On almost-Poisson structures in nonholonomic mechanics

F. Cantrijn*
Department of Mathematical Physics and Astronomy, University of Ghent
Krijgslaan 281, B-9000 Ghent, Belgium
e-mail: Frans.Cantrijn@rug.ac.be
M. de León
Instituto de Matemáticas y Física Fundamental, CSIC
Serrano 123, E-28006 Madrid, Spain
e-mail: mdeleon@fresno.csic.es
D. Martín de Diego
Departamento de Economía Aplicada (Matemáticas)
Facultad de CC. Economicas y Empresariales
Avda Valle Esgueva 6, E-47011 Valladolid, Spain
e-mail: dmartin@esgueva.eco.uva.es

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Abstract. In this paper we intend to unify different approaches to the construction of a "almost-Poisson" bracket for mechanical systems with nonholonomic constraints. This almost-Poisson structure is subsequently used to describe the phase space dynamics of a nonholonomic system. It is shown that when dealing with "non-homogeneous" constraints, the hamiltonian equations of motion can not be expressed in terms of the almost-Poisson bracket alone. This fact is illustrated in the case of mechanical systems with affine constraints. The problem of a rolling ball on a rotating table is treated as an example.

*Senior Research Associate at the Fund for Scientific Research, Flanders (Belgium)

1 Introduction

Recently, van der Schaft and Maschke [20] have presented a hamiltonian formulation of mechanical systems with linear nonholonomic constraints in terms of a kind of Poisson bracket. This bracket, which is defined on the constraint submanifold in phase space, shares the usual properties of a Poisson bracket, except for the Jacobi identity. The latter is only satisfied if the constraints are holonomic, and vice versa. This bracket formulation of nonholonomic mechanics has been taken as the starting point for a study of Poisson reduction of nonholonomic systems with symmetry by Koon and Marsden [10]. Both in [20] and [10], the bracket in question is constructed in terms of special, in general non-canonical coordinates adapted to the constraints. In [15], on the other hand, Marle has proposed an intrinsic definition for a bracket associated with general (not necessarily linear) nonholonomic constraints, by "projecting" the canonical Poisson tensor field on the phase space of the system onto the constraint submanifold. He called the resulting structure (bracket) a "pseudo-Poisson" structure (bracket).

In the framework of a unified treatment of Dirac bracket formulations for various kinds of constrained systems, i.e. degenerate lagrangian systems as well as sytems with holonomic and/or nonholonomic constraints, Ibort *et al* [6, 7] have constructed a Dirac bracket for nonholonomic systems. Their construction is based on a symplectic decomposition of the tangent bundle of the phase space along the constraint submanifold.

The purpose of the present paper is twofold. First, we wish to unify the above mentioned formulations of classical nonholonomic systems by proving that, on the constraint submanifold, the Dirac bracket of [7] and Marle's pseudo-Poisson bracket are essentially the same and that, in the case of linear constraints, they also coincide with the bracket introduced by van der Schaft and Maschke. We will call this bracket, and the corresponding geometric structure, *almost-Poisson*, where 'almost' refers to the fact that the structure (in general) fails to satisfy a certain integrability condition, namely the Jacobi identity. Secondly, it is shown that only for nonholonomic systems for which, in the lagrangian setting, the constraints can be represented by functions homogeneous in the velocities, the correponding hamiltonian equations can be completely expressed in terms of the almost-Poisson bracket. This is precisely the case of nonholonomic systems for which the hamiltonian, i.e. the energy, is conserved. For general non-homogeneous constraints, an additional term has to be added to the bracket in order to obtain the correct equations of motion.

The scheme of the paper is as follows. In Section 2 we briefly recall some elements from the geometric formulation of nonholonomic mechanical systems, mainly following the treatment presented in [11, 12]. In Section 3 we describe two possible bracket structures associated with a nonholonomic system: one corresponding to Marle's pseudo-Poisson structure and the other being a Dirac bracket. We then show that, on the constraint submanifold, both structures coincide. Using appropriate coordinates we then prove in Section 4 that, in the case of linear constraints, this bracket is precisely the one introduced by van der Schaft and Maschke. The case of mechanical systems with "non-homogeneous" constraints is treated in Section 5. Finally, in Section 6 we apply our results to the case of affine constraints and, as an example, we treat the problem of a rolling ball on a rotating table.

