# Complete separability of time-dependent second-order ordinary differential equations

F. Cantrijn[*], W. Sarlet, A. Vandecasteele,
Theoretical Mechanics Division, University of Gent
Krijgslaan 281, B–9000 Gent (Belgium)

E. Martínez
Departamento de Matemática Aplicada
Universidad de Zaragoza
E–50009 Zaragoza (Spain)

**Abstract.** Extending previous work on the geometric characterization of separability in the autonomous case, necessary and sufficient conditions are established for the complete separability of a system of time-dependent second-order ordinary differential equations. In deriving this result, extensive use is made of the theory of derivations of scalar and vector-valued forms along the projection $\pi : J^1E \to E$ of the first jet bundle of a fibre bundle $E \to \mathbb{R}$. Two illustrative examples are discussed, which fully demonstrate all aspects of the constructive nature of the theory.

Keywords: separability, time-dependent second-order equations, derivations, forms along a map
AMS classification: 34A26, 53C05, 58Fxx

Running head: Separability of time-dependent systems.

[*]Senior Research Associate at the National Fund for Scientific Research (Belgium)

# 1 Introduction

In a recent paper [9] Martínez *et al* have established necessary and sufficient conditions for complete separability of a system of autonomous second-order ordinary differential equations. These conditions, which are of an algebraic and algorithmic nature, are expressed in terms of geometric objects that are directly related to the system under consideration and, therefore, in principle can be tested on any given second-order system (cf. [12] for a comprehensive application). This certainly yields an advantage upon some other approaches to the separability problem, where the characterization of partial or complete separability of second-order systems relies on the existence of an additional structure (see e.g. [4, 5, 8]). The separability analysis presented in [9] originated from the theory of derivations of scalar and vector-valued forms along a tangent bundle projection $\tau : TM \to M$, which was developed in [10, 11] for its potential application to the geometrical study of second-order differential equations fields (SODE's).

In [13] we have extended this calculus to a time-dependent framework, where the role of the tangent bundle projection $\tau$ is taken over by the projection $\pi : \mathbb{R} \times TM \to \mathbb{R} \times M$ of an "evolution space" $\mathbb{R} \times TM$ onto the underlying "space-time manifold" $\mathbb{R} \times M$. The product manifold $\mathbb{R} \times TM$ is frequently used for treating time-dependent classical mechanical systems (see e.g. [2, 3]). In developing the theory of derivations of forms along $\pi$, however, we have carefully avoided to rely on the product structure, in order to make the theory compatible with time-dependent coordinate transformations. Therefore, all results and formulas from [13] remain valid in the more general situation where $\mathbb{R} \times M$ is replaced by an arbitrary fibre bundle $E$ over $\mathbb{R}$, and $\pi$ is the projection $\pi : J^1E \to E$.

The purpose of the present paper is to extend the separability analysis of SODE's to the time-dependent case. More precisely, we will be dealing with the following problem. Given a system of time-dependent second-order differential equations

$$\ddot{q}^i = f^i(t, q, \dot{q}) \qquad i = 1, \cdots, n, \tag{1}$$

under what conditions for the given $f^i$ does there exist a regular time- dependent coordinate transformation $Q^j = Q^j(t, q)$, such that in the new coordinates the system (1) decouples into $n$ independent second-order equations

$$\ddot{Q}^j = F^j(t, Q^j, \dot{Q}^j) \qquad j = 1, \cdots, n,$$

and if such coordinates exist, how can we construct them? Since we explicitly allow for time-dependent transformations, the problem under consideration is not merely a parametrized version of the autonomous case.

The paper is organized as follows. In Section 2 we recall some of the basic concepts and results from [13], presented in the more general setup of a jet bundle projection $\pi : J^1E \to E$. Section 3 is devoted to the notion of distribution along $\pi$ and the characterization of diagonalizability and separability of a particular class of vector-valued 1-forms along $\pi$. The separability problem for time-dependent second-order systems is then treated in Section 4 and two illustrative examples are discussed in Section 5.

## 2 Preliminaries

Let $\pi_0 : E \to \mathbb{R}$ be a fibre bundle with fibre dimension $n$, and let $\pi_1 : J^1E \to \mathbb{R}$ be its first jet bundle. For a detailed treatment of jet bundle theory we refer to [14]. Natural bundle coordinates on $E$ will be denoted by $(t, q^i)$, and the induced coordinates on $J^1E$ by $(t, q^i, \dot{q}^i)$. The coordinate transformations we will consider on $E$ are of the form

$$\tilde{t} = t, \quad \tilde{q}^i = \tilde{q}^i(t, q), \qquad i = 1, \cdots, n \tag{2}$$

with the corresponding transformations on $J^1E$ obtained via prolongation:

$$\tilde{t} = t, \quad \tilde{q}^i = \tilde{q}^i(t, q), \quad \dot{\tilde{q}}^i = \frac{\partial \tilde{q}^i}{\partial q^j}\dot{q}^j + \frac{\partial \tilde{q}^i}{\partial t}. \tag{3}$$

(Throughout this paper, the summation convention is used where appropriate). The vertical bundle to $\pi_0$ is denoted by $VE$, i.e. $VE = \{\xi \in TE \mid T\pi_0(\xi) = 0\}$, whereas $V(J^1E)$ will denote the vertical bundle to $\pi : J^1E \to E$. Recall that, when $E$ is the trivial bundle $\mathbb{R} \times M$, $J^1E$ can be identified with the extended tangent bundle $\mathbb{R} \times TM$.

In order to make the present paper reasonably self-contained, we now summarize those elements of the calculus of forms along $\pi$ [13], which will be relevant for our present purposes.

Vector fields along $\pi$ are sections of the pull-back bundle $\pi^*(TE)$ over $J^1E$. A canonically defined vector field along $\pi$ (see e.g. [14]) is the "total time-derivative operator"

$$\mathbf{T} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i}. \tag{4}$$

The $C^\infty(J^1E)$-module of vector fields along $\pi$ is denoted by $\mathcal{X}(\pi)$. Of special interest is the submodule $\overline{\mathcal{X}}(\pi)$ of sections of $\pi^*(VE)$. Every $X \in \mathcal{X}(\pi)$ has a canonical decomposition of the form

$$X = X^0\mathbf{T} + \overline{X}, \qquad \text{with } \overline{X} \in \overline{\mathcal{X}}(\pi), \tag{5}$$

which indicates that $\{\mathbf{T}, \partial/\partial q^i\}$ is the better choice for a local basis of $\mathcal{X}(\pi)$. There is a natural vertical lift operation from $\mathcal{X}(\pi)$ to $\mathcal{X}^V(J^1E)$, which induces an isomorphism between $\overline{\mathcal{X}}(\pi)$ and $\mathcal{X}^V(J^1E)$. In coordinates, if $X^0, X^i \in C^\infty(J^1E)$ are the components of $X$, with respect to the local basis just selected, we have $X^V = X^i\, \partial/\partial\dot{q}^i$. Obviously, $\mathbf{T}^V = 0$.

Sections of other appropriate pull-back bundles over $J^1E$ give rise to the graded algebra $\bigwedge(\pi) = \bigoplus_{i=0}^{n+1} \bigwedge^i(\pi)$ of scalar forms along $\pi$ and the $\bigwedge(\pi)$-module $V(\pi) = \bigoplus_{i=0}^{n+1} V^i(\pi)$ of vector-valued forms along $\pi$. Clearly, $\bigwedge^0(\pi) \equiv C^\infty(J^1E)$ and $V^0(\pi) \equiv \mathcal{X}(\pi)$. There is an isomorphism between $\bigwedge(\pi)$ and the module of semi-basic forms on $J^1E$ (which includes the submodule of contact forms). The appropriate local basis for $\bigwedge^1(\pi)$, dual to the basis $\{\mathbf{T}, \partial/\partial q^i\}$, is given by $\{dt, \theta^i\}$, where the $\theta^i = dq^i - \dot{q}^i dt$ are the contact 1-forms.

Derivations of type $i_*$ on $\bigwedge(\pi)$ are defined in the usual way (see e.g. [6]) and, in a way similar to (5), every $L \in V(\pi)$ has a unique decomposition

$$L = L^0 \otimes \mathbf{T} + \overline{L}, \quad L^0 = i_L\, dt. \tag{6}$$

The submodule of vector-valued forms for which $L^0 = 0$ is denoted by $\overline{V}(\pi)$. In particular, for the identity tensor field $I \in V^1(\pi)$, we have

$$I = dt \otimes \mathbf{T} + \overline{I}, \qquad \text{with } \overline{I} = \theta^i \otimes \frac{\partial}{\partial q^i}. \tag{7}$$

Remark: Elements of $\bigwedge(\pi)$ and $V(\pi)$ are said to be *basic* if they actually come from corresponding objects on $E$, regarded as forms along $\pi$ through composition with $\pi$.

For the survey of other derivations of interest, it is sufficient to know that derivations of $\bigwedge(\pi)$ are completely determined by their action on functions and on basic 1-forms; they can further be extended to derivations of $V(\pi)$ by specifying an action on basic vector fields in a consistent way. Type $i_*$ derivations, for example, are taken to be zero on basic vector fields. The same is true for the vertical exterior derivative $d^V$, as introduced in [13]. This canonically defined derivation of degree 1 is fully characterized by the following rules (with $F \in C^\infty(J^1E)$),

$$d^V F = \frac{\partial F}{\partial \dot{q}^i}\theta^i, \quad d^V(dt) = 0, \quad d^V\theta^i = dt \wedge \theta^i, \tag{8}$$

$$d^V\left(\frac{\partial}{\partial q^i}\right) = 0, \quad d^V\mathbf{T} = \overline{I}. \tag{9}$$

Recall that the commutator of two derivations $D_1$ and $D_2$, of degree $r_1$ and $r_2$ respectively, is a derivation of degree $r_1 + r_2$, defined by $[D_1, D_2] = D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1$. For $L \in V(\pi)$, we put $d^V_L = [i_L, d^V]$ and have in particular that $d^V_I \equiv d^V$. Using the decomposition (7) of $I$, we come across another kind of vertical exterior derivative, which will be important for the application we have in mind. Indeed, we find

$$d^V = d^V_{\overline{I}} + dt \wedge i_{\overline{I}}, \tag{10}$$

from which it follows that, in comparison with the defining relations (8) and (9), $d^V_{\overline{I}}$ coincides with $d^V$ on functions and on $\mathbf{T}$, but is zero otherwise. As a consequence, we have $d^V_{\overline{I}} \circ d^V_{\overline{I}} = 0$, a property which is not true for the full $d^V$.

