Towards a geometrical understanding of Douglas's solution of the inverse problem of the calculus of variations

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Abstract

We describe a novel approach to the study of the inverse problem of the calculus of variations, which gives new insights into Douglas's solution of the two degree of freedom case.

1 Introduction

In the Introduction to his paper 'Solution of the inverse problem of the calculus of variations' [2], published in 1941, Jesse Douglas said that the 'problem indicated in the title is one of the most important hitherto unsolved problems of the calculus of variations'. The problem is to determine whether a given system of second-order ordinary differential equations is derivable from a Lagrangian; that is to say, given a system of equations

$$\ddot{x}^i = f^i(t, x^k, \dot{x}^k) \qquad (i, k = 1, 2, \dots, n),$$

to determine whether there is a function $L(t, x^k, \dot{x}^k)$ such that these equations are equivalent to the Euler-Lagrange equations derived from L as Lagrangian. In general the

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Euler-Lagrange equations do not directly take the form above, since they will not be solved for the highest derivatives; the problem is therefore to find a so-called multiplier matrix $g_{ij}(t, x^k, \dot{x}^k)$ such that

$$g_{ij}(\ddot{x}^j - f^j) \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i}.$$

Douglas gave the solution to this problem for the case $n = 2$ only. His solution involves a detailed case-by-case examination of four main cases, some of which are further divided into several subcases, the cases being distinguished by whether or not certain quantities calculated from the functions f^i and their derivatives vanish. His approach is analytical: it consists of expressing the conditions which the multiplier must satisfy as a system of partial differential equations, and then applying the Riquier theory to determine whether these equations admit a solution. In most of the cases he is able to state whether or not a Lagrangian exists, and if so to specify the degree of arbitrariness in the solution; in the remaining cases the problem is reduced to a question of the closure of a certain 1-form.

Despite the importance of the problem, it has not been solved, in the sense that Douglas solved it, for any n greater than 2, either in general, or even in any particular subcase. This is not to say that no progress has been made in the intervening 50 years. Much effort has gone into the development of appropriate differential geometric machinery, with the result that the conditions which the multiplier has to satisfy are now well understood (see [1] and [7] for recent reviews). The fundamental point is to recognise that a system of second-order differential equations may be represented as a vector field Γ on the tangent bundle TM of a differentiable manifold M, or in the time dependent case, on $\mathbb{R} \times TM$. Several solutions to the inverse problem have been presented, which give algorithms for determining whether or not any specific system of equations is derivable from a Lagrangian. What these solutions lack, which Douglas provides for $n = 2$, is a general analysis based on quantities calculated from the functions f^i appearing in the differential equations – or in other words, in terms of quantities defined by the second-order differential equation field Γ. What Douglas's solution lacks, on the other hand, is any insight into what these quantities might represent, for instance in geometrical terms.

We believe that further progress in the solution of the inverse problem can be made with the use of the correct calculus. As a first basic principle of our approach to the inverse problem, we claim that the correct calculus is provided by a restricted version of the theory of derivations of geometric objects defined along the projection $\pi: \mathbb{R} \times TM \to \mathbb{R} \times M$, a restriction which makes it identifiable with a calculus along $\tau: \mathbb{R} \times TM \to M$. The corresponding calculus for the autonomous case, that is for geometric objects defined along the projection $\tau: TM \to M$, has been developed by Martínez, Cariñena and Sarlet [4, 5] and applied successfully by these authors in [6] to the solution of a related problem, namely the determination of necessary and sufficient conditions for a system of second-order differential equations to be separable into independent one-dimensional equations.

No essential modifications are required to adapt that calculus for use in the present discussion. In [6], and elsewhere, the importance of a certain type (1,1) tensor field along τ, the so-called Jacobi endomorphism Φ associated with a second-order differential equation field Γ, for the analysis of the properties of Γ, was emphasised. The Jacobi endomorphism is also central to the discussion of the inverse problem. Indeed, our second basic principle is that the discussion of different subcases of the inverse problem should be related in one way or another to the different possible algebraic normal forms of Φ, and that it is helpful to express the unknown multiplier g (which is a symmetric type (0,2) tensor field along τ) in terms of a basis of 1-forms along the projection τ dual to a basis of vector fields adapted to Φ – a basis of eigenvectors of Φ, for example.

Using the calculus of derivations of geometric objects defined along τ, we think we now have large parts of the Douglas paper well under control. This approach confers the following benefits.

- Douglas's explicit calculations make more sense when interpreted in these terms: for example, we can explain how the classification into cases and subcases works, and we can show where expressions which play key roles in his analysis, but which are otherwise quite mysterious, come from.

- We can derive some general results which are implicit in Douglas, but not remarked by him: for example, we can show that certain 'alternants' between the partial differential equations for the multiplier will always lead to conditions which are automatically satisfied. Furthermore, we could complete some of the unfinished details of Douglas's analysis, for example by finding the conditions under which the 1-form with which he ends his account of Case III is closed.

- By using a frame adapted to the algebraic structure of Φ, with respect to which Φ is in Jordan canonical form, we can write down the most general form of the multiplier which satisfies the algebraic conditions, and derive the detailed differential equations for the free components of the multiplier resulting from the invariance and closure conditions, in terms of certain 'structure functions' of the frame.

- We believe that we could work out a general approach to the integrability conditions, which would establish general results valid in any number of degrees of freedom, and answer the question of whether there are additional, so far unrecognised, algebraic conditions that the multiplier must satisfy (other than those introduced by the choice of a frame). It will further become clear from such an approach which, if any,

  of Douglas's results are merely 'accidents of dimension', i.e. are true for $n = 2$ only.

- We feel confident that we could make progress in analysing the three degree of freedom case; and also that we could analyse some special cases for an arbitrary number of degrees of freedom, in particular that for which Φ is diagonalizable with distinct eigenvalues, and $\nabla\Phi$ is a linear combination of Φ and the identity (corresponding to Case IIa in Douglas).

In this preliminary announcement, we shall describe our methods and results in the context of two of Douglas's cases, and indicate how we think they can be used both to give a complete reinterpretation of Douglas's paper, and to tackle some of the broader issues raised above.

It has not been possible to make this paper completely self-contained. Part of its purpose is to serve as an exegesis of Douglas's paper, so reference to that paper will be necessary in order to understand this one fully. Furthermore, it is impossible to repeat in a short space all of the results from the calculus of derivations of objects along τ which might be relevant to the inverse problem. However, we have tried to make the paper as self-explanatory as is consistent with reasonable length, by giving a brief resumé of the calculus in the next section, and summarising Douglas's results as they are needed in the following two sections.