2 Geometrical formulation of nonholonomic systems

Let Q be a n-dimensional differentiable manifold, with local coordinates (q^i). The tangent bundle TQ, with induced coordinates $(q^i, \dot{q}^i)$, is equipped with two fundamental geometrical objects [13]: the Liouville vector field Δ and the vertical endomorphism S. In natural bundle coordinates we have

$$\Delta = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}, \quad S = dq^i \otimes \frac{\partial}{\partial \dot{q}^i}.$$

Consider a lagrangian system, with lagrangian $L : TQ \to \mathbb{R}$, subjected to nonholonomic constraints, defined by a submanifold D of the velocity phase space TQ. We will assume that $\dim D = 2n - k$ and that, locally, D is described by the vanishing of k independent functions ϕ^a (the "constraint functions"). In mechanics, D will usually be a vector subbundle or an affine subbundle of TQ, i.e. the constraint functions will have either a linear or an affine dependence on the velocities. In order to derive consistent equations of motion for the constrained system, one has to introduce additional forces, the so-called constraint or reaction forces. We will always assume that the constraints satisfy d'Alembert's principle, in the linear or affine case, or Chetaev's principle, in the case of general (nonlinear) constraints. Both principles state that the virtual work (or power) of the reaction forces should vanish. For the nonlinear case, Chetaev introduced a special "ansatz" for the characterisation of virtual displacements (or velocities) which reduces to the familiar concept in the case of linear or affine constraints. (For a critical discussion of these so-called Chetaev conditions, see e.g. [18].) In geometrical terms, both d'Alembert's and Chetaev's principle imply that the constraint forces, regarded as 1-forms on TQ along D, take their values in the subbundle $S^*(TD^o)$ of T^*TQ, where TD^o denotes the annihilator of TD in T^*TQ In local coordinates the equations of motion for the nonholonomic system then read

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \lambda_a \frac{\partial \phi^a}{\partial \dot{q}^i},$$

together with the constraint equations

$$\phi^a(q^i, \dot{q}^i) = 0,$$

where the λ_a are Lagrange multipliers which have to be determined in the course of solving the problem. In an intrinsic way these equations can be written as (see [11]):

$$\begin{aligned} (i_X\omega_L - dE_L)_{|D} &\in S^*(TD^o) \\ X_{|D} &\in TD \end{aligned}$$

where ω_L is the Poincaré-Cartan 2-form defined by $\omega_L = -d(S^*(dL))$ and $E_L = \Delta(L) - L$ is the energy function.

An important class of nonholonomic systems are those for which the dilation vector field Δ is tangent to the constraint submanifold D. This condition implies that the constraint functions ϕ^a, describing D, should satisfy

$$\left(\dot{q}^i \frac{\partial \phi^a}{\partial \dot{q}^i}\right)_{|D} = 0 \ ,$$

or, using Dirac's notation for weak equality, $\dot{q}^i(\partial\phi^a/\partial\dot{q}^i) \approx 0$. For simplicity, we will henceforth refer to this type of nonholonomic constraints as the case of *homogeneous constraints*. The condition will in particular be verified if the constraint functions are homogeneous in the velocities. Linear constraints obviously belong to this category. It can be shown that the homogeneity condition implies, among others, conservation of energy, i.e. E_L is a first integral of any solution X of the equations of motion (see [4]).

Remark 2.1 In previous work (see e.g. [4, 11]), the case where Δ is tangent to D was called the case of "ideal constraints", thereby adopting the terminology used in (the translation of) the fundamental paper by Vershik and Faddeev [21]. In mechanics it is more common, however, to use the denomination "ideal constraints" whenever d'Alembert's or Chetaev's principle applies (cf. [14, 15, 19]), i.e. whenever the equations of motion can be written in the above form. Non-ideal constraints then refer, for instance, to the case of constraints with friction or to constraints realized by servomechanisms.

In the sequel we will also assume that the following *admissibility condition* holds:

$$\dim TD^o = \dim S^*(TD^o) \, .$$

This essentially means that the matrix $(\partial\phi^a/\partial\dot{q}^i)$ has rank k everywhere, which necessarily implies that $k \leq n$.

