SYMMETRIES OF SECOND-ORDER DIFFERENTIAL EQUATIONS AND DECOUPLING

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Abstract. Necessary and sufficient conditions are discussed, which characterize complete separability of second-order ordinary differential equations via symmetry properties of the system.

1. Introduction

When considering systems of second-order ordinary differential equations, it very rarely happens that special coordinates can be found in which the system will decouple into completely separate equations. Intuitively, one expects that the existence of such coordinates may be related to symmetry properties of the given system. Clearly, however, separability cannot be explained by a standard group action, originating from Lie point symmetries of the system. As a matter of fact, one could easily compose an example by mixing up a number of separate equations which do not have point symmetries. Also, the mere availability of a sufficient number of more general symmetries will not be enough. To illustrate this point, consider for example the well-known Hénon-Heiles equations

$$\begin{aligned} \ddot{q}_1 &= b\, {q_1}^2 - a\, {q_2}^2 \\ \ddot{q}_2 &= -2a\, q_1 q_2 \end{aligned}$$

and let us agree that, for simplicity, all considerations of this paper will be restricted to a strictly time-independent framework. This entails, in relation to standard works on symmetries and differential equations such as [8, 2, 9], that generators of symmetries will always appear in what is sometimes called their ‘evolutionary form’. In this sense, insisting that a and b should not be zero, the above system has no point symmetries. As a next step in an algorithmic process, one can look for generators whose leading coefficients depend linearly on the first derivatives. Then, there always is the trivial symmetry $X_1 = \dot{q}_1\,(\partial/\partial q_1) + \dot{q}_2\,(\partial/\partial q_2)$ (equivalent to time translation invariance). A second generator is detected only for two special parameter combinations: for $b = -a$ we have $X_2 = \dot q_2\,(\partial/\partial q_1) + \dot q_1\,(\partial/\partial q_2)$, whereas for $b = -6a$ one finds $X_2 = q_2 \dot q_2\,(\partial/\partial q_1) + (q_2 \dot q_1 - 2q_1\dot q_2)\,(\partial/\partial q_2)$. The system turns out to decouple only in case 1.

Our objective is to prove necessary and sufficient criteria about symmetries, which will characterize complete separability of a second-order system. Two recent contributions which are relevant for this matter will briefly be reviewed in the next two sections.

2. Calculus on TM and submersive systems

The standard differential geometric framework for studying second-order equations is a tangent bundle TM, which has a natural type (1,1) tensor field S (called the 'vertical endomorphism') (see *e.g.* [1]). In local coordinates (q, v), we have $S = dq^i \otimes (\partial/\partial v^i)$, whereas a general second-order system $\ddot q^i = f^i(q, \dot q)$ is represented by the vector field

$$\Gamma = v^i \frac{\partial}{\partial q^i} + f^i(q,v)\frac{\partial}{\partial v^i}.$$

Important in the calculus on TM is the tensor field $\mathcal{L}_\Gamma S$, which through its eigenspaces defines a splitting of the tangent space at each point of TM, *i.e.* determines a (non-linear) *connection* (see also [3]).

Kossowski and Thompson [4] recently studied so-called *submersive systems*. Submersiveness roughly refers to the existence of coordinates in which part of the system separates from the rest. Their main theorem states that submersive systems are characterized by the existence of an integrable distribution E on TM, which is *S-regular* and *invariant under $\mathcal{L}_\Gamma S$ and $\mathcal{L}^2_\Gamma S$*. Here, S-regularity means that E is obtained from *complete* and *vertical* lifts of a distribution on M.

The authors also investigate the influence of symmetries and find as sufficient condition for submersiveness the existence of an algebra $\mathcal{G}$ of Lie point symmetries of Γ, such that for each $X \in \mathcal{G}$, the vector field $[X^V, \Gamma] - X^c$ on TM is tangent to $\mathcal{G}^V$. A further result which will give us a clue for an even more restrictive situation states: if the system is Lagrangian and $X \in \mathcal{X}(M)$ determines a Noether (point) symmetry, then there is a corresponding submersion iff $[X^V, \Gamma] - X^c = \mu X^V$ for some $\mu \in C^\infty(TM)$.