2 The calculus of derivations of objects defined along $\tau\colon \mathbb{R}\times TM \to M$

It is helpful to think of geometric objects, such as vector fields and forms, defined along τ as being objects on the base manifold M, but with coefficients depending on the coordinates of $\mathbb{R}\times TM$. In order to study such objects it is necessary to adapt the properties of derivations of geometric objects defined along the tangent bundle projection. A comprehensive theory of such operators has been developed in [4] and [5]. However, very little of the detail of this theory is needed to support the calculations

of the present paper. In fact, most of the formulae of interest can be found in the preliminary section of the application to separable second-order equations [6], to which we therefore refer the reader for more background to the brief survey of concepts below.

Consider for the time being a system of autonomous second-order ordinary differential equations, represented by the vector field

$$\Gamma = v^i\,\partial/\partial x^i + f^i(x,v)\,\partial/\partial v^i$$

on TM. It is well known that Γ defines a connection on TM, in general non-linear, for which the local basis of horizontal vector fields is given by

$$H_i = \frac{\partial}{\partial x^i} - \Gamma_i^j\,\frac{\partial}{\partial v^j}, \qquad \text{with} \quad \Gamma_i^j = -\tfrac{1}{2}\,\frac{\partial f^j}{\partial v^i}.$$

Every vector field on TM has a unique expression as the sum of the horizontal lift of one vector field along τ and the vertical lift of

another. In this way we have, for arbitrary vector fields X and Y along τ, the expressions

$$\begin{aligned}[X^H, Y^V] &= (\mathrm{D}^H_X Y)^V - (\mathrm{D}^V_Y X)^H, \\ [\Gamma, X^H] &= (\nabla X)^H + (\Phi(X))^V.\end{aligned}$$

The right hand sides of these relations identify the operations which are

essential for our subsequent analysis. First of all, Φ is a type (1,1) tensor field along τ, called the *Jacobi endomorphism*, with components

$$\Phi^i_j = -\frac{\partial f^i}{\partial x^j} - \Gamma^k_j \Gamma^i_k - \Gamma(\Gamma^i_j).$$

The *dynamical covariant derivative* ∇ is a degree 0 derivation, which extends by duality to tensor fields of arbitrary type. Its action (in coordinates) is completely determined by

$$\nabla F = \Gamma(F), \quad \forall F \in C^\infty(TM),$$
$$\nabla\left(\frac{\partial}{\partial x^j}\right) = \Gamma^i_j \frac{\partial}{\partial x^i}, \quad \nabla(dx^i) = -\Gamma^i_j dx^j.$$

The *vertical* and *horizontal covariant derivatives* D^V_X and D^H_X are also derivations of degree 0. Their coordinate action is given by

$$\mathrm{D}^V_X F = X^i \frac{\partial F}{\partial v^i}, \quad \mathrm{D}^H_X F = X^i H_i(F), \quad \forall F \in C^\infty(TM),$$

$$\mathrm{D}^V_X\left(\frac{\partial}{\partial x^j}\right) = 0, \quad \mathrm{D}^H_X\left(\frac{\partial}{\partial x^j}\right) = \left(X^k \frac{\partial \Gamma^i_j}{\partial v^k}\right) \frac{\partial}{\partial x^i},$$

$$\mathrm{D}^V_X(dx^i) = 0, \quad \mathrm{D}^H_X(dx^i) = -\left(X^k \frac{\partial \Gamma^i_j}{\partial v^k}\right) dx^j.$$

Finally, we introduce an operation D^V, whose action on an arbitrary tensor field U is given by

$$\mathrm{D}^V U(X, \ldots) = \mathrm{D}^V_X U(\ldots).$$

In this framework, the inverse problem is the search for a non-singular, symmetric, type (0,2) tensor field g along τ, such that $\Phi \lrcorner\, g$ $(= g(\Phi(\cdot), \cdot))$ is symmetric; $\nabla g = 0$; and $\mathrm{D}^V g$ is symmetric. The first two of these conditions immediately give rise to a further hierarchy of algebraic relations, namely that $\nabla \Phi \lrcorner\, g$, $\nabla^2 \Phi \lrcorner\, g$, and so on, must all be symmetric. The second condition may be called the invariance condition, since it says that the multiplier must be dynamically invariant. The third condition is commonly referred to as the closure condition: it implies that a certain 2-form on TM (the 'Kähler lift of g') must be closed.

In practice, when investigating the existence of a multiplier g, one first extracts as much information as possible from the purely algebraic conditions. This is in agreement with the primary classification of subcases in Douglas, as summarised in the next section. When it comes to imposing the differential conditions, one will ultimately have to study integrability conditions. The first hierarchy of these, in our approach, will follow from commutator identities derived in the above cited work. So far as the action of the operators on vector fields is concerned, these commutator identities read:

$$\begin{aligned}
[\mathrm{D}^V_X, \mathrm{D}^V_Y] &= \mathrm{D}^V_{[X,Y]_V} \\
[\mathrm{D}^V_X, \mathrm{D}^H_Y] &= \mathrm{D}^H_{\mathrm{D}^V_X Y} - \mathrm{D}^V_{\mathrm{D}^H_Y X} + \mu_{\theta(X,Y)} \\
[\mathrm{D}^H_X, \mathrm{D}^H_Y] &= \mathrm{D}^H_{[X,Y]_H} + \mathrm{D}^V_{R(X,Y)} + \mu_{\mathrm{Rie}(X,Y)} \\
[\nabla, \mathrm{D}^V_X] &= \mathrm{D}^V_{\nabla X} - \mathrm{D}^H_X \\
[\nabla, \mathrm{D}^H_X] &= \mathrm{D}^H_{\nabla X} + \mathrm{D}^V_{\Phi(X)} - (\mathrm{D}^V \Phi)(X) - R(X, \cdot).
\end{aligned}$$

Here, R is the curvature of the connection (a vector-valued 2-form along τ), $\mathrm{Rie} = -\mathrm{D}^V R$ and $\theta(X,Y)$ is a type (1,1) tensor field along τ with components

$$\Big(\theta(X,Y)\Big)^i_j = \Big(\mathrm{D}^V_X \mathrm{D}^V_Y - \mathrm{D}^V_{\mathrm{D}^V_X Y}\Big)(\Gamma^i_j).$$

The other ingredients of these formulae are defined as follows:

$$[X,Y]_V = \mathrm{D}^V_X Y - \mathrm{D}^V_Y X, \quad [X,Y]_H = \mathrm{D}^H_X Y - \mathrm{D}^H_Y X$$

and

$$\mu_U(X) = U(X) \quad \text{for any (1,1) tensor } U \text{ along } \tau.$$

From the general properties of derivations on the $C^\infty(TM)$-module of vector fields along τ, it is easy to deduce corresponding commutator formulae for the action on functions. Finally, using the duality rule $D\langle X, \alpha\rangle = \langle DX, \alpha\rangle + \langle X, D\alpha\rangle$, which is satisfied by all derivations involved, one easily obtains related results for the action on covariant tensors. They will be given when needed.

