A differential geometric setting for mixed first- and second-order ordinary differential equations

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Abstract. A geometrical framework is presented for modelling general systems of mixed first- and second-order ordinary differential equations. In contrast to our earlier work on non-holonomic systems, the first-order equations are not regarded here as *a priori* given constraints. Two non-linear (parametrised) connections appear in the present framework in a symmetrical way and they induce a third connection via a suitable fibred product. The space where solution curves of the given differential equations live, singles out a specific projection ρ among the many fibrations in the general picture. A large part of the paper is about the development of intrinsic tools — tensor fields and derivations — for an adapted calculus along ρ. A major issue concerns the extent to which the usual construction of a linear connection associated to second-order equations fails to work in the presence of coupled first-order equations. An application of the ensuing calculus is presented.

PACS numbers: 0320, 0240

Short title: Mixed first- and second-order equations

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1 Introduction

There has recently been a considerable amount of interest in the study of non-holonomic mechanics (see [1, 2, 5, 10, 11, 12, 15, 18, 19, 24, 33] for a sample of geometrical approaches). In our earlier work [28, 31] we considered the case of Lagrangian systems subject to linear non-holonomic constraints, and we paid most of our attention to what could be called generalized Čaplygin systems, which is the case where the constraints are generated by a connection on an auxiliary bundle. One of us also discussed how some aspects of this work can be carried over to the more general situation of non-linear constraints [26, 27]. The equations under consideration then are of the form

$$\begin{aligned} \ddot{q}^\alpha &= f^\alpha(t, q^\beta, q^b, \dot{q}^\beta) \qquad && \alpha = 1, \ldots, k \\ \dot{q}^a &= g^a(t, q^\beta, q^b, \dot{q}^\beta) \qquad && a = 1, \ldots, m. \end{aligned}$$

In this earlier work, we saw how some of the constructions familiar from the geometrical study of unconstrained second-order systems could also be defined in the new situation. This is, in particular, the case for the *dynamical covariant derivative* and the *Jacobi endomorphism*, two concepts which play a key role in the geometrical analysis of second-order dynamics (see, for example, [4, 9, 14, 23, 30]). A large part of our approach to non-holonomic systems, though aimed at unravelling coordinate free properties, was carried out on the basis of coordinate calculations. A more geometrical and comprehensive construction of the basic ingredients would certainly, therefore, be of interest. It has moreover been observed in [7] that the dynamical covariant derivative is just one component of the covariant derivative operator defining some linear connection, and that the Jacobi endomorphism is in fact a component of the curvature of this connection.

These are sufficient motivations for trying to put our earlier work in a broader perspective. In doing so in the present paper, we will look at the picture from a slightly different point of view. Rather than considering the non-holonomic constraints as given, and the second-order differential equations as resulting from a larger system subjected to these constraints, we shall instead start with a mixed system of coupled first- and second-order equations treated on the same footing. There are indeed other applications which could be modelled by such type of equations (see e.g. [17]) and which need not have anything to do with the concept of constraints. The combined system of equations of the general form displayed above will be shown to define a pair of related connections on a fibre-product bundle, and the whole issue is to understand how this double structure can be put to work to detect the appropriate geometrical tools for analysing the dynamics. The most important of these tools is a collection of degree zero derivations, of which the dynamical covariant derivative is only one component. Unlike the theory developed in [7], however, the fundamental derivations will not all be of a covariant derivative type and accordingly will no longer comprise a linear connection. They do give rise, nevertheless, to corresponding exterior derivatives and associated tensor fields in much the same way as for pure second-order equations. As well as unravelling the interplay between these geometrical concepts, our further aim will be to apply this calculus to specific problems: as an example we shall prove a generalisation, directly in the appropriate framework and without relying on coordinate calculations, of some theorems about adjoint symmetries which were initiated in [28].

The structure of this paper is as follows. In Section 2 we describe the geometrical framework in which mixed equations are going to be studied, and introduce the associated (non-linear) connections. Section 3 deals with local frames adapted to these connections and discusses their curvature. Some properties of more general situations where two such parametric connections are available are developed in Appendix A. In Section 4 we examine the extent to which a linear connection, which for pure second-order systems can be used to generate the operators we need for our analysis, may also make an appearance in the present situation. Section 5 is about exterior derivative operators obtained from the derivations of degree zero introduced in the preceding section, and some aspects of the calculus related to such derivations. In the final section we apply this calculus to the description of symmetries and adjoint symmetries of mixed systems. Some further generalities about the nature of the derivations involved are briefly discussed in Appendix B.

Remark: notations in what follows are usually chosen to correspond with those in previous papers, but the present point of view will nevertheless require some changes.

2 The dynamical vector field and its two associated connections

Let E be the configuration manifold of a system, and let $\pi : E \to M$ and $\tau_0 : M \to \mathbb{R}$ be fibrations; denote the composition of the two projections by τ_1; let $k+1$ denote the dimension of M and m the fibre dimension of $E \to M$. We shall consider the pull-back manifold $\pi^* J^1\tau_0$ and denote its two projections by $\rho : \pi^* J^1\tau_0 \to E$ and $\pi_1 : \pi^* J^1\tau_0 \to J^1\tau_0$. The type of differential equations we are going to study can most directly be viewed as being given by a vector field Γ on $\pi^* J^1\tau_0$ satisfying the conditions that $\langle \Gamma, dt \rangle = 1$ (where t is the coordinate on $\mathbb{R}$) and that $S(\Gamma) = 0$ (where S is the vertical endomorphism from $J^1\tau_0$ transported to $\pi^* J^1\tau_0$). This is very similar to the framework described in [27, 28], the essential difference being that in the earlier work we considered a section of the bundle $J^1\tau_1 \to \pi^* J^1\tau_0$ as given, and took Γ to be defined instead on the image of that section, which is of course diffeomorphic to $\pi^* J^1\tau_0$.

Let coordinates on E be (t, q^α, q^a) and those on M be (t, q^α). The vertical endomorphism on $J^1\tau_0$ is

$$S = (dq^\alpha - \dot{q}^\alpha dt) \otimes \frac{\partial}{\partial \dot{q}^\alpha},$$

and the conditions satisfied by Γ restrict it to have coordinate representation

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + g^a(t, q^\beta, q^b, \dot{q}^\beta) \frac{\partial}{\partial q^a} + f^\alpha(t, q^\beta, q^b, \dot{q}^\beta) \frac{\partial}{\partial \dot{q}^\alpha}.$$

As in [27], several connections arise in this situation. One way to discover them, as we learn from the standard treatment of second-order dynamics ([6, 8, 16]) is to study the eigenspaces of the tensor field $\mathcal{L}_\Gamma S$. One readily verifies here that this tensor field has a k-dimensional eigenspace corresponding to the eigenvalue -1, another k-dimensional eigenspace with eigenvalue $+1$, and an $(m+1)$-dimensional eigenspace (containing Γ) with eigenvalue 0. Comparing this with the usual situation as described (for time-dependent systems) in [8], one is led to think of the first eigenspace, complemented with Γ, as characterizing "horizontality". There seem to remain then two different sorts of complementary vertical spaces, one of dimension k and another one of dimension m, corresponding of course to the fibration of E over $\mathbb{R}$ in two stages. To be specific, there is a connection σ on the bundle $\pi_1 : \pi^* J^1\tau_0 \to J^1\tau_0$ with a vertical projector which we shall denote by N; secondly there is a connection χ on the bundle $\rho : \pi^* J^1\tau_0 \to E$ with a vertical projector which we shall denote by P^V. The tensor field N measures the deviation from "unconstrained" second-order systems and is defined by

$$N = I - (\mathcal{L}_\Gamma S)^2 - dt \otimes \Gamma\,.$$

P^V can then be written as

$$P^V = \tfrac{1}{2}(I + \mathcal{L}_\Gamma S - N - dt \otimes \Gamma).$$

It is straightforward to check that $N^2 = N$ and $(P^V)^2 = P^V$, and that $\mathrm{Im}\, N = V\pi_1$ and $\mathrm{Im}\, P^V = V\rho$. In coordinates,

$$\begin{aligned} N &= (dq^a - B^a_\alpha dq^\alpha - (g^a - \dot{q}^\alpha B^a_\alpha)dt) \otimes \frac{\partial}{\partial q^a} \\ P^V &= (d\dot{q}^\alpha + \Gamma^\alpha_\beta dq^\beta - (f^\alpha + \dot{q}^\beta \Gamma^\alpha_\beta)dt) \otimes \frac{\partial}{\partial \dot{q}^\alpha} \end{aligned}$$

where the coefficients B^a_α and Γ^α_β are given by

$$B^a_\alpha = \frac{\partial g^a}{\partial \dot{q}^\alpha}, \quad \Gamma^\alpha_\beta = -\tfrac{1}{2}\frac{\partial f^\alpha}{\partial \dot{q}^\beta}\,;$$

the sign of Γ^α_β is chosen to be consistent with previous work in this area.

$$
\begin{tikzcd}
& J^1\rho \arrow[d, shift right] & \\
J^1\pi_1 \arrow[r, shift left] & \pi^*J^1\tau_0 \arrow[l, shift left, "\sigma"] \arrow[u, shift right, "\chi"'] \arrow[r, "\pi_1"] \arrow[d, "\rho"'] & J^1\tau_0 \arrow[d] \\
& E \arrow[r, "\pi"'] & M \arrow[d, "\tau_0"] \\
& & \mathbb{R}
\end{tikzcd}
$$

In this framework, the pull-back manifold $\pi^*J^1\tau_0$ is of course just the fibre product $E \times_M J^1\tau_0$. The connections σ and χ share a quite remarkable property: they are connections on one of the factors of the fibre product, parametrised by the other. To elaborate on this, observe for example that σ, viewed as a section of $J^1\pi_1 \to \pi^*J^1\tau_0$ and with obvious notations for the fibre coordinates of this fibration, is determined by the relations $q^a_t = g^a - \dot{q}^\alpha B^a_\alpha$, $q^a_\alpha = B^a_\alpha$, $q^a_{\dot\alpha} = 0$ (as may be seen from the coordinate representation of N). This means that σ actually takes its values in $J^1\pi \times_E \pi^*J^1\tau_0$ and so defines a map $\check\sigma : \pi^*J^1\tau_0 \to J^1\pi$. A perfectly symmetric situation applies to the connection χ on the bundle $\rho : \pi^*J^1\tau_0 \to E$ as may be seen from the lack of dq^a-terms in the coordinate expression for P^V. The general situation of two such connections on fibre-product bundles is described in Appendix A. It is shown there how the given connections give rise to a third "diagonal" connection and how the curvature of this induced connection relates to the curvatures of the original ones. In the present situation the "diagonal" connection is a connection κ on $\pi \circ \rho : \pi^*J^1\tau_0 \to M$, with horizontal projector $P^H = I - N - P^V$.

