diagonalizable*
(2) $[\nabla\Phi, \Phi] = 0$
(3) $\mathrm{C}^V_{\Phi} = 0$
(4) $R = 0$
(5) $\mathbf{t}$ *is diagonalizable*
(6) $\mathrm{C}^V_{\mathbf{t}} = 0$.

Proof: From $\mathrm{d}^V \Phi = 3R$ and $\mathrm{d}^H \Phi = \nabla R$ it follows that Φ satisfies all the assumptions of Theorem 4.5. Hence, in coordinates $(x^{A\alpha})$ which diagonalize Φ we have for $A \neq B$ that $\Gamma^{A\alpha}_{B\beta} = 0$ and $\Phi^{A\alpha}_{B\beta} = 0$, implying again that Γ projects onto every eigenmanifold. On each multidimensional eigenmanifold Φ is a constant multiple of the identity and thus Proposition 5.5 applies. The 'only if' part is easy to verify. □

Notice that conditions (1)–(3) already entail a form of partial separability of the SODE, each partial SODE corresponding to each eigenspace of Φ. Indeed, (1)–(3) imply, in accordance with Proposition 4.3, that there exist coordinates in which Φ is diagonal and $\Gamma^{A\alpha}_{B\beta} = 0$ for $A \neq B$. As repeatedly argued above, this is enough to conclude that the $f^{A\alpha}$ depend only on $(x^{A\gamma}, v^{A\gamma})$.

In practice, the most difficult condition to analyze is the matter of diagonalizability. If the SODE is defined by a Lagrangian such that the associated symmetric tensor g is positive definite then Φ is automatically diagonalizable, because Φ is symmetric with respect to g. Nevertheless, there is no general argument that ensures the diagonalizability of $\mathbf{t}$.

The above argumentation gives us the key for analyzing the separability of the Lagrangian.

Proposition 5.7. *Let Γ be a* SODE *satisfying conditions (1)–(3) of Theorem 5.6 and L a Lagrangian for Γ. For each point $q \in M$ there exists a neighbourhood $\mathcal{U}$ of q of the form $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_r$ such that L is gauge equivalent to a separated Lagrangian $L_1 + \cdots + L_r$, where $L_A \in C^\infty(T\mathcal{U}_A)$.*

Proof: Conditions (1)–(3) imply the existence of coordinates $(x^{A\alpha})$ adapted to the eigendistributions $\mathcal{D}_A$ of Φ and that Γ is partially separable $\Gamma = \sum_{A=1}^r \Gamma_A$, with Γ_A a local SODE on the A-th eigenmanifold of Φ. Let g be the symmetric tensor along τ associated to the Lagrangian, $g = \mathrm{D}^V\mathrm{D}^V L$. Then g satisfies Helmholtz conditions: (a) Φ is g-symmetric, (b) $\mathrm{D}^V g$ is symmetric, and (c) $\nabla g = 0$. From (a) it follows that the eigendistributions of Φ are g-orthogonal. Thus, g is a sum $g = \sum_{A=1}^r g_A$, with g_A the restriction of g to $\mathcal{D}_A$.

We now prove that g_A depends only on the coordinates $(x^{A\alpha}, v^{A\alpha})_{\alpha=1,\dots,d_A}$. From (b) and the block diagonal structure of g we have,

$$\mathrm{D}^V g(\frac{\partial}{\partial x^{A\alpha}}, \frac{\partial}{\partial x^{B\beta}}, \frac{\partial}{\partial x^{B\gamma}}) = \mathrm{D}^V g(\frac{\partial}{\partial x^{B\beta}}, \frac{\partial}{\partial x^{A\alpha}}, \frac{\partial}{\partial x^{B\gamma}}) = 0,$$

which is equivalent to

$$\frac{\partial}{\partial v^{A\alpha}}\, g(\frac{\partial}{\partial x^{B\beta}}, \frac{\partial}{\partial x^{B\gamma}}) = 0.$$

From equation (2.22), which by duality also holds for g, it is easy to show that (b) and (c) imply that $\mathrm{D}^H g$ is also symmetric. Using the preceding result and the fact that in the coordinates under consideration $\Gamma^{A\alpha}_{B\beta} = 0$ except for $A = B$, it then follows in the same way that

$$\frac{\partial}{\partial x^{A\alpha}}\, g(\frac{\partial}{\partial x^{B\beta}}, \frac{\partial}{\partial x^{B\gamma}}) = 0.$$

Thus, g_B projects onto the B-th eigenmanifold.