For a complete classification of derivations of $\bigwedge(\pi)$ and $V(\pi)$, one needs a connection (in the sense of Ehresmann) on the bundle $\pi : J^1E \to E$ (see [13]). Such a connection provides us with a horizontal lift operation from $\mathcal{X}(\pi)$ to $\mathcal{X}(J^1E)$, linear over $C^\infty(J^1E)$ and thus determined (locally) by the horizontal lifts of the coordinate vector fields on $E$. Setting,

$$H_i = \left(\frac{\partial}{\partial q^i}\right)^H = \frac{\partial}{\partial q^i} - \Gamma^j_i \frac{\partial}{\partial \dot{q}^j}, \quad H_0 = \left(\frac{\partial}{\partial t}\right)^H = \frac{\partial}{\partial t} - \Gamma^j_0 \frac{\partial}{\partial \dot{q}^j}, \tag{11}$$

the functions $\Gamma^j_i$ and $\Gamma^j_0$ are called connection coefficients. Given an arbitrary vector field $Z$ on $J^1E$, we now have a unique decomposition

$$Z = {X_1}^H + {X_2}^V, \tag{12}$$

with $X_1 = T\pi \circ Z \in \mathcal{X}(\pi)$ and $X_2 \in \overline{\mathcal{X}}(\pi)$. The other important point about a connection is that it enables us to define a horizontal exterior derivative $d^H$, whose action on $\bigwedge(\pi)$

and $V(\pi)$ is completely determined by

$$d^{H}F = H_i(F)\theta^i + \mathbf{T}^{H}(F)dt, \quad d^{H}(dt) = 0, \quad d^{H}\theta^i = \Gamma^i_k\theta^k \wedge dt, \tag{13}$$

$$d^{H}\left(\frac{\partial}{\partial q^i}\right) = (\Gamma^k_{0i} + \dot{q}^j\Gamma^k_{ji})dt \otimes \frac{\partial}{\partial q^k} + \Gamma^k_{ji}\theta^j \otimes \frac{\partial}{\partial q^k}, \quad d^{H}\mathbf{T} = 0, \tag{14}$$

with $\Gamma^k_{0i} = \partial\Gamma^k_0/\partial\dot{q}^i$, $\Gamma^k_{ji} = \partial\Gamma^k_j/\partial\dot{q}^i$.

Let us now pin down the case of interest for our present purposes. Second-order differential equations such as (1) are governed by a vector field

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^i\frac{\partial}{\partial q^i} + f^i(t,q,\dot{q})\frac{\partial}{\partial \dot{q}^i}, \tag{15}$$

living on $J^1E$, and intrinsically characterized by the properties $\langle\Gamma, dt\rangle = 1$ and $\langle\Gamma, \theta^i\rangle = 0$. Any such vector field (SODE) comes equipped with a connection on the bundle $\pi$, with connection coefficients

$$\Gamma^i_j = -\frac{1}{2}\frac{\partial f^i}{\partial \dot{q}^j}, \quad \Gamma^i_0 = -f^i - \dot{q}^j\Gamma^i_j. \tag{16}$$

The given vector field then itself is horizontal, in fact we have

$$\mathbf{T}^{H} = \Gamma. \tag{17}$$

Furthermore, the connection is torsionless, which in the present formalism means that $[d^{V}, d^{H}] = 0$.

A special class of derivations of degree 0 are the self-dual ones, which are characterized by the property,

$$D(\alpha(X)) = D\alpha(X) + \alpha(DX), \qquad \forall\alpha \in \textstyle\bigwedge^1(\pi),\ X \in \mathcal{X}(\pi). \tag{18}$$

Important examples of such derivations are the vertical and horizontal covariant derivatives $\mathrm{D}^{V}_X$ and $\mathrm{D}^{H}_X$. They depend in a $C^\infty(J^1E)$-linear way on the argument $X$ and can be defined, for example, by

$$\mathrm{D}^{V}_X = d^{V}_X - i_{d^{V}X}, \quad \mathrm{D}^{H}_X = d^{H}_X - i_{d^{H}X}. \tag{19}$$

Associated bracket operations on $\mathcal{X}(\pi)$ are given by (thanks to the zero torsion),

$$[X,Y]_{V} = \mathrm{D}^{V}_X Y - \mathrm{D}^{V}_Y X, \quad [X,Y]_{H} = \mathrm{D}^{H}_X Y - \mathrm{D}^{H}_Y X. \tag{20}$$

The horizontal bracket of basic vector fields coincides with the ordinary Lie bracket on $E$. Other relations of interest (proved in [13]) are, for $L \in V^1(\pi)$ and $X, Y \in \mathcal{X}(\pi)$:

$$d^{V}L(X,Y) = \mathrm{D}^{V}_X L(Y) - \mathrm{D}^{V}_Y L(X), \quad d^{H}L(X,Y) = \mathrm{D}^{H}_X L(Y) - \mathrm{D}^{H}_Y L(X). \tag{21}$$

Finally, the linearity of $\mathrm{D}^{V}_X$ and $\mathrm{D}^{H}_X$ enables us to define operators $\mathrm{D}^{V}$ and $\mathrm{D}^{H}$ (not derivations), which increase the covariant order of an arbitrary tensor field $U$ along $\pi$ by one. The defining relations of $\mathrm{D}^{V}$ and $\mathrm{D}^{H}$ read:

$$\mathrm{D}^{V}U(X,\ldots) = \mathrm{D}^{V}_X U(\ldots), \quad \mathrm{D}^{H}U(X,\ldots) = \mathrm{D}^{H}_X U(\ldots). \tag{22}$$

For coordinate calculations, it suffices that we list formulas for the action of $\mathrm{D}^{\scriptscriptstyle V}_X$ and $\mathrm{D}^{\scriptscriptstyle H}_X$ on functions and on $\mathcal{X}(\pi)$; corresponding formulas for the action on $\bigwedge^1(\pi)$ easily follow by duality. We have, with $X = X^0\mathbf{T} + X^i\,\partial/\partial q^i$,

$$\mathrm{D}^{\scriptscriptstyle V}_X F = X^{\scriptscriptstyle V}(F), \quad \mathrm{D}^{\scriptscriptstyle V}_X\left(\frac{\partial}{\partial q^i}\right) = 0, \quad \mathrm{D}^{\scriptscriptstyle V}_X\mathbf{T} = \overline{X}, \tag{23}$$

$$\mathrm{D}^{\scriptscriptstyle H}_X F = X^{\scriptscriptstyle H}(F), \quad \mathrm{D}^{\scriptscriptstyle H}_X\left(\frac{\partial}{\partial q^i}\right) = (X^j\Gamma^k_{ji} + X^0\Gamma^k_i)\frac{\partial}{\partial q^k}, \quad \mathrm{D}^{\scriptscriptstyle H}_X\mathbf{T} = 0. \tag{24}$$

It follows that, in particular, $\mathrm{D}^{\scriptscriptstyle V}_{\mathbf{T}} = 0$. On the other hand, $\mathrm{D}^{\scriptscriptstyle H}_{\mathbf{T}}$ is an important derivation for the dynamics of the given SODE; we call it the *dynamical covariant derivative* and denote it by $\nabla$. Observe in particular, in view of (17), that $\nabla F = \Gamma(F)$. Also, as $d^{\scriptscriptstyle H}\mathbf{T} = 0$, we see from (19) that $\mathrm{D}^{\scriptscriptstyle H}_{\mathbf{T}}$ coincides with $d^{\scriptscriptstyle H}_{\mathbf{T}}$; using the decomposition (7) of $I$, the analogue of (10) for the horizontal exterior derivative here becomes:

$$d^{\scriptscriptstyle H} = d^{\scriptscriptstyle H}_{\overline{I}} + dt \wedge \nabla. \tag{25}$$

As we already learned from the autonomous theory, another important concept in the study of second-order dynamical systems, is a type (1,1) tensor field called the *Jacobi endomorphism.* We have seen in [13] that, for the time-dependent framework, it is actually a component of the curvature tensor. For a general connection, the curvature is defined as a tensor field $R \in \overline{V}^2(\pi)$ and comes into the picture when one looks at the decomposition of the commutator $[d^{\scriptscriptstyle H}, d^{\scriptscriptstyle H}]$. In the case of a time-dependent SODE, the Jacobi endomorphism $\Phi \in \overline{V}^1(\pi)$ can be defined as $\Phi = i_{\mathbf{T}}R$ and has the coordinate expression,

$$\Phi = \Phi^i_j\theta^j \otimes \frac{\partial}{\partial q^i}, \quad \Phi^i_j = -\frac{\partial f^i}{\partial q^j} - \Gamma^i_k\Gamma^k_j - \Gamma(\Gamma^i_j). \tag{26}$$

In fact, it even turns out that the curvature then is entirely determined by $\Phi$, as we have

$$R = \tilde{R} + dt \wedge \Phi, \quad 3\tilde{R} = d^{\scriptscriptstyle V}_{\overline{I}}\Phi. \tag{27}$$

Other relations of interest are,

$$d^{\scriptscriptstyle H}\Phi = \nabla R, \quad d^{\scriptscriptstyle H}_{\overline{I}}\Phi = \nabla\tilde{R}, \tag{28}$$

which are equivalent in view of (25).

Concerning the commutator table of the different types of covariant derivatives we have discussed, we limit ourselves here to the formula

$$[\nabla, \mathrm{D}^{\scriptscriptstyle V}_X] = \mathrm{D}^{\scriptscriptstyle V}_{\nabla X} - \mathrm{D}^{\scriptscriptstyle H}_{\overline{X}}, \tag{29}$$

and refer to [13] for the full story.

To close this section, we want to come back to one of the statements in (14), namely the fact that $d^{\scriptscriptstyle H}\mathbf{T} = 0$. This is typically a feature of the time-dependent framework and should be of some worry for the separability analysis we are about to discuss. Indeed, in the autonomous setup, where the canonical vector field along $\tau : TM \to M$ reads

$\mathbf{T} = v^i\,\partial/\partial q^i$, $d^{\scriptscriptstyle H}\mathbf{T}$ is not zero and plays a significant role in the characterization of complete separability [9]. In fact, the tensor $\mathbf{t} = -d^{\scriptscriptstyle H}(v^i\,\partial/\partial q^i) = (\Gamma^i_j - v^k\Gamma^i_{jk})dq^j \otimes \partial/\partial q^i$ is known as the tension of the connection on $\tau : TM \to M$ and, apparently, there is no similar concept available in the calculus along $\pi : J^1E \to E$. To overcome this difficulty, we will now provide an intrinsic construction of a tension-like tensor, associated to every basic vector field with time- component equal to 1.

Let $Y$ be a vector field on $E$, satisfying: $\langle Y, dt\rangle = 1$. Its natural prolongation $Y^{(1)}$, being a vector field on $J^1E$, has a unique decomposition as in (12),

$$Y^{(1)} = {Y_1}^{\scriptscriptstyle H} + {Y_2}^{\scriptscriptstyle V}, \qquad \text{with } \; Y_2 \in \overline{\mathcal{X}}(\pi) \text{ and } Y_1 = Y \circ \pi.$$

We put

$$\mathbf{t}_Y = d^{\scriptscriptstyle V} Y_2 \in V^1(\pi).$$

In coordinates, if $Y = \partial/\partial t + u^i\,\partial/\partial q^i$, using the fact that for a SODE-connection, (16) implies that $\Gamma^i_{kj} \equiv \Gamma^i_{jk}$, $\Gamma^i_{0j} \equiv \Gamma^i_j - \dot q^k \Gamma^i_{jk}$, we find

$$\mathbf{t}_Y = \left(\Gamma^i_j - \dot q^k\Gamma^i_{jk} + u^k\Gamma^i_{jk} + \frac{\partial u^i}{\partial q^j}\right)\theta^j \otimes \frac{\partial}{\partial q^i}. \tag{30}$$

The inspiration which led to the introduction of this concept will become more transparent later on. For the time being, it suffices to make the following observations. In the neighbourhood of an arbitrary point on $E$, there exists a coordinate transformation of the form (2), which will straighten out $Y$ to the form $\partial/\partial t$. In those new coordinates $(t, \tilde q^i)$, $\mathbf{t}_Y$ will read

$$\mathbf{t}_Y = (\tilde\Gamma^i_j - \dot{\tilde q}^k\tilde\Gamma^i_{jk})\tilde\theta^j \otimes \frac{\partial}{\partial \tilde q^i}, \tag{31}$$

and so will resemble the tension from the autonomous theory. Note that if $E$ were taken to be the Cartesian product $\mathbb{R} \times M$, a tensor of the form (31) would be well defined, e.g. as $-d^{\scriptscriptstyle H}(\dot q^i\,\partial/\partial q^i)$. It would not have proper behaviour, however, when coordinate transformations of the form (2) are allowed, which do not respect the product structure of $\mathbb{R} \times M$.

