Linear connections for systems of second-order ordinary differential equations

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Abstract

We describe the construction of a linear connection associated with a second-order differential equation field, calculate its curvature, and discuss some applications.

1 Introduction

Systems of second-order ordinary differential equations of the form $\ddot{x}^i = f^i(t, x^j, \dot{x}^j)$ arise naturally in a number of contexts: geodesics (auto-parallel curves), the calculus of variations, and classical mechanics spring readily to mind. Such a system of equations may be represented as a certain type of vector field (a second-order differential equation field) on a differentiable manifold of the form $\mathbb{R} \times TM$, where M is a manifold, of dimension m, on which the x^i, $i = 1, 2, \ldots, m$ are local coordinates, and TM is its tangent bundle. By using a representation like this, one can take a geometrical approach to tackling many problems encountered in the study of systems of second-order ordinary differential equations: for example, problems concerning conditions for the existence of coordinates with respect to which the equations take a special form – in which the right-hand sides vanish, or are linear, or in which the equations decouple; the inverse problem of the calculus of variations; analysis of symmetries; and problems concerned with the qualitative behaviour of families of solutions.

In this paper, we shall describe the construction, given any second-order differential equation field, of an associated linear connection in a certain vector bundle. This linear connection is a very effective tool for the investigation of problems of the kind described above. In particular, the vanishing of the curvature of the connection is the necessary and sufficient condition for the existence of coordinates with respect to which the solution curves of the equations are straight lines. Our construction may therefore be regarded as providing a generalization of ordinary connection theory to cover types of differential equations more general than those satisfied by geodesics.

Actually, the representation of the underlying manifold as $\mathbb{R} \times TM$ is not ideal for our purposes. The reason for this is that we wish to allow t-dependent coordinate transformations (where t is the standard coordinate on $\mathbb{R}$): that is, coordinate transformations on $\mathbb{R} \times M$ of the form $(t, x^i) \mapsto (t, y^i)$ with $y^i = y^i(t, x^j)$, together with the induced coordinate transformations on $\mathbb{R} \times TM$. Such coordinate transformations do not respect the product structure which, by implication, has been picked out once for all. So instead we shall develop the theory for an $(m+1)$-dimensional manifold E which is a fibre bundle over $\mathbb{R}$, with standard fibre M. Although E will be trivial, no one trivialization of it is to be preferred to any other, and the notation will reflect this. Furthermore, by working in this way we shall ensure that all our formulas are tensorial with respect to t-dependent coordinate transformations.

So we suppose given a fibre bundle E with projection $\pi: E \to \mathbb{R}$, and we consider its first-order jet bundle, which we denote by $\pi_1^0: J^1\pi \to E$. (Our notation follows that of Saunders [26], more or less.) The fibre of $J^1\pi$ over any point $p \in E$ is an affine space modelled on $V_p\pi$, the vector subspace of T_pE consisting of those vectors vertical with respect to π, that is, tangent to the fibre of E. Given any trivialization $E \equiv \mathbb{R} \times M$, we may identify $J^1\pi$ with $\mathbb{R} \times TM$, with π_1^0 corresponding to the tangent bundle projection $\tau_M: TM \to M$. A second-order differential equation field is a vector field on $J^1\pi$ with the property that its integral curves are jets of sections of π. Using the projection $\pi_1^0: J^1\pi \to E$, we may pull back the tangent bundle $\tau_E: TE \to E$ to obtain a vector bundle $\pi_1^{0*}(\tau_E)$ over $J^1\pi$. We shall show how to construct a linear connection on $\pi_1^{0*}(\tau_E)$ whenever we are given a second-order differential equation field on $J^1\pi$. The construction depends on the fact that the second-order differential equation field determines a horizontal distribution, or non-linear connection, on $J^1\pi$. As a matter of fact the same construction will work for any horizontal distribution on $J^1\pi$, but in this paper we shall discuss only the case of a horizontal distribution coming from a second-order differential equation field.

The data, therefore, consist of the bundle $\pi: E \to \mathbb{R}$, a second-order differential equation field defined on $J^1\pi$, and the corresponding *non-linear* connection on $J^1\pi$; from these we shall construct a *linear* connection in the form of a covariant derivative operator on sections of $\pi_1^{0*}(\tau_E)$. In order to prevent confusion we shall use the term 'connection' to refer only to the linear connection which we shall construct, and refer to the given non-linear connection always in terms of the horizontal distribution which defines it.

This paper has its origins in the work of Martínez, Cariñena and Sarlet on derivations of the algebra of forms along a projection map. Initially these authors concentrated on forms along a tangent bundle projection $\tau_M: TM \to M$ [20, 21]; their methods have recently been extended to the 'time-dependent' case, which is the case considered here [25]. Though the analysis of the properties of second-order differential equation fields was one of the motives for this work, the theory developed is very comprehensive, and by no means all of its results are required for an immediate understanding of the geometrical approach to the study of second-order differential equations. The situation is analogous to that found in ordinary differential geometry, where the full theory of derivations of exterior forms developed by Frölicher and Nijenhuis is not required for the study of geodesics. The present paper contains (among other things) an exposition, in a new guise, of those of the results of Martínez et al. which are of most relevance to the study of second-order differential equation fields. As a consequence, the paper should serve as a relatively short introduction to the more general theory developed by these authors.

The connection described in this paper is related to several connections which have previously been defined in various different contexts. One type of connection, to which ours is closely related, is the Berwald connection of Finsler geometry and its generalizations, which is described e.g. by Grifone [15] and by Bejancu [5]. However, the use of this kind of connection to study geometrical properties of *general* second-order differential equation fields does not seem to have been recognised by workers in the field of Finsler geometry, though Grifone has contributed extensively to the study of the horizontal structure which is associated with a general second-order differential equation field. On the other hand, connection theory has been used to analyse the properties of a particular class of solution curves of second-order differential equations, namely the geodesics of Finsler spaces, that is, the extremals of a Lagrangian which is positively homogeneous of degree 1 in the derivative variables. Authors in this field, such as Auslander [3] and, more recently, Bao and Chern [4], have been concerned mostly with extending results of Riemannian geometry, such as the theorems of Myers and Synge, to the more general setting of Finsler geometry. One other author who has done important work in this field is P. Foulon. Foulon's work, in a way, is more closely related to ours as, apart from applications to the study of extremals of Lagrangians [13, 14], his theory of general second-order equations [12] contains elements of the idea of a linear connection which we shall develop in Section 3. In fact, his notions of *Jacobi endomorphism* and *dynamical derivation* were among the sources of inspiration for the work by Martínez, Cariñena and Sarlet referred to above.

There is an alternative approach to the construction of linear connections associated with second-order differential equation fields, which has been developed by Massa and Pagani in the context of the formulation of classical mechanics [23], and also in a purely geometrical setting by Byrnes [6]. These authors obtain an ordinary linear connection on

$T(J^1\pi)$ rather than a vector bundle connection on $\pi_1^{0*}(\tau_E)$. We claim that our approach is the better one since it avoids an almost literal duplication of effort. We shall explain how the two approaches are related in Section 3 below.

We can claim, therefore, to provide a synthesis of several approaches to the study and use of connection theory and related topics in the context of second-order differential equation fields. We claim also that our work is distinctive in several ways. In the first place, we adopt a distinctive geometric setting, namely that of the first-order jet bundle of a manifold fibred over $\mathbb{R}$, which seems to us to be the most appropriate one for the study of time-dependent second-order differential equations. General Berwald connections are defined in [5] as connections in the vertical sub-bundle of the tangent bundle of a manifold. The connections adapted to Finsler geodesics are defined on the sphere bundle [3] or the projectivised tangent bundle [4] of a manifold. Foulon [12, 13, 14] also works always in the homogeneous formalism, and his basic geometrical entity is a sphere bundle. Secondly, we give a coordinate free definition of our connection, using the Koszul conditions for covariant differentiation, where other authors use tensorial methods [5], or connection forms and structural equations [3, 4]. Thirdly, we develop the properties of our connection further than most other authors have done, and in particular we give the full Bianchi identities for its curvature. Fourthly, we demonstrate the usefulness of the connection by taking the first steps to showing how its curvature determines the intrinsic properties of the second-order differential equations on which it is based.

2 Preliminaries

In this section we shall assemble the basic facts of the geometry of E and $J^1\pi$ which are needed for our construction.

We consider first the question of trivializing E. Any trivialization determines a vector field T on E, namely the coordinate field along the $\mathbb{R}$ factor, which has the property that $\pi_* T = \partial/\partial t$. Conversely, any vector field T on E with this property determines local trivializations, in the sense that any point of E has a neighbourhood which can be made diffeomorphic to $I \times U$, where I is an open interval of $\mathbb{R}$ and U is an open subset of M, in such a way that the integral curves of T correspond to the curves $t \mapsto (t, u)$ for some fixed $u \in U$.