It is easy to see that every g_A satisfies Helmholtz conditions for the partial SODE Γ_A, so that there exists a local Lagrangian L_A for Γ_A such that $g_A = \mathrm{D}^V\mathrm{D}^V L_A$. Thus $L' = \sum_{A=1}^r L_A$ is a local Lagrangian for Γ and $g' = \mathrm{D}^V\mathrm{D}^V L' = \sum_{A=1}^r g_A = g$. Hence, $\mathrm{D}^V\mathrm{D}^V(L - L') = 0$ and L is gauge equivalent to L'. □

As a final remark note that, in general, the partial Lagrangians L_A need not be further separable even if the SODE is. This point is illustrated by the non-standard Lagrangian $L = v^1 v^2 - q^1 q^2$ for the two-dimensional harmonic oscillator.

6. Horizontal and vertical covariant Nijenhuis tensors

Our analysis of the integrability of distributions along τ defined by tensor fields was based on the vanishing of the tensors C^{V}_{U} and C^{H}_{U}. In the theory of distributions on a manifold the integrability of the eigendistributions of a tensor field is usually established in terms of the Nijenhuis tensor [8] [5]. In this section we define two tensor fields with similar properties to the properties satisfied by the Nijenhuis tensor. In order to do that we study the Nijenhuis tensor of the horizontal lift U^{H} of a tensor field along τ, defined by

$$U^{H}(X^{H}) = U(X)^{H} \qquad U^{H}(X^{V}) = U(X)^{V} \tag{6.1}$$

for every vector field X along τ, that is, U^{H} is the diagonal sum of two identical blocks equal to U. More general liftings of tensor fields along τ can be considered but the ensuing Nijenhuis tensors then become too complicated.

We recall that the Nijenhuis tensor of a type (1,1) tensor field T on a manifold P is defined by

$$N_T(V,W) = [TV,TW] - T[TV,W] - T[V,TW] + T^2[V,W] \tag{6.2}$$

for V, $W \in \mathcal{X}(P)$, or equivalently by

$$N_T(V,W) = (\mathcal{L}_{TV}T)W - (T\circ\mathcal{L}_V T)W. \tag{6.3}$$

In the case $P = TM$ and $T = U^{H}$, using (2.6) and (2.8) it is easy to see that for X, $Y \in \mathcal{X}(\tau)$ we have the following values for the Nijenhuis tensor of U^{H}

$$\begin{aligned}
N_{U^H}(X^V,Y^V) &= \{\mathrm{N}^V_U(X,Y) - \mathrm{N}^V_U(Y,X)\}^V\\
N_{U^H}(X^H,Y^V) &= \{\mathrm{N}^H_U(X,Y)\}^V - \{\mathrm{N}^V_U(Y,X)\}^H\\
N_{U^H}(X^H,Y^H) &= \{\mathrm{N}^H_U(X,Y) - \mathrm{N}^H_U(Y,X)\}^H + \{R_U(X,Y)\}^V
\end{aligned} \tag{6.4}$$

where N^H_U, N^V_U and R_U are defined by

$$\begin{aligned}
\mathrm{N}^H_U(X,Y) &= \mathrm{D}^H_{UX}(UY) - U\mathrm{D}^H_{UX}Y - U\mathrm{D}^H_X(UY) + U^2\mathrm{D}^H_X Y\\
\mathrm{N}^V_U(X,Y) &= \mathrm{D}^V_{UX}(UY) - U\mathrm{D}^V_{UX}Y - U\mathrm{D}^V_X(UY) + U^2\mathrm{D}^V_X Y\\
R_U(X,Y) &= R(UX,UY) - UR(UX,Y) - UR(X,UY) + U^2R(X,Y).
\end{aligned} \tag{6.5}$$