For completeness, we end this section by making explicit the link with the geometric picture underlying our previous publications in this area [27, 28]. Let $(p, j^1_t\gamma)$ represent an arbitrary point in $\pi^*J^1\tau_0$, where γ is a curve in M such that $\pi(p) = \gamma(t)$, and consider the point $\check\sigma(p, j^1_t\gamma) \in J^1\pi$. Let $\phi : M \to E$ be such that

$j^1_{\gamma(t)}\phi = \breve{\sigma}(p, j^1_t\gamma)$, i.e. we have $\partial\phi^a/\partial t = g^a - \dot{q}^\alpha B^a_\alpha$, $\partial\phi^a/\partial q^\alpha = B^a_\alpha$. Then $j^1_t(\phi\circ\gamma)$ is a point in $J^1\tau_1$ and, carrying out this construction for each point in $\pi^*J^1\tau_0$, it is clear that we obtain a section of $J^1\tau_1 \to \pi^*J^1\tau_0$. In [27, 28], we took the image J^1_σ of this section to be our evolution space. Whereas this is the right space to look at when we think of non-holonomic systems and want to relate, for example, a free Lagrangian system living on $J^1\tau_1$ with its reduced dynamics on the constraint submanifold, $\pi^*J^1\tau_0$ is the more natural environment when the starting point is a general mixed system of first- and second-order equations. Since both spaces are diffeomorphic, we are using ρ this time to denote the projection of $\pi^*J^1\tau_0$ (rather than J^1_σ) onto E and we are now ready to enter into an analysis of interesting operations and tensor fields along ρ.

3 Vector fields along ρ and frames adapted to the connections

Solutions of the given mixed system on $\pi^*J^1\tau_0$ will be curves in E, and studying properties of such curves will therefore give rise in a natural way to maps from $\pi^*J^1\tau_0$ to tangent vectors to E. This is why, having the calculus adapted to pure second-order equations in mind [21, 22, 30], we are now led to single out the fibration ρ. Associated to each mixed system Γ as defined above, there is a "total time derivative" vector field along ρ defined by $\mathbf{T}_\Gamma = T\rho\circ\Gamma$:

$$\mathbf{T}_\Gamma = \frac{\partial}{\partial t} + \dot{q}^\alpha\frac{\partial}{\partial q^\alpha} + g^a\frac{\partial}{\partial q^a}.$$

As in the general situation described in Appendix A, the connection σ defines a decomposition of each vector field X along ρ, although here the presence of the distinguished vector field $\mathbf{T}_\Gamma$ allows us to carry the decomposition a little further. In general we may write $X = \widehat{X} + \widetilde{X}$, but here we may also put $\widehat{X} = \langle\widehat{X}, dt\rangle\mathbf{T}_\Gamma + \overline{X}$, and as $\langle X, dt\rangle = \langle\widehat{X}, dt\rangle$ we have

$$X = \langle X, dt\rangle\mathbf{T}_\Gamma + \overline{X} + \widetilde{X}.$$

We will denote the $C^\infty(\pi^*J^1\tau_0)$-module of vector fields along ρ by $\mathcal{X}(\rho)$ and the submodules corresponding to the above decompositions by $\widehat{\mathcal{X}}(\rho)$, $\overline{\mathcal{X}}(\rho)$ and $\widetilde{\mathcal{X}}(\rho)$ respectively. Notations such as $\widehat{X}$, $\overline{X}, \widetilde{Y}, \ldots$ will always refer to vector fields along ρ, belonging to the corresponding submodule.

A local basis for $\mathcal{X}(\rho)$ is given by

$$\mathbf{T}_\Gamma, \quad X_\alpha = \frac{\partial}{\partial q^\alpha} + B^a_\alpha\frac{\partial}{\partial q^a}, \quad \frac{\partial}{\partial q^a}.$$

$\mathbf{T}_\Gamma$ and X_α, by the way, are the horizontal lifts, via the parametric connection $\check{\sigma}$, of the local basis of vector fields ($\mathbf{T} = \partial/\partial t + \dot{q}^\beta \partial/\partial q^\beta, \partial/\partial q^\alpha$) along the projection $\pi \circ \rho$. For a general X along ρ, we have

$$\overline{X} = \xi^\alpha X_\alpha, \quad \widetilde{X} = \xi^a \frac{\partial}{\partial q^a}.$$

The dual basis for $\bigwedge^1(\rho)$, the set of 1-forms along ρ, is given by

$$dt, \quad \theta^\alpha = dq^\alpha - \dot{q}^\alpha dt, \quad \eta^a = dq^a - g^a\, dt - B^a_\alpha \theta^\alpha\,.$$

By analogy, we denote the submodule spanned by the θ^α by $\overline{\bigwedge}^1(\rho)$, and the span of the η^a by $\widetilde{\bigwedge}^1(\rho)$.

For a local frame of vector fields on $\pi^* J^1\tau_0$, adapted to the connections and to the vector field Γ, we must keep track of two kinds of verticality, spanned by vector fields which we denote V_a and V_α respectively; we denote the remaining basis of horizontal fields by H_α. A suitable basis is therefore given by

$$\Gamma, \quad H_\alpha = \frac{\partial}{\partial q^\alpha} + B^a_\alpha \frac{\partial}{\partial q^a} - \Gamma^\beta_\alpha \frac{\partial}{\partial \dot{q}^\beta}, \quad V_a = \frac{\partial}{\partial q^a}, \quad V_\alpha = \frac{\partial}{\partial \dot{q}^\alpha}$$

and the dual basis of 1-forms on $\pi^* J^1\tau_0$ is

$$dt, \quad \theta^\alpha, \quad \eta^a, \quad \phi^\alpha = d\dot{q}^\alpha - f^\alpha\, dt + \Gamma^\alpha_\beta \theta^\beta.$$

With respect to these bases, we have

$$P^H = dt \otimes \Gamma + \theta^\alpha \otimes H_\alpha, \quad P^V = \phi^\alpha \otimes V_\alpha, \quad N = \eta^a \otimes V_a.$$

For computing local expressions (such as components of the Nijenhuis tensors of the various projectors which are relevant here), it is very helpful to have relations for the brackets of these basis vector fields. The most useful ones are:

$$\begin{aligned} [H_\alpha, H_\beta] &= (H_\alpha(B^b_\beta) - H_\beta(B^b_\alpha))\, V_b - (H_\alpha(\Gamma^\gamma_\beta) - H_\beta(\Gamma^\gamma_\alpha))\, V_\gamma \\ [\Gamma, H_\alpha] &= \Gamma^\beta_\alpha H_\beta + \Psi^b_\alpha\, V_b + \Phi^\beta_\alpha\, V_\beta\,, \end{aligned}$$

where we set

$$\begin{aligned} \Phi^\beta_\alpha &= -\Big(H_\alpha(f^\beta) - \Gamma^\beta_\gamma \Gamma^\gamma_\alpha + \Gamma(\Gamma^\beta_\alpha)\Big) = -\Big(X_\alpha(f^\beta) + \Gamma^\beta_\gamma \Gamma^\gamma_\alpha + \Gamma(\Gamma^\beta_\alpha)\Big) \\ \Psi^b_\alpha &= \Gamma(B^b_\alpha) - H_\alpha(g^b) - \Gamma^\beta_\alpha B^b_\beta = \Gamma(B^b_\alpha) - X_\alpha(g^b). \end{aligned}$$

We shall see in Section 5 that these expressions are the components of vector-valued forms along ρ. Needless to say, these coefficients must be related to components of the curvature of the connections involved. The curvature of a connection is often defined as being the Nijenhuis tensor of its horizontal projector or, equivalently, the Nijenhuis tensor of its vertical projector. We find that

$$\begin{aligned}
\mathcal{N}_N &= \Psi^c_\beta\, dt \wedge \theta^\beta \otimes V_c + \tfrac{1}{2}(H_\alpha(B^c_\beta) - H_\beta(B^c_\alpha))\, \theta^\alpha \wedge \theta^\beta \otimes V_c - \frac{\partial B^c_\alpha}{\partial \dot{q}^\beta}\, \theta^\alpha \wedge \phi^\beta \otimes V_c \\
\mathcal{N}_{P^V} &= \Phi^\gamma_\beta\, dt \wedge \theta^\beta \otimes V_\gamma + \tfrac{1}{2}((H_\alpha(\Gamma^\gamma_\beta) - H_\beta(\Gamma^\gamma_\alpha))\, \theta^\alpha \wedge \theta^\beta \otimes V_\gamma \\
&\quad + \left(\frac{\partial \Gamma^\gamma_\alpha}{\partial q^b}\, \theta^\alpha - \frac{\partial f^\gamma}{\partial q^b}\, dt\right) \wedge \eta^b \otimes V_\gamma.
\end{aligned}$$

It is a general property of Nijenhuis brackets of type (1,1) tensor fields that

$$\mathcal{N}_{P^V + N} = \mathcal{N}_{P^V} + \mathcal{N}_N + [P^V, N].$$

The left-hand side is nothing but the curvature of the "diagonal" connection κ; if one computes all terms in the right-hand side, there are some cancellations, the mechanism of which becomes clear from the general considerations in Appendix A.

4 In search of a linear connection

In [27, 28], the important *dynamical covariant derivative* ∇ – a degree zero derivation of the algebra of forms along the projection ρ – was detected by looking at the decomposition of the Lie derivative of horizontal and vertical lifts (of vector fields along ρ) with respect to Γ. We know from the standard second-order theory, however, that there are other paths leading to the definition of the same operator [30]. What is not clear at the moment is that an attempt to generalise these different approaches to the case of mixed systems would always lead to the same result. It is therefore important for our present purposes that we try to see how the operator ∇ arises in a more fundamental construction.

The most interesting interpretation of ∇ in the standard theory is as a component of a general linear connection associated to Γ. We will briefly review this construction as it was elaborated in [7] (following the earlier version of [20] for autonomous equations). Another approach to the same linear connection has recently been developed in [32], while related connections on the full jet space where Γ lives were introduced independently in [3, 25].

To see how the construction works for pure second-order equations, let us restrict attention to the bundle $\tau_0 : M \to \mathbb{R}$ and its jet space $J^1\tau_0$, and assume we have a second-order vector field Γ_0 on $J^1\tau_0$. We then have a (non-linear) connection on the bundle $\rho_0 : J^1\tau_0 \to M$ with horizontal projector

$$P^{H_0} = \tfrac{1}{2}(I - \mathcal{L}_{\Gamma_0} S + dt \otimes \Gamma_0),$$

and there are horizontal and vertical lift operators mapping vector fields X_0 along ρ_0 to vector fields ${X_0}^H$, ${X_0}^V$ on $J^1\tau_0$. Each vector field Z_0 on $J^1\tau_0$ gives rise to unique vector fields Z_{0V}, Z_{0H} along ρ_0 satisfying

$$Z_0 = (Z_{0H})^H + (Z_{0V})^V,$$

and Z_{0H} can be further decomposed as $Z_{0H} = \langle Z, dt\rangle \mathbf{T} + \overline{Z}_{0H}$, giving

$$Z_0 = \langle Z, dt\rangle \Gamma_0 + (\overline{Z}_{0H})^H + (Z_{0V})^V,$$

$$\langle Z_{0V}, dt\rangle = \langle \overline{Z}_{0H}, dt\rangle = 0.$$

As in [7], we can define a linear connection on $\rho_0^* TM \to J^1\tau_0$ by

$$D_{Z_0} X_0 = [P^{H_0}(Z_0), {X_0}^V]_V + [P^{V_0}(Z_0), {X_0}^H]_H + P^{H_0}(Z_0)(\langle X_0, dt\rangle)\mathbf{T}$$

for vector fields Z_0 on $J^1\tau_0$ and X_0 along ρ_0.

