Symmetries, separability and volume forms

W. Sarlet, R. Sahadevan*and F. Cantrijn†

Department of Mathematical Physics and Astronomy
University of Ghent, Krijgslaan 281, B-9000 Ghent, Belgium

Abstract. In order to clarify certain misconceptions in the literature, we discuss the details of the way determining equations for general symmetries of a Lagrangian system differ from determining equations for Noether symmetries, and establish the minimal set of extra conditions which have to be imposed in practical situations for the former to reduce to the latter. We further derive properties by which, in situations where the system is integrable through quadratic integrals in involution, the components of the corresponding Noether symmetries themselves can be used to compute the separation variables, if they exist, for the Hamilton-Jacobi equation.

PACS codes: 0320, 0240

*On leave from The Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chepauk, Chennai–600005, Tamilnadu, India

†Senior Research Associate at the Fund for Scientific Research, Flanders, Belgium

1 Introduction

The role and importance of Noether's theorem in the study of Lagrangian systems is widely recognised. It is the vehicle by which, from a physical perspective, the duality is established between symmetries and conservation laws. The mathematical interest of knowing symmetries and/or first integrals of a dynamical system is that they provide means for a reduction of the integration process; knowledge of a sufficient number of such elements may eventually lead to a form of complete integrability of the system.

Many authors have discussed ways of generating, if possible, first integrals also from general symmetries of the Lagrangian (or Hamiltonian) differential equations, i.e. symmetries of non-Noether type. The general mechanisms behind all such constructions are fairly easy to grasp if one makes use of some elementary concepts from the differential geometric description of (Lagrangian) mechanics. Indeed, leaving aside certain slight variations or generalisations, one can roughly capture the full picture in the following scheme. Noether symmetries of a Lagrangian system are symmetries of the Cartan 2-form $d\theta$ and the corresponding integral originates from the closedness of the contraction of $d\theta$ with the symmetry generator (see e.g. [28]). In the case of non-Noether symmetries, there is usually one of two mechanisms at work: if a corresponding first integral has a divergence part, then we are essentially looking at the invariance of the volume form $(d\theta)^n$, where n is the number of degrees of freedom (see e.g. [5]); alternatively, the invariant object is a type (1,1) tensor field, constructed from $d\theta$ and its Lie derivative under the action of the symmetry generator, and the first intergrals then essentially come from the Lax-type equation which expresses this invariance (see e.g. [6, 29, 8]).

Unfortunately, there is still a lot of confusion about such issues in the current literature, probably because a number of authors are still not aware or convinced of the benefits and the generality of coordinate free descriptions. To cite one (of many possible) examples where this leads to unfruitfull efforts: Lutzky recently discussed "generalisations" of his earlier work on non-Noether symmetries (see [19, 20]), in which a class of velocity-dependent symmetry generators is considered. His new results [21, 22], however, are merely particular cases of the above cited construction of an invariant type (1,1) tensor field, which is valid for all velocity-dependent symmetries.

The need for correcting some inaccurate or misleading statements lies partly at the origin of the present paper. In some of the publications of Lakshmanan and co-workers about integrability aspects of mechanical systems (see e.g. [15, 16]), consistent use of Noether's theorem was made to construct the invariants. In other papers, however, (see [13, 14, 17]), the emphasis is much more on a methodology which starts from a systematic construction of (generalised) symmetries of the differential equations (except that in the review paper [17], a large part deals with Painlevé analysis as well). The adjective generalised refers to the fact that the spirit is much in the line of the Lie approach to differential equations, but non-point symmetries are being considered. From the point of view of symmetries of the vector field generating second-order dynamics, however, there is no need to talk about anything generalised here. This is of course not the point of criticism. The point is that the authors believe that they are making use of a wider class of symmetries than just Noether symmetries, as is illustrated by the explicit statement in [13, 14] that a first integral is being constructed without recourse to Noether's theorem. Yet, the calculations are manifestly not about some volume form or some type (1,1) tensor field. Instead, a certain ad hoc ansatz is imposed on the construction of (generalised) Lie symmetries, which apparently has the effect that an invariant of the symmetry generator itself becomes a first integral of the given differential equations. Another, potentially interesting, but in the presentation of the authors rather peculiar story, is about the way separation variables (in the sense of Hamilton-Jacobi theory) can be constructed directly from the components of the symmetry generator. The claim is that such variables follow from a process of "integration of part of the characteristic equations related to the symmetry vector field". However, no justification whatsoever is given concerning reasons why the manipulations which are being carried out should have anything to do with the construction of separation variables.

In Section 2, we will show first of all that the extra anzats which is assumed in [13, 14] to find symmetries which lead to a corresponding first integral, is exactly a condition which forces these symmetries to be of Noether type. In Section 3, we will take the opportunity to add new insights into the way Noether symmetries, from a computational point of view, make up a subset of the set of all dynamical symmetries of a class of autonomous Lagrangian systems. Roughly speaking, for symmetries whose leading components are polynomial functions of the velocities of a certain degree, we will prove that the restriction to be a Noether symmetry is equivalent to the requirement that: (i) the coefficients of the terms of non-zero degree in the velocities are totally symmetric; (ii) the symmetry generator preserves energy (a necessary condition for all Noether symmetries of autonomous Lagrangian systems); (iii) the lowest order part (only) is imposed of the determining equations for arbitrary symmetry vector fields.

In Section 4, we investigate the way the Noether symmetries, corresponding to a set of n quadratic integrals in involution, might be directly related to the construction of separation variables in the sense of Hamilton-Jacobi. For a start, the calculations presented in [13, 14, 17] appear to have little to do with "integration of characteristic equations" related to the first-order partial differential operator which comes from the symmetry vector field. Instead, they are merely a set of formal manipulations which, at least in the case of two degrees of freedom discussed in [13] and for the examples under consideration, seem to produce the right answers. We will provide an explanation for this feature, which is based on Eisenhart's theorem about the intrinsic characterisation of Stäckel systems (see e.g. [12]), and shows that the manipulations under consideration can be given a theoretical backing from a factorisation property of the action of the natural volume form on the vector fields corresponding to all quadratic integrals in involution. From this point of view, it would seem that for an extension to more than two degrees of freedom, a similar manipulation of characteristic equations, as attempted in [14, 17], has no meaning whatsoever; instead, one is led to compute a determinant from the components of the symmetries and look at a factorisation property again. Section 5 contains illustrative examples.

2 Computational aspects of Noether's theorem

For convenience, we will stick to the framework of autonomous second-order differential equations (as is the case in the papers we wish to clarify).

Let $L(x^i, \dot{x}^i)$, $\quad i = 1, \ldots, n$ be the regular Lagrangian of a given second-order dynamical system $\ddot{x}^i = f^i(x, \dot{x})$, with corresponding vector field

$$\Gamma = \dot{x}^i \frac{\partial}{\partial x^i} + f^i(x, \dot{x}) \frac{\partial}{\partial \dot{x}^i}, \tag{1}$$

living on the tangent bundle TM of some manifold M. With $\Delta = \dot{x}^i \partial/\partial \dot{x}^i$ representing the Liouville vector field, $E = \Delta(L) - L$ the 'energy function' associated to L, and $\theta = (\partial L/\partial \dot{x}^i)\, dx^i$ the Poincaré-Cartan 1-form, we know that Γ is defined by

$$i_\Gamma d\theta = -dE. \tag{2}$$

If, for simplicity, we restrict ourselves to symmetry vector fields which are time-independent as well, Noether's theorem says: let $Y \in \mathcal{X}(TM)$ be a vector field with the properties

$$\mathcal{L}_Y \theta = df \qquad \text{and} \qquad Y(E) = 0, \tag{3}$$

for some function f on TM, then $F = f - \langle Y, \theta \rangle$ is a first integral, i.e. $\Gamma(F) = 0$. The function f is usually referred to as the gauge term in Noether's theorem. Every such Noether symmetry Y is effectively a symmetry of the given differential equations, i.e. we have $[\Gamma, Y] = 0$. Recall further that the first condition can equivalently be written as

$$i_Y d\theta = dF, \tag{4}$$

from which in fact it trivially follows that $\Gamma(F) = 0$ in view of (2) and $Y(E) = 0$. Conversely, if F is a first integral, the relation (4) uniquely defines a vector field Y which will automatically leave the energy function invariant. This way, we obtain a one to one correspondance between Noether symmetries and first integrals (which depends on the Lagrangian). Note in passing that (4) trivially implies that $Y(F) = 0$, but needless to say, if for a given symmetry vector field Y of Γ one searches for invariants of Y, there is no reason why these would at the same time be first integrals of the given dynamics as well.