We will call N^H_U and N^V_U the *horizontal and vertical covariant Nijenhuis tensor of U*, respectively. Equivalent definitions of these tensors are

$$\begin{aligned}
\mathrm{N}^H_U(X,Y) &= (\mathrm{D}^H_{UX}U)Y - (U\circ\mathrm{D}^H_X U)Y\\
\mathrm{N}^V_U(X,Y) &= (\mathrm{D}^V_{UX}U)Y - (U\circ\mathrm{D}^V_X U)Y,
\end{aligned} \tag{6.6}$$

which resemble the definition of the Nijenhuis tensor on a manifold, or

$$\begin{aligned} \mathrm{N}^{H}_{U}(X,Y) &= \mathrm{D}^{H}U(UX,Y) - U\mathrm{D}^{H}U(X,Y) \\ \mathrm{N}^{V}_{U}(X,Y) &= \mathrm{D}^{V}U(UX,Y) - U\mathrm{D}^{V}U(X,Y), \end{aligned} \tag{6.7}$$

showing the similarities with the definition of the tensors C^{H}_{U} and C^{V}_{U}. Their coordinate expressions are

$$\begin{aligned} \mathrm{N}^{V}_{U} &= \big\{(V_l U^k_j)U^l_i - (V_i U^l_j)U^k_l\big\}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \frac{\partial}{\partial x^k} \\ \mathrm{N}^{H}_{U} &= \big\{(H_l U^k_j + U^r_j \Gamma^k_{lr} - U^k_r \Gamma^r_{lj})U^l_i \\ &\qquad - (H_i U^l_j + U^r_j \Gamma^l_{ir} - U^l_r \Gamma^r_{ij})U^k_l\big\}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \frac{\partial}{\partial x^k}. \end{aligned} \tag{6.8}$$

Note that N^{H}_{U} and N^{V}_{U} are not skew-symmetric. Their skew-symmetric part $\mathcal{N}^{H}_{U}(X,Y)$ and $\mathcal{N}^{V}_{U}(X,Y)$, respectively, are defined in terms of the horizontal and vertical brackets as follows

$$\begin{aligned} \mathcal{N}^{H}_{U}(X,Y) &= [UX,UY]_{H} - U\,[UX,Y]_{H} - U\,[X,UY]_{H} + U^2\,[X,Y]_{H} \\ \mathcal{N}^{V}_{U}(X,Y) &= [UX,UY]_{V} - U\,[UX,Y]_{V} - U\,[X,UY]_{V} + U^2\,[X,Y]_{V}\,. \end{aligned} \tag{6.9}$$

In terms of these tensors, the skew-symmetric part $\mathcal{H}^{H}_{U}$ of the tensor field H^{H}_{U} defined in Section 4 can be expressed as

$$\mathcal{H}^{H}_{U}(X,Y) = \mathcal{N}^{H}_{U}(UX,UY) - U\mathcal{N}^{H}_{U}(UX,Y) - U\mathcal{N}^{H}_{U}(X,UY) + U^2\mathcal{N}^{H}_{U}(X,Y), \tag{6.10}$$

and a similar expression holds for the vertical counterpart.

A relation between the Nijenhuis tensor fields and the tensors C^{V}_{U} and C^{H}_{U} is given by the next proposition.

Proposition 6.1. *The following relations hold*

$$\begin{aligned} \mathrm{N}^{H}_{U}(X,Y) - \mathrm{C}^{H}_{U}(Y,X) &= \mathrm{d}^{H}U(UX,Y) - U\mathrm{d}^{H}U(X,Y) \\ \mathrm{N}^{V}_{U}(X,Y) - \mathrm{C}^{V}_{U}(Y,X) &= \mathrm{d}^{V}U(UX,Y) - U\mathrm{d}^{V}U(X,Y). \end{aligned} \tag{6.11}$$

Proof: The proof is by direct computation; for X, $Y \in \mathcal{X}(\tau)$ we have

$$\begin{aligned} \mathrm{d}^{H}U(UX,Y) - U\mathrm{d}^{H}U(X,Y) &= (\mathrm{D}^{H}_{UX}U)Y - (\mathrm{D}^{H}_{Y}U)UX \\ &\quad - (U \circ \mathrm{D}^{H}_{X}U)Y + (U \circ \mathrm{D}^{H}_{Y}U)X \\ &= (\mathrm{D}^{H}_{UX}U)Y - (U \circ \mathrm{D}^{H}_{X}U)Y \\ &\quad - (\mathrm{D}^{H}_{Y}U \circ U)X - U \circ \mathrm{D}^{H}_{Y}U)X \\ &= \mathrm{N}^{H}_{U}(X,Y) - \mathrm{C}^{H}_{U}(Y,X). \end{aligned}$$