The two essential properties which ensure that this construction does indeed define a linear connection are first, that the dependence on Z_0 is linear over $C^\infty(J^1\tau_0)$, and secondly, that D_{Z_0} acts like a derivation on the $C^\infty(J^1\tau_0)$-module of vector fields along ρ_0. As a matter of fact, the first two terms in the above formula are the ones which make the construction work in the case of autonomous second-order equations (see [20]); if one carries them over to the time-dependent framework, the linearity in Z_0 is preserved, but the derivation property is lost and the third term is precisely the correction which is needed to restore this property (while keeping the linearity in Z_0).

What we are after now is a "parametrisation" of this formula to obtain a linear connection on the bundle $\rho^* TE \to \pi^* J^1\tau_0$. One can verify that (as a first step in the parametrisation process) the formula still makes sense when Z_0 is a vector field along $\pi_1 : \pi^* J^1\tau_0 \to J^1\tau_0$, with a suitable interpretation of the Lie bracket of such vector fields, and X_0 is a vector field along $\rho_0 \circ \pi_1 : \pi^* J^1\tau_0 \to M$. It does not seem to be possible, however, to make the formula continue to work in the present situation, the difficulty coming essentially from the presence of two different kinds of verticality. What we did in our previous papers [27, 28] was, in fact, to use only the connection χ on ρ to obtain a splitting of vector fields on $\pi^* J^1\tau_0$ into a horizontal and vertical part. This way, for example, the vector fields V_a were part of the horizontal distribution. The analogue of the formula for $D_{Z_0} X_0$ above then needs a further correction to restore the derivation property, but this time the correction unfortunately destroys the linearity in the Z-argument in an irrecoverable way. The "best approximation" to a linear connection can then be achieved by restricting the deviation from linearity to the smallest possible part. For that purpose, it is necessary to make consistent use of the threefold splitting of $\mathcal{X}(\pi^* J^1\tau_0)$ provided by the projection operators P^H, P^V and N.

From now on, we distinguish three lifting operations from $\mathcal{X}(\rho)$ to $\mathcal{X}(\pi^*J^1\tau_0)$: as shown in Appendix A, the connection χ may be used to give horizontal and "diagonal" lifts, while in this context we have a vertical lift as well. We may summarise the effects of these lifts in the following table:

$$
\begin{array}{lll}
\mathbf{T}_\Gamma{}^H = \Gamma, & X_\alpha{}^H = H_\alpha, & \left(\dfrac{\partial}{\partial q^a}\right)^H = 0 \\
\mathbf{T}_\Gamma{}^D = 0, & X_\alpha{}^D = 0, & \left(\dfrac{\partial}{\partial q^a}\right)^D = V_a \\
\mathbf{T}_\Gamma{}^V = 0, & X_\alpha{}^V = V_\alpha, & \left(\dfrac{\partial}{\partial q^a}\right)^V = 0.
\end{array}
$$

Any vector field Z on $\pi^*J^1\tau_0$ may now be written uniquely as the sum of three lifts:

$$Z = (Z_H)^H + (Z_D)^D + (Z_V)^V.$$

Here, Z_D is fixed by requiring it to be vertical with respect to the projector π, Z_H is horizontal with respect to the parametrised connection $\check{\chi}$, and so is Z_V, with the additional restriction that $\langle Z_V, dt\rangle = 0$. In other words, referring to the considerations of the previous section, Z_V only has X_α-components, while Z_H may be split further as

$$Z_H = \langle Z, dt\rangle \mathbf{T}_\Gamma + \overline{Z}_H$$

where $\langle \overline{Z}_H, dt\rangle = 0$.

For later use, we also describe the dual process of lifting 1-forms along ρ to 1-forms on $\pi^*J^1\tau_0$. Any 1-form on $\pi^*J^1\tau_0$ is fully determined by its action on X^H, $\overline{X}^V$ and $\widetilde{X}^D$, with $X \in \mathcal{X}(\rho)$. For $\alpha \in \bigwedge^1(\rho)$, the three types of lifts of α are defined by

$$\alpha^H(X^H) = \alpha(X), \quad \alpha^V(\overline{X}^V) = \alpha(\overline{X}), \quad \alpha^D(\widetilde{X}^D) = \alpha(\widetilde{X}),$$

all other components being zero. This gives rise to the following table for the lifts of the local basis of $\bigwedge^1(\rho)$:

$$
\begin{array}{lll}
(dt)^H = dt, & (\theta^\alpha)^H = \theta^\alpha, & (\eta^a)^H = 0 \\
(dt)^V = 0, & (\theta^\alpha)^V = \phi^\alpha, & (\eta^a)^V = 0 \\
(dt)^D = 0, & (\theta^\alpha)^D = 0, & (\eta^a)^D = \eta^a.
\end{array}
$$

With the threefold decomposition of $\mathcal{X}(\pi^*J^1\tau_0)$ now corresponding to the three lift operations, we start our approach towards a linear connection by the provisional formula which would work if we had no "diagonal parts":

$$D_Z X \stackrel{?}{=} [P^H(Z), X^V]_V + [P^V(Z), X^H]_H + P^H(Z)(\langle X, dt\rangle)\mathbf{T}_\Gamma.$$

This formula does not, however, represent a derivation. Indeed, we find that

$$\begin{aligned} D_Z(fX) - fD_ZX &= P^H(Z)(f)(X^V)_V + P^V(Z)(f)(X^H)_H + P^H(Z)(f)\langle X, dt\rangle \mathbf{T}_\Gamma \\ &= (P^H(Z) + P^V(Z))(f)\,\widehat{X} \\ &= ((I-N)Z)(f)\,\widehat{X} \\ &= Z(f)X - Z(f)\widetilde{X} - (NZ)(f)\widehat{X}. \end{aligned}$$

We shall remedy this deficiency by adding the terms $[Z, X^D]_D + [NZ, X^H]_H$ to our provisional formula, giving a final definition of the derivation D_Z as

$$D_ZX = [P^H(Z), X^V]_V + [(P^V + N)(Z), X^H]_H + [Z, X^D]_D + P^H(Z)(\langle X, dt\rangle)\mathbf{T}_\Gamma.$$

As anticipated above, however, this operator is *not* linear over Z, and so does not represent a linear connection; we have

$$D_{fZ}X = fD_ZX - X^D(f)Z_D,$$

which shows that the deficiency has been contained in the diagonal part. Another way of looking at this deficiency is that the formula for D_ZX would define a linear connection in the usual way if we were interested in vector fields along the projection $\pi \circ \rho : \pi^*J^1\tau_0 \to M$.

Again taking the theory of unconstrained second-order equations as a model and thinking, more particularly, of the relation between the linear connection on the bundle $\rho_0^*TM \to J^1\tau_0$ and the fundamental derivations for the calculus along ρ_0, we should now discover the operations of interest for a calculus along ρ by splitting Z into its three (or better four) components. Explicitly, the way to extract degree zero derivations on forms and vector fields along ρ from the derivation D_Z goes as follows. For any X and Y in $\mathcal{X}(\rho)$, we put

$$\mathrm{D}^V_YX = D_{Y^V}X, \quad \mathrm{D}^H_YX = D_{Y^H}X, \quad \nabla X = \mathrm{D}^H_{\mathbf{T}_\Gamma}X = D_\Gamma X, \quad \mathrm{D}^D_YX = D_{Y^D}X.$$

It is clear that D^V_Y and D^H_Y are covariant derivative type operators: they depend linearly on Y. For D^D_Y on the other hand we have

$$\mathrm{D}^D_YX = [Y^D, X^H]_H + [Y^D, X^D]_D.$$

It follows that D^D_Y is covariant for its action on $\widehat{\mathcal{X}}(\rho)$, but rather acts like a Lie derivative operator, when restricted to $\widetilde{\mathcal{X}}(\rho)$. The extension of these derivations to $\bigwedge^1(\rho)$ (and subsequently to arbitrary tensor fields along ρ) is achieved via the usual duality requirement: $D\langle X, \alpha\rangle = \langle DX, \alpha\rangle + \langle X, D\alpha\rangle$.

We shall use all four of these derivations in our development of the calculus along ρ, so it is worthwhile, for the convenience of the reader, to summarise their effect on our basis vector fields and differential forms in a complete table. With $\overline{Y} = \xi^\alpha X_\alpha$, $\widetilde{Y} = \xi^a\partial/\partial q^a$, we have:

$\mathrm{D}^{V}_{\overline{Y}}\mathbf{T}_\Gamma = \overline{Y}$	$\mathrm{D}^{V}_{\overline{Y}}X_\beta = 0$	$\mathrm{D}^{V}_{\overline{Y}}\dfrac{\partial}{\partial q^b} = 0$
$\mathrm{D}^{H}_{\overline{Y}}\mathbf{T}_\Gamma = 0$	$\mathrm{D}^{H}_{\overline{Y}}X_\beta = \xi^\alpha \dfrac{\partial \Gamma^\gamma_\beta}{\partial \dot{q}^\alpha} X_\gamma$	$\mathrm{D}^{H}_{\overline{Y}}\dfrac{\partial}{\partial q^b} = -\xi^\alpha \dfrac{\partial B^c_\alpha}{\partial q^b}\dfrac{\partial}{\partial q^c}$
$\nabla \mathbf{T}_\Gamma = 0$	$\nabla X_\beta = \Gamma^\gamma_\beta X_\gamma$	$\nabla \dfrac{\partial}{\partial q^b} = -\dfrac{\partial g^c}{\partial q^b}\dfrac{\partial}{\partial q^c}$
$\mathrm{D}^{D}_{\tilde{Y}}\mathbf{T}_\Gamma = 0$	$\mathrm{D}^{D}_{\tilde{Y}}X_\beta = 0$	$\mathrm{D}^{D}_{\tilde{Y}}\dfrac{\partial}{\partial q^b} = -\dfrac{\partial \xi^a}{\partial q^b}\dfrac{\partial}{\partial q^a}$

For 1-forms the table reads:

$\mathrm{D}^{V}_{\overline{Y}}dt = 0$	$\mathrm{D}^{V}_{\overline{Y}}\theta^\beta = -\xi^\beta dt$	$\mathrm{D}^{V}_{\overline{Y}}\eta^b = 0$
$\mathrm{D}^{H}_{\overline{Y}}dt = 0$	$\mathrm{D}^{H}_{\overline{Y}}\theta^\beta = -\xi^\alpha \dfrac{\partial \Gamma^\beta_\gamma}{\partial \dot{q}^\alpha}\theta^\gamma$	$\mathrm{D}^{H}_{\overline{Y}}\eta^b = \xi^\alpha \dfrac{\partial B^b_\alpha}{\partial q^c}\eta^c$
$\nabla dt = 0$	$\nabla \theta^\beta = -\Gamma^\beta_\gamma \theta^\gamma$	$\nabla \eta^b = \dfrac{\partial g^b}{\partial q^c}\eta^c$
$\mathrm{D}^{D}_{\tilde{Y}}dt = 0$	$\mathrm{D}^{D}_{\tilde{Y}}\theta^\beta = 0$	$\mathrm{D}^{D}_{\tilde{Y}}\eta^b = -\dfrac{\partial \xi^b}{\partial q^a}\eta^a$

Finally, for functions on $\pi^* J^1\tau_0$ we have

$$\mathrm{D}^{V}_{\overline{Y}} f = Y^V(f), \quad \mathrm{D}^{H}_{\overline{Y}} f = \langle Y, dt\rangle \Gamma(f) + \overline{Y}^H(f), \quad \nabla f = \Gamma(f), \quad \mathrm{D}^{D}_{\tilde{Y}} f = Y^D(f).$$

$\nabla = \mathrm{D}^{H}_{\mathbf{T}_\Gamma}$ will be called, as in previous work, the *dynamical covariant derivative*.