We now pass to the hamiltonian description of the nonholonomic system, on the cotangent bundle T^*Q of Q [1, 9, 14]. The canonical coordinates on T^*Q are denoted by (q^i, p_i), and the cotangent bundle projection is $\pi_Q : T^*Q \to Q$. Assuming regularity of the lagrangian, we have that the lagrangian and hamiltonian formulations are locally equivalent. If we suppose, in addition, that the lagrangian L is hyperregular, then the Legendre transformation $Leg : TQ \to T^*Q, (q^i, \dot{q}^i) \mapsto (q^i, p_i = \partial L/\partial \dot{q}^i)$, is a global diffeomorphism. The constraint functions on T^*Q become $\Psi^a = \phi^a \circ Leg^{-1}$, i.e.

$$\Psi^a(q^i, p_i) = \phi^a(q^i, \frac{\partial H}{\partial p_i}),$$

where the hamiltonian $H : T^*Q \to \mathbb{R}$ is defined by $H = E_L \circ Leg^{-1}$. The equations of motion of the nonholonomic system on T^*Q can now be written as follows:

$$\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i} + \bar{\lambda}_a \frac{\partial \Psi^a}{\partial p_j} \mathcal{H}_{ji}, \end{aligned}$$

together with the constraint equations

$$\Psi^a(q, p) = 0,$$

where $\mathcal{H}_{ij}$ are the components of the inverse of the matrix $(\mathcal{H}^{ij}) = (\partial^2 H/\partial p_i \partial p_j)$. Note that

$$(\frac{\partial \Psi^a}{\partial p_j} \mathcal{H}_{ji})(q, p) = (\frac{\partial \phi^a}{\partial \dot{q}^i} \circ Leg^{-1})(q, p).$$

In the case of homogeneous constraints we have

$$\frac{\partial \Psi^a}{\partial p_j} \mathcal{H}_{ji} \frac{\partial H}{\partial p_i} \approx 0,$$

where "$\approx$" means equality on M, and it then easily follows that H is a first integral of the constrained equations of motion.

Via the Legendre map, the symplectic 2-form ω_L is Leg-related with the canonical symplectic form ω_Q on T^*Q. Let M denote the image of the constraint submanifold D under the Legendre transformation, and let F be the distribution on T^*Q along M, whose annihilator is given by

$$F^o = Leg_*(S^*(TD^o)).$$

Observe that F^o is locally generated by the k independent 1-forms

$$\mu^a = \frac{\partial \Psi^a}{\partial p_i} \mathcal{H}_{ij} dq^j, \ 1 \leq a \leq k.$$

The hamiltonian equations of motion of the nonholonomic system can then be rewritten in intrinsic form as

$$\begin{aligned} (i_X \omega_Q - dH)_{|M} &\in F^o \\ X_{|M} &\in TM. \end{aligned} \tag{1}$$

Suppose that the following *compatibility condition* (previously also called "regularity condition" [11, 12]) is satisfied: $F^{\perp} \cap TM = \{0\}$, where " $\perp$ " denotes the symplectic orthogonal with respect to ω_Q. Observe that, locally, this condition means that the matrix

$$(C^{ab}) = \left(\frac{\partial \Psi^a}{\partial p_i}\mathcal{H}_{ij}\frac{\partial \Psi^b}{\partial p_j}\right) \tag{2}$$

is regular. The compatibility condition is not too restrictive, since, taking into account the admissibility assumption, it is trivially verified by the usual systems of mechanical type (i.e. with a lagrangian of the form kinetic minus potential energy), where the $\mathcal{H}_{ij}$ represent the components of a positive definite Riemannian metric. The compatibility condition in particular guarantees the existence of a unique solution of the constrained equations of motion (1) which, henceforth, will be denoted by $X_{H,M}$.

From the compatibility condition – which, together with the admissibility condition, will always be assumed to hold – and a simple counting of dimensions, we deduce that

$$TM \oplus F^{\perp}_{|M} = T_M(T^*Q)$$

with $T_M(T^*Q)$ denoting the restriction of TT^*Q to the submanifold M. This decomposition gives rise to two complementary projectors:

$$P : T_M(T^*Q) \longrightarrow TM$$
$$Q : T_M(T^*Q) \longrightarrow F^{\perp}.$$

Observe that $F^{\perp}$ is locally generated by the vector fields Z^a, $1 \leq a \leq k$, defined by

$$i_{Z^a}\omega_Q = \mu^a .$$

In coordinates we have

$$Z^a = -\frac{\partial \Psi^a}{\partial p_i}\mathcal{H}_{ij}\frac{\partial}{\partial p_j} .$$

Herewith, the projector P locally reads:

$$P = \mathrm{id} + C_{ab}Z^a \otimes d\Psi^b$$

where C_{ab} are the components of the inverse of the matrix defined in (2). It can be proved that, under the given assumptions, the projection by P of the hamiltonian vector field X_H (solution of the unconstrained hamiltonian system, with hamiltonian H) precisely yields the (unique) solution $X_{H,M}$ of the constrained problem (1) (cf. [14] and, for the analogous result in the lagrangian case, see also [4, 11]).