It remains to explain how this calculus, which, strictly speaking, applies to autonomous situations only, can be used in the present context, in which we want to consider differential equations which may be time-dependent. Such a system of equations is represented by a vector field $\Gamma = \partial/\partial t + v^i\,\partial/\partial x^i + f^i(t,x,v)\,\partial/\partial v^i$ on $\mathbb{R} \times TM$. For general considerations, the role of the projection $\tau : TM \to M$ will normally be taken over in the time-dependent case by $\pi : \mathbb{R} \times TM \to \mathbb{R} \times M$. A corresponding theory of derivations has been developed and will soon be presented for publication (see [9] for some ideas about it). Roughly speaking, geometrical objects such as torsion or curvature may pick up an extra term in this calculus (a 'time-component'), but leaving this term apart, the coordinate expression of such an object will be similar to the one in the autonomous context, but with contact forms $dx^i - v^i dt$ replacing the dx^i we had before. A particular case of interest emerges when all relevant tensors are restricted to act only on vector fields along π which have no time-component, which could be thought of as vector fields along the projection $\tau : \mathbb{R} \times TM \to M$. The tensors Φ and g have a non-trivial action only on such vector fields. If the vector fields entering the derivation operators are also taken to be of that type, then all the formulae listed above formally remain unaltered. This is the version of the calculus required for the investigation of the inverse problem.

3 A general description of Douglas's classification

Douglas writes the system of second-order differential equations he is dealing with in the form

$$y'' = F(x, y, z, y', z'), \quad z'' = G(x, y, z, y', z').$$

Note his use of x for the independent variable, instead of the more usual t, which we have adopted except when referring directly to Douglas's paper. His division of the problem into cases is based on the properties of the matrix

$$\begin{bmatrix} A & B & C \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{bmatrix},$$

whose entries depend on F and G and their derivatives: for example,

$$A = \frac{d}{dx}F_{z'} - 2F_z - \tfrac{1}{2}F_{z'}(F_{y'} + G_{z'}),$$

the subscripts denoting partial derivatives. In fact Douglas's A, B and C are related to the components of Φ (with respect to the coordinate basis), as follows:

$$A = \Phi^1_2, \quad B = \Phi^2_2 - \Phi^1_1, \quad C = -\Phi^2_1.$$

Note that Φ is not completely determined by A, B, and C – its trace is undetermined. But this is not surprising, certainly so far as the algebraic conditions on the multiplier are concerned, since we could write Φ as the sum of its trace-free part (which is determined by A, B and C) and a multiple of the identity, and the latter does not affect the algebraic conditions. Furthermore, the operations which lead from A to A_1, and from A_1 to A_2, etc., correspond exactly to the action of the operator ∇ on type (1,1) tensor fields along the projection. Thus, for example, Douglas's condition

$$A_1 = rA, \quad B_1 = rB, \quad C_1 = rC,$$

which defines Case II, may equivalently be expressed by saying that $\nabla\Phi$ is a linear combination of Φ and the identity. In fact, the first broad classification is by the linear dependence or independence of Φ and its ∇ derivatives, as follows.

Case I: Φ is a multiple of the identity tensor I.

Case II: $\nabla\Phi$ is a linear combination of Φ and I.

Case III: $\nabla^2\Phi$ is a linear combination of $\nabla\Phi$, Φ and I.

Case IV: $\nabla^2\Phi$, $\nabla\Phi$, Φ and I are linearly independent.

Case II is further subdivided into two subcases IIa and IIb according to the diagonalizability of Φ.

Case IIa: Φ has distinct eigenvalues (real or complex).

Case IIb: The eigenvalues of Φ coincide.

The further subdivision of Cases IIa and IIb may be defined in terms of a certain type (1,2) tensor field H_Φ along τ, defined as follows:

$$H_\Phi(X,Y) = \mathrm{D}^V\Phi(\Phi X, \Phi Y) - \Phi\mathrm{D}^V\Phi(\Phi X, Y) - \Phi\mathrm{D}^V\Phi(X, \Phi Y) + \Phi^2\mathrm{D}^V\Phi(X,Y).$$

(This tensor appears to be related to one introduced by Frölicher and Nijenhuis, in [3], in the discussion of a result of Haantjes on the integrability of distributions defined by eigenvectorfields of a type (1,1) tensor field.) It turns out that in Case IIa, H_Φ has at most two independent components; the subcases are determined by whether both, one or none of these components vanishes:

Case IIa1: $H_\Phi = 0$.

Case IIa2: H_Φ has one independent component.

Case IIa3: H_Φ has two independent components.

Case IIb is also subdivided according to the vanishing of H_Φ:

Case IIb1: $H_\Phi = 0$.

Case IIb2: $H_\Phi \neq 0$.

In Case III the commutator $[\Phi, \nabla\Phi]$ is non-zero. The further subdivision of this case is determined by whether or not $[\Phi, \nabla\Phi]$ is singular.

Case III1: $\det[\Phi, \nabla\Phi] \neq 0$.

Case III2: $\det[\Phi, \nabla\Phi] = 0$.

Thus Douglas's classification of the cases and subcases of the inverse problem for $n = 2$ may be expressed entirely in terms of the properties of Φ and its covariant derivatives, in a manner that may clearly be extended to higher dimensions.

4 Case IIa

We shall next give a sketch of the way we approach the Douglas paper. To fix ideas, let us talk first about Case II, which Douglas after all calls the most interesting case. We shall also consider Case III, in the following section.

Case II is characterised for us by the fact that $\nabla\Phi$ is a combination of Φ and the identity. Case IIa is distinguished by the requirement that Φ is diagonalizable with different eigenvalues, while in Case IIb, Φ has coincident eigenvalues with just a one dimensional corresponding eigenspace, so here one must deal with a non-diagonal Jordan form. Incidentally, in Case III, where $\nabla^2\Phi$ is a combination of $\nabla\Phi$, Φ and the identity, one can have a non-singular multiplier only when Φ is diagonalizable, as we show in the next section.

We shall concentrate on Case IIa. We shall first outline our approach, and then show how to obtain Douglas's results from it.