It may be worthwhile to recall the following coordinate expressions: for any $X \in \mathcal{X}(M)$ of the form $X = X^i(q)\,(\partial/\partial q^i)$, we have $X^V = X^i(q)\,(\partial/\partial v^i) \in \mathcal{X}(TM)$ and the complete lift (or prolongation) is given by

$$X^c = X^i(q)\frac{\partial}{\partial q^i} + v^j \frac{\partial X^i}{\partial q^j}\frac{\partial}{\partial v^i}.$$

Observe further that the coordinate expression of a non-point symmetry of Γ bears some ressemblance to the latter:

$$J_\Gamma X = X^i(q,v)\frac{\partial}{\partial q^i} + \Gamma(X^i)\frac{\partial}{\partial v^i}.$$

At present, it suffices to regard the notation $J_\Gamma X$ as a way of indicating that such a symmetry is entirely determined by the object $X = X^i(q,v)\,(\partial/\partial q^i)$. The correct geometrical interpretation of this object is that X is a *vector field along the tangent bundle projection* $\tau : TM \to M$, for which we use the notation: $X \in \mathcal{X}(\tau)$.

3. Calculus along τ and separability

There are many more concepts of interest on TM, whose essential part is a corresponding object along τ. In [5] and [6], we have extensively studied the theory and classification of derivations of scalar and vector-valued forms along τ. An essential ingredient for this theory is a connection on $\tau : TM \to M$. With such a connection, one can horizontally lift vector fields along τ. In coordinates, for $X \in \mathcal{X}(\tau)$:

$$X^H = X^i H_i \in \mathcal{X}(TM), \quad H_i = \frac{\partial}{\partial q^i} - \Gamma_i^j \frac{\partial}{\partial v^j},$$

where the functions $\Gamma_i^j(q,v)$ are the connection coefficients. The vertical lift construction extends to $\mathcal{X}(\tau)$ and every $Z \in \mathcal{X}(TM)$ has a unique decomposition $Z = X^H + Y^V$, with $X, Y \in \mathcal{X}(\tau)$. Important derivations in our theory are the *horizontal* and *vertical covariant derivative*. They appear, for example, in the formula

$$[X^H, Y^V] = (\mathrm{D}_X^H Y)^V - (\mathrm{D}_Y^V X)^H,$$

and their action extends to forms along τ by duality.

As indicated before, each given Γ comes with its own connection, with coefficients $\Gamma_j^i = -\frac{1}{2}\,(\partial f^i/\partial v^j)$. Specifically for this case, two operations are most important: the *dynamical covariant derivative* ∇ and the *Jacobi endomorphism* Φ. They can be defined *e.g.* by the decomposition $\mathcal{L}_\Gamma X^H = (\nabla X)^H + (\Phi(X))^V$ and are determined in coordinates by

$$\nabla F = \Gamma(F), \text{ for functions } \ F \in C^\infty(TM)$$

$$\nabla(\partial/\partial q^j) = \Gamma_j^i\,(\partial/\partial q^i), \quad \nabla(dq^k) = -\Gamma_\ell^k\, dq^\ell$$

$$\Phi = \left(-\frac{\partial f^i}{\partial q^j} - \Gamma_k^i\, \Gamma_j^k - \Gamma(\Gamma_j^i) \right) dq^j \otimes \frac{\partial}{\partial q^i}.$$

To see the importance of these operations, observe *e.g.* that the determining equations of a (generalized) symmetry $X = X^i(q,v)\partial/\partial q^i \in \mathcal{X}(\tau)$ of Γ read $\nabla\nabla X + \Phi(X) = 0$. Also, the *curvature* of the connection

$$R = \frac{1}{2}\, R_{jk}^i\, dq^j \wedge dq^k \otimes \frac{\partial}{\partial q^i}, \quad R_{jk}^i = H_k(\Gamma_j^i) - H_j(\Gamma_k^i),$$

is obtained from Φ via a kind of vertical exterior derivative: $d^V\Phi = 3\,R$. For a link with the preceding section, note that Φ is related to the tensor field $\mathcal{L}_\Gamma^2 S$ on TM.