In coordinates, the symmetry generator Y will be of the form

$$Y = \xi^i \frac{\partial}{\partial x^i} + \Gamma(\xi^i) \frac{\partial}{\partial \dot{x}^i}, \tag{5}$$

where the leading components ξ^i are of course functions on TM, i.e. we are talking in general about more than point symmetries.

Let us first link this up with the original version of Noether's theorem, which has to do with invariance of the action integral (up to a gauge term). In that approach, again restricting ourselves to transformations which do not change the time variable, one arrives at the following necessary and sufficient conditions for a Noether symmetry:

$$\xi^i \frac{\partial L}{\partial x^i} + \dot{\xi}^i \frac{\partial L}{\partial \dot{x}^i} = \dot{f}. \tag{6}$$

This is often referred to as the Noether identity. A good reference in this respect is [18]. The right way to look at the conditions (6), from the point of view of writing down determining equations for the construction of Noether symmetries, is that (6) has to be identically satisfied for all values of the independent variables $x^i, \dot{x}^i, \ddot{x}^i$. The terms $\dot{\xi}^i$ and $\dot{f}$ stand for total time derivatives of functions of $x^i, \dot{x}^i$ and hence depend linearly on $\ddot{x}^i$. As a result, equation (6) immediately splits and is equivalent with the $n+1$ requirements:

$$\frac{\partial \xi^i}{\partial \dot{x}^j}\frac{\partial L}{\partial \dot{x}^i} = \frac{\partial f}{\partial \dot{x}^j} \qquad j = 1, \ldots, n \tag{7}$$

$$\xi^k \frac{\partial L}{\partial x^k} + \dot{x}^k \frac{\partial \xi^i}{\partial x^k}\frac{\partial L}{\partial \dot{x}^i} = \dot{x}^k \frac{\partial f}{\partial x^k}. \tag{8}$$

The full equivalence between these partial differential equations for the unknown ξ^i, f and the ones which come out of the requirements (3) has been established (also more generally for time-dependent situations) in [28]. It may be helpful, however, to make a few more observations around this point here. Multiplying Eqns. (7) by f^j, summing over j and adding the result to (8), we see right away that the system (7,8) is equivalent with

$$\frac{\partial \xi^i}{\partial \dot{x}^j}\frac{\partial L}{\partial \dot{x}^i} = \frac{\partial f}{\partial \dot{x}^j} \qquad j = 1, \ldots, n \tag{9}$$

$$Y(L) = \Gamma(f), \tag{10}$$

with Y as defined in (5). Recalling the identity

$$\mathcal{L}_\Gamma i_Y \theta = i_Y \mathcal{L}_\Gamma \theta + i_{[\Gamma, Y]}\theta$$

(which is valid for arbitrary vector fields Y, Γ and arbitrary θ), we observe that for a vector field of type (5), $[\Gamma, Y]$ is vertical, so that its contraction with the semi-basic 1-form θ is zero. We thus obtain,

$$\begin{aligned} \mathcal{L}_\Gamma i_Y \theta &= i_Y(i_\Gamma d\theta + d i_\Gamma \theta) \\ &= i_Y(-dE + d(\Delta L)) \;=\; Y(L). \end{aligned}$$

Putting $F = f - i_Y\theta$, we conclude that (10) is further equivalent to $\Gamma(F) = 0$. The point about this remark is the following. Part of the folklore in the literature is that for the explicit calculations involved in the construction of first integrals of Lagrangian systems, one can either follow Darboux' direct approach (cf. [32]), i.e. try to solve directly the partial differential equation $\Gamma(F) = 0$, or use Noether's method to construct symmetries first, and that these are different methods from the practical point of view. Of course, Noether's theorem expresses that one will lead to the other, but the above remark shows that in fact the calculations are, roughly speaking, the same.

A computation which is different, in principle, but more or less of the same complexity, is the search for solutions of the determining equations for arbitrary symmetries of Γ, i.e. vector fields Y for which $[\Gamma, Y] = 0$. Such vector fields will be of the form (5), where the ξ^i have to solve the second-order partial differential equations

$$\Gamma^2(\xi^i) = Y(f^i). \tag{11}$$

If more than just point symmetries are hoped for, an ansatz will be needed about the polynomial dependence of the ξ^i on the velocities, to make such calculations feasible. The set of solutions will always be larger than the solutions of the corresponding search for Noether symmetries with the same ansatz. The construction of general symmetries (11) is the line of approach followed in [13, 14], where the authors study Lagrangian systems with 2 and 3 degrees of freedom, for some specific cases of standard Lagrangians of the form

$$L = \frac{1}{2}\sum(\dot{x}^i)^2 - V(x). \tag{12}$$

The ansatz is that the ξ^i are linear in the velocities and the claim is somehow that with all the symmetries which are obtained "one can find, without recourse to Noether's theorem, and almost by inspection, a set of involutive first integrals by looking for invariants of the symmetry generator itself". Close inspection of their analysis reveals for a start, that in the process of solving equations (11), the authors must have taken for granted certain simplifications, because all the symmetries they obtain will turn out to be Noether symmetries anyway. Moreover, the impression is given that first integrals should follow from "integrating the equation $Y(F) = 0$". The fact that the energy integral every time appears to be one of such functions F, points in the direction of Noether symmetries (see the second condition in (3)). Finally, the rule of thumb by which the other integrals are said to appear is that we ought to expect that our symmetries are such that (in more correct notations):

$$\xi^i = \frac{\partial U}{\partial \dot{x}^i} \qquad \text{and} \qquad \Gamma(\xi^i) = -\frac{\partial U}{\partial x^i}, \tag{13}$$

for some function U (cf. Eqns. (2.7)(2.9) in [13] and (2.6)(2.8)(2.9) in [14]).

Now, for a Lagrangian of type (12), we have $\theta = \dot{x}^i\, dx^i$ and hence, with a vector field Y of type (5)

$$i_Y\,(d\dot{x}^i \wedge dx^i) = \Gamma(\xi^i)\, dx^i - \xi^i\, d\dot{x}^i.$$

It follows that the requirements (13) precisely express the condition (4) with $F = -U$, so that the whole idea is bound to be just an application of Noether's theorem, as expected.

In [16, 17], no claim is made that the construction of first integrals follows without recourse to Noether's theorem. Symmetry generators are constructed from the general requirement (11), with ξ^i which are of degree 1 or of degree 3 in the velocities. But also here, in the course of the calculations, a number of unspecified simplifications must have been imposed for the ease of finding solutions, because it turns out in the end that again all the symmetries which are found are of Noether type. In the next section, therefore, we will analyse in detail and in some generality what the minimal set of requirements is that one has to impose on general symmetries, in order for them to become Noether symmetries.

3 Noether versus general symmetries

We wish to clarify the situation about conditions which force symmetries to fall into the Noether class from the very practical point of view of actually solving determining equations. For that purpose, we will limit ourselves to purely analytical considerations and a case study for arbitrary n, which covers all the examples of symmetries mentioned in the above cited papers.