In [1, 5, 11], also the following alternative approach has been proposed. The compatibility condition is equivalent to the condition that $F \cap TM$ determines a symplectic vector bundle on M. This leads to the following symplectic decomposition of $T_M T^*Q$

$$(F \cap TM) \oplus (F \cap TM)^{\perp} = (F \cap TM) \oplus (F^{\perp} \oplus TM^{\perp}) = T_M(T^*Q)$$

with induced projectors:

$$\begin{aligned} \mathcal{P} &: T_M(T^*Q) \longrightarrow (F \cap TM) \\ \mathcal{Q} &: T_M(T^*Q) \longrightarrow (F \cap TM)^{\perp}. \end{aligned}$$

It is important to observe that the construction of these complementary projectors is similar to the one encountered in a geometric approach to the construction of Dirac brackets for singular lagrangian systems (see [2, 7]). Locally, the projector $\mathcal{P}$ is given by:

$$\mathcal{P} = \mathrm{id} - C_{ab}C_{cd}\{\Psi^b, \Psi^d\}Z^a \otimes \mu^c - C_{ab}X_{\Psi^a} \otimes \mu^b + C_{ab}Z^a \otimes d\Psi^b,$$

where $\{\ ,\ \}$ denotes the canonical Poisson bracket on T^*Q and X_{Ψ^a} are the hamiltonian vector fields on T^*Q induced by the constraint functions Ψ^a. It should be noted that, in general, the projection of the "free" hamiltonian system X_H by $\mathcal{P}$ will not produce the constrained dynamics $X_{H,M}$. However, when dealing with homogeneous constraints we have that the functions $\mu^a(X_H)$ vanish on M and, hence, in that case we obtain, along M, $\mathcal{P}(X_H) = P(X_H) = X_{H,M}$.

3 Construction of almost-Poisson tensor fields

Denote by Λ the Poisson tensor on T^*Q, induced by the canonical symplectic form ω_Q. For any two sections α, β of $T^*_M T^*Q$ we put

$$\Lambda_1(\alpha, \beta) = \Lambda(P^*(\alpha), P^*(\beta)) \ ,$$

with P the projector onto TM defined above. It is easy to see that the operator Λ_1 can be extended to a genuine 2-contravariant skew-symmetric tensor field on an open neighbourhood of M in T^*Q. Indeed, it suffices to observe that the projector P can always be extended to a type $(1,1)$ tensor field on a neighbourhood of M. In terms of Λ_1 one can now define the following bracket of functions

$$\{f, g\}_1 = \Lambda_1(df, dg) \ .$$

Similarly, using the projector $\mathcal{P}$ it is possible to define a 2-contravariant skew-symmetric tensor field Λ_2 according to the prescription

$$\Lambda_2(\alpha, \beta) = \Lambda(\mathcal{P}^*(\alpha), \mathcal{P}^*(\beta)) \ ,$$

for any two 1-forms α and β on T^*Q (defined at least on an open neighbourhood of M). The induced bracket is

$$\{f,g\}_2 = \Lambda_2(df,dg) \ .$$

From the fact that the projector $\mathcal{P}$ is associated with a symplectic decomposition of the tangent space to T^*Q in points of M, it is easy to prove that, along M,

$$\Lambda_2(\alpha,\beta) = \omega_Q(\mathcal{P}(X_\alpha),\mathcal{P}(X_\beta)) \ ,$$

and the bracket of functions then becomes

$$\{f,g\}_2 = \omega_Q(\mathcal{P}(X_f),\mathcal{P}(X_g)).$$

For later use we still note the following property.

Lemma 3.1 *For any two* 1*-forms* α *and* β*, defined in points of* M*, we have*

$$\Lambda_2(\alpha,\beta) = \omega_Q(\mathcal{P}(X_\alpha),X_\beta).$$

Proof : Observing that $X_\beta = \mathcal{P}(X_\beta)+\mathcal{Q}(X_\beta)$ and $\mathcal{P}$ and $\mathcal{Q}$ being projectors onto symplectic orthogonal spaces, the result immediately follows from the above expression for $\Lambda_2(\alpha,\beta)$.

$\boxed{QED}$

The bracket $\{\ ,\ \}_2$ was previously called a Dirac bracket (associated with the nonholonomic system), since it is obtained following a general prescription for constructing Dirac brackets (see [2] and [7]). From the definition of $\{\ ,\ \}_2$, we have that for any constraint function Ψ^a

$$\{\Psi^a,f\}_2 = 0$$

for all $f \in C^\infty(T^*Q)$. Locally, this bracket reads

$$\{f,g\}_2 = \{f,g\} + C_{ab}Z^a(g)\{f,\Psi^b\} - C_{ab}Z^a(f)\{g,\Psi^b\} + C_{ab}C_{cd}\{\Psi^b,\Psi^d\}Z^a(f)Z^c(g)$$

where the brackets on the right-hand side are again the canonical Poisson bracket on T^*Q.

The brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ being induced by bivector fields, they are obviously skew-symmetric and verify the Leibniz rule. However, in general, the Jacobi identity will not be satisfied. We will say that both tensor fields Λ_1 and Λ_2 define an *almost-Poisson structure* on (a neighbourhood of M in) T^*Q. In [7] it has been proved that, in the case of linear constraints, the bracket $\{\ ,\ \}_2$ satisfies the Jacobi identity and, hence, represents a genuine Poisson bracket, if and only if the given constraints are holonomic (that is, D is a vector subbundle of TQ, defining an integrable distribution on Q).

We now want to establish a relationship between the tensor fields Λ_1 and Λ_2 and, consequently, also between the corresponding brackets. First, we have the following property.

Lemma 3.2 *For any 1-form α on T^*Q we have*

$$P^*(\alpha_{|M}) - \mathcal{P}^*(\alpha_{|M}) \in F^o\,.$$

Proof : The result follows immediately from the local expressions of both projectors.

We can now prove that both tensor fields are in fact equal in points of M.

Proposition 3.3 *Along the constraint submanifold M we have the equality*

$$\Lambda_1 = \Lambda_2\,.$$

Proof : The proposition is a consequence of Lemma 3.2 and the fact that $\mathcal{P}$ is a projector induced by a symplectic decomposition. Indeed, if α and β are 1-forms on T^*Q we have, in points of M,

$$\begin{aligned}\Lambda_1(\alpha,\beta) &= \Lambda(P^*(\alpha), P^*(\beta))\\ &= \Lambda\left(\mathcal{P}^*(\alpha)+\alpha_1, \mathcal{P}^*(\beta)+\beta_1\right)\end{aligned}$$

where α_1 and β_1 are elements of F^o given by $\alpha_1 = P^*(\alpha) - \mathcal{P}^*(\alpha)$ and $\beta_1 = P^*(\beta) - \mathcal{P}^*(\beta)$. Note that α_1 and β_1 are semi-basic forms on $T^*Q|_M$ and, hence, $\Lambda(\alpha_1,\beta_1) = 0$. It then follows that

$$\begin{aligned}\Lambda_1(\alpha,\beta) &= \Lambda(\mathcal{P}^*(\alpha), \mathcal{P}^*(\beta))\\ &= \Lambda_2(\alpha,\beta)\,.\end{aligned}$$

$\boxed{QED}$

In the sequel, we will denote by Λ_{nh} the common restriction of the almost-Poisson structures Λ_1 and Λ_2 to M, i.e. $\Lambda_{nh} = \Lambda_1|_M = \Lambda_2|_M$. The corresponding bracket of functions on T^*Q, computed at points of M, will be denoted by $\{\ ,\ \}_{nh}$.

Suppose now that the given constraints are homogeneous, i.e. $\Delta(x) \in T_xD$ for all $x \in D$. Then, it has been proved in [7] that the bracket $\{\ ,\ \}_{nh}$ gives the evolution of the constrained dynamics in the sense that for any function $f \in C^\infty(T^*Q)$, its evolution along integral curves of $X_{H,M}$ on M is given by:

$$\dot{f} = X_{H,M}(f) = \{f, H\}_{nh}\,. \tag{3}$$

In Section 5 we will prove that property (3) only holds in the case of homogeneous constraints. We will also derive a modified expression for the right-hand side of (3) in case of general (i.e. not necessarily homogeneous) constraints.

4 Almost-Poisson brackets on the constraint submanifold

In the paper by van der Schaft and Maschke [20], an "almost-Poisson" structure is defined on the constraint submanifold of a mechanical system subjected to linear nonholonomic constraints. Recently, this almost-Poisson structure has been revisited by Koon and Marsden [10], who used it to develop a Poisson reduction theory for nonholonomic systems with symmetry. They also pointed out the importance of this structure for future developments in stability theory of nonholonomic systems.