This observation is related to the effects of (t-dependent) coordinate transformations on E, as follows. Any coordinate transformation $(x^i, t) \to (y^i, u)$, where $y^i = y^i(x^j, t)$ and $u = t$, leads to the following formulas for the new coordinate vector fields on E:

$$\frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i}\frac{\partial}{\partial x^j}, \qquad \frac{\partial}{\partial u} = \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial x^i},$$

where the functions $X^i = X^i(t, x^j)$ are determined by

$$\frac{\partial y^i}{\partial t} + X^j \frac{\partial y^i}{\partial x^j} = 0.$$

We can consider $\partial/\partial t + X^j \partial/\partial x^j$ as the local coordinate representation of a vector field T on E, which projects onto the vector field $\partial/\partial t$ on $\mathbb{R}$. As we have noted, every trivialization of E corresponds to such a vector field; the equations for y^i amount to $Ty^i = 0$, and can be solved to find the coordinate transformation with respect to which T becomes the t-coordinate field on E.

Turning now to $J^1\pi$, we note that a jet of a section of π may be regarded as a tangent vector to E which projects onto the vector $\partial/\partial t$. Thus a trivializing vector field T on E may also be regarded as a section of π_1^0. In terms of local coordinates (t, x^i, v^i), the vector field $T = \partial/\partial t + X^i(t, x^j)\partial/\partial x^i$ corresponds to the section $v^i = X^i(t, x^j)$. The coordinate transformation $(t, x^i, v^i) \mapsto (t, y^i, w^i)$ on $J^1\pi$ induced by the transformation $y^i = y^i(t, x^j)$ of E is given by $w^i = \partial y^i/\partial t + (\partial y^i/\partial x^j)v^j$. Thus choosing coordinates on E so that $T = \partial/\partial t$ is equivalent to taking the corresponding section of $J^1\pi$ as $w^i = 0$, that is, using the section to define the origin in each (affine) fibre.

The fibration $\pi_1^0 \colon J^1\pi \to E$ determines a vector sub-bundle $V\pi_1^0$ of $T(J^1\pi)$, the vertical sub-bundle; the quotient of each fibre by its vertical subspace can be identified with a tangent space to E, so we have the exact sequence of vector bundles over $J^1\pi$

$$0 \longrightarrow V\pi_1^0 \longrightarrow T(J^1\pi) \longrightarrow \pi_1^{0*}(\tau_E) \longrightarrow 0.$$

Corresponding to this is the exact sequence of modules of sections

$$0 \longrightarrow \mathcal{V}(\pi_1^0) \longrightarrow \mathcal{X}(J^1\pi) \longrightarrow \mathcal{X}(\pi_1^0) \longrightarrow 0;$$

here $\mathcal{X}(J^1\pi)$ denotes the module of vector fields on $J^1\pi$, $\mathcal{V}(\pi_1^0)$ the sub-module of vector fields vertical with respect to π_1^0, and $\mathcal{X}(\pi_1^0)$ the module of vector fields along the projection π_1^0, all of these spaces being modules over $C^\infty(J^1\pi)$.

Any section of τ_E may be pulled back to a section of $\pi_1^{0*}(\tau_E)$; that is to say, any vector field on E gives rise to an element of $\mathcal{X}(\pi_1^0)$. The elements of $\mathcal{X}(\pi_1^0)$ which arise in this way are called basic.

We observed above that each point of $J^1\pi$ may be considered as a tangent vector to E which projects onto $\partial/\partial t$. This identification may be regarded as defining a map $J^1\pi \to TE$, and therefore determines in a natural way a vector field along π_1^0, which is called the total derivative and denoted by $\mathbf{T}$; in coordinates we have

$$\mathbf{T} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i}.$$

The restriction of any element of $\mathcal{X}(\pi_1^0)$ to a section of π_1^0 determines an element of $\mathcal{X}(E)$; this remark, applied to $\mathbf{T}$, leads back to the two ways of defining a trivialization of E discussed above.

A second-order differential equation field is an element Γ of $\mathcal{X}(J^1\pi)$ which projects onto $\mathbf{T}$. We have

$$\Gamma = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} + f^i \frac{\partial}{\partial v^i}$$

where $f^i = f^i(t, x^j, v^j)$. After a (t-dependent) coordinate transformation the new f^i become

$$f'^i = \frac{\partial y^i}{\partial x^j} f^j + \frac{\partial^2 y^i}{\partial x^j \partial x^k} v^j v^k + 2 \frac{\partial^2 y^i}{\partial x^j \partial t} v^j + \frac{\partial^2 y^i}{\partial t^2}.$$

Note that the second derivatives of f^i with respect to the fibre coordinates v^j transform formally like the connection coefficients of an ordinary symmetric linear connection (though they may depend on t).

Any second-order differential equation field determines a splitting of the vector bundle exact sequence, or in other words a vector sub-bundle of $T(J^1\pi)$ which is complementary to the vertical sub-bundle $V\pi_1^0$. The corresponding distribution (vector field system) on $J^1\pi$ is called the horizontal distribution determined by Γ. The details of the construction of this horizontal distribution have been published frequently, so will not be repeated here (see for example [9, 25], and also [26, Section 5.4] for a more general formulation when the base manifold is not necessarily 1-dimensional). We content ourselves with giving the coordinate expressions for a basis $\{H_a\}$, $a = 0, 1, 2, \ldots, m$, of horizontal vector fields, which are $H_0 = \Gamma$, and $H_i = \partial/\partial x^i - \Gamma_i^j \partial/\partial v^j$ where $\Gamma_i^j = -\frac{1}{2}\partial f^j/\partial v^i$. Note in particular that the second-order differential equation field Γ is itself horizontal. The horizontal distribution will not in general be integrable.

The construction of the linear connection depends on certain features of the structure of $\pi_1^{0*}(\tau_E)$ which we now describe.

In the first place, $\pi_1^{0*}(\tau_E)$ is a direct sum of vector bundles. This is because the sub-bundle $\pi_1^{0*}(V\pi)$ (determined by vectors on E vertical with respect to π) has a naturally defined complement, spanned at each point by the total derivative $\mathbf{T}$; this is a special property of $\pi_1^{0*}(\tau_E)$ – it is not in general the case that there is a distinguished complement to $V\pi$ in TE, of course. The corresponding direct sum decomposition of the module of sections of $\pi_1^{0*}(\tau_E)$ is written

$$\mathcal{X}(\pi_1^0) \equiv \overline{\mathcal{X}}(\pi_1^0) \oplus \langle \mathbf{T} \rangle.$$

Sections in $\overline{\mathcal{X}}(\pi_1^0)$ are annihilated by dt, while the annihilators of $\mathbf{T}$ are spanned by $\{dx^i - v^i dt\}$, the contact 1-forms, these forms being regarded as local sections of the bundle dual to $\pi_1^{0*}(\tau_E)$. For any $\sigma \in \mathcal{X}(\pi_1^0)$ we write $\sigma = \overline{\sigma} + \langle \sigma, dt \rangle \mathbf{T}$, where $\overline{\sigma} \in \overline{\mathcal{X}}(\pi_1^0)$.

If F is any vector bundle over $J^1\pi$, and $\Psi : T(J^1\pi) \to F$ is a linear bundle map (over the identity) which vanishes on $V\pi_1^0$, then Ψ passes to the quotient, that is, it induces a linear bundle map $\pi_1^{0*}(\tau_E) \to F$.

As a first application of this remark, we note that the vertical endomorphism $S = (dx^i - v^i dt) \otimes \partial/\partial v^i$ of $J^1\pi$ vanishes on $V\pi_1^0$. Thus S passes to the quotient, and if we regard it as defining a linear bundle map $T(J^1\pi) \to V\pi_1^0$, then S induces a linear bundle map $\pi_1^{0*}(\tau_E) \to V\pi_1^0$. The induced map has the same coordinate representation, so its kernel is just the one-dimensional sub-bundle of $\pi_1^{0*}(\tau_E)$ spanned by $\mathbf{T}$. For any section σ of $\pi_1^{0*}(\tau_E)$ we write σ^V for the corresponding vertical vector field on $J^1\pi$. Then $\mathbf{T}^V = 0$, and $\sigma^V = \overline{\sigma}^V = (\sigma - \langle \sigma, dt \rangle \mathbf{T})^V$. Alternatively, we can regard S as defining a module isomorphism, $\overline{\sigma} \mapsto \overline{\sigma}^V$, of $\overline{\mathcal{X}}(\pi_1^0)$ with $\mathcal{V}(\pi_1^0)$.

Secondly, suppose that we have a horizontal distribution on $J^1\pi$. We shall denote by P_H the horizontal projector corresponding to the horizontal distribution: P_H is the linear bundle map (or type $(1,1)$ tensor field, or vector-valued 1-form, on $J^1\pi$) which is the projection of $T(J^1\pi)$ onto the horizontal sub-bundle along $V\pi_1^0$. Since P_H vanishes on $V\pi_1^0$ by definition, it passes to the quotient to define a linear bundle map $\pi_1^{0*}(\tau_E) \to T(J^1\pi)$, which is a bundle isomorphism of $\pi_1^{0*}(\tau_E)$ with the horizontal sub-bundle of $T(J^1\pi)$. We denote the corresponding map of sections by $\sigma \mapsto \sigma^H$.