The systems under consideration are the Lagrangian ones with a Lagrangian of the form (12). It is known (see e.g. [30, 10]) that first integrals of such a system, which polynomially depend on the velocities, can be taken to have only terms which are either all of odd or all of even degree in the velocities. We consider the even degree case and assume, to be concrete, that the degree is at most four. The leading coefficients ξ^i of corresponding Noether symmetries will thus contain only terms of odd degree in the velocities and the maximal degree is three. These will then, of course, also be the restrictions we impose on the search for general symmetries. Writing all the indices of undetermined coefficients as bottom indices for easy legibility, we look for functions ξ^i of the following form:

$$\xi^i = a_{ijkl}(x)\dot{x}^j\dot{x}^k\dot{x}^l + b_{ij}(x)\dot{x}^j. \tag{14}$$

Obviously, a_{ijkl} is assumed to be symmetric in its last three indices, but no further symmetry conditions for the coefficients in (14) must hold a priori, if the search is for general symmetries. The conditions (11) for symmetries, here more specifically of the form

$$\Gamma^2(\xi^i) + Y\left(\frac{\partial V}{\partial x^i}\right) = 0, \tag{15}$$

explicitly become:

$$\begin{aligned}
&\frac{1}{10}a_{i(jkl,mn)}\,\dot{x}^j\dot{x}^k\dot{x}^l\dot{x}^m\dot{x}^n - a_{i(mkl,j)}\frac{\partial V}{\partial x^j}\dot{x}^k\dot{x}^l\dot{x}^m - a_{ij(kl,m)}\frac{\partial V}{\partial x^j}\dot{x}^k\dot{x}^l\dot{x}^m \\
&\quad - a_{ij(kl}\frac{\partial^2 V}{\partial x^{m)}\partial x^j}\dot{x}^k\dot{x}^l\dot{x}^m + a_{jklm}\frac{\partial^2 V}{\partial x^i\partial x^j}\dot{x}^k\dot{x}^l\dot{x}^m + \frac{1}{3}b_{i(k,lm)}\dot{x}^k\dot{x}^l\dot{x}^m \\
&\quad + 6\,a_{ijkl}\frac{\partial V}{\partial x^j}\frac{\partial V}{\partial x^k}\dot{x}^l - b_{i(k,l)}\frac{\partial V}{\partial x^k}\dot{x}^l - b_{ik,l}\frac{\partial V}{\partial x^k}\dot{x}^l \\
&\quad - b_{ik}\frac{\partial^2 V}{\partial x^k\partial x^l}\dot{x}^l + b_{kl}\frac{\partial^2 V}{\partial x^i\partial x^k}\dot{x}^l \;=\; 0.
\end{aligned} \tag{16}$$

As customary, brackets around indices refer to symmetrisation. Let us, however, be precise about this to avoid confusion concerning numerical factors: the convention here is that in an expression like $a_{i(mkl,j)}$, where the object is already symmetric in k, l, m by assumption, symmetrisation simply means taking a cyclic sum over all indices inside the brackets. The determining equations for constructing all symmetries with leading components of the form (14) are obtained by putting the coefficients of the terms of degree 5, 3 and 1 in the above expression separately equal to zero.

If the symmetry is of Noether type, the ξ^i will be related to a first integral F via the relations $\xi^i = \partial F/\partial \dot{x}^i$ (strictly speaking, this is minus the F of (4)). In other words, and in agreement with our analysis in the previous section, Y will be a Noether symmetry if and only if

$$F = \frac{1}{4} a_{ijkl} \dot{x}^i \dot{x}^j \dot{x}^k \dot{x}^l + \frac{1}{2} b_{ij} \dot{x}^i \dot{x}^j + g(x) \tag{17}$$

is a first integral of Γ for some function g. This implies for a start that we have to take the functions a_{ijkl} and b_{ij} to be symmetric in all indices. Note in passing that these symmetry properties are the integrability conditions for existence of a function f which satisfies (9) and that $\Gamma(F) = 0$, as argued before, is another way of writing (10), which again shows explicitly that we are expressing here the Noether identity (6). Now, $\Gamma(F) = 0$ also gives rise to a relation of degree 5 in the velocities and the terms of order 5, 3 and 1 produce the requirements:

$$\begin{aligned} a_{(ijkl,m)} &= 0 && (18) \\ b_{(ij,k)} &= 6\, a_{ijkl} \frac{\partial V}{\partial x^l} && (19) \\ \frac{\partial g}{\partial x^j} &= b_{jk} \frac{\partial V}{\partial x^k} && (20) \end{aligned}$$

In summary, functions of the form (14) are the leading components of a Noether symmetry if and only if the a_{ijkl} and b_{ij} are completely symmetric and satisfy (18-20) for some function g. The last of the requirements will then give rise to further integrability conditions which read

$$(b_{jk,l} - b_{lk,j}) \frac{\partial V}{\partial x^k} + b_{jk} \frac{\partial^2 V}{\partial x^k \partial x^l} - b_{lk} \frac{\partial^2 V}{\partial x^k \partial x^j} = 0. \tag{21}$$

Clearly, the requirements for Noether symmetries are quite different from those for arbitrary symmetries, if only because in the general case the undetermined functions in (14) need not be totally symmetric. If one wants to pin down a minimal set of extra assumptions which will force symmetries to fall into the subcategory of Noether symmetries, imposing symmetry in all indices is obviously inevitable. Since all Noether symmetries will further preserve the energy function, we wish to investigate now to what extent $Y(E) = 0$ can count for these extra assumptions.

With $E = \frac{1}{2} \sum (\dot{x}^i)^2 + V$, we have

$$Y(E) = \dot{x}^i\, \Gamma(\xi^i) + \xi^i \frac{\partial V}{\partial x^i}$$

and for general ξ^i of the form (14), this becomes

$$\begin{aligned} Y(E) \;=\; & \dot{x}^i \left[\frac{1}{4} a_{i(jkl,m)} \dot{x}^j \dot{x}^k \dot{x}^l \dot{x}^m - 3\, a_{ijkl} \frac{\partial V}{\partial x^j} \dot{x}^k \dot{x}^l + \frac{1}{2} b_{i(j,k)} \dot{x}^j \dot{x}^k - b_{ij} \frac{\partial V}{\partial x^j} \right] \\ & + a_{ijkl} \frac{\partial V}{\partial x^i} \dot{x}^j \dot{x}^k \dot{x}^l + b_{ij} \frac{\partial V}{\partial x^i} \dot{x}^j . \end{aligned}$$

Under the assumption of fully symmetric a_{ijkl} and b_{ij}, the terms of order 1 cancel out, whereas the terms of order 5 and 3 can easily be seen to produce exactly (18) and (19).

Obviously, we need one further condition to produce the remaining requirement (20) or equivalently (21). For that purpose, we look at the lowest order terms in the general symmetry requirement (16). Originally, they produce the condition:

$$6\,a_{ijkl}\frac{\partial V}{\partial x^j}\frac{\partial V}{\partial x^k} - b_{i(l,k)}\frac{\partial V}{\partial x^k} - b_{ij,l}\frac{\partial V}{\partial x^j} - b_{ij}\frac{\partial^2 V}{\partial x^j \partial x^l} + b_{jl}\frac{\partial^2 V}{\partial x^i \partial x^j} = 0. \tag{22}$$

Making use of the symmetry assumption on the a and b functions and condition (19) which already followed from $Y(E) = 0$, one readily verifies that (22) becomes identical to (21). The results can be summarised as follows.

Proposition. *Consider a vector field Y of the form (5), where the functions $\xi^i(x, \dot{x})$ have the structure (14), then the following three conditions imply that Y will be a Noether symmetry of the Lagrangian second-order system Γ derived from the function L in (12):*

1. *the coefficients in the expression (14) for ξ^i are symmetric in all indices,*
2. *$Y(E) = 0$ (where E is the energy function associated to L),*
3. *the lowest-order terms (i.e. terms of degree 1) in the non-zero components of the vector field $[Y, \Gamma]$ cancel out.*

Remark: since Noether symmetries are symmetries, the above requirements will guarantee that the terms of degree 5 and 3 in the complicated expression (16) will automatically be zero as well.

Needless to say, the proposition is equally valid for the case of symmetry components ξ^i which are linear in the velocities, with corresponding quadratic first integrals: it suffices to put $a_{ijkl} = 0$ in all calculations above. It would not be too difficult to extend it, on the other hand, to higher-degree polynomials and to Lagrangians with a more general metric in the kinetic energy term than the standard Euclidean metric in (12). We finally point to a few other well known facts. The condition (18) in the Noether approach, which comes from the terms of highest degree does not involve the potential and hence expresses that the terms of highest degree in (17) (quartic in our case) will constitute a first integral of the pure geodesic motion produced by the kinetic energy term in L. What the structure of equations (18) further tells us is that we are looking at the components of a Killing tensor: in the present situation, the comma of course refers to taking partial derivates; if the kinetic energy metric would not be Euclidean, covariant derivatives would make their appearance there instead. To name just a few references which deal with the relation between first integrals and Killing tensors, see [31, 1, 12, 35]. As some of these references indicate, Killing tensors play a key role also in the theory about separation of variables in the Hamilton-Jacobi equation, to which we come in the next section.