An interesting decomposition of ∇ is given by

$$\nabla = \mathrm{D}_{\mathbf{T}}^H + \mathrm{D}_{\nabla\mathbf{T}}^V + \mu_{\mathbf{t}}.$$

Here $\mathbf{T} = v^i\,\partial/\partial q^i$ is the canonical vector field along τ, μ refers to some well defined derivation of algebraic type, and

$$\mathbf{t} = \left(\Gamma_j^i - \frac{\partial \Gamma_j^i}{\partial v^k}\, v^k \right) dq^j \otimes \frac{\partial}{\partial q^i}$$

is the so-called *tension* of the connection.

We have used this calculus for a comprehensive study of complete separability of second-order equations [7]. In order to formulate the main theorem of that paper, we need one more definition: for each (1,1) tensor field U along τ,

$$C^V_U(X,Y) = [\mathrm{D}^V_X U, U](Y) \qquad \forall X, Y \in \mathcal{X}(\tau).$$

When C^V_U is zero for a diagonalizable tensor U, it means that all eigendistributions can be spanned by vector fields on M.

Theorem. *There exist coordinates in which a given Γ separates into n decoupled single second-order equations iff*

$$\begin{aligned}
&\Phi \textit{ is diagonalizable}, \qquad R = 0\\
&[\nabla\Phi, \Phi] = 0, \qquad C^V_\Phi = 0\\
&\mathbf{t} \textit{ is diagonalizable}, \qquad C^V_{\mathbf{t}} = 0.
\end{aligned}$$

It is important to observe that all conditions of this theorem are algebraic and can (with computer algebra assistance) in principle be checked.

4. Symmetry properties characterizing full decoupling

Let us first try to understand the meaning of the conditions of Kossowski and Thompson in terms of the calculus along τ. Observe that for $X \in \mathcal{X}(M)$, we have $X^c = X^H + (\nabla X)^V$, $\mathcal{L}_\Gamma S\,(X^c) = -X^H + (\nabla X)^V$ and $\mathcal{L}^2_\Gamma S(X^H) = -2\,(\Phi(X))^V$. It follows that the content of their main theorem essentially says that we have an integrable distribution on M which, regarded as distribution along τ is invariant under ∇ and Φ.

Still for basic X, we have the identity $[X^V, \Gamma] - X^c = -2\,(\nabla X)^V$, so that the condition for the Noether symmetry in [4] expresses that ∇X should be proportional to X.

As explained before, point symmetries cannot be sufficient for our purposes and a natural generalization of an S-regular distribution on TM consists of a distribution spanned by $\{X^V,\, J_\Gamma X = X^H + (\nabla X)^V\}$, for some distribution span$\{X\}$ along τ. Next, since we want an extreme case of 'submersiveness', the natural extension of the condition in the Noether case seems appropriate and a strong form of integrability will have to be imposed, in which *e.g.* all symmetries commute. It will now be clear that the formulation of the theorem below is directly inspired by the results in [4]. The proof, on the other hand, will be a matter of showing that all conditions of our separability theorem are verified and will thus heavily rely on the techniques developed in [7].