5 Exterior derivatives and further aspects of the calculus along ρ

In this section, we shall see how the curvatures of the two connections σ and χ may be characterised in terms of the calculus along ρ. There are several good reasons for looking at the manisfestation of curvature in this way. Recall, for example, from the standard theory of second-order systems [21, 22, 30] that the curvature, regarded as a vector-valued 2-form along the projection on the base manifold (the map ρ_0 in the sketch of the previous section), turns out to be entirely determined by a type (1,1) tensor field Φ, the Jacobi endomorphism. Here, though, we have two curvatures, and (as may be seen in Appendix A) each splits into two components. We might therefore expect to find four such tensor fields. This is not quite what is going to happen and it will be instructive to discover that the differences can roughly be traced back to the deviation of the D_Z operator from being a linear connection, or in other words to the Lie-derivative component of the derivation $\mathrm{D}^{\scriptscriptstyle D}$.

For a start, it is obvious that vector-valued 2-forms can not be obtained from vector-valued 1-forms through derivations of degree zero, so we will have to explore first what kind of exterior derivative operations have a natural existence in the present framework. After introducing an exterior derivative corresponding to each of the D-derivations, first on $\bigwedge(\rho)$ and then, inasmuch as this is possible, also on $\mathcal{X}(\rho)$ (and thus on vector-valued forms along ρ), we compute the decomposition of the Lie brackets of the various lifted vector fields on $\pi^*J^1\tau_0$. The algebraic parts of such decompositions should be related to (and in fact determine) the curvatures of the connections σ and χ: we find three vector-valued 2-forms R_i and one symmetric vector-valued 2-tensor field G along ρ, which are shown to be in one-to-one correspondence with the different components of $\mathcal{N}_N$ and $\mathcal{N}_{P^V}$. The dt-parts of the tensors R_i give rise to type (1,1) tensor fields, which are in turn shown to determine the R_i completely. For the action on functions, since $[D_{Z_1}, D_{Z_2}](f) = D_{[Z_1,Z_2]}(f)$, the decomposition of the brackets just discussed tells us at the same time how the commutators of the various D-derivations decompose. The above equality does not hold for the action on vector fields: the difference between both sides would in fact, if D_Z were truly a linear connection, define its curvature. We will not compute these curvature-like terms in all generality here, but restrict such computations to the commutators of ∇ with $\mathrm{D}^{\scriptscriptstyle V}$, $\mathrm{D}^{\scriptscriptstyle H}$ and $\mathrm{D}^{\scriptscriptstyle D}$. Let us now develop this logical hierarchy of steps in some detail.

As usual, derivations of scalar forms (here $\bigwedge(\rho)$) are completely determined by their action on functions and 1-forms. Because of the covariant nature of $\mathrm{D}^{\scriptscriptstyle V}_X$ and $\mathrm{D}^{\scriptscriptstyle H}_X$, we can define degree 1 derivations $d^{\scriptscriptstyle V}$ and $d^{\scriptscriptstyle H}$ by $d^{\scriptscriptstyle V}f(X) = \mathrm{D}^{\scriptscriptstyle V}_X f = \mathrm{D}^{\scriptscriptstyle V}_{\overline{X}} f$ and $d^{\scriptscriptstyle H}f(X) = \mathrm{D}^{\scriptscriptstyle H}_X f = \mathrm{D}^{\scriptscriptstyle H}_{\widehat{X}} f$ for functions $f \in C^\infty(\pi^*J^1\tau_0)$ (with $X \in \mathcal{X}(\rho)$ arbitrary), and for a 1-form α along ρ, define $d^{\scriptscriptstyle V}\alpha, d^{\scriptscriptstyle H}\alpha \in \bigwedge^2(\rho)$ by

$$
\begin{aligned}
d^{\scriptscriptstyle V}\alpha(X,Y) &= (\mathrm{D}^{\scriptscriptstyle V}_X\alpha)(Y) - (\mathrm{D}^{\scriptscriptstyle V}_Y\alpha)(X) \\
d^{\scriptscriptstyle H}\alpha(X,Y) &= (\mathrm{D}^{\scriptscriptstyle H}_X\alpha)(Y) - (\mathrm{D}^{\scriptscriptstyle H}_Y\alpha)(X).
\end{aligned}
$$

It is easy to verify that these indeed have the tensorial properties of 2-forms, and that $d^{\scriptscriptstyle V}$ and $d^{\scriptscriptstyle H}$ have the required derivation characteristics.

On the other hand, in view of the Lie derivative aspect in $\mathrm{D}^{\scriptscriptstyle D}_X$, we define $d^{\scriptscriptstyle D}$ by $d^{\scriptscriptstyle D}f(X) = \mathrm{D}^{\scriptscriptstyle D}_X f = \mathrm{D}^{\scriptscriptstyle D}_{\widetilde{X}} f$ and

$$
d^{\scriptscriptstyle D}\alpha(X,Y) = (\mathrm{D}^{\scriptscriptstyle D}_{\widetilde{X}}\alpha)(Y) - (\mathrm{D}^{\scriptscriptstyle D}_{\widetilde{Y}}\alpha)(X) - \alpha(\mathrm{D}^{\scriptscriptstyle D}_{\widetilde{X}}\widetilde{Y}).
$$

The covariant part of $\mathrm{D}^{\scriptscriptstyle D}$ helps to verify that this construction depends f-linearly on X and Y.

For practical purposes, we list the coordinate formulas for the exterior derivatives of functions and 1-forms along ρ. We have:

$$d^V f = V_\alpha(f)\,\theta^\alpha, \quad d^H f = \Gamma(f)\,dt + H_\alpha(f)\,\theta^\alpha, \quad d^D f = V_a(f)\,\eta^a,$$

and the table for the actions on basis 1-forms reads:

$d^V(dt) = 0$	$d^V\theta^\alpha = dt \wedge \theta^\alpha$	$d^V\eta^a = 0$
$d^H(dt) = 0$	$d^H\theta^\alpha = -\Gamma^\alpha_\beta\, dt \wedge \theta^\beta$	$d^H\eta^a = \dfrac{\partial g^a}{\partial q^b}\, dt \wedge \eta^b + \dfrac{\partial B^a_\alpha}{\partial q^b}\theta^\alpha \wedge \eta^b$
$d^D(dt) = 0$	$d^D\theta^\alpha = 0$	$d^D\eta^a = 0$

Since we actually have exterior derivatives of vector-valued forms in mind, we ought to extend the action of these operations to vector fields along ρ if possible. We may define

$$d^V X(Y) = \mathrm{D}^V_Y X, \quad d^H X(Y) = \mathrm{D}^H_Y X, \quad d^D \widehat{X}(Y) = \mathrm{D}^D_{\widetilde{Y}} \widehat{X},$$

where it has to be emphasised that d^D can be defined only on $\widehat{\mathcal{X}}(\rho)$, because of the non-linearity in $\widetilde{Y}$ of the action of $\mathrm{D}^D_{\widetilde{Y}}$ on $\widetilde{\mathcal{X}}(\rho)$. The table of coordinate expressions of interest here becomes (in a slightly different arrangement):

$d^V\mathbf{T}_\Gamma = \theta^\alpha \otimes X_\alpha$	$d^H\mathbf{T}_\Gamma = 0$	$d^D\mathbf{T}_\Gamma = 0$
$d^V X_\alpha = 0$	$d^H X_\alpha = \Gamma^\beta_\alpha\, dt \otimes X_\beta + \dfrac{\partial \Gamma^\gamma_\beta}{\partial \dot q^\alpha}\,\theta^\beta \otimes X_\gamma$	$d^D X_\alpha = 0$
$d^V \dfrac{\partial}{\partial q^a} = 0$	$d^H\left(\dfrac{\partial}{\partial q^a}\right) = -\dfrac{\partial g^b}{\partial q^a}\, dt \otimes \dfrac{\partial}{\partial q^b} - \dfrac{\partial B^b_\alpha}{\partial q^a}\,\theta^\alpha \otimes \dfrac{\partial}{\partial q^b}$	–

To see a manifestation of the curvatures at the level of the calculus along ρ, one has to compute Lie brackets of the various types of lifts of vector fields along ρ. We repeat that the main idea behind such computations is that relevant operations on $\mathcal{X}(\rho)$ will become apparent by looking at the decomposition of the resulting vector fields on $\pi^*J^1\tau_0$. Such a procedure in fact could be used to define the D-derivations to some extent, while the curvature components are expected to make their appearance in the algebraic parts of the decomposition.

It is appropriate to introduce the following bracket operations on the various submodules of $\mathcal{X}(\rho)$:

$$\begin{aligned}
[\overline{X}, \overline{Y}]_V &= \mathrm{D}^V_{\overline{X}}\overline{Y} - \mathrm{D}^V_{\overline{Y}}\overline{X} \\
[\widehat{X}, \widehat{Y}]_H &= \mathrm{D}^H_{\widehat{X}}\widehat{Y} - \mathrm{D}^H_{\widehat{Y}}\widehat{X} \\
[\widetilde{X}, \widetilde{Y}]_D &= \mathrm{D}^D_{\widetilde{X}}\widetilde{Y} - \mathrm{D}^D_{\widetilde{Y}}\widetilde{X}.
\end{aligned}$$

A straightforward calculation then yields the following results:

$$
\begin{aligned}
[\overline{X}^{V}, \overline{Y}^{V}] &= ([\overline{X}, \overline{Y}]_{V})^{V} \\
[\widetilde{X}^{D}, \widetilde{Y}^{D}] &= ([\widetilde{X}, \widetilde{Y}]_{D})^{D} \\
[\overline{X}^{V}, \widetilde{Y}^{D}] &= \left(\mathrm{D}^{V}_{\overline{X}} \widetilde{Y}\right)^{D} - \left(\mathrm{D}^{D}_{\widetilde{Y}} \overline{X}\right)^{V} \\
[\widehat{X}^{H}, \widehat{Y}^{H}] &= ([\widehat{X}, \widehat{Y}]_{H})^{H} + \left(R_1(\widehat{X}, \widehat{Y})\right)^{D} + \left(R_2(\widehat{X}, \widehat{Y})\right)^{V} \\
[\widetilde{X}^{D}, \widehat{Y}^{H}] &= \left(\mathrm{D}^{D}_{\widetilde{X}} \widehat{Y}\right)^{H} - \left(\mathrm{D}^{H}_{\widehat{Y}} \widetilde{X}\right)^{D} + \left(R_3(\widetilde{X}, \widehat{Y})\right)^{V} \\
[\overline{X}^{V}, \widehat{Y}^{H}] &= \left(\mathrm{D}^{V}_{\overline{X}} \widehat{Y}\right)^{H} - \left(\mathrm{D}^{H}_{\widehat{Y}} \overline{X}\right)^{V} + \left(G(\overline{X}, \widehat{Y})\right)^{D}.
\end{aligned}
$$