4 Symmetries and separability

Separation of variables in the Hamilton-Jacobi equation is a problem with a long history. It is not our intention to even attempt a short survey of the history here. For that we refer to specialists in the field, like Benenti and Kalnins and Miller, who have contributed to the subject with numerous publications of which we cite just a few here: [3, 11, 12]. Probably the best known classical result is Stäckel's theorem about orthogonally separable systems. It is a key result which tells you how separable systems look like in the coordinates in which indeed the separation works (generalisations by Benenti cover a similar characterisation for the non-orthogonal case). It does not tell you, however, how to construct such coordinates, or in fact whether such coordinates exist (see also [23]). The first fundamental result about existence of separation variables is Eisenhart's theorem [9], which is in the first place an existence theorem, but also has constructive features which are perhaps underestimated. Eisenhart's theorem essentially gives an intrinsic characterisation of the existence of a point transformation which will transform a Hamiltonian system into Stäckel form. Very little is known about the existence of more general canonical transformations which have such effect. A remarkable example of a non-point transformation which transforms a Hamiltonian system into separable form is contained in [4]. In [34], this example is put into a more general framework by the introduction of what the authors call quasi-point transformations. An equivalent result for the same example, by the way, has been obtained through techniques of algebraic geometry. See in that respect [27] and references therein. In any event, even in the case of point transformations, there is still room for new insights about tests which can tell whether a given system is separable and how one can find the right coordinates explicitly. As a preliminary remark, although Hamilton-Jacobi is of course, strictly speaking, about Hamiltonian systems, we will continue to describe the situation for the corresponding Lagrangian setting. This is certainly justified for systems with a Lagrangian of type (12), for which velocities and momenta are the same.

In [13, 14, 17], it is claimed that a direct calculation on the components of symmetry vector fields can produce separation variables, at least in the case of functions ξ^i which are linear in the velocities. In a programme in which one starts the analysis of a given system by constructing symmetries, in particular Noether symmetries and the corresponding first integrals, it is certainly a good idea to investigate whether the symmetries contain by themselves information about the construction of separation variables. Roughly, what the authors do goes as follows. Consider the characteristic equations for the partial differential equation $Y(F) = 0$, which read, for the case $n = 2$ say,

$$\frac{dx^1}{\xi^1} = \frac{dx^2}{\xi^2} = \frac{d\dot{x}^1}{\Gamma(\xi^1)} = \frac{d\dot{x}^2}{\Gamma(\xi^2)}, \tag{23}$$

and assume that the functions ξ^i are of the form

$$\xi^i = a_{ij}(x)\dot{x}^j, \qquad \text{with} \quad a_{ij} = a_{ji}. \tag{24}$$

Then, the first equality can be written in the form

$$a_{2j}\dot{x}^j\, dx^1 - a_{1j}\dot{x}^j\, dx^2 = 0.$$

Thinking of $\dot{x}^j$ as standing for dx^j/dt and multiplying by dt, the formal quadratic expression

$$a_{2j}\, dx^j dx^1 - a_{1j}\, dx^j dx^2 = 0 \tag{25}$$

is obtained. Solving this, for example, algebraically for dx^2/dx^1 and treating these as first-order differential equations, it so happens that their integration seems to produce the separation variables. The authors call this procedure "integrating part of the characteristic equations". However, when integrating characteristic equations such as (23), the x^i and $\dot{x}^i$ ought to be treated as independent variables, so the formal manipulation we just sketched has nothing to do with that. In any event, the authors do not give any reason why this manipulation should have something to do with the construction of separation variables. The lack of a supporting theory puts the authors on even more shaky grounds, when they attempt similar manipulations on some 3-dimensional systems, because there is then more than one symmetry involved and it is in general not clear how one could, even in principle, handle a set of such formal quadratic expressions.

We will develop arguments here to show first that the formal manipulations mentioned above are indeed bound to produce separation variables for $n = 2$. These arguments will show at the same time that the route to generalisations to more than two degrees of freedom is quite different from the calculations performed in [14, 17]. As a preliminary remark, observe that in manipulating the quadratic expression (25), it makes no difference whether one writes dx^i or $\dot{x}^i$, and in fact it will be more appropriate for the developments below to think of it as

$$a_{2j}\,\dot{x}^j\dot{x}^1 - a_{1j}\,\dot{x}^j\dot{x}^2 = 0. \tag{26}$$

Since Eisenhart's theorem concerning separability is about Killing tensors, and such objects are related to first integrals which in turn correspond to Noether symmetries, it should not come as a surprise that we will seek an explanation which starts from Eisenhart. We first state the theorem more or less in the form in which it is cited in [12].

Eisenhart's theorem. *A necessary and sufficient condition that a kinetic energy Lagrangian $L = \frac{1}{2}g_{ij}(x)\,\dot{x}^i\dot{x}^j$ (or better its corresponding Hamiltonian) can be given the Stäckel form is that there exist $n-1$ Killing tensors a^γ_{ij}, $\gamma = 1, \ldots, n-1$, such that:*

1. *the tensors $\{g_{ij}, a^\gamma_{ij}\}$ are linearly independent,*
2. *all roots of the (generalised) eigenvalue problem $\det(a^\gamma - \lambda^\gamma g) = 0$ are distinct for all γ (a^γ stands for the matrix with components a^γ_{ij}),*
3. *there exist new coordinates y^i such that the vectors $X_{(k)}$, with components $X^j_{(k)} = \partial x^j/\partial y^k$ are common eigenvectors of all a^γ, i.e. we have*

$$\left(a^\gamma_{ij} - \lambda^\gamma_{(k)}\,g_{ij}\right)\,X^j_{(k)} = 0, \qquad k = 1, \ldots, n, \qquad \gamma = 1, \ldots, n-1. \tag{27}$$

A dual version of this theorem is mentioned in [2]. If (g^{kl}) denotes the inverse of the matrix of the given metric tensor and a^{ij}_γ the contravariant version of the Killing tensors obtained by raising indices in the usual way, then the contravariant version of the third requirement in the theorem says that there are n common orthogonal closed eigenforms $\alpha^{(k)}$:

$$\left(a^{ij}_\gamma - \lambda^{(k)}_\gamma\,g^{ij}\right)\,\alpha^{(k)}_j = 0, \qquad \text{with} \quad d\alpha^{(k)} = 0. \tag{28}$$

Locally then, $\alpha^{(k)} = dy^k$ for some functions y^k, which define, as in the other version, separation coordinates. It is worth mentioning in this context that Benenti now even has a theorem about the characterisation of separable systems in which only one Killing tensor is needed (see [3]), but this result will not be directly relevant to our further discussion.

Naturally, for the standard metric we are considering in Lagrangians of type (12), the covariant and contravariant versions of the Killing tensors are the same, so if we construct eigenvectors in the algebraic sense, we have the freedom of interpreting them either as vector fields or as 1-forms. Strictly speaking, Eisenhart's theorem is about geodesic motions or, in other words, about Lagrangians without potential term. However, it is well known since Levi-Civita that for a Hamiltonian of type $T+V$ to be separable, it is necessary that the kinetic energy part by itself is separable. Moreover, if the full Lagrangian or Hamiltonian has a quadratic first integral, which is the sum of a quadratic part and terms of degree zero, it is the quadratic part only which defines the corresponding symmetry as in (24) and, as said before, this quadratic part determines a Killing tensor for the geodesic motion coming from the kinetic energy term. As a result, for the sake of identifying separation variables and trying to relate them to symmetries, we can, without loss of generality, omit the potential term alltogether. There are simple criteria then (see again [3], for example, or even the original theorem of Stäckel) to check, once separation variables have been detected, whether the potential term has a structure which is compatible with the separation.

We now start examining what the implications are of the conditions on the a^{γ}_{ij} in Eisenhart's theorem, for the symmetry components ξ^i which correspond to these via (24). It is clear that a symmetry vector field Y as in (5) is completely determined by its first term

$$X = \xi^i \frac{\partial}{\partial x^i}, \tag{29}$$

which looks like a vector field on the base manifold M, whose components, however, are functions on TM. This is called a vector field along the projection $\tau : TM \to M$. So-called semi-basic forms on TM likewise can be regarded as differential forms along τ. For an adapted calculus about forms and fields along τ and an application to separability of second-order equations, we refer to [24, 25, 26]. We will, however, not really need that sophisticated machinery here. Recall though, that there exists a canonically defined vector field along τ, namely the 'total time derivative operator'

$$\mathbf{T} = \dot{x}^i \frac{\partial}{\partial x^i} \tag{30}$$

which, in the present context of Lagrangians of type (12), is in fact precisely the vector field along τ determining the Noether symmetry corresponding to the energy integral.