Our intention is to relate the almost-Poisson tensor Λ_{nh}, defined in the previous section, with the bracket defined by van der Schaft and Maschke. First, we note that the tensor field Λ_{nh} naturally induces a bivector field Λ_M on the constraint submanifold M according to:

$$\Lambda_M(x)(\alpha_M(x), \beta_M(x)) = \Lambda_{nh}(x)(\alpha(x), \beta(x)) \tag{4}$$

for all $x \in M$, where α_M and β_M are 1-forms on M, and α and β are arbitrary extensions of, respectively, α_M and β_M to T^*Q. It is routine to verify that Λ_M is indeed well-defined. This tensor field now induces a bracket on $C^\infty(M)$, namely:

$$\{f, g\}_M = \Lambda_M(df, dg) = \Lambda_{nh}(dF, dG) = \{F, G\}_{nh}, \tag{5}$$

where f and g are smooth functions on M, and F and G are arbitrary smooth extensions of them to a neighbourhood of M. Bearing in mind the definition of the tensor field Λ_1, which coincides with Λ_{nh} on M, one easily recognizes that Λ_M precisely corresponds to the pseudo-Poisson structure defined by Marle [15].

The construction of the bracket introduced by van der Schaft and Maschke [20], can be summarized as follows (see also [10]). Consider a mechanical system with a regular lagrangian L and subjected to linear nonholonomic constraints. Passing to the phase space T^*Q, with canonical coordinates (q^i, p_i), assume the constraints are represented by a submanifold M of T^*Q, and let H be the hamiltonian of the system. First, new bundle coordinates $(q^i, \tilde{p}_\alpha, \tilde{p}_A)$ are introduced on T^*Q, with $i = 1, \ldots n$, $\alpha = 1, \ldots, n-k$ and $A = n-k+1, \ldots n$, where $2n - k = \dim M$. These coordinates are selected in such a way that the constraint submanifold M is locally defined by

$$\frac{\partial \tilde{H}}{\partial \tilde{p}_A} = 0\,,$$

where $\tilde{H}(q, \tilde{p}) := H(q, p(q, \tilde{p}))$, and such that the equations of motion can be obtained free of lagrangian multipliers. Expressing the Poisson tensor field Λ on T^*Q in terms of these new (non-canonical) coordinates, its component matrix reads

$$(\Lambda^{rs}) = \begin{pmatrix} \{q^i, q^j\} & \{q^i, \tilde{p}_\beta\} & \{q^i, \tilde{p}_B\} \\ \{\tilde{p}_\alpha, q^j\} & \{\tilde{p}_\alpha, \tilde{p}_\beta\} & \{\tilde{p}_\alpha, \tilde{p}_B\} \\ \{\tilde{p}_A, q^j\} & \{\tilde{p}_A, \tilde{p}_\beta\} & \{\tilde{p}_A, \tilde{p}_B\} \end{pmatrix}$$

in which we have, of course, that $\{q^i, q^j\} = 0$. Note that, by construction, $(q^i, \tilde{p}_\alpha)$ induce local coordinates on M. Upon deleting the last k columns and last k rows from the previous matrix, one then obtains a $(2n-k) \times (2n-k)$ matrix function which, when restricted to points of M, represents the component matrix of a skew-symmetric contravariant 2-tensor field on M. The latter precisely yields the (almost-)Poisson structure considered by van der Schaft and Maschke. The point now is to observe that the procedure of "truncating" the Poisson matrix (Λ^{rs}) in the way just described, geometrically corresponds to passing from Λ to $\Lambda_1 = \Lambda_{nh}$ using the projector P. Restricting everything to the constraint submanifold M, it then follows that the almost-Poisson matrix of van der Schaft and Maschke is precisely the component matrix of the bivector field Λ_M defined above. We will make this now more explicit in the case of a system of mechanical type.

Consider a system with a lagrangian of the form $L = T - V$, with kinetic energy T being derived from a Riemannian metric on Q, and where $V = V(q)$ is a potential energy. Assume the system is subjected to linear nonholonomic constraints $\phi^a := \mu^a_i(q)\dot{q}^i = 0$, $(a = 1, \ldots, k)$, defining a vector subbundle D of TQ. The hamiltonian of the system then is of the form

$$H = \frac{1}{2} g^{ij} p_i p_j + V(q)$$

where the g^{ij} are the components of the inverse of the kinetic energy matrix (g_{ij}), and the constraint functions on T^*Q read

$$\Psi^a = \mu^a_i \frac{\partial H}{\partial p_i}, \; 1 \leq a \leq k\,.$$

In accordance with the construction of van der Schaft and Maschke, we now consider the following change of coordinates

$$\tilde{p}_\alpha = X^i_\alpha p_i \;\text{ and }\; \tilde{p}_{n-k+a} = Y^i_a p_i$$