Thus the splitting of the bundle exact sequence determined by a second-order differential equation field Γ carries over the direct sum decomposition of $\pi_1^{0*}(\tau_E)$, to give a three-way split: at the level of sections we may write

$$\mathcal{X}(J^1\pi) \equiv \overline{\mathcal{X}}(\pi_1^0)^V \oplus \mathcal{X}(\pi_1^0)^H \equiv \overline{\mathcal{X}}(\pi_1^0)^V \oplus \overline{\mathcal{X}}(\pi_1^0)^H \oplus \langle \Gamma \rangle,$$

since $\mathbf{T}^H = \Gamma$. Thus every vector field ξ on $J^1\pi$ may be written $\xi = (\xi_V)^V + (\xi_H)^H$ with $\xi_V \in \overline{\mathcal{X}}(\pi_1^0)$ and $\xi_H \in \mathcal{X}(\pi_1^0)$. Further, we may write $\xi_H = \overline{\xi_H} + \langle \xi, dt \rangle \mathbf{T}$, with $\overline{\xi_H} \in \overline{\mathcal{X}}(\pi_1^0)$. Furthermore, for any $\sigma \in \mathcal{X}(\pi_1^0)$ we have $(\sigma^V)_V = \sigma - \langle \sigma, dt \rangle \mathbf{T}$, while $(\sigma^H)_H = \sigma$.

For any $\rho, \sigma \in \mathcal{X}(\pi_1^0)$, we have $[\rho^V, \sigma^V]_H = 0$, since the vertical distribution on $J^1\pi$

is of course integrable. But $[\rho^H, \sigma^H]_V$ will not be zero in general. We define a map $R: \mathcal{X}(\pi_1^0) \times \mathcal{X}(\pi_1^0) \to \overline{\mathcal{X}}(\pi_1^0)$ by

$$R(\rho, \sigma) = [\rho^H, \sigma^H]_V.$$

Then R is $C^\infty(J^1\pi)$-bilinear and skew-symmetric. It measures the departure of the horizontal distribution from integrability. Bearing in mind the direct sum decomposition of $\pi_1^{0*}(\tau_E)$, it is convenient also to define a map $\Phi: \overline{\mathcal{X}}(\pi_1^0) \to \overline{\mathcal{X}}(\pi_1^0)$ by

$$\Phi(X) = R(\mathbf{T}, X) = [\Gamma, X^H]_V, \qquad X \in \overline{\mathcal{X}}(\pi_1^0);$$

for reasons which will become apparent later, Φ is called the Jacobi endomorphism of Γ. It is $C^\infty(J^1\pi)$-linear. We shall denote the restriction of R to $\overline{\mathcal{X}}(\pi_1^0) \times \overline{\mathcal{X}}(\pi_1^0)$ by $\tilde{R}$.

The objects R, Φ and $\tilde{R}$ may also be regarded as tensor fields. In the case of R we could simply say that it is a type $(1, 2)$ tensor field along π_1^0; but this would not be strictly correct for the others. So we shall adopt the following terminology. For any vector bundle F, we shall call a section of any tensor bundle constructed from F a tensor (of the appropriate type) on F. Then R is a type $(1, 2)$ tensor on $\pi_1^{0*}(\tau_E)$, while Φ and $\tilde{R}$ are tensors on $\pi_1^{0*}(V\pi)$, of types $(1, 1)$ and $(1, 2)$ respectively.

We can give a more explicit formula for R when its arguments are basic. Notice that if A is a vector field on E, and we form A^H (regarding A as an element of $\mathcal{X}(\pi_1^0)$), then A is π_1^0-related to A^H. It follows that for any two vector fields A and B on E, $[A, B]$ is π_1^0-related to $[A^H, B^H]$, and therefore that $[A^H, B^H] - [A, B]^H$ is vertical with respect to π_1^0. Thus

$$[A^H, B^H] - [A, B]^H = R(A, B)^V.$$

One further result, involving the bracket of Γ with a vertical vector field, will be needed in the next section: it follows directly from the formulas for the horizontal distribution given earlier (see [9]) that for any $X \in \overline{\mathcal{X}}(\pi_1^0)$,

$$[\Gamma, X^V]_H = -X.$$

3 The linear connection defined

We now define the linear connection on $\pi_1^{0*}(\tau_E)$, by specifying the associated covariant derivative operator on $\mathcal{X}(\pi_1^0)$.

Theorem 1 *The operator* $\mathrm{D}_\xi\colon \mathcal{X}(\pi_1^0) \to \mathcal{X}(\pi_1^0)$ *defined as follows*

$$\mathrm{D}_\xi\sigma = [P_H(\xi), \sigma^V]_V + [P_V(\xi), \sigma^H]_H + P_H(\xi)\langle\sigma, dt\rangle\mathbf{T},$$

where $P_V = I - P_H$ *is the vertical projector, is a covariant derivative.*

Proof To show that this operator is indeed a covariant derivative, we have merely to establish that it obeys the correct rules when its arguments are multiplied by functions. Note first that for any $f \in C^\infty(J^1\pi)$,

$$\begin{aligned}\mathrm{D}_{f\xi}\sigma &= [fP_H(\xi), \sigma^V]_V + [fP_V(\xi), \sigma^H]_H + fP_H(\xi)\langle\sigma, dt\rangle\mathbf{T}\\ &= f\mathrm{D}_\xi\sigma - (\sigma^V f)(P_H\xi)_V - (\sigma^H f)(P_V\xi)_H = f\mathrm{D}_\xi\sigma,\end{aligned}$$

since the terms involving derivatives of f also involve $(P_H\xi)_V$ and $(P_V\xi)_H$, both of which are zero. On the other hand,

$$\begin{aligned}\mathrm{D}_\xi(f\sigma) &= [P_H(\xi), f\sigma^V]_V + [P_V(\xi), f\sigma^H]_H + P_H(\xi)\langle f\sigma, dt\rangle\mathbf{T}\\ &= f\mathrm{D}_\xi\sigma + (P_H(\xi)f)(\sigma^V)_V + (P_V(\xi)f)(\sigma^H)_H + (P_H(\xi)f)\langle\sigma, dt\rangle\mathbf{T}\\ &= f\mathrm{D}_\xi\sigma + (\xi f)\sigma,\end{aligned}$$

where we have used the fact that $(\sigma^V)_V = \sigma - \langle\sigma, dt\rangle\mathbf{T}$. ■

So, given a horizontal distribution on $J^1\pi$, we can define a linear connection on $\pi_1^{0*}(\tau_E)$. In particular,

$$\mathrm{D}_\xi\mathbf{T} = [(\xi_V)^V, \Gamma]_H + P_H(\xi)\langle\mathbf{T}, dt\rangle\mathbf{T} = \xi_V.$$

Furthermore, for any $X \in \overline{\mathcal{X}}(\pi_1^0)$ we have

$$\mathrm{D}_\xi X = [P_H(\xi), X^V]_V + [P_V(\xi), X^H]_H.$$

It is important to note that $\mathrm{D}_\xi X$ has no $\mathbf{T}$ component. In fact, for any section σ of $\pi_1^{0*}(\tau_E)$ we have

$$\langle\mathrm{D}_\xi\sigma, dt\rangle = \xi\langle\sigma, dt\rangle.$$

To see this, observe first that for any vertical vector field ζ on $J^1\pi$, $\mathcal{L}_\zeta(dt) = 0$. Thus $\langle[P_V(\xi), \sigma^H], dt\rangle = P_V(\xi)\langle\sigma^H, dt\rangle = P_V(\xi)\langle\sigma, dt\rangle$, whence it follows that $\langle\mathrm{D}_\xi\sigma, dt\rangle =$

$P_V(\xi)\langle\sigma, dt\rangle + P_H(\xi)\langle\sigma, dt\rangle = \xi\langle\sigma, dt\rangle$. So in particular, if $\xi\langle\sigma, dt\rangle = 0$, then $\langle \mathrm{D}_\xi\sigma, dt\rangle = 0$ also. Thus $\overline{\mathcal{X}}(\pi_1^0)$ is mapped to itself by covariant differentiation; or in other words, the connection induces a linear connection on the sub-bundle $\pi_1^{0*}(V\pi)$ of $\pi_1^{0*}(\tau_E)$.