# 3 Diagonalizable and separable type (1,1) tensor fields

From the treatment of the autonomous case [9], we recall that complete separability of a system of second-order differential equations depends, among other things, on the properties of the eigendistributions of its Jacobi endomorphism $\Phi$. In particular, diagonalizability of $\Phi$ was the first prerequisite and will again be a necessary condition in the time-dependent case. As a matter of fact, if a time-dependent second-order system like (1) is completely separated, it is easily seen from (16) and (26) that the matrix $(\Phi^i_j)$ is diagonal.

Guided by the analogy with the autonomous case, this section will be devoted to the study of such concepts as (integrable) distributions along $\pi$ and diagonalizability of (a class of) vector-valued 1-forms along $\pi$. The general setup is that of the previous section, with $\pi_0 : E \to \mathbb{R}$ a fibre bundle with $n$-dimensional fibres, and whereby the affine bundle $\pi : J^1E \to E$ is supposed to be equipped with a torsionless connection (i.e. a SODE connection).

**Definition 3.1** *A $d$-dimensional distribution $\mathcal{D}$ along $\pi$ is a smooth assignment of a $d$-dimensional subspace $\mathcal{D}(m)$ of $T_{\pi(m)}E$ for every $m \in J^1E$. A distribution $\mathcal{D}$ along $\pi$ is called vertical if $\mathcal{D}(m)$ is a subspace of $V_{\pi(m)}E$ (the vertical tangent space to $\pi_0$ at $\pi(m)$).*

Alternatively, a distribution (resp. vertical distribution) along $\pi$ can be seen as a vector subbundle of the pull-back bundle $\pi^*(TE)$ (resp. $\pi^*(VE)$) over $J^1E$. In the above definition, 'smooth' means that in a neighbourhood $U$ of each point $m \in J^1E$, one can find $d$ independent vector fields $X_i$ along $\pi$ such that $\mathcal{D}(m')$ is spanned by $\{X_i(m') : i = 1, \cdots, d\}$ for every $m' \in U$. A vector field $X$ along $\pi$ is said to belong to $\mathcal{D}$ if $X(m) \in \mathcal{D}(m)$ at each point, and we then simply write $X \in \mathcal{D}$.

**Definition 3.2** *A (vertical) distribution $\mathcal{D}$ along $\pi$ is called basic if there exists a (vertical) distribution $\mathcal{E}$ on $E$ such that $\mathcal{D}(m) = \mathcal{E}(\pi(m))$ for each $m \in J^1E$. $\mathcal{D}$ is called integrable if it is basic and if the associated (vertical) distribution on $\mathcal{E}$ is integrable in the sense of Frobenius.*

The distributions which will play a role in the separability analysis are vertical ones. We will therefore confine ourselves here to vertical distributions along $\pi$, although many of the subsequent considerations also apply (or trivially extend) to the more general distributions.

From definition 3.2 it readily follows that a vertical distribution $\mathcal{D}$ along $\pi$ is basic if and only if it is locally spanned by basic vector fields belonging to $\overline{\mathcal{X}}(\pi)$. Now, we know from (23) that the operator $\mathrm{D}^V_X$ vanishes on basic vector fields. Hence, if a vertical distribution $\mathcal{D}$ along $\pi$ is basic, we will have $\mathrm{D}^V_X(\mathcal{D}) \subset \mathcal{D}$ for all $X \in \mathcal{X}(\pi)$, or equivalently, since $\mathrm{D}^V_{\mathbf{T}} = 0$, $\mathrm{D}^V_{\overline{X}}(\mathcal{D}) \subset \mathcal{D}$ for all $\overline{X} \in \overline{\mathcal{X}}(\pi)$. The converse is equally true and can be proved in exactly the same way as in the autonomous case. Next, since on basic vector fields the horizontal bracket (20) coincides with the ordinary Lie bracket on $E$ and involutivity is the necessary and sufficient condition for integrability of an ordinary distribution, one can easily see that a basic vertical distribution along $\pi$ is integrable if and only if it is $[\,,\,]_H$-invariant. Summarizing, we have shown that the following result holds.

**Proposition 3.3** *Let $\mathcal{D}$ be a vertical distribution along $\pi$. Then: (i) $\mathcal{D}$ is basic if and only if it is $\mathrm{D}^V_X$-invariant for every $X \in \overline{\mathcal{X}}(\pi)$; (ii) $\mathcal{D}$ is integrable if and only if it is basic and $[\,,\,]_H$-invariant.* □

A *co-distribution $\mathcal{D}^*$ along $\pi$* is a smooth assignment of a subspace $\mathcal{D}^*(m)$ of $T^*_{\pi(m)}E$, of fixed dimension, for each $m \in J^1E$. A co-distribution along $\pi$ is called basic if it is spanned by a co-distribution $\mathcal{E}^*$ on $E$, and it is called integrable if it is basic and the associated co-distribution $\mathcal{E}^*$ on $E$ is integrable in the sense of Frobenius. Similar to the autonomous case it is rather straightforward to check (using local representations) that a co-distribution $\mathcal{D}^*$ along $\pi$ is basic if and only if it is $\mathrm{D}^V_X$-invariant for every $X \in \overline{\mathcal{X}}(\pi)$. Next, assume $\mathcal{D}^*$ is basic. Then, $\mathcal{D}^*$ is locally spanned by some independent basic 1-forms $\beta^i$ along $\pi$. Now, $d^H$ was constructed in such a way that, for basic forms along $\pi$, it coincides with the exterior derivative $d$ on $\bigwedge(E)$. So, for $\alpha = a_i(t,q,\dot{q})\beta^i \in \mathcal{D}^*$, we have $d^H\alpha = d^H a_i \wedge \beta^i + a_i\, d\beta^i$. From the definition of integrability it then easily follows that a basic co-distribution $\mathcal{D}^*$ is integrable if and only if it is $d^H$-closed, in the sense that for any $\alpha \in \mathcal{D}^*$, $d^H\alpha$ belongs to the ideal generated by $\mathcal{D}^*$. Finally, we observe that, given any vertical distribution $\mathcal{D}$ along $\pi$, its annihilator $\mathcal{D}^\perp = \{\alpha \in \bigwedge^1(\pi) \mid \alpha(X) = 0$ for all $X \in \mathcal{D}\}$ determines a co-distribution $\mathcal{D}^*$ along $\pi$ which contains the basic 1-form $dt$. Taking all this into account, the proof of the following proposition merely requires some minor modifications with respect to the proof of lemma 3.4 in [9], and we therefore omit it. It is to be emphasized, however, that the coordinate transformations involved in obtaining the desired result are all of type (2), i.e. they are time-dependent without changing the time-coordinate itself.

**Proposition 3.4** *Let $\mathcal{D}_A$ $(A = 1, \cdots, r)$ be $r$ complementary vertical distributions along $\pi$, i.e. $V_{\pi(m)}E = \bigoplus_{A=1}^r \mathcal{D}_A(m)$ for all $m \in J^1E$, such that each $\mathcal{D}_A$ is integrable. Assume furthermore that every sum of $\mathcal{D}_A$'s is $[\ ,\ ]_H$- closed. Then, for each $m \in J^1E$ there exist local coordinates $(t, q^{A\alpha})_{A=1,\cdots,r;\alpha=1,\cdots,\dim\mathcal{D}_A}$ defined on an open neighbourhood of $\pi(m)$, such that each $\mathcal{D}_A$ is spanned by $\{\partial/\partial q^{A\alpha}\}_{\alpha=1,\cdots,\dim\mathcal{D}_A}$.* □

The distributions we are interested in for the purpose of studying separability of second-order systems, are those determined by the eigenspaces of a certain class of vector-valued 1-forms along $\pi$. More precisely, we define the set

$$\tilde{V}^1(\pi) = \{U \in V^1(\pi) \mid i_{\mathbf{T}}U = 0 \text{ and } U(\mathcal{X}(\pi)) \subset \overline{\mathcal{X}}(\pi)\}.$$

Clearly, $\tilde{V}^1(\pi)$ is a $C^\infty(J^1E)$-submodule of $V^1(\pi)$ and its elements are locally of the form

$$U = U^i_j \theta^j \otimes \frac{\partial}{\partial q^i}. \tag{32}$$

Note, in particular, that the Jacobi endomorphism $\Phi$ of a SODE belongs to $\tilde{V}^1(\pi)$. A tensor field $U \in \tilde{V}^1(\pi)$ determines, at each point $m \in J^1E$, a linear endomorphism of the vertical tangent space $V_{\pi(m)}E$. This linear map is represented by the $(n \times n)$-matrix $(U^i_j(m))$.

**Definition 3.5** *A type (1,1) tensor field $U \in \tilde{V}^1(\pi)$ is said to be diagonalizable if: (i) for each $m \in J^1E$, the linear map $U(m)|_{V_{\pi(m)}E} : V_{\pi(m)}E \to V_{\pi(m)}E$ is diagonalizable, in the sense that the real Jordan normal form of $(U^i_j(m))$ is diagonal; (ii) there (locally) exist smooth functions $\mu_A$ such that $\mu_A(m)$ is an eigenvalue of $U(m)|_{V_{\pi(m)}E}$; (iii) the rank of $\mu_A\overline{I} - U$ is constant.*

The functions $\mu_A$ are called eigenfunctions of $U$. The eigendistribution associated with $\mu_A$ is the distribution $\mathcal{D}_A$ along $\pi$, determined by $\mathcal{D}_A(m) = \ker(U - \mu_A\overline{I})(m)|_{V_{\pi(m)}E}$. $\mathcal{D}_A$ is a vertical distribution and at each $m$, $V_{\pi(m)}E$ is the direct sum of all $\mathcal{D}_A(m)$.

In what follows, we will use $r$ to denote the number of eigenfunctions of a diagonalizable $U \in \tilde{V}^1(\pi)$ and $d_A$ to denote the dimension of $\mathcal{D}_A$.