As expected, curvature components arise when the distributions are not integrable, i.e. when "horizontal" vector fields are involved. The tensor fields R_1, R_2 and R_3 are vector-valued 2-forms along ρ. Less expected is that G on the other hand is a symmetric vector-valued 2-tensor. The coordinate expressions of these tensor fields read as follows:

$$
\begin{aligned}
R_1 &= \Psi^c_\beta \, dt \wedge \theta^\beta \otimes \frac{\partial}{\partial q^c} + \tfrac{1}{2}(H_\alpha(B^c_\beta) - H_\beta(B^c_\alpha))\, \theta^\alpha \wedge \theta^\beta \otimes \frac{\partial}{\partial q^c} \\
R_2 &= \Phi^\gamma_\beta \, dt \wedge \theta^\beta \otimes X_\gamma + \tfrac{1}{2}((H_\alpha(\Gamma^\gamma_\beta) - H_\beta(\Gamma^\gamma_\alpha))\, \theta^\alpha \wedge \theta^\beta \otimes X_\gamma \\
R_3 &= -\frac{\partial f^\gamma}{\partial q^b}\, dt \wedge \eta^b \otimes X_\gamma + \frac{\partial \Gamma^\gamma_\alpha}{\partial q^b}\, \theta^\alpha \wedge \eta^b \otimes X_\gamma \\
G &= \frac{\partial^2 g^a}{\partial \dot q^\alpha \partial \dot q^\beta}\, \theta^\alpha \otimes \theta^\beta \otimes \frac{\partial}{\partial q^a}.
\end{aligned}
$$

The relationship between these tensors and the curvatures $\mathcal{N}_N$ and $\mathcal{N}_{P^V}$ is now obvious. A kind of vertical lift of $R_2 + R_3$ gives $\mathcal{N}_{P^V}$. The same procedure applied to R_1 gives rise to the first part of $\mathcal{N}_N$. The relationship between the second part of $\mathcal{N}_N$ and G is more subtle: it is of the same type as the so-called Kähler lift which relates a symmetric tensor field determined by the Hessian matrix of a Lagrangian to the Poincaré-Cartan 2-form. We refer to [22] and [30] for this construction and do not elaborate on it further here.

Observe in passing that the two different meanings we have given to vector fields carrying a subscript V, D or H (one for the decomposition of vector fields on $\pi^* J^1\tau_0$ and one for brackets of vector fields along ρ) are consistent. Indeed we see from the above relations that $[\overline{X}, \overline{Y}]_V = ([\overline{X}^V, \overline{Y}^V])_V$ and likewise for the other two notations.

We are now ready to define three type (1,1) tensor fields which are obtained simply from the dt-part of the curvature tensors R_i. Explicitly, let $\Psi = i_{\mathbf{T}_\Gamma} R_1$, $\Phi = i_{\mathbf{T}_\Gamma} R_2$ and $\Lambda = i_{\mathbf{T}_\Gamma} R_3$. The components of Ψ and Φ were already listed in Section 3, but we repeat the full expressions here because of the importance of these tensor fields:

$$
\begin{aligned}
\Psi &= \left(\Gamma(B^c_\beta) - X_\beta(g^c)\right)\theta^\beta \otimes \frac{\partial}{\partial q^c} \\
\Phi &= -\left(\Gamma(\Gamma^\gamma_\beta) + X_\beta(f^\gamma) + \Gamma^\alpha_\beta \Gamma^\gamma_\alpha\right)\theta^\beta \otimes X_\gamma \\
\Lambda &= -\frac{\partial f^\gamma}{\partial q^b}\,\eta^b \otimes X_\gamma .
\end{aligned}
$$

One of the striking features of these tensors is that they actually determine the curvature 2-forms R_i completely. Indeed, it is fairly easy to verify in coordinates that:

$$
\begin{aligned}
R_1 &= \tfrac{1}{2}(d^V \Psi + dt \wedge \Psi) \\
R_2 &= \tfrac{1}{3}(d^V \Phi + 2dt \wedge \Phi) \\
R_3 &= \tfrac{1}{2}(d^V \Lambda + 2dt \wedge \Lambda) .
\end{aligned}
$$

Remark: the tensor field Φ defined in our earlier work ([27, 28]) was in fact the sum of the Φ and Λ which are introduced here. The analysis of the next section will illustrate that the present option is more appropriate, but a sufficient argument for choosing it would be that the non-zero parts of the coefficient matrices of the two tensors have different dimensions.

The next step which one logically makes in building up a calculus along ρ is to look at the commutators of the fundamental degree zero derivations introduced in the previous section. In fact, the above bracket relations already carry the information for these commutators inasmuch as their action on functions is concerned. Indeed, it follows from the definitions that on functions f we have:

$$
\begin{aligned}
[\mathrm{D}^V_{\overline{X}}, \mathrm{D}^V_{\overline{Y}}](f) &= \mathrm{D}^V_{[\overline{X},\overline{Y}]_V}(f) \\
[\mathrm{D}^D_{\widetilde{X}}, \mathrm{D}^D_{\widetilde{Y}}](f) &= \mathrm{D}^D_{[\widetilde{X},\widetilde{Y}]_D}(f) \\
[\mathrm{D}^V_{\overline{X}}, \mathrm{D}^D_{\widetilde{Y}}](f) &= \left(\mathrm{D}^D_{\mathrm{D}^V_{\overline{X}}\widetilde{Y}} - \mathrm{D}^V_{\mathrm{D}^D_{\widetilde{Y}}\overline{X}}\right)(f) \\
[\mathrm{D}^H_{\widehat{X}}, \mathrm{D}^H_{\widehat{Y}}](f) &= \left(\mathrm{D}^H_{[\widehat{X},\widehat{Y}]_H} + \mathrm{D}^D_{R_1(\widehat{X},\widehat{Y})} + \mathrm{D}^V_{R_2(\widehat{X},\widehat{Y})}\right)(f) \\
[\mathrm{D}^D_{\widetilde{X}}, \mathrm{D}^H_{\widehat{Y}}](f) &= \left(\mathrm{D}^H_{\mathrm{D}^D_{\widetilde{X}}\widehat{Y}} - \mathrm{D}^D_{\mathrm{D}^H_{\widehat{Y}}\widetilde{X}} + \mathrm{D}^V_{R_3(\widetilde{X},\widehat{Y})}\right)(f) \\
[\mathrm{D}^V_{\overline{X}}, \mathrm{D}^H_{\widehat{Y}}](f) &= \left(\mathrm{D}^H_{\mathrm{D}^V_{\overline{X}}\widehat{Y}} - \mathrm{D}^V_{\mathrm{D}^H_{\widehat{Y}}\overline{X}} + \mathrm{D}^D_{G(\overline{X},\widehat{Y})}\right)(f).
\end{aligned}
$$

In Appendix B, building on our experience for second-order systems, we sketch a few aspects of what a complete theory of derivations of forms along ρ would tell us about their canonical decomposition. It would follow from such a theory that the difference between the two sides of each of the above relations among degree zero derivations (when they are allowed to act on vector fields and forms as well) can merely be an "algebraic derivation", i.e. a derivation vanishing on functions. Such a derivation (of degree 0) is determined by a type (1,1) tensor field, say Q, and then written as μ_Q. Its action on vector fields $U \in \mathcal{X}(\rho)$ or 1-forms $\alpha \in \bigwedge^1(\rho)$ is given by $Q(U)$ and $-Q(\alpha)$ respectively. So, to complete the picture about commutators of D-derivations, all that needs to be done, in principle, is to compute the $Q_{X,Y}$ tensor for each case.

In fact, there is a lot more that can be said about these tensors. As we learn from [7], if the founding father of the D-derivations, i.e. the operation D_Z introduced in the preceding section, really had been a linear connection on the bundle $\rho^*TE \to \pi^*J^1\tau_0$, then its curvature would be defined by $curv\,(Z_1, Z_2) = [D_{Z_1}, D_{Z_2}] - D_{[Z_1,Z_2]}$. This implies that the different tensor fields $Q_{X,Y}$ we are talking about here would essentially constitute the components of the curvature of this linear connection. In the present situation, D_Z falls short of defining a linear connection. The effect of the deviation from a linear connection is that the tensor fields $Q_{X,Y}$ will not in all cases depend tensorially on X and Y as well: sometimes derivatives of the components of X and Y will be involved. For the first two commutators in the above list there are no terms of type μ_Q, i.e. the corresponding Q-tensor is zero. Computing the Q-tensors for the other cases is a rather messy enterprise and so we will abstain from it in the general case. We will instead pay more attention to the special case where one of the operands in the commutator is the dynamical covariant derivative $\nabla = \mathrm{D}^H_{\mathbf{T}_\Gamma}$. Incidentally, it is possible to obtain all such commutator relations in a coordinate free way by using the Jacobi identity, applied to suitable combinations of derivations. Often, however, the result will follow more quickly from a coordinate calculation.

The following results are obtained:

$$\begin{aligned}
[\nabla, \mathrm{D}^V_{\overline{X}}] &= -\,\mathrm{D}^H_{\overline{X}} + \mathrm{D}^V_{\nabla\overline{X}} \\
[\nabla, \mathrm{D}^D_{\widetilde{X}}] &= \mathrm{D}^D_{\nabla\widetilde{X}} + \mathrm{D}^V_{\Lambda(\widetilde{X})} + \mu_{Q^D_{\widetilde{X}}} \\
[\nabla, \mathrm{D}^H_{\widehat{X}}] &= \mathrm{D}^H_{\nabla\widehat{X}} + \mathrm{D}^V_{\Phi(\widehat{X})} + \mathrm{D}^D_{\Psi(\widehat{X})} + \mu_{Q^H_{\widehat{X}}}\,,
\end{aligned}$$

where the (1,1) tensors $Q^D_{\widetilde{X}}$ and $Q^H_{\widehat{X}}$ are given by

$$\begin{aligned}
Q^D_{\widetilde{X}} &= i_{\widetilde{X}} R_3 - i_\Lambda d^V \widetilde{X} \\
Q^H_{\widehat{X}} &= 2\, i_{\widehat{X}} R_2 - \mathrm{D}^V_{\widehat{X}}\Phi - 2\langle \widehat{X}, dt\rangle \Phi \\
&\quad + i_{\widehat{X}}\Xi + \Psi \circ d^D \widehat{X} - \Lambda \circ (\widehat{X} \lrcorner G).
\end{aligned}$$

Here Ξ is a vector-valued 2-form along ρ, which can be constructed out of G and

Λ:

$$\Xi(X,Y) = (\Lambda \lrcorner G)(X,Y) - (\Lambda \lrcorner G)(Y,X).$$

In coordinates:

$$\Xi = \frac{\partial f^\gamma}{\partial q^b}\frac{\partial B^c_\alpha}{\partial \dot{q}^\gamma}\,\theta^\alpha \wedge \eta^b \otimes \frac{\partial}{\partial q^c}.$$

6 The calculus along ρ at work

In this section we wish to develop, as an application of the basic intrinsic operations which have been introduced so far, a theory of adjoint symmetries for mixed first and second-order equations. The idea will be that, with the most essential tools at hand, everything else should follow without needing to use coordinate expressions. Infinitesimal generators of symmetry transformations of differential equations are more familiar than adjoint symmetries, so we shall first derive the characterisation of symmetries within the framework of vector fields along ρ (which is the most economical way for writing the determining equations in a coordinate free way), and then proceed to the theory of adjoint symmetries by duality.