The second relation is proved in a similar way. □

Corollary 6.2. *Let U be a tensor field such that $\mathrm{d}^{H}U = 0$ (resp. $\mathrm{d}^{V}U = 0$). Then $\mathrm{N}^{H}_{U} = 0$ (resp. $\mathrm{N}^{V}_{U} = 0$) if and only if $\mathrm{C}^{H}_{U} = 0$ (resp. $\mathrm{C}^{V}_{U} = 0$).* □

As a consequence, some of the theorems in Sections 4 and 5 can be reformulated in terms of the covariant Nijenhuis tensors. For instance, in Theorem 4.5 condition (1) can be replaced by $\mathrm{N}^{V}_{U} = 0$, in Proposition 5.5 condition $\mathrm{C}^{V}_{\mathbf{t}} = 0$ can be replaced by $\mathrm{N}^{V}_{\mathbf{t}} = 0$, and in Theorem 5.6 conditions (3) and (6) can be replaced by $\mathrm{N}^{V}_{\Phi} = 0$ and $\mathrm{N}^{V}_{\mathbf{t}} = 0$ (but notice that the remarks following this last theorem are then no longer true). For two type (1,1) tensor fields U and W along τ, we can introduce covariant Nijenhuis type tensor fields which (as in the standard theory) reduce to the definitions (6.6) (up to a factor 2), when U and W coincide. Explicitly we put

$$\mathrm{N}^{V}_{U,W}(X,Y) = \big(\mathrm{D}^{V}_{UX}W + \mathrm{D}^{V}_{WX}U - W\circ \mathrm{D}^{V}_{X}U - U\circ \mathrm{D}^{V}_{X}W\big)(Y), \tag{6.12}$$

and similarly for $\mathrm{N}^{H}_{U,W}$.

Proposition 6.3. *The following relations hold*

$$\nabla \mathrm{N}^{V}_{U} = \mathrm{N}^{V}_{U,\nabla U} - \mathrm{N}^{H}_{U} \tag{6.13}$$

$$\begin{aligned} 3R_U(X,Y) = {}& \mathrm{N}^{V}_{U}(X,\Phi Y) - \mathrm{N}^{V}_{U}(Y,\Phi X) - \Phi \mathrm{N}^{V}_{U}(X,Y) + \Phi \mathrm{N}^{V}_{U}(Y,X)\\ &+ (\mathrm{D}^{V}_{UX}[\Phi,U])Y - (U\circ \mathrm{D}^{V}_{X}[\Phi,U])Y\\ &- (\mathrm{D}^{V}_{UY}[\Phi,U])X - (U\circ \mathrm{D}^{V}_{Y}[\Phi,U])X - [\Phi,U]\{\mathrm{d}^{V}U(X,Y)\}. \end{aligned} \tag{6.14}$$

Proof: The proof is by direct computation. For the first one use equation (2.22). For the second one use $\mathrm{d}^{V}\Phi = 3R$. □

For a diagonalizable tensor field U, if $X \in \mathcal{D}_A$ and $Y \in \mathcal{D}_B$ we have the following values of $\mathrm{N}^{V}_{U}(X,Y)$ and $R_U(X,Y)$:

$$\begin{aligned} \mathrm{N}^{V}_{U}(X,Y) &= (\mu_A - U)(\mu_B - U)\mathrm{D}^{V}_{X}Y + (\mu_A-\mu_B)(\mathrm{D}^{V}_{X}\mu_B)\,Y\\ R_U(X,Y) &= (\mu_A - U)(\mu_B - U)R(X,Y). \end{aligned} \tag{6.15}$$

It follows that N^{V}_{U} vanishes if and only if

$$\begin{aligned} &\mathrm{D}^{V}_{X}Y \in \mathcal{D}_A + \mathcal{D}_B \qquad && \forall X \in \mathcal{D}_A,\ \forall\, Y \in \mathcal{D}_B\\ &\mathrm{D}^{V}_{X}\mu_B = 0 && \forall X \in \mathcal{D}_A,\ A \neq B, \end{aligned} \tag{6.16}$$

and that $R_U = 0$ iff $R(X,Y) \in \mathcal{D}_A + \mathcal{D}_B$.