**Proposition 3.6** *Let $D$ be a self-dual derivation such that $D(\overline{\mathcal{X}}(\pi)) \subset \overline{\mathcal{X}}(\pi)$. Then, the eigendistributions $\mathcal{D}_A$ of a diagonalizable $U \in \tilde{V}^1(\pi)$ are invariant under $D$ if and only if the commutator $[DU, U]|_{\overline{\mathcal{X}}(\pi)} = 0$.*

**Proof:** $D$ being a self-dual derivation, we obtain for every $X \in \mathcal{D}_A$:

$$\begin{aligned} DU(X) &= D(U(X)) - U(DX) \\ &= (D\mu_A)X + \mu_A(DX) - U(DX). \end{aligned}$$

Since $X \in \overline{\mathcal{X}}(\pi)$ it follows that also $DX \in \overline{\mathcal{X}}(\pi)$. Therefore, the previous relation can be rewritten as

$$DU(X) = (\mu_A \overline{I} - U)(DX) + (D\mu_A)(X), \tag{33}$$

from which one can deduce that

$$[DU, U](X) = (\mu_A \overline{I} - U)^2(DX),$$

showing that $DX \in \mathcal{D}_A$ if and only if $[DU, U](X) = 0$. Since this holds for every $X \in \mathcal{D}_A$ and for all eigendistributions $\mathcal{D}_A$, the result readily follows. □

**Corollary 3.7** *The eigendistributions $\mathcal{D}_A$ of $U$ are basic if and only if $[\mathrm{D}^{\mathrm{v}}_X U, U]|_{\overline{\mathcal{X}}(\pi)} = 0$ for all $X \in \overline{\mathcal{X}}(\pi)$.*

**Proof:** From (23) we learn that $\mathrm{D}^{\mathrm{v}}_X(\overline{\mathcal{X}}(\pi)) \subset \overline{\mathcal{X}}(\pi)$ for any $X \in \mathcal{X}(\pi)$, and a combination of propositions 3.3 and 3.6 then immediately leads to the desired result. □

Under the assumptions of proposition 3.6, equation (33) implies

$$DU(X) = (D\mu_A)X$$

for every $X \in \mathcal{D}_A$. However, a stronger requirement on $D$ is needed to make sure that also $DU$ belongs to $\tilde{V}^1(\pi)$.

**Lemma 3.8** *Let $D$ be a self-dual derivation such that $D\mathbf{T}$ is proportional to $\mathbf{T}$ (i.e. $\overline{D\mathbf{T}} = 0$) and $D(\overline{\mathcal{X}}(\pi)) \subset \overline{\mathcal{X}}(\pi)$, then $D(\tilde{V}^1(\pi)) \subset \tilde{V}^1(\pi)$. Moreover, in such a case we also have: for $L \in \tilde{V}^1(\pi)$, $[DL, L]|_{\overline{\mathcal{X}}(\pi)} = 0$ if and only if $[DL, L] = 0$.*

**Proof:** Using the properties of $D$ and the definition of $\tilde{V}^1(\pi)$, we immediately see that for every $L \in \tilde{V}^1(\pi)$

$$i_{\mathbf{T}}(DL) = -i_{D\mathbf{T}}L = 0$$

and

$$DL(X) = D(L(X)) - L(DX) \in \overline{\mathcal{X}}(\pi)$$

for all $X \in \mathcal{X}(\pi)$. Hence, $DL \in \tilde{V}^1(\pi)$. The second part follows from $[DL, L](\mathbf{T}) = 0$. □

Assuming $D$ verifies the conditions of the lemma, $D$ leaves the eigendistributions of a diagonalizable $U \in \tilde{V}^1(\pi)$ invariant, if and only if $[DU, U] = 0$. It then follows that $DU$ is also diagonalizable, with eigenfunctions $D\mu_A$.

**Definition 3.9** *A diagonalizable $U \in \tilde{V}^1(\pi)$ is said to be diagonalizable in coordinates if for each $m \in J^1E$ there exist local bundle coordinates $(t, q^i)$ in a neighbourhood of $\pi(m)$, such that (the local representation (32) of) $U$ is diagonal with respect to the induced coordinates $(t, q^i, \dot{q}^i)$ on $J^1E$.*

In order to establish criteria for diagonalizability in coordinates, we first introduce some new tensorial objects along $\pi$, in full analogy with the autonomous case. For general $L \in \tilde{V}^1(\pi)$ we define an object $C_L^V$ as follows

$$C_L^V(X, Y) = [\mathrm{D}_X^V L, L](Y) \text{ for all } X, Y \in \overline{\mathcal{X}}(\pi),$$
$$C_L^V(Z, \mathbf{T}) = C_L^V(\mathbf{T}, Z) = 0 \text{ for all } Z \in \mathcal{X}(\pi).$$

It is straightforward to check that $C_L^V$ is a type (1,2) tensor field along $\pi$ and $C_L^V(X, Y) \in \overline{\mathcal{X}}(\pi)$ for all $X, Y \in \mathcal{X}(\pi)$. In coordinates, with $L = L^i_j \theta^j \otimes \partial/\partial q^i$, $C_L^V$ reads

$$C_L^V = \left( \frac{\partial L^k_m}{\partial \dot{q}^i} L^m_j - \frac{\partial L^m_j}{\partial \dot{q}^i} L^k_m \right) \theta^i \otimes \theta^j \otimes \frac{\partial}{\partial q^k}.$$

Replacing $\mathrm{D}_X^V$ by $\mathrm{D}_X^H$, we obtain a similar type (1,2) tensor field $C_L^H$ along $\pi$. Next, for $L \in \tilde{V}^1(\pi)$ and arbitrary $X, Y \in \mathcal{X}(\pi)$ we put

$$H_L^H(X, Y) = C_L^H(L^2(X), Y) - 2LC_L^H(L(X), Y) + L^2 C_L^H(X, Y).$$

This determines another type (1,2) tensor field along $\pi$ and we also have

$$H_L^H(\mathbf{T}, X) = H_L^H(X, \mathbf{T}) = 0, \tag{34}$$

for all $X \in \mathcal{X}(\pi)$. (A similar construction, using $C_L^V$ instead of $C_L^H$, produces a tensor field $H_L^V$ which, however, will not be used in the sequel).

Returning now to a diagonalizable $U \in \tilde{V}^1(\pi)$ we note first of all that $C_U^V = 0$ is equivalent to $[\mathrm{D}_X^V U, U]|_{\overline{\mathcal{X}}(\pi)} = 0$ for every $X \in \overline{\mathcal{X}}(\pi)$. Next, taking into account (34), we see that $H_U^H$ is symmetric if and only if its restriction to $\overline{\mathcal{X}}(\pi) \times \overline{\mathcal{X}}(\pi)$ is symmetric. Then, following exactly the same reasoning as in the proof of theorem 4.2 in [9], it can be shown that $H_U^H$ is symmetric if and only if each $\mathcal{D}_A$ and every sum of $\mathcal{D}_A$'s is $[\ ,\ ]_H$-closed. By virtue of propositions 3.3, 3.4 and 3.6, we may already conclude that $C_U^V = 0$ and $H_U^H$ symmetric imply the existence of local coordinates $(t, q^{A\alpha})$, with $A = 1, \cdots, r$ and $\alpha = 1, \cdots, d_A$, such that $\mathcal{D}_A$ is spanned by $\{\partial/\partial q^{A\alpha}\}$. In terms of these coordinates, $U$ is of the form (with summation over $A$)

$$U = \mu_A(t, q, \dot{q}) \left( \sum_{\alpha=1}^{d_A} \theta^{A\alpha} \otimes \frac{\partial}{\partial q^{A\alpha}} \right) \tag{35}$$

and, thus, is diagonal. Conversely, if $U$ is diagonalizable in coordinates it is readily verified, using (35), that $C_U^V = 0$ and $H_U^H$ is symmetric. Summarizing, we may draw the following conclusion.

**Proposition 3.10** *A diagonalizable $U \in \tilde{V}^1(\pi)$ is diagonalizable in coordinates if and only if $C^{V}_U = 0$ and $H^{H}_U$ is symmetric.* $\square$

From (24) it follows that the dynamical covariant derivative $\nabla = \mathrm{D}^{H}_{\mathbf{T}}$ satisfies the conditions of lemma 3.8. Moreover, the relation (29) implies that whenever a distribution along $\pi$ is both $\nabla$- and $\mathrm{D}^{V}_X$-invariant for some $X \in \overline{\mathcal{X}}(\pi)$, then it is also $\mathrm{D}^{H}_X$-invariant. We can now state the following important result.

**Theorem 3.11** *If a diagonalizable $U \in \tilde{V}^1(\pi)$ satisfies the conditions*

$$C^{V}_U = 0 \quad \textit{and} \quad [\nabla U, U] = 0.$$

*then $U$ diagonalizes in coordinates $(t, q^{A\alpha})$ such that the connection coefficients $\Gamma^{A\alpha}_{B\beta}$ vanish for $A \neq B$. The converse is also true.*

**Proof:** Using proposition 3.10, and taking into account the previous considerations, the proof of the 'if'-part of the theorem is completely analogous to the one given for the corresponding property in the autonomous case (cf. [9], proposition 4.3). Conversely, assume $U$ is diagonalizable in coordinates $(t, q^{A\alpha})$ such that $\Gamma^{A\alpha}_{B\beta} = 0$ for $A \neq B$. That $C^{V}_U = 0$ already follows from proposition 3.10. The additional statement about the connection coefficients moreover shows that the eigendistributions of $U$ are $\nabla$-invariant. It then follows from proposition 3.6 and lemma 3.8 that $[\nabla U, U] = 0$. $\square$

A diagonalizable $U \in \tilde{V}^1(\pi)$ is called *separable* if it is diagonalizable in coordinates and if in such coordinates, each eigenfunction $\mu_A$ depends on $t$ and on the corresponding coordinates $(q^{A\alpha}, \dot{q}^{A\alpha})_{\alpha=1,\cdots,d_A}$ only. Before establishing sufficient conditions for separability, we first prove a useful intermediate result.

**Lemma 3.12** *For each $L \in \tilde{V}^1(\pi)$ and arbitrary $X, Y \in \mathcal{X}(\pi)$, we have*

$$d^{V}_{\overline{I}} L(X,Y) = d^{V} L(\overline{X}, \overline{Y}); \quad d^{H}_{\overline{I}} L(X,Y) = d^{H} L(\overline{X}, \overline{Y}). \tag{36}$$

**Proof:** Knowing that $i_{\overline{I}}$ acts as the identity on $\tilde{V}^1(\pi)$, we have

$$d^{V}_{\overline{I}} L = [i_{\overline{I}}, d^{V}] L = i_{\overline{I}} d^{V} L - d^{V} L.$$

Using the decomposition (5) for $X$ and $Y$, it follows that

$$\begin{aligned} d^{V}_{\overline{I}} L(X,Y) &= d^{V} L(\overline{I}(X), Y) + d^{V} L(X, \overline{I}(Y)) - d^{V} L(X,Y) \\ &= d^{V} L(\overline{X}, Y) + d^{V} L(X, \overline{Y}) - d^{V} L(\overline{X}, \overline{Y}) \\ &\quad - X^0 d^{V} L(\mathbf{T}, \overline{Y}) - Y^0 d^{V} L(\overline{X}, \mathbf{T}) \\ &= d^{V} L(\overline{X}, \overline{Y}). \end{aligned}$$

The same computation applies to $d^{H}_{\overline{I}}$. $\square$

We now arrive at the main result of this section.