Let $Z \in \mathcal{X}(\pi^* J^1\tau_0)$ be a dynamical symmetry of the given dynamics Γ. As usual, two symmetries should be regarded as equivalent if their difference is a multiple of Γ. This means that we can concentrate, without loss of generality, on a representative of the class for which $\mathcal{L}_\Gamma Z = 0$. Such a representative will not have a Γ-component so that, in accordance with the general discussion of Section 4, Z will have a unique decomposition of the form

$$Z = \overline{X}^H + \widetilde{X}^D + \overline{Y}^V,$$

for some $\overline{X}, \widetilde{X}, \overline{Y} \in \mathcal{X}(\rho)$.

From the brackets of lifted vector fields computed at the beginning of the previous section, in the special case where one of the vector fields is $\Gamma = \mathbf{T}_\Gamma{}^H$, we obtain

$$\begin{aligned} \mathcal{L}_\Gamma \overline{X}^V &= -\overline{X}^H + (\nabla\overline{X})^V \\ \mathcal{L}_\Gamma \widehat{X}^H &= (\nabla\widehat{X})^H + \Psi(\widehat{X})^D + \Phi(\widehat{X})^V \\ \mathcal{L}_\Gamma \widetilde{X}^D &= (\nabla\widetilde{X})^D + \Lambda(\widetilde{X})^V. \end{aligned}$$

It now follows that $\mathcal{L}_\Gamma Z = 0$ is equivalent to specifying that $\overline{Y} = \nabla\overline{X}$, and requiring that $\overline{X}$ and $\widetilde{X}$ must be solutions of the mixed first and second-order partial differential equations

$$\nabla^2\overline{X} + \Phi(\overline{X}) + \Lambda(\widetilde{X}) = 0,$$

$$\nabla\widetilde{X} + \Psi(\overline{X}) = 0.$$

Essentially, therefore, looking for a symmetry means looking for a vector field $X \in \mathcal{X}(\rho)$, of the form $X = \overline{X} + \widetilde{X}$, satisfying the above two conditions. Note that $\Phi(X) = \Phi(\overline{X})$, $\Lambda(X) = \Lambda(\widetilde{X})$ and $\Psi(X) = \Psi(\overline{X})$, and also that the left-hand side of the first equation takes values in $\overline{\mathcal{X}}(\rho)$, whereas this is $\widetilde{\mathcal{X}}(\rho)$ for the second equation. As a result, the two conditions can equivalently be cast into the single condition

$$\nabla^2\overline{X} + \nabla\widetilde{X} + (\Phi + \Lambda + \Psi)(X) = 0\,.$$

The dualisation of this picture is easily obtained using the standard procedure by which one defines the adjoint of a linear partial differential operator. Hence, we adopt here the following definition for the concept of adjoint symmetries.

Definition. An adjoint symmetry of Γ is a 1-form $\alpha \in \bigwedge^1(\rho)$, of the form $\alpha = \overline{\alpha} + \tilde{\alpha} = a_\beta\theta^\beta + c_b\eta^b$, satisfying

$$\nabla^2\overline{\alpha} - \nabla\tilde{\alpha} + (\Phi + \Lambda + \Psi)(\alpha) = 0.$$

It is easy to see, for instance from the coordinate expressions, that the submodules $\overline{\bigwedge}^1(\rho)$ and $\tilde{\bigwedge}^1(\rho)$ are invariant under ∇ and Φ. On the other hand, Λ has $\tilde{\bigwedge}^1(\rho)$ in its kernel and maps $\overline{\bigwedge}^1(\rho)$ into $\tilde{\bigwedge}^1(\rho)$, whereas Ψ does the reverse. As a result, the single condition of the definition is equivalent to the two conditions

$$\nabla^2\overline{\alpha} + \Phi(\overline{\alpha}) + \Psi(\tilde{\alpha}) = 0$$

$$-\nabla\tilde{\alpha} + \Lambda(\overline{\alpha}) = 0.$$

Observe that, in [27, 28], the adjoint symmetry condition did not split into two separate conditions; this was due to the fact that $\Phi + \Lambda$ was regarded there as a single tensor (called Φ). It is clear that the present situation is more elegant, if only because of the similarity with the picture for symmetries. Comparison with the line of approach adopted in [28] shows that, if α is an adjoint symmetry of Γ, then $\phi = \overline{\alpha}^{V} + \tilde{\alpha}^{D} - (\nabla\overline{\alpha})^{H}$ will be a Γ-invariant 1-form on $\pi^*J^1\tau_0$ with Γ in its kernel. More strongly, one can show that this formula establishes a bijective correspondence between 1-forms ϕ satisfying the conditions $\mathcal{L}_\Gamma\phi = 0$ and $i_\Gamma\phi = 0$, and adjoint symmetries in the sense defined here. Hence, in particular, all first integrals of the system should correspond to certain adjoint symmetries, and these will be of the form $d^V F + d^D F$ for some function F.

When we wish to substitute an α of this form in the adjoint symmetry condition, it is clear that we will need to know how the dynamical covariant derivative ∇ commutes with the exterior derivatives. Since the exterior derivatives were defined in terms of the D-derivations, it is fairly easy to obtain this information from the commutator relations for $[\nabla, \mathrm{D}^V_{\overline{X}}]$, $[\nabla, \mathrm{D}^D_{\widetilde{X}}]$ and $[\nabla, \mathrm{D}^H_{\widehat{X}}]$. We limit ourselves to the action on scalar forms along ρ. From the defining relation of $d^V\alpha$, for example, we find:

$$\nabla d^V\alpha(X,Y) + d^V\alpha(\nabla X, Y) + d^V\alpha(X, \nabla Y) = \left(\mathrm{D}^V_{\overline{X}}\nabla\alpha - \mathrm{D}^H_{\overline{X}}\alpha + \mathrm{D}^V_{\overline{\nabla X}}\alpha\right)(Y)$$
$$+ \mathrm{D}^V_{\overline{X}}\alpha(\nabla Y) - \left(\mathrm{D}^V_{\overline{Y}}\nabla\alpha - \mathrm{D}^H_{\overline{Y}}\alpha + \mathrm{D}^V_{\overline{\nabla Y}}\alpha\right)(X) - \mathrm{D}^V_{\overline{Y}}\alpha(\nabla X).$$

Subtracting

$$d^V\nabla\alpha(X,Y) = \mathrm{D}^V_{\overline{X}}\nabla\alpha(Y) - \mathrm{D}^V_{\overline{Y}}\nabla\alpha(X),$$

it then readily follows that, on $\bigwedge(\rho)$,

$$[\nabla, d^V] = -d^H + dt \wedge \nabla.$$

The other two commutators are somewhat more involved; we obtain (see Appendix B for notations)

$$\begin{aligned} [\nabla, d^D] &= d^V_\Lambda + i_{R_3} - 2\,dt \wedge i_\Lambda \\ [\nabla, d^H] &= d^V_\Phi + d^D_\Psi + 2\,i_{R_2} - 2\,dt \wedge i_\Phi - i_\Xi\,. \end{aligned}$$

If one were to extend these commutator relations to vector-valued forms (whenever that makes sense), there would again be additional algebraic terms arising. We shall, however, not need these extensions for our present purposes.

Now let α be an adjoint symmetry for which $\overline{\alpha} = d^VF$ and $\tilde{\alpha} = d^DF$. Then

$$\nabla d^VF = d^V\nabla F - d^HF + (\nabla F)dt,$$

from which it follows, applying ∇ again and using the commutator $[\nabla, d^H]$, that

$$\nabla^2 d^VF = \nabla(d^V\nabla F + (\nabla F)dt) - d^H\nabla F - \Phi(d^VF) - \Psi(d^DF).$$

Putting $L = \nabla F$, we conclude that the second-order requirement for α to be an adjoint symmetry is equivalent to:

$$d^HL = \nabla(d^VL + L\,dt).$$

Similarly, the first-order condition is found to be equivalent to

$$d^DL = 0.$$

In coordinates, the first condition says that $\Gamma(V_\beta(L)) = X_\beta(L)$, but the right-hand side, in view of the second condition reduces further. Altogether we obtain

$$\Gamma\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) = \frac{\partial L}{\partial q^\alpha}, \quad \text{with} \quad \frac{\partial L}{\partial q^b} = 0\,.$$

Therefore, provided that the Hessian matrix $V_\beta V_\alpha(L)$ is regular, the conclusion is that the right-hand sides f^α of the second-order equations do not depend on the q^b coordinates and that these equations actually come from Euler-Lagrange equations. It is easy also to formulate a converse statement, so that we arrive at the following theorem.

Theorem. If Γ has an adjoint symmetry of the form $\alpha = d^{\scriptscriptstyle V}F + d^{\scriptscriptstyle D}F$, and if $V_\beta V_\alpha(L)$ is regular (where $L = \nabla F$), then the second-order differential equations coming from Γ are decoupled from the first-order ones and are the Lagrange equations with Lagrangian L. Conversely, if the second-order equations are Lagrange equations with a Lagrangian $L(t, q^\beta, \dot{q}^\beta)$ of the form $L = \nabla F$ for some function $F \in C^\infty(\pi^* J^1\tau_0)$, then $\alpha = d^{\scriptscriptstyle V}F + d^{\scriptscriptstyle D}F$ is an adjoint symmetry.

This result is of course not quite the one which was predicted when we referred to adjoint symmetries related to first integrals. Let us see now how the generation of first integrals is in some sense a particular case. What is already obvious from the preceding analysis is that for any first integral F with $\nabla F = 0$, the 1-form $d^{\scriptscriptstyle V}F + d^{\scriptscriptstyle D}F$ is an adjoint symmetry. Conversely, however, the situation may be more complicated, since a Lagrangian $L = \nabla F$ (which is not regular) may be a first integral in disguise through the customary gauge freedom. We have seen above that for $\overline{\alpha} = d^{\scriptscriptstyle V}F$,

$$\nabla\overline{\alpha} + d^{\scriptscriptstyle H}F = d^{\scriptscriptstyle V} L + L\, dt, \quad \text{with} \quad L = \nabla F.$$

The right-hand side is like a Poincaré-Cartan form

$$\theta_L = \frac{\partial L}{\partial \dot{q}^\beta}\, \theta^\beta + L\, dt.$$

When L is a total time-derivative of some function f, this necessarily has to be a function on M, and an easy way to express this is to say that θ_L, regarded as a "semi-basic" 1-form on $\pi^* J^1\tau_0$, is exact (or closed for local results). We therefore come to the following statement.