Let X_i, $i = 1, 2$, be two vector fields along τ which are independent eigenvectors of Φ; and consider the dual basis $\{\theta^i\}$ of 1-forms along τ. Then the only immediate algebraic condition, namely that $\Phi \lrcorner\, g$ is symmetric, simply means that g must be of the form

$$\rho_1\theta^1 \otimes \theta^1 + \rho_2\theta^2 \otimes \theta^2.$$

With the appropriate choice for X_1 and X_2, we can explicitly reproduce all the relevant formulae in Douglas (leaving the integrability

conditions aside for a moment), as we shall show briefly below. But first, we describe some more general considerations.

It follows from the dependence of $\nabla\Phi$ on Φ and the identity that ∇X_i is proportional to X_i. We can in principle, without loss of generality, scale the X_i in such a way that $\nabla X_i = 0$. (This is not the scaling chosen by Douglas, but is natural for theoretical work once one has recognised the geometrical structure.) Then the differential condition $\nabla g = 0$ is equivalent to saying that the ρ_i actually have to be first integrals.

Next come the closure conditions. To express them in terms of our basis, we note that for each i and j we may write $\mathrm{D}^{\mathrm{V}}_{X_i}X_j$ as a linear combination of the X_k. We therefore introduce 'structure functions' τ^k_{ij} which characterise the vertical covaraiant derivatives of the X_j:

$$\mathrm{D}^{\mathrm{V}}_{X_i}X_j = \tau^k_{ij}X_k, \qquad \text{or equivalently} \quad \mathrm{D}^{\mathrm{V}}_{X_i}\theta^j = -\tau^j_{ik}\theta^k.$$

In principle, these functions can be computed for every choice of the eigenvectors; but it is preferable to treat them as 'known' quantities, which are however unspecified. The further classification of subcases may be expressed in terms of these structure functions. The closure condition gives rise to two equations which express vertical derivatives of the ρ_i as certain linear combinations of the ρ_i with coefficients given in terms of the τ^k_{ij}. Explicitly, these equations, which are the analogues of Equations (10.8) in Douglas, read

$$\begin{aligned}
\mathrm{D}^V_{X_2}\rho_1 &= (2\tau^1_{21} - \tau^1_{12})\rho_1 - \tau^2_{11}\rho_2, \\
\mathrm{D}^V_{X_1}\rho_2 &= (2\tau^2_{12} - \tau^2_{21})\rho_2 - \tau^1_{22}\rho_1.
\end{aligned}$$

In our analysis, therefore, the separated Case IIa1 (the case where these equations decouple) is defined by $\tau^1_{22} = \tau^2_{11} = 0$ (which is the analogue of Equation (10.11) of Douglas).

The vanishing of these components has a tensorial meaning (although the τ functions of course are not tensor components). There is a relation between the quantities τ^1_{22} and τ^2_{11} and the 'generalised Haantjes tensor' H_Φ, which is derived as follows. The vector fields X_i are eigenvectors of Φ, with eigenvalues ξ_i say. It follows from the formula for H_Φ given earlier that

$$\begin{aligned}
H_\Phi(X_1, X_1) &= (\xi_1 I - \Phi)^2 \mathrm{D}^V\Phi(X_1, X_1) \\
&= (\xi_1 I - \Phi)^2 (\mathrm{D}^V_{X_1}\Phi)(X_1) \\
&= (\xi_1 I - \Phi)^2 (\mathrm{D}^V_{X_1}(\Phi(X_1)) - \Phi(\mathrm{D}^V_{X_1}X_1)) \\
&= (\xi_1 I - \Phi)^3 \mathrm{D}^V_{X_1}X_1 \\
&= (\xi_1 - \xi_2)^3 \tau^2_{11},
\end{aligned}$$

since $\xi_1 I - \Phi$ annihilates multiples of X_1. Similarly

$$H_\Phi(X_2, X_2) = (\xi_2 - \xi_1)^3 \tau^1_{22}.$$

A similar calculation for $H_\Phi(X_1, X_2)$ shows that this vector may be expressed as $(\xi_1 I - \Phi)(\xi_2 I - \Phi)$ operating on something, and is therefore zero.

This result is particularly interesting because, being tensorial, it will have the same significance in arbitrary higher dimension. In fact the vanishing of H_Φ will not only ensure that the equations for the vertical derivatives of the components of the multiplier decouple, but at the same time it will prevent certain extra algebraic conditions (which are void in dimension 2) from turning up.

Before pushing this line of reasoning a bit further, let us show that by making the appropriate choice for the eigenvectors of Φ we obtain exactly the analytical expressions of Douglas.

We can represent Φ as a matrix (with respect to the coordinate basis) as follows:

$$\Phi = \begin{bmatrix} \frac{1}{2}(T - B) & A \\ -C & \frac{1}{2}(T + B) \end{bmatrix},$$

with $T = \operatorname{tr}\Phi$. The eigenvalue equation for Φ is

$$\xi^2 - T\xi + \tfrac{1}{4}(T^2 - (B^2 - 4AC)) = 0,$$

and its eigenvalues are $\frac{1}{2}(T \pm \sqrt{B^2 - 4AC})$. Now Douglas's subdivision into Cases IIa and IIb depends on whether $B^2 - 4AC$ is non- zero or zero; and this corresponds exactly to whether the eigenvalues of Φ are distinct or coincident. Note that the eigen*values* of Φ are not directly related to the quantities λ and μ occuring in Douglas's analysis: in fact these quantities, the roots of the quadratic equation $A\xi^2 + B\xi + C = 0$, are related instead to the eigen*vectors* of Φ.

In Case IIa, $B^2 - 4AC \neq 0$, and Φ has distinct eigenvalues. Douglas does not distinguish between the cases of real and complex eigenvalues, and the following discussion has been based on the real case (rather as Douglas's seems to have been). It may easily be checked that

$$X = \frac{1}{(\lambda - \mu)}\begin{bmatrix} 1 \\ -\mu \end{bmatrix} \quad \text{and} \quad Y = \frac{1}{(\mu - \lambda)}\begin{bmatrix} 1 \\ -\lambda \end{bmatrix}$$

are eigenvectors of Φ; these eigenvectors are scaled so that the dual basis of row vectors is

$$\theta = [\lambda \quad 1] \quad \text{and} \quad \phi = [\mu \quad 1].$$

With this choice the corresponding symmetric bilinear forms $\theta \otimes \theta$ and $\phi \otimes \phi$ are represented by the matrices

$$\theta\theta^T = \begin{bmatrix} \lambda^2 & \lambda \\ \lambda & 1 \end{bmatrix} \quad \text{and} \quad \phi\phi^T = \begin{bmatrix} \mu^2 & \mu \\ \mu & 1 \end{bmatrix}.$$