Theorem. *A Γ with n degrees of freedom is completely separable iff there exist n independent vector fields X_i along τ, such that*:

1. $\nabla X_i = \mu_i\, X_i$ *for some* $\mu_i \in C^\infty(TM)$
2. $J_\Gamma X_i$ *is a symmetry of* Γ *for all* i
3. $[J_\Gamma X_i, J_\Gamma X_j] = 0$ *and* $[J_\Gamma X_i, X^V_j] = 0, \quad i \neq j$
4. $S([J_\Gamma X_i, X^V_i])$ *is proportional to* X^V_i.

PROOF. From 1 and 2, it follows that $\Phi(X_i) = \lambda_i X_i$, with $\lambda_i = -\nabla\mu_i - {\mu_i}^2$. Hence Φ is diagonalizable and condition 1 further implies that the eigendistributions are ∇-invariant, so that $[\nabla\Phi, \Phi] = 0$. To proceed further, we need the following relations for general $X, Y \in \mathcal{X}(\tau)$:

$$
\begin{aligned}
[J_\Gamma X, Y^V] &= -\,(\mathrm{D}^V_Y X)^H + (\mathrm{D}^H_X Y + \mathrm{D}^V_{\nabla X} Y - \mathrm{D}^V_Y \nabla X)^V \\
[J_\Gamma X, J_\Gamma Y] &= (\mathrm{D}^H_X Y - \mathrm{D}^H_Y X + \mathrm{D}^V_{\nabla X} Y - \mathrm{D}^V_{\nabla Y} X)^H \\
&\quad + (R(X,Y) + \mathrm{D}^H_X \nabla Y - \mathrm{D}^H_Y \nabla X + \mathrm{D}^V_{\nabla X} \nabla Y - \mathrm{D}^V_{\nabla Y} \nabla X)^V.
\end{aligned}
$$

They are easy to obtain from general bracket relations in [7]. It follows from the first of these and condition 3 that $\mathrm{D}^V_{X_j} X_i = 0$ $(j \neq i)$ and

$$\mathrm{D}^H_{X_i} X_j - (\mathrm{D}^V_{X_j} \mu_i) X_i = 0 \quad (*).$$

It will be clear by now that the summation convention is being used only where appropriate.

The second relation and condition 3 next imply two further properties: the horizontal part, using $(*)$, gives rise to $\mathrm{D}^V_{X_j}\mu_i = 0$ and $\mathrm{D}^H_{X_j} X_i = 0$; the vertical part subsequently results in

$$R(X_i, X_j) + (\mathrm{D}^H_{X_i}\mu_j) X_j - (\mathrm{D}^H_{X_j}\mu_i) X_i = 0 \quad (**).$$

We next wish to exploit the relationship between the Jacobi endomorphism and the curvature. First of all, we have

$$d^V\Phi(X_i, X_j) = (\mathrm{D}^V_{X_i}\Phi)(X_j) - (\mathrm{D}^V_{X_j}\Phi)(X_i) = (\mathrm{D}^V_{X_i}\lambda_j) X_j - (\mathrm{D}^V_{X_j}\lambda_i) X_i\,.$$

Now $\mathrm{D}^V_{X_j}\lambda_i = -\mathrm{D}^H_{X_j}\mu_i$. This follows from the relation between the λ_i and μ_i, using the property $\mathrm{D}^V_{X_j}\mu_i = 0$ and the general commutator relation

$$[\mathrm{D}^V_X, \nabla] = \mathrm{D}^H_X - \mathrm{D}^V_{\nabla X} \quad (***)$$

established in [7]. We thus obtain:

$$3\,R(X_i, X_j) = d^V\Phi(X_i, X_j) = -\,(\mathrm{D}^H_{X_i}\mu_j) X_j + (\mathrm{D}^H_{X_j}\mu_i) X_i = R(X_i, X_j),$$

where the last equality follows from $(**)$. Since the X_i form a local basis of $\mathcal{X}(\tau)$, we conclude that $R = 0$.