Furthermore, it follows from this calculation that we may express the covariant derivative as

$$\mathrm{D}_\xi\sigma = [P_H(\xi), \sigma^V]_V + \overline{[P_V(\xi), \sigma^H]_H} + \xi\langle\sigma, dt\rangle\mathbf{T}.$$

Using the expression for H_i given earlier, we find that the covariant derivatives are given in terms of local bases $\{\partial/\partial v^i, H_i, \Gamma\}$ of $\mathcal{X}(J^1\pi)$ and $\{\partial/\partial x^i, \mathbf{T}\}$ of $\mathcal{X}(\pi_1^0)$ as follows.

$$\begin{aligned}
&\mathrm{D}_{\partial/\partial v_i}\left(\frac{\partial}{\partial x^j}\right) = 0, \qquad && \mathrm{D}_{H_i}\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial\Gamma_i^k}{\partial v^j}\frac{\partial}{\partial x^k}, \qquad && \mathrm{D}_\Gamma\left(\frac{\partial}{\partial x^j}\right) = \Gamma_j^k\frac{\partial}{\partial x^k},\\
&\mathrm{D}_{\partial/\partial v_i}\mathbf{T} = \frac{\partial}{\partial x^i}, && \mathrm{D}_{H_i}\mathbf{T} = 0, && \mathrm{D}_\Gamma\mathbf{T} = 0.
\end{aligned}$$

If we write these entirely in terms of coordinate vector fields we obtain the following formulas.

$$\begin{aligned}
\mathrm{D}_{\partial/\partial v^i}\left(\frac{\partial}{\partial x^j}\right) &= 0\\
\mathrm{D}_{\partial/\partial x^i}\left(\frac{\partial}{\partial x^j}\right) &= \frac{\partial\Gamma_i^k}{\partial v^j}\frac{\partial}{\partial x^k},\\
\mathrm{D}_{\partial/\partial t}\left(\frac{\partial}{\partial x^j}\right) &= \left(\Gamma_j^k - v^l\frac{\partial\Gamma_l^k}{\partial v^j}\right)\frac{\partial}{\partial x^k}\\
\mathrm{D}_{\partial/\partial v^i}\left(\frac{\partial}{\partial t}\right) &= 0\\
\mathrm{D}_{\partial/\partial x^i}\left(\frac{\partial}{\partial t}\right) &= \left(\Gamma_i^k - v^j\frac{\partial\Gamma_i^k}{\partial v^j}\right)\frac{\partial}{\partial x^k}\\
\mathrm{D}_{\partial/\partial t}\left(\frac{\partial}{\partial t}\right) &= -\left(f^k + 2v^j\Gamma_j^k - v^jv^l\frac{\partial\Gamma_j^k}{\partial v^l}\right)\frac{\partial}{\partial x^k}.
\end{aligned}$$

Notice that the covariant derivatives of both $\partial/\partial x^j$ and $\partial/\partial t$ with respect to $\partial/\partial v^i$ vanish. It follows that a necessary and sufficient condition for $\sigma \in \mathcal{X}(\pi_1^0)$ to be basic (that is, to be a vector field on E) is that $\mathrm{D}_\zeta\sigma = 0$ whenever ζ is vertical with respect to π_1^0.

The covariant derivative has been defined in terms of the bracket operation on vector fields on $J^1\pi$. It so happens that because of the special relationships between sections of $\pi_1^{0*}(\tau_E)$ and vector fields on $J^1\pi$, it is possible to turn this round so as to express the bracket in terms of covariant derivatives. We shall explain next how this is done.

It is easy to verify that for any $\rho, \sigma \in \mathcal{X}(\pi_1^0)$, each of the three different expressions obtained by substituting V or H for the asterisks in $[\rho^*, \sigma^{**}] - ((\mathrm{D}_{\rho^*}\sigma)^{**} - (\mathrm{D}_{\sigma^{**}}\rho)^*)$ is tensorial with respect to the arguments ρ and σ. To evaluate them in general, therefore, it is sufficient to do so when ρ and σ are chosen from a local basis of sections of $\pi_1^{0*}(\tau_E)$. Using the explicit expressions for the covariant derivatives given above, and expressing ρ and σ in terms of their components with respect to the direct sum decomposition of $\mathcal{X}(\pi_1^0)$, we obtain the following results; here $X, Y \in \overline{\mathcal{X}}(\pi_1^0)$.

$$\begin{aligned}
&[X^V, Y^V] = (\mathrm{D}_{X^V} Y)^V - (\mathrm{D}_{Y^V} X)^V \\
&[X^H, Y^V] = (\mathrm{D}_{X^H} Y)^V - (\mathrm{D}_{Y^V} X)^H \\
&[X^H, Y^H] = (\mathrm{D}_{X^H} Y)^H - (\mathrm{D}_{Y^H} X)^H + (\tilde{R}(X, Y))^V \\
&[\Gamma, Y^V] = (\mathrm{D}_\Gamma Y)^V - (\mathrm{D}_{Y^V} \mathbf{T})^H = (\mathrm{D}_\Gamma Y)^V - Y^H \\
&[\Gamma, Y^H] = (\mathrm{D}_\Gamma Y)^H - (\mathrm{D}_{Y^H} \mathbf{T})^H + (\Phi(Y))^V = (\mathrm{D}_\Gamma Y)^H + (\Phi(Y))^V.
\end{aligned}$$

Any reader who is familiar with the work originated by Martínez, Cariñena and Sarlet will have noticed that these equations are formally similar to equations satisfied by the derivations D_X^V, D_X^H and ∇ which appear in their papers (see [21, 22], and [25] for the time-dependent case at hand). In fact there is a direct correspondence between these operators and the various components of the covariant derivative that we have defined, which our notation is intended to reflect. The operators defined in [25] are derivations of the tensor algebra of $\pi_1^{0*}(\tau_E)$. The correspondence between the derivations of covariant type in [25] and our covariant derivative is given by

$$\mathrm{D}_X^V = \mathrm{D}_{X^V}, \qquad \mathrm{D}_X^H = \mathrm{D}_{X^H}, \qquad \nabla = \mathrm{D}_\Gamma.$$

To put this another way, we can express the covariant derivative in terms of the derivations in [25] in the form

$$\mathrm{D}_\xi = \mathrm{D}_{\xi_V}^V + \mathrm{D}_{\overline{\xi_H}}^H + \langle \xi, dt \rangle \nabla.$$

We shall stick to our initial notation on the whole; but it will be convenient occasionally to take advantage of this correspondence with the operators first introduced by Martínez et al. to make use of their vertical and horizontal covariant differentials D^V and D^H. For example, the condition for $\sigma \in \mathcal{X}(\pi_1^0)$ to be basic may be written $\mathrm{D}^V \sigma = 0$.

It follows from the formula above for $[X^H, Y^H]$, and the observations made earlier about the corresponding bracket when X and Y are basic, that for any vector fields A and B on E, $[A, B] = \mathrm{D}_{A^H} B - \mathrm{D}_{B^H} A$.

Note one important consequence of the formulas relating brackets and covariant derivatives: all brackets of vector fields on $J^1\pi$ can be expressed in terms of covariant derivatives and the 'curvature' R of the non-linear connection defined by the given horizontal distribution. The formulas can be combined together to make this more apparent, as follows. For any vector fields ξ, η on $J^1\pi$ we have

$$\begin{aligned}[\xi,\eta] &= \left[(\xi_V)^V + (\xi_H)^H, (\eta_V)^V + (\eta_H)^H\right] \\ &= \{\mathrm{D}_\xi\eta_V - \mathrm{D}_\eta\xi_V + R(\xi_H,\eta_H)\}^V + \{\mathrm{D}_\xi\eta_H - \mathrm{D}_\eta\xi_H\}^H.\end{aligned}$$

We may express this result in the following alternative form.

$$\begin{aligned}&\mathrm{D}_\xi\eta_V - \mathrm{D}_\eta\xi_V - [\xi,\eta]_V = -R(\xi_H,\eta_H) \\ &\mathrm{D}_\xi\eta_H - \mathrm{D}_\eta\xi_H - [\xi,\eta]_H = 0.\end{aligned}$$

These equations are reminiscent of those which define the torsion of an ordinary linear connection; it seems natural, therefore, to regard the tensor $-R$ associated with the horizontal distribution as being the vertical component of the torsion of the linear connection on $\pi_1^{0*}(\tau_E)$. The horizontal component of the torsion vanishes because the horizontal distribution is defined by a second-order differential equation field.

Covariant differentiation can be extended to tensors on $\pi_1^{0*}(\tau_E)$, and on $\pi_1^{0*}(V\pi)$, in the usual way. Note that there are two kinds of covariant differentials, D^V and D^H. In particular, if K is a type $(1,k)$ tensor on $\pi_1^{0*}(V\pi)$, then we can define type $(1,k+1)$ tensors $\mathrm{D}^V K$ and $\mathrm{D}^H K$ on $\pi_1^{0*}(V\pi)$ by

$$D^*K(X_1,X_2,\dots X_k;X) = (D^*_X K)(X_1,X_2,\dots X_k);$$

the asterisk stands for either V or H. The rule for determining when an element of $\mathcal{X}(\pi_1^0)$ is basic applies to tensors on $\pi_1^{0*}(V\pi)$ too: the necessary and sufficient condition for such a tensor K to be basic (that is, to come from a tensor field on E) is that $\mathrm{D}^V K = 0$.