Theorem. Let $\alpha = d^{\scriptscriptstyle V}F + d^{\scriptscriptstyle D}F$ be an adjoint symmetry of Γ, such that $\nabla\overline{\alpha} + d^{\scriptscriptstyle H}F$, regarded as 1-form on $\pi^* J^1\tau_0$, is (locally) of the form df. Then $F - f$ is a first integral of Γ. Conversely, every first integral can be obtained through such a procedure: if F is a first integral, then $d^{\scriptscriptstyle V}F + d^{\scriptscriptstyle D}F$ is an adjoint symmetry.

It is worthwhile highlighting a quite striking feature of these results. Compared to the standard theory of second-order equations, the second theorem is the type of result one expects: it produces a general mechanism by which first integrals can be generated, and this mechanism is likely to reduce to the more familiar relationship between symmetries and conservation laws (Noether's theorem) whenever the given system of differential equations is self-adjoint (cf. [29]). The situation covered by the first theorem, however, is less expected, because it first of all creates circumstances in which the second-order equations are decoupled from the first-order ones, and then the additional statement is that the second-order equations will be of Lagrangian type.

Appendix A: Connections on fibre-product bundles

Suppose we have two bundles $\mu_0 : Y \to M$ and $\nu_0 : Z \to M$ over the same base manifold. We may consider the fibre-product manifold $Y \times_M Z$, and the projections of this manifold onto its components define two further bundles $\nu : Y \times_M Z \to Y$ and $\mu : Y \times_M Z \to Z$. We shall be interested in connections defined on these two new bundles, and the way such connections interact with each other. (For the situation described in the main body of the paper, the base manifold M has an additional fibration over $\mathbb{R}$, Y and Z are the manifolds E and $J^1\tau_0$ respectively, and their fibred product over M is $\pi^* J^1\tau_0$.)

In general, a connection on (say) μ is a section of the first jet bundle $J^1\mu \to Y \times_M Z$. In the present situation, though, we may define a distinguished submanifold $(J^1\mu)_0$ of $J^1\mu$ and consider only those sections which take their values in this submanifold. To construct the submanifold, consider those (local) sections ψ of μ which are "projectable" to μ_0, in the sense that there are corresponding sections ψ_0 of μ_0 satisfying $\psi_0 \circ \nu_0 = \nu \circ \psi$. We shall define $(J^1\mu)_0$ as the submanifold of jets $j^1_p\psi$ admitting a representative section which is projectable: it is clear that this submanifold has a natural identification with $J^1\mu_0 \times_M Z$. We shall call a section σ of $J^1\mu_0 \times_M Z \to Y \times_M Z$ a *projectable connection on* μ: the name is appropriate because the composition of σ followed by projection on the first factor of $J^1\mu_0 \times_M Z$ gives a map $\check{\sigma} : Y \times_M Z \to J^1\mu_0$ which is rather like a connection on μ_0 parametrised by Z. Conversely, starting from $\check{\sigma}$ we may recover σ in full by specifying that the second component of the image should be given by the identity on Z.

Now suppose we have two projectable connections σ and χ on μ and ν respectively. As the connections are projectable, we may combine them to give a connection κ on the composite bundle $\lambda = \nu_0 \circ \mu : Y \times_M Z \to M$, using the natural identification of $J^1\lambda$ with the fibre product $J^1\mu_0 \times_M J^1\nu_0$: we just specify that κ should be given by $\kappa(x, y, z) = (\check{\sigma}(x, y, z), \check{\chi}(x, y, z))$. Conversely, starting with κ, we may obtain $\check{\sigma}$ and $\check{\chi}$ by projecting on the first and second factor respectively, and hence recover σ and χ.

$$
\begin{tikzcd}
J^1\mu_0 \times_M J^1\nu_0 \cong J^1\lambda \arrow[dr, shift left] & Y \times_M J^1\nu_0 \arrow[d, shift right] & \\
J^1\mu \supset J^1\mu_0 \times_M Z \arrow[r, shift left] \arrow[d] & Y \times_M Z \arrow[l, shift left, "\sigma"] \arrow[ul, shift left, "\kappa"'] \arrow[u, shift right, "\chi"'] \arrow[r, "\mu"] \arrow[dl, "\check{\sigma}"] \arrow[d, "\nu"'] \arrow[dr, "\lambda"] & Z \arrow[d, "\nu_0"] \\
J^1\mu_0 \arrow[r] & Y \arrow[r, "\mu_0"'] & M
\end{tikzcd}
$$

In terms of the horizontal and vertical tangent vectors associated with each connection, we find that

$$\mathrm{Hor}\ \kappa \quad = \quad \mathrm{Hor}\ \sigma \cap \mathrm{Hor}\ \chi$$

whereas

$$\begin{aligned} \mathrm{Hor}\ \sigma &= \mathrm{Hor}\ \kappa \oplus V\nu \\ \mathrm{Hor}\ \chi &= \mathrm{Hor}\ \kappa \oplus V\mu. \end{aligned}$$

Take local coordinate systems x^A on M, (x^A, y^a) on Y, (x^A, z^α) on Z, and (x^A, y^a, z^α) on $Y \times_M Z$. The horizontal projector of a general connection σ on μ would be given in these coordinates by

$$P^H_\sigma = dx^A \otimes \left(\frac{\partial}{\partial x^A} + \sigma^a_A \frac{\partial}{\partial y^a} \right) + dz^\alpha \otimes \left(\frac{\partial}{\partial z^\alpha} + \sigma^a_\alpha \frac{\partial}{\partial y^a} \right),$$

but if σ is a projectable connection then the coefficients σ^a_α all vanish, so that

$$P^H_\sigma = dx^A \otimes \left(\frac{\partial}{\partial x^A} + \sigma^a_A \frac{\partial}{\partial y^a} \right) + dz^\alpha \otimes \frac{\partial}{\partial z^\alpha}.$$

Similarly

$$P^H_\chi = dx^A \otimes \left(\frac{\partial}{\partial x^A} + \chi^\alpha_A \frac{\partial}{\partial z^\alpha} \right) + dy^a \otimes \frac{\partial}{\partial y^a},$$

and we also find that

$$P^H_\kappa = dx^A \otimes \left(\frac{\partial}{\partial x^A} + \sigma^a_A \frac{\partial}{\partial y^a} + \chi^\alpha_A \frac{\partial}{\partial z^\alpha} \right).$$

These formulas become easier to read if we use bases of vector fields and differential forms adapted to the connections, rather than the coordinate bases. For the basis of vector fields we shall use

$$H_A = \frac{\partial}{\partial x^A} + \sigma^a_A \frac{\partial}{\partial y^a} + \chi^\alpha_A \frac{\partial}{\partial z^\alpha}, \quad \frac{\partial}{\partial y^a}, \quad \frac{\partial}{\partial z^\alpha}$$

and for the dual basis of forms we shall use

$$dx^A, \quad \eta^a = dy^a - \sigma^a_A dx^A, \quad \phi^\alpha = dz^\alpha - \chi^\alpha_A dx^A.$$

The formulas for the horizontal projectors then become

$$\begin{aligned} P^H_\sigma &= dx^A \otimes H_A + \phi^\alpha \otimes \frac{\partial}{\partial z^\alpha}, \\ P^H_\chi &= dx^A \otimes H_A + \eta^a \otimes \frac{\partial}{\partial y^a}, \\ P^H_\kappa &= dx^A \otimes H_A \end{aligned}$$

so that

$$P^H_\kappa + P^V_\sigma + P^V_\chi = I$$

where P^V_σ, P^V_χ are the corresponding vertical projectors.

We may also consider the horizontal projectors of the "parametrised connections" $\check\sigma$ and $\check\chi$. These will be tensor fields along ν and μ respectively, and in coordinates on Y, Z will be given by

$$\begin{aligned} P^H_{\check\sigma} &= dx^A \otimes \left(\frac{\partial}{\partial x^A} + \sigma^a_A \frac{\partial}{\partial y^a}\right) \\ P^H_{\check\chi} &= dx^A \otimes \left(\frac{\partial}{\partial x^A} + \chi^\alpha_A \frac{\partial}{\partial z^\alpha}\right). \end{aligned}$$

We shall be particularly interested in using these connections to decompose vector fields and curvature tensors. It is clear that any vector field on $Y \times_M Z$ may be written as the sum of three components by using the decomposition of the identity tensor given above. We shall, however, be more concerned with vector fields X along the projection ν, and the parametrised connection $\check\sigma$ may be used to split such a vector field into two components $\widehat{X}$, $\widetilde{X}$ according to the rule

$$\widehat{X}_p = (P^H_{\check\sigma})_p(X_p), \qquad \widetilde{X}_p = (P^V_{\check\sigma})_p(X_p)$$

at each point $p \in Y \times_M Z$.

We now have three vector fields along ν: the original one X, and its two components $\widehat{X}$, $\widetilde{X}$ which are, respectively, horizontal and vertical with respect to σ. But to any vector field along ν we may apply the horizontal lift defined by the other connection χ, giving a lifted vector field on $Y \times_M Z$. We shall denote the χ-horizontal lift of $\widehat{X}$ by X^H, and the χ-horizontal lift of $\widetilde{X}$ by X^D. The reasoning behind the notation is that X^H is horizontal with respect to both σ and χ (and hence, also, with respect to κ), whereas X^D is horizontal with respect to χ but vertical with respect to σ, and so may be thought of as "diagonal". In coordinates, if

$$X = \xi^A \left(\frac{\partial}{\partial x^A} + \sigma^a_A \frac{\partial}{\partial y^a} \right) + \xi^a \frac{\partial}{\partial y^a},$$

then

$$\widehat{X} = \xi^A \left(\frac{\partial}{\partial x^A} + \sigma^a_A \frac{\partial}{\partial y^a} \right), \qquad \widetilde{X} = \xi^a \frac{\partial}{\partial y^a},$$

and

$$X^H = \xi^A H_A, \qquad X^D = \xi^a \frac{\partial}{\partial y^a}.$$

We can see a similar phenomenon when looking at the curvatures of the three connections: each splits naturally into two components. Calculating the Nijenhuis tensor of the horizontal projector of σ,

we find that

$$\begin{aligned} \mathcal{N}_\sigma &= \tfrac{1}{2} \left(H_A(\sigma^a_B) - H_B(\sigma^a_A) \right) dx^A \wedge dx^B \otimes \frac{\partial}{\partial y^a} + \frac{\partial \sigma^a_B}{\partial z^\alpha} \phi^\alpha \wedge dx^B \otimes \frac{\partial}{\partial y^a} \\ &= \widehat{\mathcal{N}_\sigma} + \widetilde{\mathcal{N}_\sigma} \end{aligned}$$

where $\widetilde{\mathcal{N}_\sigma} = P^V_\chi \lrcorner \mathcal{N}_\sigma$ and $\widehat{\mathcal{N}_\sigma} = \mathcal{N}_\sigma - \widetilde{\mathcal{N}_\sigma}$. The relationship between the three curvatures may then be given by the formula

$$\mathcal{N}_\kappa = \widehat{\mathcal{N}_\sigma} + \widehat{\mathcal{N}_\chi}.$$

It is interesting to note that, whereas the connection κ completely determines the two projectable connections σ and χ, it is not the case that the curvature of κ determines the curvatures of σ and χ: two curvatures are always needed to determine the third.