These are the expressions for the singular symmetric bilinear forms which Douglas derives as the intersection of the plane $AL + BM + CN = 0$ with the 'critical cone' $LM - N^2 = 0$, when the latter is parametrised in the peculiar way he chooses. The general solution of the equation $g\Phi = \Phi^T g$, which is the matrix

form of the symmetry condition for the multiplier, may therefore be written in this case

$$g = \rho\theta\theta^T + \sigma\phi\phi^T = \begin{bmatrix} \rho\lambda^2 + \sigma\mu^2 & \rho\lambda + \sigma\mu \\ \rho\lambda + \sigma\mu & 1 \end{bmatrix},$$

where ρ and σ have the same meanings as in Douglas's paper. Douglas does not choose to scale X and Y so that $\nabla X = \nabla Y = 0$. However, it remains true that ∇X is proportional to X and ∇Y to Y; or equivalently that $\nabla\theta$ is proportional to θ and $\nabla\phi$ to ϕ. In fact it is easy to compute $\nabla\theta$ and $\nabla\phi$, using the definition of ∇; expressing the results in Douglas's notation, we find that

$$\nabla\theta = \tfrac{1}{2}(F_{z'}\lambda + G_{z'})\theta, \quad \nabla\phi = \tfrac{1}{2}(F_{z'}\mu + G_{z'})\phi.$$

Now consider the condition $\nabla g = 0$. In terms of ρ and σ this condition becomes

$$\left(\frac{d\rho}{dx}\right)\theta\otimes\theta + \rho(\nabla\theta\otimes\theta + \theta\otimes\nabla\theta) + \left(\frac{d\sigma}{dx}\right)\phi\otimes\phi + \sigma(\nabla\phi\otimes\phi + \phi\otimes\nabla\phi) = 0.$$

(Remember that on functions $\nabla = \Gamma = d/dt$, and that in Douglas's notation the independent variable is not t but x.) It follows that

$$\left(\frac{d\rho}{dx} + (F_{z'}\lambda + G_{z'})\right)\theta\otimes\theta + \left(\frac{d\sigma}{dx} + (F_{z'}\mu + G_{z'})\right)\phi\otimes\phi = 0.$$

Since $\theta\otimes\theta$ and $\phi\otimes\phi$ are linearly independent,

$$\frac{d\rho}{dx} + (F_{z'}\lambda + G_{z'})\rho = 0 \quad \text{and} \quad \frac{d\sigma}{dx} + (F_{z'}\mu + G_{z'})\sigma = 0;$$

these are precisely the equations found by Douglas (Equations (10.7) of his paper).

Using the explicit expressions for X, Y, θ and ϕ given earlier we find that, for example,

$$\mathrm{D}^V_Y\theta = \frac{1}{(\mu-\lambda)}[\lambda_{y'} - \lambda\lambda_{z'} \quad 0].$$

Indeed, we can now recognise that expressions of the form $(\cdot)_{y'} - \lambda(\cdot)_{z'}$ and $(\cdot)_{y'} - \mu(\cdot)_{z'}$, which occur so frequently in Douglas's analysis, arise as vertical covariant derivatives along Y and X respectively. The components of the vertical covariant derivatives of the basis 1-forms are given by

$$\begin{aligned}
\langle X, \mathrm{D}^V_X\theta\rangle &= -\langle Y, \mathrm{D}^V_X\theta\rangle = \frac{\lambda_{y'} - \mu\lambda_{z'}}{(\lambda-\mu)^2}\\
\langle X, \mathrm{D}^V_X\phi\rangle &= -\langle Y, \mathrm{D}^V_X\phi\rangle = \frac{\mu_{y'} - \mu\mu_{z'}}{(\lambda-\mu)^2} = -\frac{\beta}{\lambda-\mu}\\
\langle X, \mathrm{D}^V_Y\theta\rangle &= -\langle Y, \mathrm{D}^V_Y\theta\rangle = -\frac{\lambda_{y'} - \lambda\lambda_{z'}}{(\lambda-\mu)^2} = \frac{\alpha}{\lambda-\mu}\\
\langle X, \mathrm{D}^V_Y\phi\rangle &= -\langle Y, \mathrm{D}^V_Y\phi\rangle = -\frac{\mu_{y'} - \lambda\mu_{z'}}{(\lambda-\mu)^2};
\end{aligned}$$

here α and β are the functions defined by Douglas in Equations (10.9). In effect, we have found here explicit expressions for the structure functions τ^k_{ij}. The equation satisfied by ρ (i.e. the equation for $\mathrm{D}^V_{X_2}\rho_1$ in our notation) may therefore be written

$$\begin{aligned}
\rho_{y'} - \lambda\rho_{z'} &= \left(\frac{\lambda_{y'} - \mu\lambda_{z'} - 2(\lambda_{y'} - \lambda\lambda_{z'})}{\lambda-\mu}\right)\rho + \beta\sigma\\
&= \left(\frac{(\lambda-\mu)\lambda_{z'} + \lambda\lambda_{z'} - \lambda_{y'}}{\lambda-\mu}\right)\rho + \beta\sigma\\
&= (\lambda_{z'} + \alpha)\rho + \beta\sigma.
\end{aligned}$$

The equation

$$\sigma_{y'} - \mu\sigma_{z'} = (\mu_{z'} - \beta)\sigma - \alpha\rho$$

follows similarly. These are Douglas's Equations (10.8).

The next stage in Douglas's considerations is the computation of 'alternants'. To give a rudimentary idea of how this works in our approach, let us take the separated Case IIa1 and look at the set of equations so far to be satisfied by ρ_1, namely (in our notation)

$$\nabla\rho_1 = 0, \quad \mathrm{D}^V_{X_2}\rho_1 = (2\tau^1_{21} - \tau^1_{12})\rho_1.$$