For simplicity, we first discuss the case $n=2$ in some detail, for which there is just one Killing tensor a_{ij} with distinct eigenvalues. It is perhaps instructive to show via Eisenhart's theorem why in such a case separability always works. Using the covariant version, for example, we have two 1-dimensional, integrable eigendistributions and their sum is also integrable by dimension. It follows from the Frobenius theorem that there exist coordinates y^i such that $\partial/\partial y^1$ spans the first distribution and $\partial/\partial y^2$ the second. This in turn means that in each of the original eigendistributions, one can find an element such that the two selected eigenvectors commute and hence can simultaneously be straightened out. These elements are of course the eigenvectors with components of the form $\partial x^j/\partial y^k$ featuring in Eisenhart's theorem. Likewise, referring to the contravariant version, 1-forms in dimension 2 always have an integrating factor, so that we can always make the $\alpha^{(k)}$ locally exact, yielding again the desired separation variables. Note that these ideas can in fact be used constructively to find the separation variables for two degrees of freedom systems with a known second quadratic integral (see next section).

Now, if $\alpha^{(k)} = \alpha_i^{(k)} dx^i$, $k = 1, 2$ are the eigenforms and we consider their formal product to produce a quadratic expression in the dx^i, we know that this expression will become (up to a factor) the formal product $dy^1 dy^2$, when expressed in the right coordinates. Hence, if there is any truth in the expectation that the manipulations described about the quadratic expression (25) should indeed lead to the identification of separation variables y^i, there should be a link between (25) and the formal product of the $\alpha^{(k)}$ (to be interpreted properly, for example, as the symmetric tensor product). This is what we shall establish now, taking account of the earlier remark that one can write just as well a quadratic expression in $\dot{x}^i$ instead of dx^i, meaning that we then think of

$$\alpha_i^{(1)} \alpha_j^{(2)} \, \dot{x}^i \dot{x}^j. \tag{31}$$

Expressing that the $\alpha^{(k)}$ are eigenforms of the matrix a_{ij}, we have

$$a_{ij} \alpha_j^{(k)} = \lambda^{(k)} \alpha_i^{(k)}, \qquad k = 1, 2, \qquad i = 1, 2,$$

or equivalently, since these are functions of the coordinates only,

$$\dot{x}^i a_{ij} \alpha_j^{(k)} = \lambda^{(k)} \alpha_i^{(k)} \dot{x}^i, \qquad k = 1, 2. \tag{32}$$

In the left-hand side, we recognise the contraction of the basic 1-form $\alpha^{(k)}$, regarded as 1-form along τ, with the vector field X along τ, determined by (29) and (24). The right-hand side contains the contraction of $\alpha^{(k)}$ with $\mathbf{T}$. In other words, the equality (32) has the intrinsic interpretation:

$$\langle X, \alpha^{(k)} \rangle = \lambda^{(k)} \, \langle \mathbf{T}, \alpha^{(k)} \rangle, \qquad k = 1, 2, \tag{33}$$

from which it follows that

$$(\alpha^{(1)} \wedge \alpha^{(2)})(\mathbf{T}, X) = (\lambda^{(2)} - \lambda^{(1)}) \langle \mathbf{T}, \alpha^{(1)} \rangle \langle \mathbf{T}, \alpha^{(2)} \rangle. \tag{34}$$

The final point to observe now is that $\alpha^{(1)} \wedge \alpha^{(2)}$, being a volume form (on an open set), is proportional to the standard volume form $dx^1 \wedge dx^2$, and

$$(dx^1 \wedge dx^2)(\mathbf{T}, X) = \xi^2 \dot{x}^1 - \xi^1 \dot{x}^2 = a_{2j} \, \dot{x}^j \dot{x}^1 - a_{1j} \, \dot{x}^j \dot{x}^2. \tag{35}$$

Collecting the results we thus conclude: (35) shows that the quadratic expression (26) (which was obtained through manipulation of characteristic equations in [13]) can be interpreted as coming from the action of a volume form on the symmetry generators

$(\mathbf{T}, X)$; as such, it is bound to be proportional to the right-hand side of (34) when one thinks of the eigenforms of the matrix (a_{ij}) to construct a volume form; since the eigenvalues $\lambda^{(k)}$ are further assumed to be different in Eisenhart's theorem, this right-hand side is in turn proportional to the expression (31); finally, Eisenhart's theorem guarantees that separation coordinates exist, in which (31) will simply factorise as $\dot{y}^1\dot{y}^2$ (always up to a multiplicative function).

From a practical point of view, once a second quadratic integral has been found (or its corresponding linear Noether symmetry) and the eigenvalues of the matrix (a_{ij}) are different, the above result somehow provides an indirect way of making use, constructively, of Eisenhart's theorem: without bothering about computing eigenforms, we compute a determinant, namely the function $(dx^1 \wedge dx^2)(\mathbf{T}, X)$ which gives rise to a quadratic form in the $\dot{x}^i$ and one can try to find separation variables by factorising this expression into the product of two linear expressions. For two degrees of freedom, in fact, this idea can be made stronger because there is a converse result: any such factorisation, with factors which are linearly independent, will give rise to separation variables. Indeed, suppose we can write the determinant in question as follows

$$a_{2j}\,\dot{x}^j\dot{x}^1 - a_{1j}\,\dot{x}^j\dot{x}^2 = (\alpha_i^{(1)}\dot{x}^i)(\alpha_j^{(2)}\dot{x}^j).$$

Identification of the coefficients of corresponding terms in both sides then gives:

$$a_{21} = \alpha_1^{(1)}\alpha_1^{(2)}, \quad a_{22} - a_{11} = \alpha_1^{(1)}\alpha_2^{(2)} + \alpha_2^{(1)}\alpha_1^{(2)}, \quad a_{12} = -\alpha_2^{(1)}\alpha_2^{(2)}.$$

Remembering of course that $a_{12} = a_{21}$ it follows that

$$\begin{aligned}
a_{11}\alpha_1^{(1)} + a_{12}\alpha_2^{(1)} &= \left(a_{11} + \alpha_1^{(2)}\alpha_2^{(1)}\right)\,\alpha_1^{(1)}, \\
a_{21}\alpha_1^{(1)} + a_{22}\alpha_2^{(1)} &= \left(a_{22} - \alpha_2^{(2)}\alpha_1^{(1)}\right)\,\alpha_2^{(1)}, \\
a_{11}\alpha_1^{(2)} + a_{12}\alpha_2^{(2)} &= \left(a_{11} + \alpha_1^{(1)}\alpha_2^{(2)}\right)\,\alpha_1^{(2)}, \\
a_{21}\alpha_1^{(2)} + a_{22}\alpha_2^{(2)} &= \left(a_{22} - \alpha_2^{(1)}\alpha_1^{(2)}\right)\,\alpha_2^{(2)}.
\end{aligned}$$

This proves that, with

$$\begin{aligned}
\lambda^{(1)} &= a_{11} + \alpha_1^{(2)}\alpha_2^{(1)} = a_{22} - \alpha_2^{(2)}\alpha_1^{(1)}, \\
\lambda^{(2)} &= a_{11} + \alpha_1^{(1)}\alpha_2^{(2)} = a_{22} - \alpha_2^{(1)}\alpha_1^{(2)},
\end{aligned}$$

the $\alpha_i^{(1)}$ and $\alpha_i^{(2)}$ give the components of an eigenform, and we have $\lambda^{(1)} \neq \lambda^{(2)}$ in view of the assumed linear independence. The two eigenforms have an integrating factor by dimension and the resulting new variables will be separation variables by Eisenhart's theorem.