In the light of our previous results we can now re-examine the characterization of separable systems given by Ferrario *et al* in [3], which in fact inspired the present work. An important difference between our main Theorem 5.6 and the theorem proved by Ferrario *et al* is that the latter depends on the existence of a tensor field T on TM with certain properties. As such it should be possible to relate it to results discussed in Section 4. The question how to obtain a T with the desired properties is a different matter and has only clearly been answered for second-order systems with a multi-Lagrangian description. Theorem 5.6 on the contrary does not a priori rely on the availability of a Lagrangian. The theorem stated by Ferrario *et al* in fact was not quite correct. A corrected version of it would read as follows:

Let Γ be a SODE *and T be a diagonalizable tensor field on TM with doubly degenerate eigenvalues such that $S \circ T = T \circ S$. Assume further that $\mathcal{L}_\Gamma T = 0$, that the Nijenhuis tensor of T vanishes and that the Nijenhuis bracket of S and T takes vertical values. Then Γ is separable.*

Under the conditions stated in the theorem, the tensor T is of the form $T = U^{H}$ for some tensor field U along τ which has n different eigenvalues and satisfies $\nabla U = 0$ and $[\Phi, U] = 0$, by virtue of the invariance of T. The vanishing of the Nijenhuis tensor is equivalent to $\mathrm{N}_U^{H} = 0$, $\mathrm{N}_U^{V} = 0$ and $R_U = 0$. But in view of Proposition 6.3 the vanishing of N_U^{V} implies $\mathrm{N}_U^{H} = 0$ and $[\Phi, U] = 0$ further yields $R_U = 0$. The condition on the Nijenhuis bracket of S and T can be shown to be equivalent to $\mathrm{d}^{V} U = 0$.

Summarizing, the assumptions in the theorem of Ferrario *et al* essentially mean that U satisfies

(1) U has n different eigenvalues,
(2) $\nabla U = 0$,
(3) $\mathrm{N}_U^{V} = 0$,
(4) $\mathrm{d}^{V} U = 0$,
(5) $[\Phi, U] = 0$.

Since as a consequence of (2.22) we have $[\mathrm{d}^{V}, \nabla] = \mathrm{d}^{H}$, it follows that also $\mathrm{d}^{H} U$ vanishes, and thus U satisfies all conditions of Theorem 4.5. Therefore, U is separable. Moreover, condition (5) implies that Φ is also diagonalizable and the eigenspaces of Φ and U coincide. Thus Φ is diagonal in the coordinates in which U is separated. Since in these coordinates the matrix of the coefficients of the connection is diagonal, the separability of Γ follows.

With our present understanding of the results of Section 4, the preceding theorem can in fact be generalized as follows:

Proposition 6.4. *Let U be a diagonalizable tensor field along τ satisfying*

(1) *U has n different eigenvalues,*

(2) $[\nabla U, U] = 0$,
(3) $\mathrm{C}_U^V = 0$, *and*
(4) $[\Phi, U] = 0$.

Then Γ *is separable.*

Proof: (2) and (3) imply that U is diagonalizable in coordinates. In the coordinates in which U is diagonal, (1) and (4) imply that Φ is also diagonal and $\Gamma^i_j = 0$ for $i \neq j$. Thus Γ is separable. □

Notice that if U has $r < n$ different eigenvalues, the result remains true for a partial separability of the SODE.

7. EXAMPLES

As a first example we consider the class of so called mechanical systems. They are described by a Lagrangian L on $T\mathbf{R}^n$ of the form

$$L = \frac{1}{2}\delta_{ij} v^i v^j - V(x^1, \ldots, x^n).$$

The SODE corresponding to the Euler-Lagrange equations is

$$\Gamma = \sum_{i=1}^{n} \left(v^i \frac{\partial}{\partial x^i} - \frac{\partial V}{\partial x^i}\frac{\partial}{\partial v^i} \right).$$

The induced connection is flat and the coefficients of the connection vanish so that $\mathbf{t} = 0$. The Jacobi endomorphism is given by

$$\Phi^i_j = \frac{\partial^2 V}{\partial x^i \partial x^j}.$$

It is a symmetric tensor and therefore is diagonalizable. Since $\mathrm{D}^V \Phi = 0$ we also have $\mathrm{C}^V_\Phi = 0$. It follows that Γ is separable if and only if $[\nabla\Phi, \Phi] = 0$, which in coordinates reads

$$\sum_{l=1}^{n} \frac{\partial^3 V}{\partial x^i \partial x^k \partial x^l}\frac{\partial^2 V}{\partial x^l \partial x^j} = \sum_{l=1}^{n} \frac{\partial^3 V}{\partial x^j \partial x^k \partial x^l}\frac{\partial^2 V}{\partial x^l \partial x^i}.$$