Our linear connection on $\pi_1^{0*}(\tau_E)$ can easily be lifted to a corresponding linear connection on $J^1\pi$ by using the decomposition of a vector field on $J^1\pi$ in terms of sections of $\pi_1^{0*}(\tau_E)$ to define its covariant derivative, as follows. Let ξ and η be vector fields on $J^1\pi$; write η as $(\eta_V)^V + (\eta_H)^H$ and put $\nabla_\xi\eta = (\mathrm{D}_\xi\eta_V)^V + (\mathrm{D}_\xi\eta_H)^H$. It is easy to see that ∇ is the covariant derivative operator of a connection on $J^1\pi$. What we obtain this way is essentially equivalent to the linear connection involved in the work of Massa and Pagani [23] and Byrnes [6]. There is, however, one minor difference, which is that our connection will not make all components of $\nabla_\xi\Gamma$ equal to zero. This is due to the fact that we have

$\mathrm{D}^V_\xi \mathbf{T} = \xi_V$, a property which arises naturally in our formalism (see also [25]). All the other connection coefficients will be found to coincide on the full space $J^1\pi$. Needless to say, there is a big advantage in our approach in terms of the economy of the number of formulas and calculations, because the effect of lifting the construction to $J^1\pi$ is merely to reproduce each of our formulas twice, once with a superscript V and once with a superscript H. We are convinced also that our construction is the more elegant and fundamental.

4 Curvature

We now turn our attention to the curvature of the linear connection which we have constructed. The curvature is a $C^\infty(J^1\pi)$-trilinear map $curv\colon \mathcal{X}(J^1\pi) \times \mathcal{X}(J^1\pi) \times \mathcal{X}(\pi^0_1) \to \mathcal{X}(\pi^0_1)$ which is skew-symmetric in its first two arguments. It is defined by

$$curv(\xi, \eta) = [\mathrm{D}_\xi, \mathrm{D}_\eta] - \mathrm{D}_{[\xi,\eta]},$$

in the usual way.

It is easy to calculate how the curvature acts on $\mathbf{T}$ directly from the definition, using the formulas $\mathrm{D}_{X^V}\mathbf{T} = X$, $\mathrm{D}_{X^H}\mathbf{T} = \mathrm{D}_\Gamma \mathbf{T} = 0$, and the expressions for $[\Gamma, \xi]$. The only non-zero components of $curv(\cdot\,, \cdot)\mathbf{T}$ are

$$curv(X^H, Y^H)\mathbf{T} = -\mathrm{D}_{[X^H,Y^H]}\mathbf{T} = -\tilde{R}(X, Y);$$
$$curv(\Gamma, X^H)\mathbf{T} = -\mathrm{D}_{[\Gamma,X^H]}\mathbf{T} = -\Phi(X).$$

It is not immediately obvious how to calculate the other components of the curvature; however, there is much that can be found out about them indirectly, as we now show.

The Jacobi identity for vector fields on $J^1\pi$, that is, $\sum[\xi, [\eta, \zeta]] = 0$ (where, here and below, $\sum$ means cyclic sum), imposes identities on the curvature components, which are effectively first Bianchi identities for the connection.

Theorem 2 *The curvature satisfies the first Bianchi identities*

$$\textstyle\sum \left(curv(\xi, \eta)\zeta_V + (\mathrm{D}_\zeta R)(\xi_H, \eta_H)\right) = 0,$$
$$\textstyle\sum curv(\xi, \eta)\zeta_H = 0.$$

Proof Using the formulas which give the vertical and horizontal components of the bracket of two vector fields on $J^1\pi$ in terms of covariant derivatives, we see that

$$\begin{aligned}
[\xi,[\eta,\zeta]]_V &= \mathrm{D}_\xi\left\{\mathrm{D}_\eta\zeta_V - \mathrm{D}_\zeta\eta_V + R(\eta_H,\zeta_H)\right\} - \mathrm{D}_{[\eta,\zeta]}\xi_V + R(\xi_H,\mathrm{D}_\eta\zeta_H) - R(\xi_H,\mathrm{D}_\zeta\eta_H) \\
&= \mathrm{D}_\xi\mathrm{D}_\eta\zeta_V - \mathrm{D}_\xi\mathrm{D}_\zeta\eta_V - \mathrm{D}_{[\eta,\zeta]}\xi_V + (\mathrm{D}_\xi R)(\eta_H,\zeta_H) \\
&\quad + R(\mathrm{D}_\xi\eta_H,\zeta_H) + R(\eta_H,\mathrm{D}_\xi\zeta_H) + R(\xi_H,\mathrm{D}_\eta\zeta_H) - R(\xi_H,\mathrm{D}_\zeta\eta_H),
\end{aligned}$$

and that

$$[\xi,[\eta,\zeta]]_H = \mathrm{D}_\xi\left\{\mathrm{D}_\eta\zeta_H - \mathrm{D}_\zeta\eta_H\right\} - \mathrm{D}_{[\eta,\zeta]}\xi_H.$$

Taking the cyclic sum and using these two formulas gives the identities quoted above. ■

By making the various possible substitutions for ξ, η, ζ, corresponding to the direct sum decomposition of $\mathcal{X}(J^1\pi)$, in these first Bianchi identities, and using the results for $curv(\cdot\,,\cdot)\mathbf{T}$, we obtain the following information about the curvature. First, a number of the curvature components automatically vanish:

$$curv(X^V,Y^V) = curv(\Gamma,X^V) = 0; \qquad curv(X^V,Y^H)\mathbf{T} = 0.$$

Next, we see that two components of the curvature are simply covariant derivatives of R:

$$\begin{aligned}
&curv(Y^H,Z^H)X = -(\mathrm{D}_{X^V}R)(Y,Z); \\
&curv(Y^H,\Gamma)X = (\mathrm{D}_{X^V}R)(\mathbf{T},Y).
\end{aligned}$$

But $R(\mathbf{T},X) = \Phi(X)$, from which it follows that $(\mathrm{D}_{X^V}R)(\mathbf{T},Y) = (\mathrm{D}_{X^V}\Phi)(Y) - R(X,Y)$; we may therefore write the second of these as

$$curv(Y^H,\Gamma)X = (\mathrm{D}_{X^V}\Phi)(Y) - R(X,Y).$$

From the Bianchi identities we next obtain some identities for R and Φ and their covariant derivatives:

$$\begin{aligned}
&\textstyle\sum(\mathrm{D}_{X^H}R)(Y,Z) = 0; \\
&(\mathrm{D}_\Gamma R)(X,Y) = (\mathrm{D}_{X^H}R)(\mathbf{T},Y) - (\mathrm{D}_{Y^H}R)(\mathbf{T},X) = (\mathrm{D}_{X^H}\Phi)(Y) - (\mathrm{D}_{Y^H}\Phi)(X); \\
&3R(X,Y) = (\mathrm{D}_{X^V}\Phi)(Y) - (\mathrm{D}_{Y^V}\Phi)(X).
\end{aligned}$$

The only remaining Bianchi identities are those involving terms like $curv(X^H, Y^H)Z$ or $curv(X^V, Y^H)Z$. There is one identity for the former, $\sum curv(Y^H, Z^H)X = 0$. The two identities involving the latter taken together state that this quantity is completely symmetric in its arguments X, Y and Z.

We may regard both $curv(X^V, Y^H)Z$ and $curv(X^H, Y^H)Z$ as defining type (1,3) tensors on $\pi_1^{0*}(V\pi)$ (or, preferably, type (1,1) tensor valued 2-covariant tensors on $\pi_1^{0*}(V\pi)$). When considering them in this guise we write

$$curv(X^V, Y^H)Z = \theta(X, Y)Z;$$
$$curv(X^H, Y^H)Z = Rie(X, Y)Z.$$

Then θ is completely symmetric, while $Rie(X, Y)$ is skew-symmetric in X and Y and is given in terms of R by

$$Rie(X, Y)Z = -(\mathrm{D}_{Z^V} R)(X, Y) = -(\mathrm{D}_{Z^V} \tilde{R})(X, Y).$$

The notation θ and Rie is taken from the papers of Martínez et al.; the results obtained here are derived in their work also, but by different means. However, it is of interest to observe exactly how these results are related, through the Bianchi identities. (In fact, our use of this notation differs in detail from that adopted by Martínez et al. in their account of the time-dependent case, in [25]. Our tensor θ is the same as the one in [25]. On the other hand, Rie in [25] is a type (1,3) tensor on $\pi_1^{0*}(\tau_E)$, given by $Rie(\rho, \sigma) = curv(\rho^H, \sigma^H)$; our Rie is the restriction of this to $\pi_1^{0*}(V\pi)$.)

All components of $curv$ have now been expressed in terms of $\pi_1^{0*}(V\pi)$ tensors. We summarize our results as follows.