Let us now move on to the case $n = 3$. Clearly, if we follow the same line of thought the generalisation will not lead to a set of quadratic expressions but rather to one cubic expression, obtained from the computation of a determinant again and having a factorisation property as before. A sketch of the reasoning goes as follows. Assume we have obtained two extra quadratic integrals in involution or, equivalently, their corresponding Noether symmetries which are determined by vector fields along τ, say

$$X_\gamma = \xi^k_\gamma \frac{\partial}{\partial x^k}, \qquad \text{with} \quad \xi^k_\gamma = a_{\gamma\, kl}\dot{x}^l \qquad \gamma = 1, 2. \tag{36}$$

In case of separability, following Eisenhart, there must be closed 1-forms $\alpha^{(i)}$, $i = 1, 2, 3$, which are common eigenforms of both matrices a_γ. Denoting the eigenvalues of a_γ, which are assumed to be different, by $\lambda^{(k)}_\gamma$, we thus have, in the same way as (32) was derived and rewritten in the form (33):

$$\langle X_\gamma, \alpha^{(k)}\rangle = \lambda^{(k)}_\gamma \langle \mathbf{T}, \alpha^{(k)}\rangle, \qquad k = 1, 2, 3, \qquad \gamma = 1, 2. \tag{37}$$

It follows that

$$\left(\alpha^{(1)} \wedge \alpha^{(2)} \wedge \alpha^{(3)}\right)(\mathbf{T}, X_1, X_2) = \rho \,\langle \mathbf{T}, \alpha^{(1)}\rangle \langle \mathbf{T}, \alpha^{(2)}\rangle \langle \mathbf{T}, \alpha^{(3)}\rangle, \tag{38}$$

with

$$\rho = \lambda^{(1)}_1 \left(\lambda^{(2)}_2 - \lambda^{(3)}_2\right) + \lambda^{(2)}_1 \left(\lambda^{(3)}_2 - \lambda^{(1)}_2\right) + \lambda^{(3)}_1 \left(\lambda^{(1)}_2 - \lambda^{(2)}_2\right). \tag{39}$$

The function ρ is non-zero because of the first assumption in Eisenhart's theorem. The left-hand side of (38), coming from a volume form, is proportional to the determinant $(dx^1 \wedge dx^2 \wedge dx^3)(\mathbf{T}, X_1, X_2)$ which can be computed directly from the symmetry components. The right-hand side of (38) then expresses that this cubic expression in the $\dot{x}^i$ can be factorised as the product of three linear expressions and even in such a way that this product becomes simply $\dot{y}^1\dot{y}^2\dot{y}^3$ in a set of separation variables. For $n > 2$ there is no converse in the sense that the existence of a factorisation of the determinant by itself does not necessarily imply that each of the factors will be integrable. The results of our analysis can be summarised as follows.

Proposition. *Assume that X_1 and X_2 of the form (36) define Noether symmetries of a Lagrangian system (12) with three degrees of freedom. Then, if we are in a situation where corresponding separation variables for the Hamilton-Jacobi equation exist, they can be obtained from factorising the determinant*

$$\begin{vmatrix} \dot{x}^1 & \dot{x}^2 & \dot{x}^3 \\ \xi^1_1 & \xi^2_1 & \xi^3_1 \\ \xi^1_2 & \xi^2_2 & \xi^3_2 \end{vmatrix} \tag{40}$$

into the product of three factors which are linear in the velocities and integrable.

Observe that the potential energy function V is not present in this statement: we repeat that the potential is irrelevant as long as one wants to find out the nature of separation variables if they exist; having found them, one is assured of separability of the kinetic energy part and then it is relatively easy to check whether the form of the potential is such that also the full system will be separable (see next section).

We have proved above a similar result for $n = 2$ and it is obvious that it can actually be done for arbitrary n. But beyond $n = 3$, it is doubtful that this procedure (as perhaps any other) could be implementable in practice. We will show in the next section, however, that up to three degrees of freedom, there are indeed different ways of making constructive use of this proposition. Our point of view in this respect is that this amounts to making constructive use of the existence theorem by Eisenhart.

On the theoretical side, it looks worthwhile for a future study to go more deeply into the potential role and relevance of volume forms for the characterisation of separability. A generalisation of the present considerations to non-standard metrics looks fairly straightforward. A more difficult issue is the question whether anything similar could be done related to non-quadratic integrals, where a transformation to separation variables will not be a point transformation (cf. the already cited example in [4]).

Before closing this section, let us make clear why one should not expect any meaningful conclusion from manipulations of quadratic expressions such as (26) when n is greater than 2. Indeed, thinking of the case $n = 3$, for example, and the matrix a_{ij} related to one of the symmetry generators, (26) can still be generated by a computation as in (35), and the related computation (34) will still produce a quadratic expression which factorises in the new variables guaranteed by Eisenhart's theorem. The point then is, however, that $\alpha^{(1)} \wedge \alpha^{(2)}$ will in general contain terms in all $dx^i \wedge dx^j$, or alternatively, $dx^1 \wedge dx^2$ will be a combination of all $\alpha^{(i)} \wedge \alpha^{(j)}$ and hence will by itself not factorise.

5 Illustrative examples

An example of an integrable and separable system with two degrees of freedom which can be found e.g. in one of the tables in [17] is the Lagrangian system (12) with potential (we will write here x, y for the coordinates instead of x^1, x^2)

$$V(x,y) = a\,(x^2 + 4y^2) + b\,(x^4 + 12x^2y^2 + 16y^4)\,, \tag{41}$$

for which a second quadratic integral is of the form

$$F = (x\dot{y} - y\dot{x})\dot{x} + 2\,(a + 4by^2 + 2bx^2)x^2y\,. \tag{42}$$

For the sake of illustrating a number of points discussed in the previous section, we will first of all limit ourselves to quartic potentials ($a = 0$) and in fact assume that the class of admissible quartic potentials, in the sense that there exists a first integral with $(x\dot{y} - y\dot{x})\dot{x}$ as quadratic part in the velocities, has as yet to be discovered. Let us say, for example, that we aim at finding potentials of the form

$$V = c_1x^4 + c_2x^2y^2 + c_3y^4\,, \tag{43}$$

with c_i to be determined.

When we think of quadratic integrals, the Noether requirements (18-20) reduce to

$$a_{(ij,k)} = 0, \qquad \frac{\partial g}{\partial x^j} = a_{jk}\frac{\partial V}{\partial x^k}. \tag{44}$$

Our starting point is the assumption that we have found from the highest-order terms the quadratic part of the above first integral F as particular solution, or in other words, the symmetry generator (as vector field along τ):

$$X = (x\dot{y} - 2y\dot{x})\frac{\partial}{\partial x} + x\dot{x}\frac{\partial}{\partial y}\,. \tag{45}$$

The potential V plays no role in obtaining this part, which determines a Killing tensor with components

$$A = (a_{ij}) \;=\; \begin{pmatrix} -2y & x \\ x & 0 \end{pmatrix}. \tag{46}$$

Our objective is not only to find potentials for which a second quadratic integral exists, but also to construct separation coordinates. The determination of the potential can then further be suspended and separation variables, if they exist, should follow from the following procedure. According to Eisenhart's theorem, we construct a set of eigenvectors of the matrix A; they read:

$$(x, y + r) \quad \text{and} \quad (x, y - r) \qquad \text{with} \quad r = (x^2 + y^2)^{1/2}.$$

If we treat these as components of eigenvector fields, say

$$X_1 = x\frac{\partial}{\partial x} + (y + r)\frac{\partial}{\partial y}, \qquad X_2 \;=\; x\frac{\partial}{\partial x} + (y - r)\frac{\partial}{\partial y}\,, \tag{47}$$

then it is easy to verify that this happens to be a basis of commuting vector fields for the eigendistributions, so that they can simultaneously be straightened out. A coordinate transformation which does this is and therefore defines separation variables reads:

$$u = \ln\frac{x}{\sqrt{r - y}}, \qquad v \;=\; \ln\sqrt{r - y}\,.$$

However, any further transformation which does not couple these coordinates will preserve separability. We may, therefore, just as well take

$$u = \sqrt{r + y}, \qquad v \;=\; \sqrt{r - y}\,. \tag{48}$$

In agreement with the theoretical discussion of the previous section, we could here also treat the eigenvectors of A as components of eigenforms, and one then readily finds integrating factors which bring these into the form du and dv, with u and v as defined by (48). Finally, the new idea which we have introduced with regard to separation variables, namely as coming from a factorisation of the determinant $\dot{x}\xi^2 - \dot{y}\xi^1$, can be seen to work as follows. The quadratic expression (35) here becomes:

$$x\dot{x}^2 - 2y\dot{x}\dot{y} - x\dot{y}^2 = \frac{1}{x}\left(x\dot{x} + (y + r)\dot{y}\right)\left(x\dot{x} + (y - r)\dot{y}\right)\,.$$