where $1 \leq a \leq k$ and $1 \leq \alpha \leq n-k$. Here X^i_α are components of a basis of vector fields X_α spanning the vector subbundle D, and $Y^i_a = \delta_{ab}\mu^b_j g^{ji}$, with δ_{ab} the Kronecker delta, are the components of k vector fields spanning a complementary subbundle of TQ. Note that, in general, the transformation $(q,p) \to (q,\tilde{p})$ is not a canonical transformation. In these new coordinates, the constraint functions can be rewritten as

$$\Psi^a = C^{ab} \frac{\partial \tilde{H}}{\partial \tilde{p}_{n-k+b}}$$

with $\tilde{H}(q,\tilde{p}) := H(q,p(q,\tilde{p}))$, which reduces to

$$\Psi^a = \tilde{p}_{n-k+a}\,.$$

Hence, the constraint submanifold M is here locally defined by $\tilde{p}_{n-k+a} = 0$.

In the coordinates $(q^i, \tilde{p}_\alpha, \tilde{p}_{n-k+a})$, the components of the canonical Poisson tensor field Λ on T^*Q are determined by the brackets

$$\begin{aligned}
&\{q^i,q^j\} = 0 && \{q^i,\tilde{p}_\alpha\} = X^i_\alpha \\
&\{q^i,\tilde{p}_{n-k+a}\} = Y^i_a && \{\tilde{p}_\alpha,\tilde{p}_\beta\} = X^i_\beta p_j \frac{\partial X^j_\alpha}{\partial q^i} - X^i_\alpha p_j \frac{\partial X^j_\beta}{\partial q^i} \\
&\{\tilde{p}_{n-k+a},\tilde{p}_\beta\} = X^i_\alpha p_j \frac{\partial Y^j_a}{\partial q^i} - Y^i_a p_j \frac{\partial X^j_\alpha}{\partial q^i} && \{\tilde{p}_{n-k+a},\tilde{p}_{n-k+b}\} = Y^i_b p_j \frac{\partial Y^j_a}{\partial q^i} - Y^i_a p_j \frac{\partial Y^j_b}{\partial q^i}
\end{aligned}$$

where, on the right-hand sides, $p_j = p_j(q,\tilde{p})$. Now, following van der Schaft and Maschke, upon truncating the Poisson matrix by leaving out the last k rows and last k columns, and putting $\tilde{p}_{n-k+a} = 0$ for $a = 1,\ldots,k$, we obtain a matrix which defines an almost-Poisson structure on M:

$$J_M(q^i,\tilde{p}_\alpha) = \begin{pmatrix} \{q^i,q^j\} & \{q^i,\tilde{p}_\beta\} \\ \{\tilde{p}_\alpha,q^j\} & \{\tilde{p}_\alpha,\tilde{p}_\beta\} \end{pmatrix}.$$

The brackets on the right-hand side are expressed in terms of the coordinates $(q^i,\tilde{p}_\alpha)$ and, obviously, we still have that all the brackets $\{q^i,q^j\}$ identically vanish. On the other hand, in terms of the (non-canonical) coordinates $(q,\tilde{p})$, the bracket $\{\ ,\ \}_{nh}$, defined in the previous section, becomes

$$\begin{aligned}
&\{q^i,q^j\}_{nh} = 0 && \{q^i,\tilde{p}_\alpha\}_{nh} = X^i_\alpha \\
&\{q^i,\tilde{p}_{n-k+a}\}_{nh} = 0 && \{\tilde{p}_\alpha,\tilde{p}_\beta\}_{nh} = X^i_\beta p_j \frac{\partial X^j_\alpha}{\partial q^i} - X^i_\alpha p_j \frac{\partial X^j_\beta}{\partial q^i} \\
&\{\tilde{p}_{n-k+a},\tilde{p}_\beta\}_{nh} = 0 && \{\tilde{p}_{n-k+a},\tilde{p}_{n-k+b}\}_{nh} = 0\,.
\end{aligned}$$

From this local expression it immediately follows that the above matrix J_M precisely yields the component matrix of Λ_M in the induced coordinates $(q^i,\tilde{p}_\alpha)$ on M and, therefore, the bracket constructed by van der Schaft and Maschke equals the bracket $\{\ ,\ \}_M$ defined at the beginning of this section. We thus have that for any two functions f and g on M

$$\{f,g\}_M = \left(\left(\frac{\partial f}{\partial q^i}\right)^T, \left(\frac{\partial f}{\partial \tilde{p}_\alpha}\right)^T \right) J_M \begin{pmatrix} \frac{\partial g}{\partial q^i} \\ \frac{\partial g}{\partial \tilde{p}_\alpha} \end{pmatrix}$$

where ^{T} denotes the transpose of a matrix.