Theorem 3 *The curvature components are given by*

$$\begin{aligned}
&curv(X^V, Y^V)Z = 0 && curv(X^V, Y^V)\mathbf{T} = 0\\
&curv(X^V, Y^H)Z = \theta(X, Y)Z && curv(X^V, Y^H)\mathbf{T} = 0\\
&curv(X^H, Y^H)Z = Rie(X, Y)Z && curv(X^H, Y^H)\mathbf{T} = -\tilde{R}(X, Y)\\
&curv(\Gamma, Y^V)Z = 0 && curv(\Gamma, Y^V)\mathbf{T} = 0\\
&curv(\Gamma, Y^H)Z = -(\mathrm{D}_{Z^V}\Phi)(Y) - \tilde{R}(Y, Z) && curv(\Gamma, Y^H)\mathbf{T} = -\Phi(Y).
\end{aligned}$$

The following identities are satisfied

$$
\begin{aligned}
&\theta(X,Y)Z = \theta(Y,X)Z = \theta(X,Z)Y;\\
&Rie(X,Y)Z = -(\mathrm{D}_{Z^V}\tilde{R})(X,Y);\\
&3\tilde{R}(X,Y) = (\mathrm{D}_{X^V}\Phi)(Y) - (\mathrm{D}_{Y^V}\Phi)(X);\\
&\textstyle\sum Rie(X,Y)Z = 0;\\
&\textstyle\sum(\mathrm{D}_{X^H}\tilde{R})(Y,Z) = 0;\\
&(\mathrm{D}_\Gamma\tilde{R})(X,Y) = (\mathrm{D}_{X^H}\Phi)(Y) - (\mathrm{D}_{Y^H}\Phi)(X).
\end{aligned}
$$

The fourth of these is actually a consequence of the second and the third.

Note that once θ and Φ are known, the other components of the curvature are determined. In particular, the necessary and sufficient conditions for the curvature to be identically zero are that $\theta = 0$ and $\Phi = 0$.

The occurrence of $\Phi(X)$ as the component $curv(X^H,\Gamma)\mathbf{T}$ may help to explain why Φ is called the Jacobi endomorphism – if the reader is reminded thereby of the equation of geodesic deviation.

Theorem 4 *Suppose that ξ is a vector field defined along an integral curve of Γ by Lie transport, so that $\mathcal{L}_\Gamma\xi = 0$. Then*

$$\mathrm{D}_\Gamma^2\overline{\xi_H} + \Phi(\overline{\xi_H}) = 0.$$

Proof We have

$$\xi = (\xi_V)^V + (\overline{\xi_H})^H + \langle\xi, dt\rangle\Gamma,$$

and therefore

$$\mathcal{L}_\Gamma\xi = (\mathrm{D}_\Gamma\xi_V)^V - {\xi_V}^H + (\mathrm{D}_\Gamma\overline{\xi_H})^H + \Phi(\overline{\xi_H})^V + (\Gamma\langle\xi, dt\rangle)\Gamma.$$

Setting this equal to zero and equating like components gives

$$
\begin{aligned}
&\Gamma\langle\xi, dt\rangle = 0\\
&\xi_V = \mathrm{D}_\Gamma\overline{\xi_H}\\
&\mathrm{D}_\Gamma\xi_V + \Phi(\overline{\xi_H}) = 0,
\end{aligned}
$$

which leads directly to the desired result. ■

Thus the equation $\mathrm{D}_\Gamma^2 X + \Phi(X) = 0$ is a generalization of the Jacobi equation for geodesics (the equation of geodesic deviation). This formula is equivalent to the one derived by Foulon [12, 14].

5 The second Bianchi identity

The fact that there is a first Bianchi identity is a special feature of this particular structure. But every linear connection on a vector bundle leads to a Bianchi identity, here called the second Bianchi identity, which comes from the Jacobi identity for the covariant derivative operators. It may be written

$$\sum \left([\mathrm{D}_\xi, \mathit{curv}(\eta,\zeta)] + \mathit{curv}(\xi,[\eta,\zeta]) \right) = 0.$$

This can be broken down into components by making various substitutions for ξ, η, ζ as before; we have to consider what happens when the curvature terms act on $\mathbf{T}$, as well as on a section of $\pi_1^{0*}(V\pi)$, and so for each choice of ξ, η, ζ we obtain two identities. These express relations between the tensors θ, Rie, $\tilde{R}$, and Φ, and their covariant derivatives, which are likely to be important for applications; we therefore derive them below. In the following list we indicate the choice of ξ, η, ζ and the argument as follows: ξ, η, ζ; σ. We have omitted all those cases in which the identity is vacuous because all terms vanish identically.

1. X^V, Y^V, Z^H; W:

   $$(\mathrm{D}_{Y^V}\theta)(X,Z)W = (\mathrm{D}_{X^V}\theta)(Y,Z)W.$$

2. X^V, Y^V, Γ; Z: the identity is automatically satisfied by virtue of the symmetry of θ.

3. X^V, Y^H, Z^H; W:

   $$(\mathrm{D}_{X^V}\mathit{Rie})(Y,Z)W = (\mathrm{D}_{Y^H}\theta)(X,Z)W - (\mathrm{D}_{Z^H}\theta)(X,Y)W.$$

4. X^V, Y^H, Z^H; $\mathbf{T}$: we obtain the known identity $\mathit{Rie}(Y,Z)X = -(\mathrm{D}_{X^V}\tilde{R})(Y,Z)$.

5. X^V, Y^H, Γ; Z:

   $$\begin{aligned}(\mathrm{D}_\Gamma\theta)(X,Y)Z &+ (\mathrm{D}_{X^V}\mathrm{D}_{Z^V}\Phi)(Y) - \left(\mathrm{D}_{(\mathrm{D}_{X^V}Z)^V}\Phi\right)(Y) \\ &+ (\mathrm{D}_{X^V}\tilde{R})(Y,Z) + \mathit{Rie}(X,Y)Z = 0.\end{aligned}$$

6. X^V, Y^H, Γ; **T**: the identity is automatically satisfied as a consequence of the formula for $curv(\Gamma, X^H)Y$.

7. X^H, Y^H, Z^H; W:

$$\sum \Big((\mathrm{D}_{X^H} Rie)(Y,Z)W - \theta(X, \tilde{R}(Y,Z))W \Big) = 0$$

(the cyclic sum being taken over X, Y and Z).

8. X^H, Y^H, Z^H; **T**: we get the cyclic identity $\sum(\mathrm{D}_{X^H}\tilde{R})(Y,Z) = 0$.

9. X^H, Y^H, Γ; Z:

$$\begin{aligned}(\mathrm{D}_\Gamma Rie)(X,Y)Z \; & - (\mathrm{D}_{Y^H}\mathrm{D}_{Z^V}\Phi)(X) + \Big(\mathrm{D}_{(\mathrm{D}_{Y^H}Z)^V}\Phi\Big)(X) \\ & + (\mathrm{D}_{X^H}\mathrm{D}_{Z^V}\Phi)(Y) - \Big(\mathrm{D}_{(\mathrm{D}_{X^H}Z)^V}\Phi\Big)(Y) \\ & - \theta(\Phi(X),Y)Z + \theta(X,\Phi(Y))Z - (\mathrm{D}_{Z^H}\tilde{R})(X,Y) = 0.\end{aligned}$$

10. X^H, Y^H, Γ; **T**: the identity reduces to the known formula for $\mathrm{D}_\Gamma\tilde{R}$.

The rather formidable looking identities obtained at numbers 5 and 9 of this list may be simplified considerably if it is recognised that the differential operators acting on the Φ terms are second covariant differentials (see, for example, [17, Chapter III, Section 2]). For any type $(1,k)$ tensor K on $\pi_1^{0*}(V\pi)$ we define type $(1,k+2)$ tensors $D^*D^{**}K$ on $\pi_1^{0*}(V\pi)$ (where the asterisks stand for V or H) by

$$(D^*D^{**}K)(\ldots;X,Y) = (D_{Y^*}(D^{**}K))(\ldots;X).$$

It is easy to see that

$$(D^*D^{**}K)(\ldots;X,Y) = (D_{Y^*}(D_{X^{**}}K))(\ldots) - (D_{(D_{Y^*}X)^{**}}K))(\ldots).$$

In order to simplify the identity involving $(\mathrm{D}_\Gamma\theta)(X,Y)Z$ (number 5) we do not just apply this result in the obvious way; we also express $\tilde{R}$ and Rie in terms of covariant differentials of Φ. We have

$$(\mathrm{D}_{Z^V}\tilde{R})(X,Y) = -Rie(X,Y)Z = \tfrac{1}{3}\{(\mathrm{D}^V\mathrm{D}^V\Phi)(Y;X,Z) - (\mathrm{D}^V\mathrm{D}^V\Phi)(X;Y,Z)\},$$

from which it follows that

$$(\mathrm{D}_\Gamma\theta)(X,Y)Z + \tfrac{1}{3}\sum(\mathrm{D}^V\mathrm{D}^V\Phi)(X;Y,Z) = 0.$$

Note that since $curv(X^V, Y^V) = 0$, for any tensor K, $(\mathrm{D}^V\mathrm{D}^V K)(\dots; X, Y)$ is symmetric in X and Y. The cyclic sum is therefore completely symmetric, as is $(\mathrm{D}_\Gamma\theta)(X,Y)Z$.