**Theorem 3.13** *If $U \in \tilde{V}^1(\pi)$ is diagonalizable and satisfies the following conditions: (i) $C^V_U = 0$, (ii) $[\nabla U, U] = 0$, (iii) $d^V_I U = 0$, (iv) $d^H_I U = 0$, then $U$ is separable and, moreover, its degenerate eigenfunctions are functions of $t$ only.*

**Proof:** In view of theorem 3.11, the conditions (i) and (ii) imply that $U$ is diagonalizable in coordinates. Consider now two eigenfunctions $\mu_A$ and $\mu_B$ of $U$ and take $X \in \mathcal{D}_A$, $Y \in \mathcal{D}_B$. We know that $X$ and $Y$ belong to $\overline{\mathcal{X}}(\pi)$. Applying (36) and (21), and taking into account assumptions (iii) and (iv) we obtain

$$\mathrm{D}^V_X U(Y) - \mathrm{D}^V_Y U(X) = 0, \quad \mathrm{D}^H_X U(Y) - \mathrm{D}^H_Y U(X) = 0. \tag{37}$$

As argued before, (i) and (ii) imply that the eigendistributions are $\mathrm{D}^V_{\overline{Z}}$- and $\nabla$-invariant and therefore, from (29), also $\mathrm{D}^H_{\overline{Z}}$-invariant. It then easily follows from (37) that

$$(\mathrm{D}^V_X \mu_B)Y - (\mathrm{D}^V_Y \mu_A)X = 0, \tag{38}$$

$$(\mathrm{D}^H_X \mu_B)Y - (\mathrm{D}^H_Y \mu_A)X = 0. \tag{39}$$

First, assume $A \neq B$. Then, $X$ and $Y$ are necessarily independent and (38) implies $\mathrm{D}^V_Y \mu_A = 0$ for every $Y \in \mathcal{D}_B$ and for all $B$, with $B \neq A$. Hence, in coordinates which diagonalize $U$, $\mu_A$ is independent of all velocity coordinates $\dot{q}^{B\beta}$ $(B \neq A; \beta = 1, \cdots, d_B)$. Similarily, (39) implies that $\mathrm{D}^H_Y \mu_A = 0$ for every $Y \in \mathcal{D}_B$ $(B \neq A)$. Taking into account that $\Gamma^{A\alpha}_{B\beta} = 0$ for $A \neq B$ (cf. theorem 3.11), we conclude that $\mu_A$ is also independent of all coordinates $q^{B\beta}$ $(B \neq A; \beta = 1, \cdots, d_B)$. Consequently, for each eigenfunction $\mu_A$ of $U$ we have that $\mu_A = \mu_A(t, q^{A\alpha}, \dot{q}^{A\alpha})$.

Next, assume $\mu_A$ is a degenerate eigenfunction. Choosing independent elements $X$ and $Y$ of $\mathcal{D}_A$, it follows from (38) and (39), with $A = B$, that $\mathrm{D}^V_X \mu_A = \mathrm{D}^H_X \mu_A = 0$ for all $X \in \mathcal{D}_A$. Hence, $\mu_A$ depends on $t$ only. □

REMARK: As in the autonomous case, one can prove that separability of a diagonalizable $U$ is already satisfied under the weaker assumptions: (i) $C^V_U = 0$, (ii) $[\nabla U, U] = 0$, (iii) $d^V U(UX, Y) = d^V U(X, UY)$, (iv) $d^H U(UX, Y) = d^H U(X, UY)$ for all $X, Y \in \overline{\mathcal{X}}(\pi)$. Under these conditions, however, one does not have the additional property that degenerate eigenfunctions only depend on $t$.

# 4 Separability of time-dependent second-order differential equations

Let $\Gamma$ be a SODE defined on $J^1E$, with local representation (15) and associated connection (16).

**Definition 4.1** *$\Gamma$ is said to be separable at a point $m \in J^1E$ if there exist local bundle coordinates $(t, \tilde{q}^i)$, defined on an open neighbourhood of $\pi(m)$, such that in the induced coordinates $(t, \tilde{q}^i, \dot{\tilde{q}}^i)$, the functions $\tilde{f}^i = \langle \Gamma, d\dot{\tilde{q}}^i \rangle$ depend on $t$ and on the corresponding coordinates $\tilde{q}^i$ and $\dot{\tilde{q}}^i$ only. $\Gamma$ is called separable if it is separable at each point $m \in J^1E$.*

Stated otherwise, a SODE is separable if locally the associated system of (time-dependent) second-order ordinary differential equations admits a full decoupling into $n$ separate one-dimensional second-order equations.

In order to appreciate the privileged role which will be attributed to the Jacobi endomorphism $\Phi$ in what follows, we mention the following property which is fairly easy to prove. If two different SODE's give rise to the same $\Gamma^i_j$ (not the same $\Gamma^i_0$) and the same Jacobi endomorphism $\Phi$, then separability of one implies separability of the other.

As observed before, the Jacobi endomorphism $\Phi$ of a SODE $\Gamma$ is an element of $\tilde{V}^1(\pi)$. Consequently, the results from the previous section can be applied to it. We now first present some conditions which already guarantee a form of 'partial separability' of a SODE.

**Proposition 4.2** *Let $\Gamma$ be a* SODE *on $J^1E$ and assume the following conditions are verified: (i) $\Phi$ is diagonalizable, (ii) $C^{\mathrm{v}}_\Phi = 0$, (iii) $[\nabla\Phi, \Phi] = 0$, (iv) $\tilde{R} = 0$. Then, in a neighbourhood of each point there exist local bundle coordinates in terms of which the system separates into single equations, one for each 1-dimensional eigenspace of $\Phi$, and into individual subsystems, one for each degenerate eigenvalue of $\Phi$, which is then necessarily a function of time only.*

**Proof:** By virtue of (27) and (28) we see that the condition $\tilde{R} = 0$ implies $d^{\mathrm{v}}_{\overline{I}}\Phi = 0$ and $d^{\mathrm{H}}_{\overline{I}}\Phi = 0$. Theorem 3.13 then tells us that $\Phi$ is separable and its degenerate eigenfunctions depend on $t$ only. In local coordinates $(t, q^{A\alpha}, \dot{q}^{A\alpha})$ which realize the separation of $\Phi$, we know that $\Gamma^{A\alpha}_{B\beta} = 0$ for $A \neq B$ and $\Phi$ is of the form (35). Using the local expressions (16) and (26) it is then straightforward to verify that the second-order system indeed decouples in the way described above. □

REMARK: A converse to this proposition is true in the following sense: if a second-order system decouples into separate blocks and for each subsystem of dimension greater than one, the corresponding Jacobi endomorphism is of the form $\mu(t)\overline{I}$, then the conditions (i) to (iv) are satisfied.

**Corollary 4.3** *If the conditions of proposition 4.2 are verified and all eigenvalues of $\Phi$ are nondegenerate, then $\Gamma$ is separable.* □

Now, in case $\Phi$ has degenerate eigenvalues, further conditions will have to be invoked in order to assure (complete) separability of $\Gamma$. To fix the ideas, assume $\mu_A(t)$ is a degenerate eigenfunction of $\Phi$ with a $d_A$-dimensional eigendistribution $\mathcal{D}_A$. Recall that $\mathcal{D}_A$ is a vertical integrable distribution along $\pi$, and let $\mathcal{E}_A$ denote the underlying vertical distribution on $E$ (cf. Section 3, definition 3.2). Through each point of $E$ then passes a $(d_A+1)$-dimensional submanifold $\mathcal{U}_A$ which is fibred over (a connected subset of) $\mathbb{R}$. The fibres of $\mathcal{U}_A$ are integral manifolds of $\mathcal{E}_A$ and coordinates on $\mathcal{U}_A$ are given by $(t, q^{A\alpha})_{\alpha=1,\cdots,d_A}$. The subsystem corresponding to $\mu_A$ then determines a SODE on $J^1\mathcal{U}_A$ and, by construction, its Jacobi endomorphism reads $\mu_A(t)\sum_{\alpha=1}^{d_A}\theta^{A\alpha}\otimes\partial/\partial q^{A\alpha}$ (no summation over $A$). As a result we see that the problem we have to tackle consists in finding conditions for the separability of a SODE whose Jacobi-endomorphism is a (time-dependent) multiple of the 'identity' $\overline{I}$. It is precisely at this point that, in the autonomous case, the notion of tension enters the picture. As pointed out at the end of Section 2, in the time-dependent framework there is no natural analogue of the tension available and the closest we can get to a tension-like object is by looking at the tensor field $\mathbf{t}_Y$ (30), associated to a vector field $Y$ on $E$ of the form $Y = \partial/\partial t + u^i(t,q)\,\partial/\partial q^i$. Observe, by the way, that $\mathbf{t}_Y$ belongs to $\tilde{V}^1(\pi)$.

We now first explain the general idea behind the subsequent analysis. Suppose the given $\Gamma$ is completely separable. Then, in coordinates which realize the separation, the matrix with components $\Gamma^i_j - \dot{q}^k\Gamma^i_{jk}$ will be diagonal (with eigenvalues depending on $t$ and at most separate coordinates). In the coordinates under consideration, this matrix can be thought of as being the coefficient matrix of the tensor $\mathbf{t}_Y$ for $Y = \partial/\partial t$. The idea is to express this intrinsically by the existence of a $\mathbf{t}_Y$ which satisfies at least the criteria of theorem 3.11, and which perhaps will be such that a coordinate transformation which diagonalizes $\mathbf{t}_Y$ will at the same time straighten out the vector field $Y$. Now, the conditions of theorem 3.11 are not strong enough to ensure that, in diagonalizing $\mathbf{t}_Y$, all $\Gamma^i_j$ for $i \neq j$ will become zero. To make sure that this happens we will assume in addition that all eigenfunctions of $\mathbf{t}_Y$ are different. The point is that this can somehow be done without loss of generality, because, if $\Gamma$ is separable and if the original matrix we thought of would have degenerate eigenvalues, a suitable scale transformation can bring us to a situation where the system is still totally decoupled and the corresponding matrix has all different eigenfunctions. This is the content of the lemma we prove first.