Again, one can see that the potential is irrelevant in this process of detecting possible separation variables. One way of determining subsequently whether the potential has an appropriate form for separability could go as follows: having found separation variables already, one could first express the given system in these new coordinates and then verify for what values of the parameters the potential satisfies the criteria of Stäckel's theorem. Alternatively, there is an intrinsic characterisation of the condition which the potential should satisfy, which therefore can be verified directly in the original coordinates. This characterisation can be found in [3], and says that the 1-form $K \cdot dV$ should be closed (for all of the Killing tensors K entering Eisenhart's theorem). For the present example, this reduces to the requirement that the functions $a_{ij}\partial V/\partial x^j$ should be the components (locally) of an exact 1-form. Not surprisingly, this is exactly the remaining Noether condition in (44), or equivalently, the condition (21). For the Killing tensor A given in (46), we have

$$A \cdot dV = \left(-2y\frac{\partial V}{\partial x} + x\frac{\partial V}{\partial y} \right) dx + x\frac{\partial V}{\partial x}\, dy,$$

and expressing that $d(A \cdot dV)$ should be zero, for a V of the form (43), reduces to the condition

$$(12c_1 - c_2)x^3 + (8c_2 - 6c_3)xy^2 = 0.$$

It follows that we must have $c_2 = 12c_1$ and $c_3 = 16c_1$, which is in agreement with the quartic part in the potential function (41).

It is worth mentioning that another, in fact well-known, admissible potential could have been discovered along the lines of this example. Indeed, the quadratic part $(x\dot{y} - y\dot{x})\dot{x}$ of the function F in (42) is the velocity-dependent part of one of the components of the Runge-Lenz vector in the reduced (planar) Kepler problem. One easily verifies that the Kepler potential also satisfies the above requirement $d(A\cdot dV) = 0$ and we further conclude that (48) defines separation variables for the Kepler problem as well. A good reference where this example is encountered in the context of linking symmetries to Killing tensors is [7].

For three degrees of freedom, it is not so obvious how one could use in practice the factorisation property of the determinant (40) to find separation variables. We will nevertheless illustrate two ways in which one could make use of this result for constructive purposes. We shall write x, y, z for the coordinates x^i now.

Looking once more at the rich collection of examples which is contained in [17], assume we have a Lagrangian of type (12) which has an angular momentum type first integral, the square of which provides a first quadratic integral, say

$$F_1 = (y\dot{z} - z\dot{y})^2 . \tag{49}$$

Assume another integral is of the form

$$F_2 = (y\dot{x} - x\dot{y})\dot{y} + (z\dot{x} - x\dot{z})\dot{z} + \cdots . \tag{50}$$

As before, we leave the terms of degree zero, both in the Lagrangian and in F_2, out of the discussion for the time being. For the corresponding Noether symmetries, we have

$$\xi_1^1 = 0, \qquad \xi_1^2 \; = \; -2z(y\dot{z} - z\dot{y}), \qquad \xi_1^3 \; = \; 2y(y\dot{z} - z\dot{y}) , \tag{51}$$

$$\xi_2^1 = y\dot{y} + z\dot{z}, \qquad \xi_2^2 \; = \; y\dot{x} - 2x\dot{y}, \qquad \xi_2^3 \; = \; z\dot{x} - 2x\dot{z} . \tag{52}$$

It is obvious from the second row (51) in the determinant (40) that there will be factor $y\dot{z} - z\dot{y}$. The remaining quadratic factor then is

$$(y\dot{y} + z\dot{z})^2 + 2x(y\dot{y} + z\dot{z})\dot{x} - (y^2 + z^2)\dot{x}^2\,,$$

and it is not too hard to see that this factorises as

$$(y\dot{y} + z\dot{z} + (x+r)\dot{x})(y\dot{y} + z\dot{z} + (x-r)\dot{x})\,,$$

with $r = (x^2 + y^2 + z^2)^{1/2}$. It turns out that these three factors are integrable and identify separation variables u, v, w, defined by

$$u = \sqrt{r+x}, \qquad v \;=\; \sqrt{r-x}, \qquad w = \arctan\left(\frac{y}{z}\right).$$

However, we should emphasize that our theoretical arguments in Section 4 do not allow to draw this conclusion at all. Indeed, it is easy to see that the matrix corresponding to F_1 has a double eigenvalue zero, so that we are not in the situation of Eisenhart's theorem from which our results were derived. The reason is that the essential first integral behind F_1 is the linear expression $y\dot{z} - z\dot{y}$, which corresponds to a Killing vector (not a Killing 2-tensor). A rudimentary explanation about the reasons why the above determinant calculation happens to produce the right answers anyway, is that the Killing vectors for a separable Hamiltonian in the end correspond to ignorable coordinates (see [2]). As a result, the system will immediately reduce to a lower dimensional one, and the remaining quadratic expression which we factorised above can then be expected to be in direct correspondance with the quadratic form coming from this lower dimensional system. Note in passing that this situation should not be confused with the well-known property (see e.g. [31]) that in spaces of constant curvature, every Killing 2-tensor is trivial, in the sense of being expressible as a linear combination of symmetrised tensor products of Killing vectors. From the point of view of the present analysis, there is no problem with this form of 'triviality' as long as the regularity assumptions of Eisenhart's theorem are satisfied. For now, let us again sketch how one can subsequently find appropriate potentials which preserve the separability.

Assume we want to find suitable potentials again in the class of homogeneous functions of degree 4 and, to simplify matters, more particularly of the form

$$V = c_1x^4 + c_2y^4 + c_3z^4 + d_1x^2y^2 + d_2x^2z^2 + d_3y^2z^2\,. \tag{53}$$

The 1-form $A \cdot dV$ which comes from the Killing tensor related to F_2 reads

$$(yV_y + zV_z)\,dx + (yV_x - 2xV_y)\,dy + (zV_x - 2xV_z)\,dz\,.$$

Expressing its closure for a V of type (53), we find restrictions on the undetermined constants, which reduce to $c_1 = 16c_2,\ c_3 = c_2,\ d_1 = d_2 = 12c_2,\ d_3 = 2c_2$. This way, up to a constant factor c_2, the potential is bound to be of the form

$$V = 16x^4 + (y^2 + z^2)^2 + 12x^2(y^2 + z^2)\,. \tag{54}$$

We observe then that this potential is already invariant under the action of the Killing vector $y\partial/\partial z - z\partial/\partial y$ related to F_1. It follows that F_1 will be a quadratic integral for the full system as well.

Let us finally illustrate how one can now further determine the terms in F_2 which are indicated by dots. With the potential (54), the 1-form $A \cdot dV$ becomes

$$4(y^2+z^2)(6x^2+y^2+z^2)\,dx + 16x(x^2+y^2+z^2)(ydy+zdz)\,.$$

In agreement with the Noether condition (20), putting this expression equal to dg should produce the remaining function in the quadratic integral. We thus find

$$g = 4x(y^2+z^2)^2 + 8x^3(y^2+z^2)\,. \tag{55}$$

For a third illustration now, assume we start from the following two Killing tensors of the standard metric in dimension 3,

$$\begin{aligned} F_1 &= (x\dot{y}-y\dot{x})^2 + (x\dot{z}-z\dot{x})^2 + (y\dot{z}-z\dot{y})^2 + (\dot{x}^2-\dot{z}^2) & (56)\\ F_2 &= (y\dot{z}-z\dot{y})^2 - (x\dot{y}-y\dot{x})^2 - (\dot{x}^2+\dot{z}^2). & (57) \end{aligned}$$