¿From the $[\nabla, \mathrm{D}^V_X]$–commutator relation (applied to functions), it follows that ρ_1 must satisfy the new equation

$$\mathrm{D}^H_{X_2}\rho_1 = -\nabla(2\tau^1_{21} - \tau^1_{12})\rho_1.$$

Again, with the choice of eigenvectors made by Douglas, we would discover in the left hand side the origin of an expression like $\rho_y - \lambda\rho_z$. The commutators $[\nabla, \mathrm{D}^H_X]$ and $[\mathrm{D}^V_X, \mathrm{D}^H_X]$ subsequently produce algebraic conditions, namely homogeneous linear equations for ρ_1. Thus, integrability in this case means that the coefficients in these equations must vanish. Verifying that they do should follow from 'structure equations'; these are identities satisfied by the functions τ^k_{ij} which, for example, can be obtained from the same commutator relations, this time applied to the eigenvectors X_i themselves. It is obvious that this is a tedious job and that one does not want to go through such an analysis for every separate subcase (as Douglas does). We defer further considerations on this problem to Section 5. The general results which are derived there will constitute one of the major points of progress with respect to Douglas. At this stage, however, it is already worth observing another interesting difference. In computing alternants, Douglas makes a point of working with the independent operators $\left\{\frac{d}{dx}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right\}$ instead of the five coordinate derivatives $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right\}$. What our more geometrical approach indicates is that in fact it is still more advantageous to go further and replace $\left\{\frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right\}$ by the combinations which constitute $\{\mathrm{D}^H_{X_1}, \mathrm{D}^H_{X_2}, \mathrm{D}^V_{X_1}, \mathrm{D}^V_{X_2}\}$.

Note in passing: the fact that the $[\nabla, \mathrm{D}^H_{X_2}]$–integrability condition for ρ_1 is satisfied has a nice geometrical interpretation – it means that $(2\tau^1_{21} - \tau^1_{12})X_2$ generates a symmetry of Γ. Likewise, in the similar analysis for the separate equations for ρ_2, one discovers that $(2\tau^2_{12} - \tau^2_{21})X_1$ generates a symmetry.

5 Case III

First, note that if R and S are any two matrices which are symmetric with respect to a scalar product g, then the commutator $[R, S]$ is skew-symmetric with respect to g:

$$g(u, [R, S]v) = g(Ru, Sv) - g(Su, Rv) = -g([R, S]u, v).$$

It follows that $g([\Phi, \nabla\Phi]\cdot, \cdot)$ is a 2-form, and therefore in dimension 2 is a multiple of a fixed non-zero 2-form. If, following Douglas, we set

$$\Delta_1 = BC_1 - CB_1, \quad \Delta_2 = CA_1 - AC_1, \quad \Delta_3 = AB_1 - BA_1,$$

then

$$[\Phi, \nabla\Phi] = \begin{bmatrix} \Delta_2 & \Delta_3 \\ -\Delta_1 & -\Delta_2 \end{bmatrix} = \Delta, \quad \text{say.}$$

We can therefore write

$$g\Delta = \hat{\rho} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where $\hat{\rho}$ is a scalar. Thus provided that $\det \Delta = \Delta_1\Delta_3 - {\Delta_2}^2 \neq 0$, we have

$$g = \frac{\hat{\rho}}{\det \Delta} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -\Delta_2 & -\Delta_3 \\ \Delta_1 & \Delta_2 \end{bmatrix} = \rho \begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_2 & \Delta_3 \end{bmatrix},$$

where $\rho = \hat{\rho}/\det \Delta$. This is the form of the multiplier given by Douglas in Equation (17.3).

Having sufficiently established the link with Douglas in the previous paragraph, we shall now follow independently our own line of approach, much as we did for Case IIa above. We shall show for a start that in Case III, Φ must be diagonalizable. Assume otherwise, that Φ is not diagonalizable, and let X_1, X_2 be a basis for its Jordan canonical form: $\Phi(X_1) = \xi X_1$, $\Phi(X_2) = A\,X_1 + \xi X_2$, with A any non-zero function. The first algebraic condition – that $\Phi \lrcorner\, g$ is symmetric – implies that $g(X_1, X_1) = 0$. So, with respect to a 1-form basis θ^i dual to the X_i, g must take the form

$$g = \sigma(\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1) + \mu\theta^2 \otimes \theta^2,$$

where $\sigma \neq 0$ since g must be nonsingular. It follows that the next algebraic condition – that $\nabla\Phi \lrcorner\, g$ is symmetric – is equivalent to $g(\nabla X_1, X_1) = 0$. If we put $\nabla X_i = \nu_i^j X_j$, then this second requirement says that $\nu_1^2\,\sigma = 0$, i.e. $\nu_1^2 = 0$. We can therefore rescale X_1 to have $\nabla X_1 = 0$. Using the fact that the dimension is 2, it is easy to show that it follows that one can find functions a and b such that $\nabla\Phi = a\Phi + bI$. We conclude from this that if Φ is not diagonalizable, we are necessarily in Case II (or I).

Assume therefore that Φ is diagonalizable, so that the first algebraic condition implies that g is of the form $g = \rho\theta^1 \otimes \theta^1 + \sigma\theta^2 \otimes \theta^2$. The second algebraic condition is now equivalent to $g(\nabla X_1, X_2) + g(X_1, \nabla X_2) = 0$. This tells us that $\nu_1^2\sigma + \nu_2^1\rho = 0$, or in other words that g is of the form

$$g = \mu(\nu_1^2\theta^1 \otimes \theta^1 - \nu_2^1\theta^2 \otimes \theta^2),$$

for some unknown μ, where none of the functions in this expression can be zero if g is to be nonsingular. We have again derived the result that in this case there is just one independent coefficient in g, namely μ. One further simplification which we can certainly achieve in this case is to rescale the X_i in such a way that $\nabla X_1 = \nu_1^2 X_2$ and $\nabla X_2 = \nu_2^1 X_1$.

The next thing to observe is the distinction between Case III and Case IV. Case III is the case where $\nabla^2\Phi$ is a linear combination of $\nabla\Phi$, Φ and the identity, so that the third algebraic condition should be void. But this condition, which is equivalent to $g(\nabla^2 X_1, X_2) + 2\,g(\nabla X_1, \nabla X_2) + g(X_1, \nabla^2 X_2) = 0$, implies that

$$\mu(\nu_1^2 \nabla \nu_2^1 - \nu_2^1 \nabla \nu_1^2) = 0,$$

which is void only if the quantity in brackets is zero. In the opposite case, Case IV, we get $\mu = 0$ and there is clearly no solution to the inverse problem. The Case III situation allows us a final simplification: it actually means that ν_2^1/ν_1^2 is a first integral (which is in principle known). Setting $\nu_2^1 = \nu_1^2\,\alpha$, and putting the common factor in front and absorbing it into the unknown μ, we may finally regard g as being of the form:

$$g = \mu(\theta^1 \otimes \theta^1 - \alpha\,\theta^2 \otimes \theta^2).$$

where α is a first integral.