**Lemma 4.4** *Let $\Gamma$ be a* SODE *on $J^1E$ with $\Phi = \mu(t)\overline{I}$ and assume $\Gamma$ is separable. Then one can always find local bundle coordinates $(t,q^i)$ on $E$ such that in terms of the induced coordinates $(t,q^i,\dot{q}^i)$, the matrix $(\Gamma^i_j - \dot{q}^k\Gamma^i_{jk})$ is diagonal, with all diagonal elements different.*

**Proof:** Let $(t,q^i,\dot{q}^i)$ be local coordinates which realize the separation of $\Gamma$, so that the corresponding second-order system reads

$$\ddot{q}^i = f^i(t,q^i,\dot{q}^i), \qquad i = 1,\cdots,n. \tag{40}$$

In these coordinates we see that $\Gamma^i_j = -\frac{1}{2}\partial f^i/\partial\dot{q}^j = 0$ for $i \neq j$ and each $\Gamma^i_i$ is a function of $t, q^i$ and $\dot{q}^i$ only. Hence, the matrix $(\Gamma^i_j - \dot{q}^k\Gamma^i_{jk})$ is already diagonal, with diagonal elements $\phi_{(i)}$ say. Whenever two of the diagonal elements are equal, they can depend at most on $t$. Assume, for instance, $k$ of the $\phi_{(i)}$'s are equal. Without loss of generality we may take $\phi_{(1)} = \phi_{(2)} = \cdots = \phi_{(k)} = \phi(t)$. Consider then a coordinate transformation $(t,q^i) \to (t,\tilde{q}^i)$ of the form

$$q^\alpha = \rho^\alpha(t)\tilde{q}^\alpha \quad (\alpha = 1,\cdots,k), \qquad q^\beta = \tilde{q}^\beta \quad (\beta = k+1,\cdots,n)$$

with $\rho^\alpha$ arbitrary but non-vanishing smooth functions of $t$. This is a regular transformation of type (2) which, moreover, preserves the decoupled structure of the second-order system (40). In fact, in the new coordinates, the system reads

$$\ddot{\tilde{q}}^\alpha = \tilde{f}^\alpha(t,\tilde{q}^\alpha,\dot{\tilde{q}}^\alpha), \quad \ddot{\tilde{q}}^\beta = f^\beta(t,\tilde{q}^\beta,\dot{\tilde{q}}^\beta)$$

with

$$\tilde{f}^\alpha(t,\tilde{q}^\alpha,\dot{\tilde{q}}^\alpha) = -\frac{\ddot{\rho}^\alpha}{\rho^\alpha}\tilde{q}^\alpha - \frac{2\dot{\rho}^\alpha}{\rho^\alpha}\dot{\tilde{q}}^\alpha + \frac{1}{\rho^\alpha} f^\alpha(t,\rho^\alpha\tilde{q}^\alpha,\dot{\rho}^\alpha\tilde{q}^\alpha + \rho^\alpha\dot{\tilde{q}}^\alpha).$$

Clearly, the matrix $(\tilde{\Gamma}^i_j - \dot{\tilde{q}}^k\tilde{\Gamma}^i_{jk})$, with $\tilde{\Gamma}^i_j = -\frac{1}{2}\partial\tilde{f}^i/\partial\dot{\tilde{q}}^j$, is still diagonal and its diagonal elements are given by

$$\tilde{\phi}_{(a)} = \phi(t) + \frac{\dot{\rho}^\alpha}{\rho^\alpha} + \dot{\rho}^\alpha\tilde{q}^\alpha\frac{\partial\Gamma^\alpha_\alpha}{\partial\dot{q}^\alpha}(t,\rho^\alpha\tilde{q}^\alpha,\dot{\rho}^\alpha\tilde{q}^\alpha + \rho^\alpha\dot{\tilde{q}}^\alpha)$$

for $\alpha = 1,\cdots,k$ and $\tilde{\phi}_{(\beta)} = \phi_{(\beta)}(t,\tilde{q}^\beta,\dot{\tilde{q}}^\beta)$ for $\beta = k+1,\cdots,n$. From this it is easily seen that an appropriate choice of the functions $\rho^\alpha(t)$ will make all the $\phi_{(\alpha)}$'s $(\alpha = 1,\cdots,k)$ differ from each other. $\square$

We can now establish necessary and sufficient conditions for the separability of a SODE whose Jacobi endomorphism is a multiple of $\overline{I}$.

**Theorem 4.5** *Let $\Gamma$ be a* SODE *on $J^1E$ for which $\Phi = \mu(t)\overline{I}$. Then, $\Gamma$ is separable if and only if there exists in a neighbourhood of each point of $E$, a vector field $Y$, with $\langle Y, dt\rangle = 1$, such that the associated tensor field $\mathbf{t}_Y$ has the following properties: $\mathbf{t}_Y$ is diagonalizable with all eigenvalues different, $C^V_{\mathbf{t}_Y} = 0$ and $[\nabla\mathbf{t}_Y,\mathbf{t}_Y] = 0$.*

**Proof:** Consider an arbitrary point $m \in J^1E$ and suppose there exists a vector field $Y$, defined on a neighbourhood of $\pi(m)$, for which the conditions of the theorem hold. In view of theorem 3.11, $\mathbf{t}_Y$ is then diagonalizable in coordinates and in coordinates which diagonalize $\mathbf{t}_Y$, we have $\Gamma^i_j = 0$ for $i \neq j$. From the assumption on $\Phi$ and its general coordinate expression (26), it readily follows that, in the coordinates under consideration, the second-order system corresponding to $\Gamma$ completely decouples. Hence, $\Gamma$ is separable at $m$ and since this holds for each point $m \in J^1E$, $\Gamma$ is separable.
Conversely, assume $\Gamma$ is separable. From lemma 4.4 we then know there exist local coordinates $(t,q^i,\dot{q}^i)$ in terms of which the matrix $(\Gamma^i_j - \dot{q}^k\Gamma^i_{jk})$ is diagonal with all its diagonal entries different. A vector field $Y$ with the desired properties clearly is given by $\partial/\partial t$. Indeed, in agreement with (31), $\mathbf{t}_Y$ is represented, in the coordinates under consideration, by the above diagonal matrix and the vanishing of $C^V_{\mathbf{t}_Y}$ and $[\nabla\mathbf{t}_Y,\mathbf{t}_Y]$ then follows from theorem 3.11. $\square$

We finally recast the preceding analysis into the following overall statement.

**Theorem 4.6** *A* SODE *$\Gamma$ on $J^1E$ is separable if and only if the following conditions hold:*

*(i)* $\Phi$ *is diagonalizable,*

*(ii)* $C^V_\Phi = 0$,

*(iii)* $[\nabla\Phi,\Phi] = 0$,

*(iv)* $\tilde{R} = 0$,

*(v) for each degenerate eigenvalue $\mu_A$ of $\Phi$ there locally exists a submanifold $\mathcal{U}_A$ of $E$, fibred over (a connected subset of) $\mathbb{R}$, and a vector field $Y$ on $\mathcal{U}_A$ with $\langle Y, dt\rangle = 1$, such that the corresponding $\mathbf{t}_Y$ is diagonalizable, with all eigenvalues nondegenerate, and satisfies $C^{\prime V}_{\mathbf{t}_Y} = 0$, $[\nabla \mathbf{t}_Y, \mathbf{t}_Y] = 0$.* □

Before elaborating in some more detail on the practical implementation of this theory in the next section, a further remark is in order, to clarify that the final stage of the separability analysis is indeed fully compatible with the general idea behind the use of tension-like tensors $\mathbf{t}_Y$, as explained before lemma 4.4.

In situations where theorem 4.5 comes in, any coordinate transformation which diagonalizes $\mathbf{t}_Y$ will take care of the ultimate decoupling of $\Gamma$, though it will generally not at the same time straighten out $Y$. However, if we now write $\partial/\partial t + u^i\,\partial/\partial q^i$ for the representation of $Y$ in the new coordinates and take into account that $\mathbf{t}_Y$ is diagonal while $\Gamma^i_j = 0$ for $i \neq j$, we see from the coordinate expression (30) that $\partial u^i/\partial q^j$ will be zero for $i \neq j$. Hence, the vector field $Y$ itself gives rise to decoupled first-order equations and therefore, it is possible to introduce a further coordinate transformation which will straighten out $Y$, while not affecting the state of decoupling of $\Gamma$.

# 5 Applications

Although theorem 4.6 yields a global characterization of separability, in practical applications one will mainly be concerned with the purely local problem of investigating whether or not a given system of second-order differential equations completely decouples. In the autonomous case, the conditions for separability are of a purely algebraic nature, expressed entirely in terms of objects which can directly be computed from the second-order system under consideration (cf. [9, 12]). In the time-dependent case, however, this is only true as far as conditions (i) to (iv) of theorem 4.6 are concerned. Condition (v) is of a different nature and will be more difficult to implement in any practical procedure for testing separability.

If for a given SODE $\Gamma$, conditions (i) to (iv) of theorem 4.6 are verified, we know from proposition 4.1 that $\Gamma$ at least partially separates. Moreover, the theory provides us with a way of constructing local coordinates $(t, q^{A\alpha})$ in terms of which the partial decoupling of the second-order system takes places. Indeed, such coordinates can be obtained through the 'simultaneous' integration of the integrable vertical distributions on $E$ which generate the complementary eigendistribution of $\Phi$. If all eigenvalues of $\Phi$ happen to be nondegenerate, we are done, and the system is completely decoupled in those coordinates. In case, however, $\Phi$ has some degenerate eigenvalues, condition (v) has to be invoked for every subsystem which is still internally coupled. It will give rise to a system of non-linear partial differential equations for the functions $u^i(t, q)$ which define the vector field $Y$. In a favourable situation, all we need is a particular solution of these equations which gives rise to a tensor of type (30) with non-degenerate eigenvalues. The hard case is the one where the conclusion about separability is bound to be negative. For example, if a particular solution for the $u^i$ is found, which happens to produce a $\mathbf{t}_Y$ with degenerate eigenvalues, it may be hard to prove that no other solution exists which will yield a $\mathbf{t}_Y$ with non-degenerate eigenvalues.

Before embarking on some examples now, it is worth noting that, although diagonalizability of $\Phi$ is the first condition in the theoretical discussion, it will be one of the last to impose in practice. The reason is that computing the Jordan normal form of $\Phi$ may be quite laborious, so that all restrictions ensuing from the other algebraic conditions may help to alleviate this task. The general procedure in practical applications is to impose the conditions $\tilde{R}=0$, $C^V_\Phi=0$ and $[\nabla\Phi,\Phi]=0$ in that order and, if there is still some freedom left, then investigate the diagonalizability of $\Phi$. With $\Phi$ as in (26), a coordinate expression for $\tilde{R}$, according to (27) is given by

$$\tilde{R}^i_{jk}=\frac{1}{3}\left(\frac{\partial\Phi^i_k}{\partial\dot{q}^j}-\frac{\partial\Phi^i_j}{\partial\dot{q}^k}\right), \tag{41}$$

For convenience, we also list coordinate expressions for the components of $C^V_\Phi$ and $[\nabla\Phi,\Phi]$,

$$(C^V_\Phi)^i_{jk} = \frac{\partial\Phi^i_\ell}{\partial\dot{q}^j}\Phi^\ell_k-\Phi^i_\ell\frac{\partial\Phi^\ell_k}{\partial\dot{q}^j}, \tag{42}$$

$$\begin{aligned}[][\nabla\Phi,\Phi]^i_j &= \Gamma(\Phi^i_k)\Phi^k_j-\Phi^i_k\Gamma(\Phi^k_j)+\Gamma^i_k\Phi^k_\ell\Phi^\ell_j \\ &\quad +\Phi^i_k\Phi^k_\ell\Gamma^\ell_j-2\Phi^i_k\Gamma^k_\ell\Phi^\ell_j.\end{aligned} \tag{43}$$

Finally, in case $\Phi$ has degenerate eigenvalues one has to deal with condition (v) of theorem 4.6, which is not an entirely algebraic matter.