The second example we consider is the second-order equation in $\mathbf{R}^2$

$$\begin{cases} \ddot{x}_1 = -c_1 x_1 + b x_1^2 - a x_2^2 \\ \ddot{x}_2 = -c_2 x_2 - 2m x_1 x_2, \end{cases}$$

where a, b, m, c_1 and c_2 are real numbers. This system generalizes the Hénon-Heiles model, to which it reduces if all parameters are equal to 1. Its complete integrability has been discussed recently in [13]. The coefficients of the connection vanish, and the Jacobi endomorphism is given by

$$\Phi = \begin{pmatrix} c_1 - 2bx_1 & 2ax_2 \\ 2mx_2 & c_2 + 2mx_1 \end{pmatrix}.$$

Since Φ is basic $\mathrm{D}^V\Phi = 0$ and $R = 0$. Moreover $[\nabla\Phi, \Phi] = 0$ iff either $a = m = 0$ or $c_2 = c_1$ and $m+b = 0$. In the first case the system is already decoupled in the coordinates (x_1, x_2). In the second case, one verifies that Φ is diagonalizable iff $ab < 0$ and the system separates in coordinates $(X_1 = bx_1 + sx_2, X_2 = bx_1 - sx_2)$, where $s^2 = -ab$.

As a third example we will analyze the separability of the system

$$\begin{cases} \ddot{x}_1 = f(\dot{x}_2) \\ \ddot{x}_2 = 0, \end{cases}$$

that appears in Douglas' paper on the inverse problem of Lagrangian mechanics [2]. For this system $\Phi = 0$, so that we only have to analyze the conditions on $\mathbf{t}$, which has the expression

$$\mathbf{t} = -\frac{1}{2}\begin{pmatrix} 0 & f'(\dot{x}_2) - f''(\dot{x}_2)\dot{x}_2 \\ 0 & 0 \end{pmatrix}$$

This tensor is diagonalizable iff it vanishes, i.e. if f' is homogeneous of degree 1. It follows that the system is separable if and only if f is of the form $f(\dot{x}_2) = a\dot{x}_2^2 + b$ for some constants a, $b \in \mathbf{R}$, and the SODE is a spray plus a constant vertical vector field. A pair of parallel 1-forms with respect to the induced (linear) connection is easily found:

$$\alpha_1 = \mathrm{d}x_1 - ax_2\mathrm{d}x_2 = \mathrm{d}(x_1 - \frac{1}{2}ax_2^2)$$
$$\alpha_2 = \mathrm{d}x_2,$$

so that in coordinates $(X_1 = x_1 - \frac{1}{2}ax_2^2, X_2 = x_2)$ the system decouples.

As a final application we will indirectly prove a theorem by Hojman and Ramos [6] which states that a two-dimensional mechanical system admits alternative Lagrangians if and only if it is separable. Knowing that alternative Lagrangians lead to a suitable type (1,1) tensor field on TM, which in turn relates to a tensor field along τ, we will cover the theorem in [6] by showing that a two-dimensional mechanical system admits a diagonalizable type (1,1) tensor field along τ such that $[U, \Phi] = 0$ and $[\nabla U, U] = 0$ if and only if it is separable (of course, we exclude the case $U = fI$). We can distinguish two cases. If Φ is a multiple of the identity then $\partial^2 V/\partial x_1 \partial x_2 = 0$ and $\partial^2 V/\partial x_1 \partial x_1 = \partial^2 V/\partial x_2 \partial x_2$ from which it follows that the system is a harmonic oscillator (plus constant forces). In the opposite case, i.e. Φ is not a multiple of the identity, there are two 1-dimensional eigendistributions of Φ which are basic since Φ is basic. Since $[\Phi, U] = 0$ we have that the eigendistributions of Φ are also the eigendistributions of U. From $[\nabla U, U] = 0$ it follows that they are integrable and hence the system is separable.

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