We finally turn to condition 4. It implies, from the horizontal part of $[J_\Gamma X_i, X_i^V]$ that $\mathrm{D}^V_{X_i} X_i$ is proportional to X_i, which together with $\mathrm{D}^V_{X_j} X_i = 0$ $(i \neq j)$ means that each eigendistribution is basic, *i.e.* $C^V_\Phi = 0$. The commutator relation $(***)$ subsequently shows that also $\mathrm{D}^H_{X_i} X_i$ is proportional to X_i. Using the decomposition of the operator ∇ and knowing that $\mathsf{t}(X) = \mu_{\mathsf{t}}(X)$, one fairly easily obtains that the tension is also diagonal with respect to the basis X_i and since we already know that this is a basic distribution, it follows that $C^V_{\mathsf{t}} = 0$. All conditions of the separability theorem thus turn out to be satisfied.

The converse is trivial: if the system is separable, it suffices to think of each of the X_i as coming from a (generalized) symmetry of each separate equation. This completes the proof. $\square$

5. Concluding remarks

Our goal was to characterize separability by symmetry properties of the equations. It may look a bit unsatisfactory that the conditions of the preceding theorem are so strong, particularly since, in the separating coordinates, the symmetries under consideration will be symmetries of each of the individual equations. One can formulate conditions which look much milder, but given the fact that differential equations locally always have (generalized) symmetries, translating the milder conditions to those symmetries will make the stronger conditions hold true anyway. To be specific, a milder looking theorem, which contains the minimal requirements for separability, says the following.

Theorem. *For complete separability of a second-order system, it is necessary and sufficient that n independent $X_i \in \mathcal{X}(\tau)$ exist, such that*:

1. $\nabla X_i = \mu_i\, X_i$
2. $\Phi(X_i) = \lambda_i\, X_i$
3. $\mathrm{D}^{\mathrm{V}}_{Z} X_i = \sigma_i(Z)\, X_i, \quad \forall Z \in \mathcal{X}(\tau)$, with $\sigma_i \in C^\infty(TM)$.

For a sketch of the proof, observe first that conditions 1 and 3, using $(***)$, imply that also $\mathrm{D}^{\mathrm{H}}_{Z} X_i$ is proportional to X_i for all $Z \in \mathcal{X}(\tau)$. As a result (see the argumentation in [7]), the basic vector fields which span the eigendistributions of Φ constitute a number of distributions on M which are simultaneously integrable. Therefore, there exist new coordinates with respect to which each of the X_i is in fact a multiple of $\partial/\partial q^i$. Condition 1 then implies $\Gamma^i_j = 0$ for $i \neq j$ and condition 2 subsequently tells us that also $\partial f^i/\partial q^j = 0$ for $i \neq j$. The complete separation of Γ follows. □

With conditions 1 and 2 of this result, $J_\Gamma X_i$ need not necessarily be a symmetry, but we can always pass to one via multiplication by a suitable function. Having done this, condition 3 again looks milder, because it does not impose that these symmetries must in the end be the ones corresponding to separate equations. One can easily show, however, that they can at most differ from those by a factor which is then necessarily a first integral. Hence, if multiplicative first integrals are factored out, the stronger integrability conditions of the former theorem will hold true.

Concerning examples where one happens to know a sufficient number of independent symmetries, a strategy for checking separability could start by looking for a suitable set of linear combinations which will make the ∇X_i proportional to X_i. Applying this idea to the Hénon-Heiles example, it turns out that for $b = -a$, $X_1 + X_2$ and $X_1 - X_2$ are the only such combinations. The subsequent conditions appear to be satisfied. For the case $b = -6a$ on the other hand, no combination can be obtained which verifies the first requirement.

Recall, finally, that the Hénon-Heiles system, for $b = -6a$, is known to be separable in the sense of Hamilton-Jacobi. For future developments, it is therefore hoped that our approach may provide new insight, at the level of symmetry properties of the equations, concerning the difference between equations which completely decouple, are separable in the sense of Hamilton-Jacobi or are completely integrable.

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