## Example 1.

Consider the following system of two coupled second-order ordinary differential equations (for notational convenience, coordinate indices are henceforth denoted as subscripts):

$$\begin{aligned}\ddot{q}_1 &= k(t)\dot{q}_1+\ell_1(t)\dot{q}_2+a(t)q_1^2-b(t)q_2^2-c(t)q_1,\\ \ddot{q}_2 &= \ell_2(t)\dot{q}_1+k(t)\dot{q}_2+2a(t)q_1q_2-c(t)q_2,\end{aligned} \tag{44}$$

where $a,b,c,k,\ell_1$ and $\ell_2$ are as yet arbitrary functions of time. The purpose now is to identify the conditions one has to impose on these functions in order that (44) be completely separable. For the system under consideration we easily find (see (16)) $\Gamma^1_1=\Gamma^2_2=-k/2$, $\Gamma^1_2=-\ell_1/2$, $\Gamma^2_1=-\ell_2/2$ and the components of $\Phi$ read:

$$\begin{aligned}\Phi^1_1 &= \Phi^2_2=-2aq_1+c+\frac{1}{2}\dot{k}-\frac{1}{4}k^2-\frac{1}{4}\ell_1\ell_2,\\ \Phi^1_2 &= 2bq_2+\frac{1}{2}(\dot{\ell}_1-k\ell_1),\\ \Phi^2_1 &= -2aq_2+\frac{1}{2}(\dot{\ell}_2-k\ell_2).\end{aligned}$$

Since the $\Phi^i_j$'s are independent of the velocities, it is readily seen from (41) and (42) that $\tilde{R}$ and $C^V_\Phi$ identically vanish. We next consider the condition $[\nabla\Phi,\Phi]=0$. Its components

(43) here become polynomials in the coordinates and velocities, so the coefficients of the various independent monomials must separately be put equal to zero. In principle, this is a straightforward, but rather tedious job. The type of calculations and manipulations involved, lends itself perfectly well, however, to the use of computer algebra. We have made use of procedures in REDUCE to assist us in the present (and subsequent) computations. The vanishing of (43) eventually leads to the following set of restrictions on the functions $a, b, k, \ell_1$ and $\ell_2$:

$$a(\dot{\ell}_1 - k\ell_1) + b(\dot{\ell}_2 - k\ell_2) = 0, \tag{45}$$
$$a\dot{b} - \dot{a}b = 0, \tag{46}$$
$$b(a\ell_1 + b\ell_2) = 0, \tag{47}$$
$$a(a\ell_1 + b\ell_2) = 0, \tag{48}$$
$$\dot{b}(\dot{\ell}_2 - k\ell_2) - b\frac{d}{dt}(\dot{\ell}_2 - k\ell_2) + \dot{a}(\dot{\ell}_1 - k\ell_1) - a\frac{d}{dt}(\dot{\ell}_1 - k\ell_1) = 0, \tag{49}$$
$$b(2\dot{\ell}_1\ell_2 - \ell_1\dot{\ell}_2 - k\ell_1\ell_2) + a\ell_1(\dot{\ell}_1 - k\ell_1) = 0, \tag{50}$$
$$a(2\ell_1\dot{\ell}_2 - \dot{\ell}_1\ell_2 - k\ell_1\ell_2) + b\ell_2(\dot{\ell}_2 - k\ell_2) = 0, \tag{51}$$
$$(\dot{\ell}_2 - k\ell_2)\frac{d}{dt}(\dot{\ell}_1 - k\ell_1) - (\dot{\ell}_1 - k\ell_1)\frac{d}{dt}(\dot{\ell}_2 - k\ell_2) = 0, \tag{52}$$
$$(\dot{\ell}_2\ell_1 - \dot{\ell}_1\ell_2)(\dot{\ell}_2 - k\ell_2) = 0, \tag{53}$$
$$(\dot{\ell}_2\ell_1 - \dot{\ell}_1\ell_2)(\dot{\ell}_1 - k\ell_1) = 0. \tag{54}$$

For the further analysis we start by distinguishing the following four cases: (1) $ab \neq 0$, (2) $a \neq 0, b = 0$, (3) $a = 0, b \neq 0$, (4) $a = b = 0$.

(1) $ab \neq 0$.
Conditions (46–48) then yield

$$b(t) = \rho a(t), \qquad \ell_1(t) = -\rho\ell_2(t), \tag{55}$$

for some nonzero real constant $\rho$, and it turns out then that all relations (45–54) are satisfied. We now have to investigate diagonalizability of $\Phi$. Computing its eigenvalues $\mu_i$, we obtain

$$\begin{aligned} \mu_{1,2} &= \frac{1}{2}\left[\Phi^1_1 + \Phi^2_2 \pm \sqrt{(\Phi^1_1 - \Phi^2_2)^2 + 4\Phi^1_2\Phi^2_1}\right] \\ &= \frac{1}{4}\left[-8aq_1 + 2\dot{k} - k^2 + \rho\ell_2^2 + 4c \pm 2\sqrt{-\rho}(4aq_2 - \dot{\ell}_2 + k\ell_2)\right]. \end{aligned}$$

Since diagonalizability here refers to the real Jordan normal form, we must require $\rho < 0$. $\Phi$ then has two distinct real eigenvalues and, hence, is diagonalizable.

Summarizing, if (55) holds for some $\rho < 0$ and $a(t) \neq 0$, the conditions (i)–(iv) of the separability theorem 4.6 are verified and since the eigenvalues of $\Phi$ are nondegenerate, the given SODE (44) decouples. A regular transformation which diagonalizes $\Phi$ and, simultaneously, establishes the decoupling of (44), is given by

$$Q_1 = q_1 + \sqrt{-\rho}\, q_2, \qquad Q_2 = q_1 - \sqrt{-\rho}\, q_2.$$

The transformed system (44) becomes

$$\begin{aligned}\ddot{Q}_1 &= [k(t)+\sqrt{-\rho}\,\ell_2(t)]\dot{Q}_1 + a(t)Q_1^2 - c(t)Q_1,\\ \ddot{Q}_2 &= [k(t)-\sqrt{-\rho}\,\ell_2(t)]\dot{Q}_2 + a(t)Q_2^2 - c(t)Q_2.\end{aligned}$$

Note that the functions $a, c, k$ and $\ell_2$ are still competely arbitrary.

(2) $a \neq 0$, $b = 0$.
From (48) it follows that $\ell_1 = 0$ and then all other relations (45–54) are automatically verified. $\Phi$ now is of the form

$$\Phi = \begin{pmatrix} \mu & 0 \\ \Phi_1^2 & \mu \end{pmatrix}, \qquad \text{with } \Phi_1^2 \neq 0,$$

and hence is not diagonalizable. Therefore, in this case (44) can not be decoupled.

(3) $a = 0$, $b \neq 0$.
This case is similar to the previous one. We now have $\ell_2 = 0$ and again it turns out that $\Phi$ is not diagonalizable and, hence, (44) is not separable.

(4) $a = b = 0$.
Conditions (45–51) are trivially satisfied. The eigenvalues of $\Phi$ now read

$$\mu_{1,2} = \frac{1}{4}\left[2\dot{k} - k^2 - \ell_1\ell_2 + 4c \pm \sqrt{(\dot{\ell}_1 - k\ell_1)(\dot{\ell}_2 - k\ell_2)}\right]$$

and for these to be real we must have $(\dot{\ell}_1 - k\ell_1)(\dot{\ell}_2 - k\ell_2) \geq 0$.

Assume first $(\dot{\ell}_1 - k\ell_1)(\dot{\ell}_2 - k\ell_2) > 0$. Then $\Phi$ is diagonalizable with two distinct real eigenvalues. From (53) (or (54)) we further deduce that $\dot{\ell}_1\ell_2 = \ell_1\dot{\ell}_2$, i.e. $\ell_1(t) = \nu\ell_2(t)$ for some real constant $\nu$ which, in view of the above assumption, must be strictly positive. Note that relation (52) is then also fulfilled and we are thus again in a situation where the conditions (i)–(iv) of theorem 4.6 are verified. Since, moreover, $\Phi$ has two distinct eigenvalues, the system (44) must be competely separable. A transformation which diagonalizes $\Phi$ is of the same form as in case (1), with $\nu = -\rho$, and we find ourselves in fact in the corresponding subcase of decoupled equations for $Q_1$ and $Q_2$.

Assume on the contrary that $(\dot{\ell}_1 - k\ell_1)(\dot{\ell}_2 - k\ell_2) = 0$. Taking into account the relations (53),(54) and the requirement that $\Phi$ be diagonalizable, it follows that $\dot{\ell}_1 - k\ell_1$ and $\dot{\ell}_2 - k\ell_2$ both have to be zero, which implies

$$\ell_1(t) = \alpha_1 \exp(\int k\,dt), \qquad \ell_2(t) = \alpha_2 \exp(\int k\,dt),$$

for some real constants $\alpha_1, \alpha_2$. Condition (52) is then also satisfied and thus $[\nabla\Phi, \Phi]$ identically vanishes. In this case, $\Phi$ is already diagonal in the given coordinates, with equal diagonal elements. The given second-order system (44) now takes the form

$$\begin{aligned}\ddot{q}_1 &= k(t)\dot{q}_1 + \alpha_1\ell(t)\dot{q}_2 - c(t)q_1,\\ \ddot{q}_2 &= \alpha_2\ell(t)\dot{q}_1 + k(t)\dot{q}_2 - c(t)q_2,\end{aligned} \tag{56}$$

with $\ell(t) = \exp(\int k\, dt)$, and so is not yet decoupled, unless $\alpha_1 = \alpha_2 = 0$. Hence, in case at least one of the $\alpha_i$'s is nonzero, we are in a situation where condition (v) of theorem 4.6 has to be invoked. More precisely, assuming $\alpha_1^2 + \alpha_2^2 \neq 0$, (56) will be separable if and only if functions $u_1(t, q)$ and $u_2(t, q)$ can be found, regarded as components of a vector field $Y = \partial/\partial t + u_i \partial/\partial q_i$, such that the tensor field

$$\begin{aligned}\mathbf{t}_Y &= \left(\frac{\partial u_1}{\partial q_1} - \frac{1}{2}k\right)\theta_1 \otimes \frac{\partial}{\partial q_1} + \left(\frac{\partial u_1}{\partial q_2} - \frac{1}{2}\alpha_1 \ell\right)\theta_2 \otimes \frac{\partial}{\partial q_1} + \left(\frac{\partial u_2}{\partial q_1} - \frac{1}{2}\alpha_2 \ell\right)\theta_1 \otimes \frac{\partial}{\partial q_2} \\ &\quad + \left(\frac{\partial u_2}{\partial q_2} - \frac{1}{2}k\right)\theta_2 \otimes \frac{\partial}{\partial q_2},\end{aligned}$$

is diagonalizable, with distinct eigenvalues and satisfies $C^V_{\mathbf{t}_Y} = 0$ and $[\nabla \mathbf{t}_Y, \mathbf{t}_Y] = 0$. It is easily seen that the condition $C^V_{\mathbf{t}_Y} = 0$ is always verified here. The commutator condition, however, yields a set of non-linear partial differental equations for the $u_i$'s. The most obvious particular solution which can be surmised simply from the expression for $\mathbf{t}_Y$, is given by $u_1 = \frac{1}{2}\alpha_1 \ell q_2$, $u_2 = \frac{1}{2}\alpha_2 \ell q_1$. Unfortunately, this choice gives rise to a $\mathbf{t}_Y$ with a degenerate eigenvalue $-\frac{1}{2}k$. A more interesting particular solution is found to be,

$$\begin{aligned}u_1 &= \frac{1}{2}k q_1 + \alpha_1\left(\frac{1}{2}\ell + 1\right)q_2, \\ u_2 &= \frac{1}{2}k q_2 + \alpha_2\left(\frac{1}{2}\ell + 1\right)q_1,\end{aligned} \tag{57}$$

producing a $\mathbf{t}_Y$ of the form

$$\mathbf{t}_Y = \alpha_1 \theta_2 \otimes \frac{\partial}{\partial q_1} + \alpha_2 \theta_1 \otimes \frac{\partial}{\partial q_2}.$$

In case $\alpha_1\alpha_2 > 0$, this $\mathbf{t}_Y$ has two distinct real eigenvalues, namely $\pm\sqrt{\alpha_1\alpha_2}$. A transformation which diagonalizes $\mathbf{t}_Y$ is given by

$$\begin{aligned}Q_1 &= \sqrt{\alpha_1\alpha_2}\, q_1 + \alpha_2 q_2, \\ Q_2 &= \sqrt{\alpha_1\alpha_2}\, q_1 + \alpha_1 q_2.\end{aligned}$$

The transformed system (56) reads

$$\begin{aligned}\ddot{Q}_1 &= \left[k(t) + \sqrt{\alpha_1\alpha_2}\exp(\int k\, dt)\right]\dot{Q}_1 - c(t)Q_1, \\ \ddot{Q}_2 &= \left[k(t) - \sqrt{\alpha_1\alpha_2}\exp(\int k\, dt)\right]\dot{Q}_2 - c(t)Q_2\end{aligned}$$

and is completely decoupled.