The identity involving $(\mathrm{D}_\Gamma Rie)(X,Y)Z$ (number 9) may be simplified in a somewhat similar way. First of all, it may be written

$$\begin{aligned}(\mathrm{D}_\Gamma Rie)(X,Y)Z &- (\mathrm{D}^H\mathrm{D}^V\Phi)(X;Z,Y) + (\mathrm{D}^H\mathrm{D}^V\Phi)(Y;Z,X)\\ &- \theta(\Phi(X),Y)Z + \theta(X,\Phi(Y))Z - (\mathrm{D}_{Z^H}\tilde{R})(X,Y) = 0.\end{aligned}$$

A further simplification arises if we reverse the order of the second covariant differentials; this introduces curvature terms, the relevant component of the curvature being the one involving θ. It so happens that the new terms in θ and Φ which are introduced cancel with those already present, when the symmetries of θ are taken into account. The final form of the identity is therefore

$$(\mathrm{D}_\Gamma Rie)(X,Y)Z - (\mathrm{D}_{Z^H}\tilde{R})(X,Y) = (\mathrm{D}^V\mathrm{D}^H\Phi)(X;Y,Z) - (\mathrm{D}^V\mathrm{D}^H\Phi)(Y;X,Z).$$

However, this identity is not really new: it may be derived directly from one of the first Bianchi identities by covariant differentiation. The tensor Rie satisfies the identity $Rie(X,Y)Z = -(\mathrm{D}_{Z^V}\tilde{R})(X,Y)$. If this is covariantly differentiated with respect to Γ; the order of the derivatives on the right hand side reversed using the formula for the curvature; and the resulting terms involving $(\mathrm{D}_\Gamma\tilde{R})(X,Y)$ replaced with terms in Φ by means of the identity $(\mathrm{D}_\Gamma\tilde{R})(X,Y) = (\mathrm{D}_{X^H}\Phi)(Y) - (\mathrm{D}_{Y^H}\Phi)(X)$, then the second Bianchi identity above is obtained.

Similarly, the identity $\sum\left((\mathrm{D}_{X^H}Rie)(Y,Z)W - \theta(X,R(Y,Z))W\right) = 0$ (number 7 above) may be obtained by differentiating $Rie(X,Y)W = -(\mathrm{D}_{W^V}\tilde{R})(X,Y)$ covariantly with respect to Z^H, interchanging the order of differentiation, and using the cyclic identity $\sum(\mathrm{D}_{X^H}\tilde{R})(Y,Z) = 0$.

Finally, identity 3, $(\mathrm{D}_{Y^H}\theta)(X,Z)W - (\mathrm{D}_{Z^H}\theta)(X,Y)W = (\mathrm{D}_{X^V}Rie)(Y,Z)W$, may be obtained by covariantly differentiating with respect to Γ the identity $(\mathrm{D}_{Y^V}\theta)(X,Z)W - (\mathrm{D}_{Z^V}\theta)(X,Y)W = 0$ (which is equivalent to the first identity by virtue of the symmetry of θ), interchanging the order of differentiation, and using the formula for $\mathrm{D}_\Gamma\theta$ from identity 5.

There remain, therefore, only two essentially new independent results, as follows.

Theorem 5 *The second Bianchi identity for the curvature is equivalent to the following* *two identities relating the tensors θ and Φ:*

$$(\mathrm{D}_{Y^V}\theta)(X,Z)W = (\mathrm{D}_{X^V}\theta)(Y,Z)W$$
$$(\mathrm{D}_\Gamma\theta)(X,Y)Z + \tfrac{1}{3}\textstyle\sum(\mathrm{D}^V\mathrm{D}^V\Phi)(X;Y,Z) = 0.$$

6 Vanishing curvature

We now derive some results concerned with the consequences of the vanishing of the curvature, or of certain components of it, which illustrate the significance of the connection in the study of second-order differential equations.

Theorem 6 *The linear connection has zero curvature (is flat) if and only if about every point of E there is a local trivialization, and adapted coordinates (t,x^i), such that with respect to these coordinates*

$$\Gamma = \frac{\partial}{\partial t} + v^i\frac{\partial}{\partial x^i},$$

so that the corresponding system of differential equations takes the form $\ddot{x}^i = 0$.

Proof If $curv = 0$ then there is a parallel field of frames of $\pi_1^{0*}(\tau_E)$, say $\{\sigma_a\}$, where $a = 0,1,2,\ldots,m$. That is to say, the $\sigma_a \in \mathcal{X}(\pi_1^0)$ are everywhere linearly independent and satisfy $\mathrm{D}_\xi\sigma_a = 0$ for all vector fields ξ on $J^1\pi$. They are determined up to replacement by linear combinations with constant coefficients. Now

$$0 = \langle \mathrm{D}_\xi\sigma_a, dt\rangle = \xi\langle\sigma_a, dt\rangle,$$

so that $\langle\sigma_a, dt\rangle$ is constant for each a; taking advantage of the freedom of choice of the σ_a we may ensure that $\langle\sigma_0, dt\rangle = 1$, while $\langle\sigma_i, dt\rangle = 0$ for $i = 1,2,\ldots,m$. Thus in particular $\sigma_i = X_i \in \overline{\mathcal{X}}(\pi_1^0)$. Since $\mathrm{D}^V\sigma_a = 0$, the σ_a are basic, that is, they define vector fields on E; and in particular, σ_0 projects onto $\partial/\partial t$ and therefore defines a local trivialization of E about any point. We may therefore introduce local coordinates on E, adapted to the local trivialization, such that $\sigma_0 = \partial/\partial t$. Now for any vector fields A and B on E, we have $[A,B] = \mathrm{D}_{A^H}B - \mathrm{D}_{B^H}A$; and therefore if A and B (considered as elements of $\mathcal{X}(\pi_1^0)$) are parallel, then $[A,B] = 0$. We conclude that the X_i are independent of t, and therefore define local vector fields on M. Furthermore, $[X_i, X_j] = 0$ for every i and j, so there are local coordinates x^i on M such that $X_i = \partial/\partial x^i$. By inspecting the coordinate formulas for the covariant derivative given earlier, we see that with respect to the coordinates (t, x^i) (whose corresponding coordinate vector fields, considered as elements of $\mathcal{X}(\pi_1^0)$, are parallel) the functions $f^i(t, x^j, \dot{x}^j)$ in the definition of Γ all vanish. Thus if *curv* $= 0$ the corresponding system of second-order differential equations is just $\ddot{x}^i = 0$.

The converse is obvious. ∎

In practical terms, the test conditions which have to be checked in order to apply this theorem are just $\theta = 0$ and $\Phi = 0$.

A slightly less restrictive condition on the curvature also leads to an interesting result. As we have already noted, covariant derivatives of sections of $\pi_1^{0*}(V\pi)$ (that is, sections of $\pi_1^{0*}(\tau_E)$ which are vertical with respect to π) remain so; we may therefore define a linear connection on $\pi_1^{0*}(V\pi)$ by restriction. The curvature of this connection is simply the restriction of $curv(\xi, \eta)$ to the vertical sub-bundle; in other words, it is given by the components of *curv* of the form $curv(\xi, \eta)Z$ where $Z \in \overline{\mathcal{X}}(\pi_1^0)$. The vanishing of just these components is thus a well-defined condition.

Theorem 7 *The linear connection restricted to $\pi_1^{0*}(V\pi)$ has zero curvature (is flat) if and only if about every point of E there is a local trivialization, and adapted coordinates (t, x^i), such that with respect to these coordinates*

$$\Gamma = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} + f^i(t, x^j) \frac{\partial}{\partial v^i},$$

so that the corresponding system of equations takes the form $\ddot{x}^i = f^i(t, x^j)$ (the right-hand sides being independent of $\dot{x}^j$).

Proof The argument proceeds in spirit as before, except that now we are assured merely of a covariant constant field of frames of $\pi_1^{0*}(V\pi)$. But this still ensures that, with respect to any local trivialization of E, for each t there is a local field of frames $\{X_i(t)\}$ on the standard fibre M for which the vector fields $X_i(t)$ pairwise commute (where the t is treated as a parameter). We may therefore choose local coordinates x^i on each fibre so that the coordinate vector fields, considered as elements of $\mathcal{X}(\pi_1^0)$, are parallel, as before; this choice of coordinates defines the required trivialization. Note, however, it is no longer the case that $\partial/\partial t$ is parallel, or that any trivialization consistent with the choice of these affine coordinates will make it so. From the expressions for the covariant derivatives of the $\partial/\partial x^i$ we see that $\Gamma^i_j = 0$, which is to say that $\partial f^i/\partial v^j = 0$, as required. ∎

The conditions of this theorem may be expressed as follows: $\theta = 0$, $Rie = 0$ and $(\mathrm{D}_{X^V}\Phi)(Y) = \tilde{R}(X, Y)$. From the last of these, and from the identity $3\tilde{R}(X, Y) = (\mathrm{D}_{X^V}\Phi)(Y) - (\mathrm{D}_{Y^V}\Phi)(X)$, it follows that $\tilde{R}(X, Y) = 0$; all components of the curvature therefore vanish, except $curv(X^H, \Gamma)\mathbf{T} = \Phi(X)$. Furthermore, $\mathrm{D}_{X^V}\Phi(Y) = 0$ for all X and Y, so that Φ is basic. In order to test whether a system of equations can be converted to the form $\ddot{x}^i = f^i(t, x^j)$ by a change of coordinates, therefore, it is necesary to check only whether $\theta = 0$ and Φ is basic.