Next, let us move on to the differential conditions. Firstly, $\nabla g = 0$ is equivalent to $\nabla\mu = 0$. The closure conditions,

$$\mathrm{D}^V_{X_1} g(X_1, X_2) = \mathrm{D}^V_{X_2} g(X_1, X_1), \quad \mathrm{D}^V_{X_1} g(X_2, X_2) = \mathrm{D}^V_{X_2} g(X_2, X_1),$$

explicitly become

$$\mathrm{D}^V_{X_1}\mu = a_1\,\mu, \quad \mathrm{D}^V_{X_2}\mu = a_2\,\mu,$$

where in terms of the structure functions τ^k_{ij}, defined as before, we have

$$a_1 = \alpha^{-1}\Big(\alpha(2\,\tau^2_{12} - \tau^2_{21}) + \tau^1_{22} - \mathrm{D}^V_{X_1}\alpha\Big),$$

$$a_2 = (2\,\tau^1_{21} - \tau^1_{12} + \alpha\tau^2_{11}).$$

As in the previous section, the commutator identity $[\nabla, \mathrm{D}^V_X] = \mathrm{D}^V_{\nabla X} - \mathrm{D}^H_X$ produces new equations for the $\mathrm{D}^H_{X_i}\mu$, say

$$\mathrm{D}^H_{X_1}\mu = b_1\,\mu, \quad \mathrm{D}^H_{X_2}\mu = b_2\,\mu,$$

where the b_i are known functions.

At this point, we can already produce a statement about the existence of a Lagrangian, which is equivalent to the one in Douglas. For this purpose we must interpret our calculations in the full time-dependent formalism. We will choose the forms dual to the X_i to satisfy $\langle \mathbf{T}, \theta^i \rangle = 0$, where $\mathbf{T} = \partial/\partial t + v^i\,\partial/\partial x^i$ is the canonical vector field along $\pi : \mathbb{R} \times TM \to \mathbb{R} \times M$. In this way, with appropriately defined vertical and horizontal lifts, $\{dt, \theta^{i^V}, \theta^{i^H}\}$ will be a basis of 1-forms on $\mathbb{R} \times TM$ dual to $\{\Gamma, X_i{}^V, X_i{}^H\}$. With respect to such a basis, for any function φ on $\mathbb{R} \times TM$

$$d\varphi = (\nabla\varphi)dt + (\mathrm{D}^V_{X_i}\varphi)\theta^{i^V} + (\mathrm{D}^H_{X_i}\varphi)\theta^{i^H}.$$

The overall integrability condition can now be stated, as follows: a Lagrangian will exist provided that the form $a_i\theta^{i^V} + b_i\theta^{i^H}$ is closed.

The question is whether we could actually do better than Douglas here, which would be the case if we could pin down more precisely the obstructions to this 1-form being closed. To do so would involve considering the integrability conditions which would arise, as in the other cases, from calculating alternants (to use Douglas's terminology) between the various differential conditions which must be satisfied by the multiplier. We turn to this issue in the following, final, section of the paper.

6 The first hierarchy of integrability conditions

Remembering the tedious calculations referred to in Section 4, it should be clear that a better way of approaching the integrability problem will be to go back to the original conditions on g and use the commutator identities on them directly. We have carried out this procedure completely for the first set of commutators, that is to say, for all the commutators $[D_1, D_2]$ where D_1 and D_2 are chosen from ∇, $\mathrm{D}^V_{X_i}$ and $\mathrm{D}^H_{X_j}$. The results are summarised below. This analysis is valid for arbitrary dimension. Furthermore, it is independent of the eigenspace structure of Φ; in other words, the X_i represent any local basis of vector fields along τ.

We start with the invariance condition

$$\nabla g = 0$$

and the closure condition

$$\mathrm{D}^V_{X_i} g(X_j, X_k) = \mathrm{D}^V_{X_k} g(X_j, X_i).$$

Using the comutator relation

$$[\nabla, \mathrm{D}^V_{X_i}]g = \mathrm{D}^V_{\nabla X_i} g - \mathrm{D}^H_{X_i} g,$$

it follows easily that g must satisfy

$$\mathrm{D}^H_{X_i} g(X_j, X_k) = \mathrm{D}^H_{X_k} g(X_j, X_i).$$

We next proceed in the same way by applying ∇ to this relation, using the identity

$$[\nabla, \mathrm{D}^H_{X_i}]g = \mathrm{D}^H_{\nabla X_i} g + \mathrm{D}^V_{\Phi(X_i)} g - 2i_{i_{X_i}R}\, g + i_{\mathrm{D}^V_{X_i}\Phi}\, g,$$

where for any (1,1) tensor A, $i_A g(X,Y) = g(AX,Y) + g(X,AY)$. In the calculations which result one has to make use of the closure conditions, the fact that $\mathrm{D}^V_X(\Phi \lrcorner g)$ will be symmetric for any X and the property $3\,R(X,Y) = \mathrm{D}^V_X\Phi(Y) - \mathrm{D}^V_Y\Phi(X)$ (see [5]). It then follows that we must have

$$g(R(X_k,X_i),X_j) + g(R(X_i,X_j),X_k) + g(R(X_j,X_k),X_i) = 0.$$

These are extra algebraic relations which must be satisfied by a multiplier (in addition to those involving Φ and its ∇ derivatives). They were first derived in coordinate form in [8] and are also mentioned in [1]. They are void in dimension 2.

We next apply $\mathrm{D}^V_{X_l}$ to the original closure conditions and use the commutator identity

$$[\mathrm{D}^V_{X_k}, \mathrm{D}^V_{X_l}] = \mathrm{D}^V_{[X_k,X_l]_V}.$$

All terms in the resulting expression cancel by virtue of the closure conditions and the commutation relation, so no new condition arises.