Also this case is inspired by an example in [17] and fits as well into the category of systems described in [33]. The idea again is that these should become the leading parts of quadratic integrals of a Lagrangian system of type (12) for some suitable potential. If we first want to verify whether the ultimate system and its integrals will correspond to a case of separability in the Hamilton-Jacobi sense, then we could try our luck with the factorisation property of the determinant (40). Needless to say, we could also try to address directly the conditions of Eisenhart's theorem. The determinant (40) for this case gives rise to a fairly complicated cubic expression in $\dot{x}, \dot{y}, \dot{z}$ and it is not obvious at first sight how one could think of a way to write it as a product of three linear factors. Nevertheless, we know that there are only a finite number of coordinate systems in $\mathbb{R}^3$ in which the standard metric is separable (the eleven types first derived by Eisenhart [9] and reconstructed also e.g. in [3]). With some experience in the field one can readily rule out a number of these possibilities and then a direct computation on the determinant (40) does become feasible, as we shall now illustrate. One of the more difficult types of coordinates to work with is for example the class (IV_1) in [9]. If we denote the new coordinates by u, v, w again, they are defined by relations of the following type

$$x^2 = \frac{(\alpha-u)(\alpha-v)(\alpha-w)}{(\alpha-\beta)(\alpha-\gamma)}, \tag{58}$$

$$y^2 = \frac{(\beta-u)(\beta-v)(\beta-w)}{(\beta-\alpha)(\beta-\gamma)}, \tag{59}$$

$$z^2 = \frac{(\gamma-u)(\gamma-v)(\gamma-w)}{(\gamma-\alpha)(\gamma-\beta)}, \tag{60}$$

for some constants α, β, γ, and the domain of definition is determined by the requirement $\alpha > u > \beta > v > \gamma > w$. It is then a fairly straightforward exercise to transform the cubic polynomial in $\dot{x}, \dot{y}, \dot{z}$ into a corresponding cubic expression in $\dot{u}, \dot{v}, \dot{w}$ via the derivatives of (58-60). If there indeed exists a transformation of this type which will give rise to separation variables, the new cubic polynomial should reduce to just one term, namely the monomial $\dot{u}\dot{v}\dot{w}$. Doing such a calculation is not as formidable as it may at first sound: first of all, it is typically a symbolic calculation which computer algebra packages can handle; secondly, to see whether there is any chance for this to work, it suffices to test first just one term, i.e. to check for example whether the coefficient of $\dot{u}^3$ can be made to vanish. We have done this for the above situation and found that (58-60) does produce a factorisation of the determinant (40), provided we choose α, β, γ in such a way that

$$\alpha - \beta = 1, \qquad \gamma - \beta = -1. \tag{61}$$

As before, one can separately compute what kind of potentials are acceptable and in that process one will find at the same time what extra functions then have to be added to the expressions (56)(57), in order to obtain quadratic integrals for the full Lagrangian. The procedure works exactly as in the preceding case. We limit ourselves to the observation that it is not possible for this case to find an acceptable potential which is homogeneous of degree four in the coordinates. One does obtain, however, the following class of potentials (with A an arbitrary constant):

$$V = A(x^2 + y^2 + z^2) + (y^2 + 2z^2) + (x^2 + y^2 + z^2)^2. \tag{62}$$

Acknowledgements. RS would like to thank the University of Ghent for a research grant which made this work possible. WS and FC further thank the Fund for Scientific Research, Flanders, Belgium, for continuing support.

References

[1] S. Benenti, Stäckel systems and Killing tensors, *Note de Matematica* Vol.IX-Suppl. (1989) 39–58.

[2] S. Benenti, Separation of variables in the geodesic Hamilton-Jacobi equation, In: *Symplectic Geometry and Mathematical Physics*, P. Donato, C. Duval, J. Elhadad, G.M. Tuynman (Eds.), Progress in Mathematics 99 (Birkhäuser) (1991).

[3] S. Benenti, Intrinsic characterization of the variable separation in the Hamilton-Jacobi equation, *J. Math. Phys.* **38** (1997) 6578–6602.

[4] M. Blaszak and S. Wojciechowski, A generalized Henon-Heiles system and related integrable Newton equations, *J. Math. Phys.* **35** (1994) 1693–1709.

[5] M. Crampin, Symmetries and invariants of volume preserving flows, *Phys. Lett.* **79A** (1980) 138–140.

[6] M. Crampin, A note on non-Noether constants of motion, *Phys. Lett.* **95A** (1983) 209–212.

[7] M. Crampin, Hidden symmetries and Killing tensors, *Rep. Math. Phys.* **20** (1984) 31–40.

[8] M. Crampin, G. Marmo and C. Rubano, Conditions for the complete integrability of a dynamical system admitting alternative Lagrangians, *Phys. Lett.* **97A** (1983) 88–90.

[9] L.P. Eisenhart, Separable systems of Stäckel, *Ann. Math.* **35** (1934) 284–305.

[10] J. Hietarinta, Direct methods for the search of the second invariant, *Phys. Reports* **147** (1987) 87–154.

[11] E.G. Kalnins, *Separation of variables for Riemannian spaces of constant curvature*, Pitman Monographs 28 (1986).

[12] E.G. Kalnins and W. Miller, Killing tensors and variable separation for Hamilton-Jacobi and Helmholtz equations, *SIAM J. Math. Anal.* **11** (1980) 1011–1026.

[13] M. Lakshmanan and M. Senthil Velan, Direct integration of generalized Lie symmetries of nonlinear Hamiltonian systems with two degrees of freedom: integrability and separability, *J. Phys. A* **25** (1992) 1259–1272.

[14] M. Lakshmanan and M. Senthil Velan, Direct integration of generalized Lie or dynamical symmetries of three degrees of freedom nonlinear Hamiltonian systems: integrability and separability, *J. Math. Phys.* **33** (1992) 4068–4077.

[15] R. Sahadevan and M. Lakshmanan, Invariance and integrability: Hénon-Heiles and two coupled quartic anharmonic oscillator systems, *J. Phys. A* **19** (1986) L949–L954.

[16] M. Lakshmanan and R. Sahadevan, Generalized Lie symmetries and complete integrability of certain nonlinear Hamiltonian systems with three degrees of freedom, *J. Math. Phys.* **32** (1991) 75–83.

[17] M. Lakshmanan and R. Sahadevan, Painlevé analysis, Lie symmetries, and integrability of coupled nonlinear oscillators of polynomial type, *Phys. Reports* **224** (1993) 1–93.

[18] J.D. Logan, *Invariant Variational Principles*, Academic Press (New York) (1977).

[19] M. Lutzky, Origin of non-Noether invariants, *Phys. Lett.* **75A** (1979) 8–10.

[20] M. Lutzky, New classes of conserved quantities associated with non-Noether symmetries, *J. Phys. A* **15** (1982) L87–L91.

[21] M. Lutzky, Remarks on a recent theorem about conserved quantities, *J. Phys. A* **28** (1996) L637–L638.

[22] M. Lutzky, Conserved quantities and velocity-dependent symmetries in Lagrangian mechanics, *Int. J. Non-Linear Mech.* **33** (1997) 393–396.

[23] I. Marshall and S. Wojciechowski, When is a Hamiltonian system separable, *J. Math. Phys.* **29** (1988) 1338–1346.

[24] E. Martínez, J.F. Cariñena and W. Sarlet, Derivations of differential forms along the tangent bundle projection, *Diff. Geometry and its Applications* **2** (1992) 17–43.

[25] E. Martínez, J.F. Cariñena and W. Sarlet, Derivations of differential forms along the tangent bundle projection II, *Diff. Geometry and its Applications* **3** (1993) 1–29.

[26] E. Martínez, J.F. Cariñena and W. Sarlet, Geometric characterization of separable second-order equations, *Math. Proc. Camb. Phil. Soc.* **113** (1993) 205–224.

[27] V. Ravoson, L. Gavrilov and R. Caboz, Separability and Lax pairs for Hénon-Heiles system, *J. Math. Phys.* **34** (1993) 2385–2393.

[28] W. Sarlet and F. Cantrijn, Generalizations of Noether's theorem in classical mechanics, *SIAM Review* **23** (1981) 467–494.

[29] W. Sarlet and M. Crampin, Some recent results on symmetries of Lagrangian systems re-examined, *J. Phys. A* **18** (1985) 2849–2855.

[30] G. Thompson, Polynomial constants of motion in flat space, *J. Math. Phys.* **25** (1984) 3474–3478.

[31] G. Thompson, Killing tensors in spaces of constant curvature, *J. Math. Phys.* **27** (1986) 2693–2699.

[32] E.T. Whittaker, *A treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge Univ. Press (1964).

[33] S. Wojciechowski, On a Lax-type representation and separability of the anisotropic harmonic oscillator in a radial quartic potential, *Lett. Nuovo Cimento* **41** (1984) 361–369.

[34] S. Wojciechowski and A.V. Tsiganov, Quasi-point separation of variables for the Henon-Heiles system and a system with a quartic potential, *J. Phys. A* **29** (1996) 7769–7778.

[35] N.M.J. Woodhouse, Killing tensors and the separation of the Hamilton-Jacobi equation, *Commun. Math. Phys.* **44** (1975) 39–44.