Further progress in this general direction has been made by Martínez and Cariñena [19], in the autonomous case, where the theory is developed on a tangent bundle $\tau_M : TM \to M$. Their results, so far as the identification of the components of the curvature, and the Bianchi identities, are concerned, comprise the subset of our results which would be obtained by restricting all arguments to $\overline{\mathcal{X}}(\pi_1^0)$. The results of the present paper therefore constitute a significant generalization of those of [19]. However, Martínez and Cariñena show further that the necessary and sufficient condition for the existence of local coordinates on M with respect to which a given system of second-order differential equations $\ddot{x}^i = f^i(x^j, \dot{x}^j)$ is linear in the $\dot{x}^j$, so that f^i takes the form $f^i(x^j, v^j) = A^i_j(x^k)v^j + b^i(x^k)$, is that the linear connection induced by the corresponding second-order differential equation field is flat; and that the equations are linearizable in both variables if and only if, in addition, the Jacobi endomorphism is parallel (its covariant derivatives all vanish). Now linearity in the v^j corresponds to the vanishing of $\partial\Gamma^i_j/\partial v^k$, and as we pointed out earlier, the transformation rule for this object is formally the same in the time-dependent and time-independent cases. It follows that the same results hold, mutatis mutandis, in the time-dependent case. Thus the necessary and sufficient conditions for the linearizability of a given system of second-order differential equations, that is, for the existence of coordinates with respect to which the system takes the form $\ddot{x}^i = A^i_j(t)\dot{x}^j + B^i_j(t)x^j + a^i(t)$, are that $\theta = 0$ and $\mathrm{D}_{X^*}\Phi(Y) = 0$ for all X and Y.

It is not easy to make direct comparisons between our results and those on similar topics which have been obtained previously. First of all, most of the existing literature dealing with linearization or other forms of simplification of second-order differential equations is concerned with single equations, not systems (see, for example, [2, 16, 27]). Secondly, the coordinate transformations which have previously been used in the process of simplification are of a type which makes no distinction between the coordinates t and x, and thus requires a projectivized space for its geometrical description. Our results are more restrictive than these in the sense that we allow only coordinate transformations which preserve the distinction between dependent and independent variables; but on the other hand they are much more general in the sense that they are valid for systems of differential equations.

Clearly, if we specialize our results to the case $m = 1$, we should obtain statements which, interpreted analytically, are subcases of those in the aforementioned literature. It will be instructive to explore a couple of these instances.

Arnold [2], in his brief discussion of normal forms of second-order differential equations, mentions the general rule by which such an equation transforms under a transformation of the dependent variable (Chapter 1, §6 B.3). The corresponding analysis in our notation of the conditions for an equation to be transformable into the free particle equation $\ddot{x} = 0$, goes as follows. Consider a single second-order differential equation $\ddot{y} = f(t, y, \dot{y})$, with corresponding vector field Γ). Suppose that this equation transforms into $\ddot{x} = 0$ under a transformation of the form $x = G(t, y)$. The function G must be such that $G_y = \partial G/\partial y \neq 0$, and it must satisfy $\Gamma^2(G) = 0$, or equivalently $f(t, y, \dot{y}) = -(\Gamma(G_t) + \dot{y}\,\Gamma(G_y))/G_y$. Thus f will necessarily be quadratic in $\dot{y}$, so that the tensor θ will be zero. Starting from a general quadratic expression

$$f(t, y, \dot{y}) = A(t, y) + B(t, y)\dot{y} + C(t, y)\dot{y}^2,$$

one can ask directly for the conditions that there should exist a function $G(t, y)$ such that f has the required form. A standard analysis of the integrability conditions for the partial differential equations to which this question gives rise shows that the conditions are that $2C_t = B_y$ and $2(B_t - 2A_y) = B^2 - 4AC$. On the other hand, it is easy to compute the single component of the tensor field Φ for this situation: we find that

$$4\Phi^1_1 = 2(B_t - 2A_y) + 4AC - B^2 + 2(2C_t - B_y)\dot{y}.$$

Thus the results of the analysis in this special case are in perfect agreement with our general Theorem 6.

For a second comparison of our results with those in the literature, we consider the problem of linearizability in the case of a single equation. Grissom *et al* [16], who use Cartan's method of equivalence to study the problem, claim that $\ddot{y} = f(t, y, \dot{y})$ is linearizable if and only if f is cubic in $\dot{y}$,

$$f(t, y, \dot{y}) = A(t, y) + B(t, y)\dot{y} + C(t, y)\dot{y}^2 + D(t, y)\dot{y}^3,$$

and its coefficients satisfy the following two conditions:

$$\begin{aligned} &2C_{ty} - B_{yy} - 3D_{tt} + 3AD_y + 6A_yD - 3B_tD - 3BD_t - B_yC + 2CC_t = 0, \\ &B_{ty} - C_{tt} - 3A_tD - 3AD_t + 3A_yC + 3AC_y + BC_t - 2BB_y = 0. \end{aligned}$$

According to our Theorem 7 the necessary and sufficient conditions for linearizability are that $\theta = 0$, $\mathrm{D}_{X^V}\Phi(Y) = 0$ and $\mathrm{D}_{X^H}\Phi(Y) = 0$ for all $X, Y \in \overline{\mathcal{X}}(\pi^0_1)$. The first of these again implies that f must be quadratic in $\dot{y}$; but this is not in conflict with [16] as the cubic dependence on $\dot{y}$ is an extra freedom coming from their permitted freedom to transform the independent variable t. The other two conditions, in this case of one degree of freedom, mean simply that the single component of Φ cannot depend on $\dot{y}$ or on y, respectively. In the case in which $D = 0$ we can read these conditions off directly from the formula above for Φ^1_1; they are $B_y = 2C_t$ and $B_{ty} - 2A_{yy} + 2A_yC + 2AC_y - BB_y = 0$. Using the first, the second may be written

$$C_{tt} - A_{yy} + A_yC + AC_y - BC_t = 0.$$

Now, with $D = 0$ and $B_y = 2C_t$, the first condition of Grissom *et al* is satisfied; but the second one reduces to $B_{ty} - C_{tt} + 3A_yC + 3AC_y + BC_t - 2BB_y = 0$, or equivalently

$$C_{tt} + 3A_yC + 3AC_y - 3BC_t = 0.$$

This is evidently incompatible with our condition, so one or the other must be wrong. But the latter would be satisfied for $C = 0$ and would therefore imply that every equation of the form $\ddot{y} = A(t, y) + 3B(t)\dot{y}$ is linearizable in y and $\dot{y}$. Clearly, for arbitrary $A(t, y)$, this cannot be true.

7 Discussion

The results discussed in this paper represent the coming together of two strands of analysis of second-order differential equation fields, one using non-linear connection theory [8, 18, 28], the other being the adaptation of the Frölicher-Nijenhuis theory of derivations of forms to take account of the additional structure that arises when one is dealing with a tangent bundle or similar manifold [20, 21, 25]. The overall effect of all these developments is that we now have ready to hand a collection of very effective tools for the study of second-order differential equations and related matters. To end the paper we shall briefly mention some of the work that has been, and is being, done to apply these tools to the solution of specific problems in this field.

In addition to the results concerning the existence of special coordinates noted above, the methods described in this paper have been used to good effect in the study of at least two other problems. One is the search for conditions under which the second-order differential equations are completely separable. This problem is again concerned with conditions for the existence of coordinates with respect to which the equations take a special form, namely that they decouple into m independent equations each involving one dependent variable only. This problem has been solved completely, in [22] for the autonomous case, and quite recently in [7] for the time-dependent case. The conditions are somewhat more complicated than those discussed in this paper, but they may be expressed in terms of the linear connection and its curvature also.

The second problem in question is the inverse problem of the calculus of variations, which seeks the conditions for a system of second-order differential equations to be equivalent to the Euler-Lagrange equations of some Lagrangian function. This is a long standing problem (see [1, 24] for recent reviews), whose complete solution remains elusive; but the use of the methods described in this paper gives promise of new and illuminating results. As an example of the possibilities we cite recent work [10] on the re-evaluation of one of the classic papers in the field. In 1941 Douglas [11] gave a complete solution of the inverse problem with $m = 2$. His results appear to be exhaustive, but his methods are entirely analytical in nature, so until recently it has been very difficult to form any intuitive understanding of his work. In [10], however, almost all of Douglas's paper is shown to be readily interpretable in terms of the linear connection and its associated tensors and operators. Thus certain algebraic conditions that arise in Douglas's classification of types of equations turn out to be expressible directly in terms of the Jordan normal form of the Jacobi endomorphism; certain unexplained but recurring combinations of first derivatives are merely covariant differentiations; and, most striking of all, certain complicated expressions which arise in the analysis of integrability conditions are nothing else than the second covariant differentials which appeared in our second Bianchi identities.

No comparable solution to Douglas's for any case $m > 2$ has ever appeared in print – or even been attempted, so far as we know. This is due no doubt to the deterrent effect of the complexity of Douglas's paper. But our success in interpreting Douglas's work geometrically emboldens us to hope that progress can be made with this problem in future.

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