The computation resulting from acting on $\mathrm{D}^H_{X_i} g(X_j,X_k) = \mathrm{D}^H_{X_k} g(X_j,X_i)$ with $\mathrm{D}^H_{X_l}$ proceeds along much the same lines, but there are of course more terms involved, because on g

$$[\mathrm{D}^H_{X_i}, \mathrm{D}^H_{X_j}] = \mathrm{D}^H_{[X_i,X_j]_H} + \mathrm{D}^V_{R(X_i,X_j)} - i_{\mathrm{Rie}(X_i,X_j)}.$$

We obtain first a relation of the form

$$\begin{aligned} 0 \;=\;& g(\mathrm{Rie}(X_l,X_i)X_j,X_k) + g(\mathrm{Rie}(X_l,X_i)X_k,X_j) - \mathrm{D}^V_{X_k} g(X_j,R(X_l,X_i)) \\ &+ \mathrm{D}^H_{X_l}\mathrm{D}^H_{X_k} g(X_j,X_i) - \mathrm{D}^H_{\mathrm{D}^H_{X_l}X_k} g(X_j,X_i) \\ &- \mathrm{D}^H_{X_i}\mathrm{D}^H_{X_k} g(X_j,X_l) + \mathrm{D}^H_{\mathrm{D}^H_{X_i}X_k} g(X_j,X_l). \end{aligned}$$

We now make use of the property $\mathrm{Rie} = -\mathrm{D}^V R$. After some obvious cancellations the condition reduces to the following form:

$$\begin{aligned} 0 \;=\;& \mathrm{D}^V_{X_j} g(R(X_i,X_l),X_k) + \mathrm{D}^V_{X_j} g(R(X_l,X_k),X_i) + \mathrm{D}^V_{X_j} g(R(X_k,X_i),X_l) \\ &+ g(\mathrm{D}^V_{X_j} R(X_i,X_l),X_k) + g(\mathrm{D}^V_{X_j} R(X_l,X_k),X_i) + g(\mathrm{D}^V_{X_j} R(X_k,X_i),X_l) \\ &+ g(\mathrm{D}^V_{X_k} R(X_i,X_l),X_j) + g(\mathrm{D}^V_{X_i} R(X_l,X_k),X_j) + g(\mathrm{D}^V_{X_l} R(X_k,X_i),X_j). \end{aligned}$$

The last three terms cancel out in view of the relation between R and Φ. It is then easy to verify that the remaining terms consist precisely of those that are left when the extra algebraic conditions involving R obtained earlier are acted upon by $\mathrm{D}^V_{X_j}$. Surprisingly, therefore, the conditions coming from the commutator of two horizontal covariant derivatives are also always satisfied, by virtue of the previously derived conditions.

The final alternant between the closure conditions arises from applying $\mathrm{D}^V_{X_l}$ to the equation $\mathrm{D}^H_{X_i} g(X_j,X_k) = \mathrm{D}^H_{X_k} g(X_j,X_i)$, or equivalently $\mathrm{D}^H_{X_l}$ to the equation $\mathrm{D}^V_{X_i} g(X_j,X_k) =$

$\mathrm{D}^V_{X_k} g(X_j, X_i)$. In this case it is not possible to eliminate the second derivatives; however, by using the commutator

$$[\mathrm{D}^V_{X_i}, \mathrm{D}^H_{X_j}] = \mathrm{D}^H_{\mathrm{D}^V_{X_i} X_j} - \mathrm{D}^V_{\mathrm{D}^H_{X_j} X_i} - i_{\theta(X_i, X_j)}$$

we can derive an integrability condition involving the second derivatives of the multiplier, which may be described conveniently in terms of a certain differential operator $A(X_i, X_j)$ defined as follows:

$$A(X_i, X_j) = \mathrm{D}^V_{X_i} \mathrm{D}^H_{X_j} - \mathrm{D}^V_{X_j} \mathrm{D}^H_{X_i} - \mathrm{D}^H_{[X_i, X_j]_V}.$$

Note that the second derivative terms in $A(X_i, X_j)$ do not form a commutator, though the operator is clearly skew-symmetric in X_i and X_j. However, $A(X_i, X_j)$ is $C^\infty(\mathbb{R} \times TM)$-linear in both X_i and X_j, though it is not a tensor because it is not $C^\infty(\mathbb{R} \times TM)$-linear in its operation on vector fields along τ. Using the commutation relation for $\mathrm{D}^V_{X_i}$ and $\mathrm{D}^H_{X_j}$ we can express $A(X_i, X_j)$ in a couple of different ways:

$$\begin{aligned} A(X_i, X_j) &= -\{\mathrm{D}^H_{X_i} \mathrm{D}^V_{X_j} - \mathrm{D}^H_{X_j} \mathrm{D}^V_{X_i} - \mathrm{D}^V_{[X_i, X_j]_H}\} \\ &= \mathrm{D}^V_{X_i} \mathrm{D}^H_{X_j} - \mathrm{D}^H_{X_i} \mathrm{D}^V_{X_j} - \mathrm{D}^H_{\mathrm{D}^V_{X_i} X_j} + \mathrm{D}^V_{\mathrm{D}^H_{X_i} X_j} + i_{\theta(X_i, X_j)}. \end{aligned}$$

We can establish the identity

$$\begin{aligned} &(A(X_i, X_j) g)(X_k, X_l) + g(\theta(X_k, X_l) X_i, X_j) - g(X_i, \theta(X_k, X_l) X_j) \\ &\quad \equiv -\{(A(X_k, X_l) g)(X_i, X_j) + g(\theta(X_i, X_j) X_k, X_l) - g(X_k, \theta(X_i, X_j) X_l)\}, \end{aligned}$$

modulo derivatives of the closure conditions. The left hand side is skew-symmetric in X_i and X_j but symmetric in X_k and X_l, while the right hand side is symmetric in X_i and X_j but skew-symmetric in X_k and X_l; it follows from this that each must separately be zero, modulo derivatives of the closure conditions: thus if g is to satisfy the conditions for being a multiplier we must have

$$(A(X_i, X_j) g)(X_k, X_l) + g(\theta(X_k, X_l) X_i, X_j) - g(X_i, \theta(X_k, X_l) X_j) = 0.$$

Note that this condition is vacuous if either $X_i = X_j$ or $X_k = X_l$; in the two degree of freedom case, therefore, there is only one non-trivial component, namely

$$(A(X_1, X_2) g)(X_1, X_2) + g(\theta(X_1, X_2) X_1, X_2) - g(X_1, \theta(X_1, X_2) X_2) = 0.$$

It will be seen from this analysis that most of the integrability conditions are always satisfied in dimension 2. As a matter of fact, in each case Douglas constructs the alternants in the same order as we have given them above, but of course considering only those that are relevant to the case in hand. In every case he finds, first, that taking the alternant corresponding to $[\nabla, \mathrm{D}^V_X]$ has the effect of turning derivatives of the components of the multiplier with respect to y' and z' (vertical derivatives) into similar derivatives with respect to y and z (horizontal derivatives). Next, he finds that the alternant of these new equations with the invariance equations, which corresponds to $[\nabla, \mathrm{D}_X^H]$, disappears identically. In fact all

of Douglas's integrability conditions, which he obtained by arduous calculations on a case-by-case basis, are covered by the results described above, with the exception of the final alternant required in Case I, which actually belongs to the second hierarchy and is therefore not included in the above discussion. The efficiency of our approach, compared with his, should be apparent; as should be the possibilities that it gives for analysing cases in higher dimensions.

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