Appendix B: Derivations of forms along ρ

For the general definition of derivations of scalar and vector-valued forms along ρ, we refer to the similar concepts along the tangent bundle projection in [21]. Following the pattern of the standard theory of Fröhlicher and Nijenhuis [13], one is naturally interested in a classification of such derivations. We give a sketch here of how this works in the present situation, without giving any proofs.

If $L \in V^r(\rho)$ denotes a vector-valued r-form along ρ, the meaning of the derivation i_L, of degree $r-1$, is familiar. We will denote by d^V_L, d^D_L and d^H_L the derivations of degree r obtained via the commutator of i_L and the exterior derivatives defined in Section 5. In fact, each of these derivations will be relevant only when L takes values in $\overline{\mathcal{X}}(\rho)$, $\widetilde{\mathcal{X}}(\rho)$ and $\widehat{\mathcal{X}}(\rho)$ respectively. We will denote this by the corresponding symbol on L.

One can prove that every derivation D of $\bigwedge(\rho)$ has a unique decomposition in the form:

$$D = i_{L_1} + d^V_{\overline{L}_2} + d^D_{\widetilde{L}_3} + d^H_{\widehat{L}_4}.$$

Looking for such decompositions is one of the ways by which derivations can lead to the discovery of new tensor fields. We illustrate this by investigating the decomposition of the commutators of the exterior derivatives. For a start we have

$$d^V \circ d^V = dt \wedge d^V, \qquad d^D \circ d^D = 0,$$

and

$$[d^V, d^D] = 0, \qquad [d^V, d^H] = 0.$$

It will not be a surprise that curvature components arise when we look at the remaining commutators involving d^H. We obtain

$$\begin{aligned} [d^H, d^D] &= -i_{dt\wedge R_3} + d^V_{R_3} \\ d^H \circ d^H = \tfrac{1}{2}[d^H, d^H] &= -i_{d^V R_2 + \Sigma} + d^D_{R_1} + d^V_{R_2}. \end{aligned}$$

Here, the apparently new vector-valued 3-form Σ turns out to be derived from the vector-valued 2-form Ξ introduced in Section 5:

$$\Sigma = \tfrac{1}{2}(d^V \Xi + dt \wedge \Xi).$$

In this respect, it is also worthwhile to observe the following properties of the R_i tensors which are easily obtained from their expressions in terms of the (1,1) tensors Ψ, Φ, Λ:

$$\begin{aligned} d^V R_1 &= 0 \\ d^V R_2 &= -dt \wedge R_2 \\ d^V R_3 &= -dt \wedge R_3. \end{aligned}$$

We do not enter into the extension of these derivations to vector-valued forms as they are not always defined on the complete set $V(\rho)$. The situation is different when it concerns a derivation of degree zero, as such a derivation can always be extended to vector fields (and then to all tensor fields) by the duality rule.

Let D be an arbitrary derivation of $\bigwedge(\rho)$ of degree 0. According to the general decomposition result, we know that there exist vector fields $\overline{X}, \widehat{Y}, \widetilde{Z} \in \mathcal{X}(\rho)$ and some $L_1 \in V^1(\rho)$, such that

$$D = i_{L_1} + d^V_{\overline{X}} + d^H_{\widehat{Y}} + d^D_{\widetilde{Z}}.$$

Now, from the definition of the exterior derivatives we easily obtain that for $\omega \in \bigwedge^1(\rho)$,

$$(d^V_{\overline{X}}\omega)(Y) = (\mathrm{D}^V_{\overline{X}}\omega)(Y) + \omega(d^V\overline{X}(Y)).$$

A similar property holds for $d^H_{\widehat{Y}}$, but the situation is different for d^D, where one finds

$$d^D_{\widetilde{Z}} = \mathrm{D}^D_{\widetilde{Z}}.$$

It follows that

$$D = \mathrm{D}^V_{\overline{X}} + \mathrm{D}^H_{\widehat{Y}} + \mathrm{D}^D_{\widetilde{Z}} - i_Q,$$

with $-Q = L_1 + d^V\overline{X} + d^H\widehat{Y}$ not depending on $\widetilde{Z}$. If we next extend the action of D by duality, the term $-i_Q$ (which vanishes on vector fields by definition) is replaced by, say, a_Q, an algebraic derivation vanishing on forms and acting on vector fields as $a_Q(X) = Q(X)$. It follows that every self-dual derivation D of degree 0 has a unique decomposition into four self-dual components, namely

$$D = \mathrm{D}^V_{\overline{X}} + \mathrm{D}^H_{\widehat{Y}} + \mathrm{D}^D_{\widetilde{Z}} + \mu_Q\,,$$

with $\mu_Q = a_Q - i_Q$. The algebraic derivation μ_Q vanishes on functions. Therefore, once the action of a self-dual derivation of degree zero is known on functions, its complete determination is a matter of finding the type (1,1) tensor field Q in the above decomposition. This is the technique which has been used to describe, in Section 5, the properties of the self-dual derivations which are obtained through the commutators of the basic D-derivations introduced in Section 4.

Acknowledgements. This research was partially supported by NATO Collaborative Research Grant No. CRG 940195. W.S. and F.C. further thank the Fund for Scientific Research – Flanders (Belgium) for continuing support.

References

[1] L. Bates and J. Sniatycki, Nonholonomic reduction, *Rep. on Math. Phys.* **32** (1993) 99–115.

[2] A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden and R.M. Murray, Nonholonomic mechanical systems with symmetry, *Arch. Rat. Mech. Anal* (1996) to appear.

[3] G. Byrnes, A complete set of Bianchi identities for tensor fields along the tangent bundle projection, *J. Phys. A: Math. Gen.* **27** (1994) 6617–6632.

[4] F. Cantrijn, W. Sarlet, A. Vandecasteele and E. Martínez, Complete separability of time-dependent second-order equations, *Acta Applic. Mathematicae* **42** (1996) 309–334.

[5] F. Cardin and M. Favretti, On nonholonomic and vakonomic dynamics of mechanical systems with nonintegrable constraints, *J. Geom. Phys.* **18** (1996) 295–325.

[6] M. Crampin, Tangent bundle geometry for Lagrangian dynamics, *J. Phys. A: Math. Gen.* **16** (1983) 3755–3772.

[7] M. Crampin, E. Martínez and W. Sarlet, Linear connections for systems of second-order ordinary differential equations, *Ann. Inst. H. Poincaré, Phys. Théor.* **65** (1996) 223–249.

[8] M. Crampin, G.E. Prince and G. Thompson, A geometrical version of the Helmholtz conditions in time-dependent Lagrangian dynamics, *J. Phys. A: Math. Gen.* **17** (1984) 1437–1447.

[9] M. Crampin, W. Sarlet, E. Martínez, G.B. Byrnes and G.E. Prince, Towards a geometrical understanding of Douglas's solution of the inverse problem of the calculus of variations, *Inverse Problems* **10** (1994) 245–260.

[10] R. Cushman, D. Kemppainen, J. Sniatycki and L. Bates, Geometry of nonholonomic constraints, *Rep. on Math. Phys.* **36** (1995) 275–286.

[11] M. de León and D. Martín de Diego, On the geometry of non-holonomic Lagrangian systems, *J. Math. Phys.* **37** (1996) 3389–3414.

[12] M. de León and D. Martín de Diego, Non-holonomic mechanical systems in jet bundles, *Extracta Mathematicae* **11** (1996) 127–139.

[13] A. Frölicher and A. Nijenhuis, Theory of vector-valued differential forms, *Proc. Ned. Acad. Wetensch. Sér. A* **59** (1956) 338–359.

[14] P. Foulon, Géometrie des équations differentielles du second ordre, *Ann. Inst. H. Poincaré Phys. Théor.* **45** (1986) 1–28.

[15] G. Giachetta, Jet methods in nonholonomic mechanics, *J. Math. Phys.* **33** (1992) 1652–1665.

[16] J. Grifone, Structure presque tangente et connexions I, *Ann. Inst. Fourier* **22** No 1 (1972) 287–334.

[17] L.A. Ibort and J. Marín Solano, On the inverse problem of the calculus of variations for a class of coupled dynamical systems *Inverse Problems* **7** (1991) 713–725.

[18] J. Koiller, Reduction of some classical non-holonomic systems with symmetry *Arch. Rational Mech. Anal.* **118** (1992) 113–148.

[19] C.-M. Marle, Reduction of constrained mechanical systems and stability of relative equilibria, *Commun. Math. Phys.* **174** (1995) 295–318.

[20] E. Martínez and J.F. Cariñena, Geometric characterization of linearisable second-order differential equations, *Math. Proc. Camb. Phil. Soc.* **119** (1996) 373–381.

[21] E. Martínez, J.F. Cariñena and W. Sarlet, Derivations of differential forms along the tangent bundle projection, *Diff. Geometry and its Applications* **2** (1992) 17–43.

[22] E. Martínez, J.F. Cariñena and W. Sarlet, Derivations of differential forms along the tangent bundle projection II, *Diff. Geometry and its Applications* **3** (1993) 1–29.

[23] E. Martínez, J.F. Cariñena and W. Sarlet, Geometric characterization of separable second-order equations, *Math. Proc. Camb. Phil. Soc.* **113** (1993) 205–224.

[24] E. Massa and E. Pagani, Classical dynamics of non-holonomic systems: a geometric approach, *Ann. Inst. H. Poincaré* **55** (1991) 511–544.

[25] E. Massa and E. Pagani, Jet bundle geometry, dynamical connections, and the inverse problem of Lagrangian mechanics, *Ann. Inst. H. Poincaré Phys. Théor.* **61** (1994) 17–62.

[26] W. Sarlet, A direct geometrical construction of the dynamics of non-holonomic Lagrangian systems, *Extracta Mathematicae* **11** (1996) 202–212.

[27] W. Sarlet, The geometry of mixed first and second-order differential equations with applications to non-holonomic mechanics, in: *Differential Geometry and Applications*, Proc. 6th Int. Conf. Brno (August 28 – September 1, 1995) J. Janyška, I. Kolář and J. Slovák eds., (1996) 641–650.

[28] W. Sarlet, F. Cantrijn and D.J. Saunders, A geometrical framework for the study of non-holonomic Lagrangian systems, *J. Phys. A: Math. Gen.* **28** (1995) 3253–3268.

[29] W. Sarlet, G.E. Prince and M. Crampin, Adjoint symmetries for time-dependent second-order equations, *J. Phys. A: Math. Gen.* **23** (1990) 1335–1347.

[30] W. Sarlet, A. Vandecasteele, F. Cantrijn and E. Martínez, Derivations of forms along a map: the framework for time-dependent second-order equations, *Diff. Geometry and its Applications*, **5** (1995) 171–203.

[31] D.J. Saunders, W. Sarlet and F. Cantrijn A geometrical framework for the study of non-holonomic Lagrangian systems: II, *J. Phys. A: Math. Gen.* **29** (1996) 4265–4274.

[32] D.J. Saunders, A new approach to the linear connection associated with second-order differential equation fields, preprint (1996).

[33] A.M. Vershik, Classical and non-classical dynamics with constraints, in: Yu.G. Borisovich and Yu.E. Gliklikh, eds., *Global Analysis — Studies and Applications I*, Lecture Notes in Math. **1108** (Springer-Verlag, Berlin) (1984) 278–301.