In case $\alpha_1\alpha_2 \leq 0$ (but $\alpha_1^2 + \alpha_2^2 \neq 0$), the solution (57) for the $u_i$'s is no longer suitable for then $\mathbf{t}_Y$ has either complex eigenvalues or a degenerate (zero) eigenvalue. For the sake of brevity, we will not pursue the full analysis of this subcase, but conclude instead by pointing out one special case with $\alpha_1\alpha_2 = 0$, for which (56) is separable. Assume, for instance, $\alpha_1 = 0$, $\alpha_2 \neq 0$ and $k \neq 0$ but constant. Then, a suitable solution for the $u_i$'s is provided by $u_1 = (k/2)q_1$, $u_2 = kq_2 + (\alpha_2\ell(t)/4)q_1$, where $\ell(t) = \exp(kt)$. The corresponding $\mathbf{t}_Y$ is then diagonalizable with eigenvalues 0 and $k/2$. Hence, (56) can again be decoupled.

Another special case of (56), with $\alpha_1\alpha_2 < 0$, will be treated in the next example.

## Example 2.

Consider the following autonomous second-order system, depending on one real parameter $\lambda$,

$$\ddot{q}_1 = -\lambda \dot{q}_2, \qquad \ddot{q}_2 = \lambda \dot{q}_1, \tag{58}$$

with $\lambda \neq 0$. This is a special case of (56), with $k = c = 0$ and $\alpha_2 = -\alpha_1 = \lambda$. The reason for taking this simple system is that it exhibits the following interesting feature: when treated entirely within the autonomous framework, i.e. applying the results of [9], it turns out that (58) is not separable, whereas, as we will show, it can be decoupled when allowing for time-dependent transformations.

The connection coefficients $\Gamma^i_j$ and the components of the Jacobi endomorphism, corresponding to (58), are given by $\Gamma^1_1 = \Gamma^2_2 = 0$, $\Gamma^1_2 = \lambda/2$, $\Gamma^2_1 = -\lambda/2$ and $\Phi^1_1 = \Phi^2_2 = \lambda^2/4$, $\Phi^1_2 = \Phi^2_1 = 0$. Obviously, these expressions are the same, regardless of whether (58) is treated within the autonomous or within the time-dependent framework. It is straightforward to check that, in both cases, the algebraic conditions on $\Phi$ are satisfied. However, $\Phi$ has a degenerate eigenvalue. The tension $\mathbf{t}$, corresponding to (58) in the autonomous framework, has no real eigenvalues. Consequently, according to the theory in [9], (58) cannot be decoupled by means of a time-independent transformation.

In the time-dependent framework, on the other hand, we are led to investigate the tension-like object

$$\mathbf{t}_Y = \frac{\partial u_1}{\partial q_1}\theta_1 \otimes \frac{\partial}{\partial q_1} + \left(\frac{\partial u_1}{\partial q_2} + \frac{\lambda}{2}\right)\theta_2 \otimes \frac{\partial}{\partial q_1} + \left(\frac{\partial u_2}{\partial q_1} - \frac{\lambda}{2}\right)\theta_1 \otimes \frac{\partial}{\partial q_2} + \frac{\partial u_2}{\partial q_2}\theta_2 \otimes \frac{\partial}{\partial q_2}.$$

It manifestly satisfies the condition $C^V_{\mathbf{t}_Y} = 0$. The partial differential equations for the $u_i$'s, resulting from the commutator condition $[\nabla \mathbf{t}_Y, \mathbf{t}_Y] = 0$, admit the following particular solution

$$u_1 = -2q_1 - \left[\frac{\lambda}{2} + \mathrm{tg}\,(\lambda t)\right] q_2, \quad u_2 = \left[\frac{\lambda}{2} - \mathrm{tg}\,(\lambda t)\right] q_1.$$

With this choice, $\mathbf{t}_Y$ turns out to be diagonalizable and has two distinct eigenvalues. A transformation which does the job is given by

$$\begin{aligned} Q_1 &= [1 - \cos(\lambda t)]q_1 - \sin(\lambda t) q_2, \\ Q_2 &= \sin(\lambda t) q_1 + [1 - \cos(\lambda t)]q_2. \end{aligned}$$

(This transformation is regular for all $t \in \mathbb{R}\backslash\{2k\pi/\lambda; k \in \mathbb{Z}\}$). In the new coordinates the system (58) reads

$$\begin{aligned} \ddot{Q}_1 &= \frac{\lambda}{1 - \cos(\lambda t)}[\sin(\lambda t)\dot{Q}_1 - \lambda Q_1], \\ \ddot{Q}_2 &= \frac{\lambda}{1 - \cos(\lambda t)}[\sin(\lambda t)\dot{Q}_2 + \lambda Q_2], \end{aligned}$$

which indeed proves the point that the given autonomous system can be decoupled by means of a time-dependent transformation.

**Example 3.**

As a final example, consider the autonomous system

$$\ddot{q}_1 = 0, \qquad \ddot{q}_2 = b\dot{q}_1^3 \tag{59}$$

with $b$ a nonzero real constant. One easily verifies that for this system $\Phi \equiv 0$, and so conditions (i) to (iv) of theorem 4.6 are automatically satisfied. Since $\Phi$ has a degenerate (zero) eigenvalue, we are again forced to look for a suitable $\mathbf{t}_Y$. The condition $C^V_{\mathbf{t}_Y} = 0$ here leads to the following relations for the $u_i$'s:

$$\frac{\partial u_1}{\partial q_2} = 0, \qquad \frac{\partial u_1}{\partial q_1} = \frac{\partial u_2}{\partial q_2}. \tag{60}$$

Taking these into account, it turns out that $[\nabla \mathbf{t}_Y, \mathbf{t}_Y]$ identically vanishes. The general solution of (60) reads:

$$u_1 = \psi(t, q_1), \qquad u_2 = (\partial\psi/\partial q_1)q_2 + \nu(t, q_1),$$

where $\psi$ and $\nu$ are arbitrary functions, and $\mathbf{t}_Y$ then becomes

$$\mathbf{t}_Y = \begin{pmatrix} \partial\psi/\partial q_1 & 0 \\ \mu & \partial\psi/\partial q_1 \end{pmatrix}$$

with $\mu = 3/2(b\dot{q}_1^2) - 3b\psi\dot{q}_1 + (\partial^2\psi/\partial q_1^2)q_2 + 2(\partial\nu/\partial q_1)$. It is clear that $\mathbf{t}_Y$ can never have two different eigenvalues; moreover, with $b \neq 0$, it is not even diagonalizable. We thus encounter a situation here where no vector field $Y$ exists, satisfying the requirements of theorem 4.5. The system (59) is not separable.

As can be seen already on the above examples, the conditions of theorem 4.6 impose some very severe restrictions on a SODE. This, of course, should not come as a surprise since complete separability remains, generically speaking, a rather exceptional feature. In that respect it would certainly be interesting to investigate whether the above formalism can also be used to derive necessary and sufficient conditions for some form of partial separability (or "submersiveness" in the sense of [4, 8]). Finally, in case of a Lagrangian system, it may be of interest to study the interrelationship between complete separability of the Euler-Lagrange equations and separability in the sense of Hamilton-Jacobi (see e.g. [1, 7]) for the corresponding Hamiltonian system.

Acknowledgements. This research was partially supported by NATO Collaborative Research Grant (no. CRG 940195). F.C. and W.S. further thank the Belgian National Fund for Scientific Research for continuing support.

# References

[1] Benenti, S.: Separation of Variables in the Geodesic Hamilton-Jacobi Equation, in P. Donato *et al.* (eds.), *Symplectic Geometry and Mathematical Physics*, Progress in Math. **99**, Birkhäuser, Basel, 1991, pp. 1–35.

[2] Crampin, M., Prince, G.E. and Thompson, G.: A geometrical version of the Helmholtz conditions in time-dependent Lagrangian dynamics, *J. Phys. A: Math. Gen.* **17** (1984), 1437–1447.

[3] de León, M. and Rodrigues, P.R.: *Methods of Differential Geometry in Analytical Mechanics*, Mathematics Studies **158**, North-Holland, Amsterdam, 1989.

[4] de León, M. and Martín de Diego, D.: Nonautonomous submersive second-order differential equations and Lie symmetries, *Int. J. Theor. Phys.* **33** (1994), 1759–1781.

[5] Ferrario, C., Lo Vecchio, G., Marmo, G. and Morandi, G.: A separability theorem for dynamical systems admitting alternative Lagrangians, *J. Phys. A: Math. Gen.* **20** (1987), 3225–3236.

[6] Frölicher, A. and Nijenhuis, A.: Theory of vector valued differential forms, *Proc. Ned. Acad. Wetensch. Sér. A* **59** (1956), 338–359.

[7] Kalnins, E.G.: *Separation of Variables for Riemannian Spaces of Constant Curvature*, Pitman Monographs in Pure and Appl. Math. **28**, Longman Sci. & Technical, Essex, 1986.

[8] Kossowski, M. and Thompson, G.: Submersive second-order differential equations, *Math. Proc. Camb. Phil. Soc.* **110** (1991), 207–224.

[9] Martínez, E., Cariñena, J.F. and Sarlet, W.: Geometric characterization of separable second-order differential equations, *Math. Proc. Camb. Phil. Soc.* **113** (1993), 205–224.

[10] Martínez E., Cariñena, J.F. and Sarlet, W.: Derivations of differential forms along the tangent bundle projection, *Diff. Geometry and its Applications* **2** (1992), 17–43.

[11] Martínez, E., Cariñena, J.F. and Sarlet, W.: Derivations of differential forms along the tangent bundle projection II, *Diff. Geometry and its Applications* **3** (1993), 1–29.

[12] Sarlet, W., Cantrijn, F. and Martínez, E.: Application of the theory of separability of second-order differential equations, *J. Nonlinear Math. Phys.* **1** (1994), 5–24.

[13] Sarlet, W., Vandecasteele, A., Cantrijn, F. and Martínez, E.: Derivations of forms along a map: the framework for time-dependent second-order equations, Preprint, 1993.

[14] Saunders, D.J.: *The Geometry of Jet Bundles*, London Math. Soc. Lect. Notes Series **142**, Cambridge University Press, 1989.