As mentioned before, in general, Λ_M is not a Poisson tensor, i.e. its Schouten-Nijenhuis bracket $[\Lambda_M,\Lambda_M]$ will be different from zero. In the case of linear nonholonomic constraints, Koon and Marsden have made the interesting observation that the Schouten-Nijenhuis bracket of Λ_M yields an expression involving the curvature of an Ehresmann connection which one can always, at least locally, associate with the given constraint distribution (cf. [10]). In fact, it turns out that $[\Lambda_M, \Lambda_M]$ vanishes if and only if this underlying connection has zero curvature, i.e. if and only if the corresponding (horizontal) distribution is integrable. This is precisely the case when the given linear constraints are holonomic.

5 The case of general constraints

We now consider the case of a system with a hyperregular lagrangian, subjected to non-holonomic constraints locally defined by functions $\phi^a(q, \dot{q})$ on TQ which need not verify the homogeneity condition, but such that all other assumptions made above still hold (such as the admissibility and compatibility condition). It turns out that, on the hamiltonian side, we then need to modify expression (3) for the evolution of a function along the integral curves of the constrained system.

We know that the constrained dynamics is given by $P(X_H) = X_{H,M}$ and, for general (non-homogeneous) constraints, this vector field differs from $\mathcal{P}(X_H)$. The following proposition relates $X_{H,M}(f)$ and $\{f, H\}_{nh}$ for an arbitrary function $f \in C^\infty(T^*Q)$.

Proposition 5.1 *For any function $f \in C^\infty(T^*Q)$ we have, along M,*

$$\dot{f} = X_{H,M}(f) = R_H(f) + \{f, H\}_{nh}$$

where the vector field R_H is given by $R_H = P(X_H) - \mathcal{P}(X_H)$.

Proof : We have that

$$\begin{aligned} X_{H,M}(f) &= P(X_H)(f) \\ &= (P(X_H) - \mathcal{P}(X_H))(f) + \mathcal{P}(X_H)(f) \\ &= R_H(f) + \mathcal{P}^*(df)(X_H) \\ &= R_H(f) + \Lambda(\mathcal{P}^*(df), dH). \end{aligned}$$

Taking into account Lemma 3.1, we then obtain that

$$\begin{aligned} X_{H,M}(f) &= R_H(f) + \Lambda_{nh}(df, dH) \\ &= R_H(f) + \{f, H\}_{nh}. \end{aligned}$$

$\boxed{QED}$

Remark 5.2 There is an alternative way for computing the time evolution of a function f without having to make explicit use of the projector $\mathcal{P}$. (We are indebted to one of the referees for pointing this out to us). Indeed, one easily finds that, along M,

$$\begin{aligned} X_{H,M}(f) &= P(X_H)(f) \\ &= < P^*(df), X_H > \\ &= \Lambda(P^*(df), P^*(dH)) + \Lambda(P^*(df), dH - P^*(dH)) \\ &= \{f, H\}_{nh} + \Lambda(P^*(df), dH - P^*(dH)). \end{aligned}$$

To see that this produces the same result as the one derived above, note that, using Proposition 3.3 and Lemma 3.1, one may readily verify that

$$\Lambda(P^*(df), dH - P^*(dH)) =< (P^* - \mathcal{P}^*)(df), X_H >= R_H(f)\,.$$

From its definition it follows that R_H can be extended to a genuine vector field defined on an open neighbourhood of M. Using the notations introduced above, its local expression reads

$$R_H = -C_{ab}Z^a(H)P(X_{\Psi^b}) = C_{ab}\frac{\partial \Psi^a}{\partial p_i}\mathcal{H}_{ij}\frac{\partial H}{\partial p_j}\left(X_{\Psi^b} + C_{cd}\{\Psi^d, \Psi^b\}Z^c\right).$$

The Leg^{-1}-related vector field on TQ is

$$\bar{R} = \bar{C}_{ab}\frac{\partial \phi^a}{\partial \dot{q}^i}\dot{q}^i Leg^*(P(X_{\Psi^b}))$$

where $\bar{C}_{ab} = C_{ab} \circ Leg$. Observe that the vector fields R_H and $\bar{R}$ are tangent to the constraint submanifolds M and D in T^*Q and TQ, respectively, and that both vanish when the constraints verify the homogeneity condition (i.e. when the dilation vector field Δ is tangent to D).

We may conclude that the evolution of an observable for a nonholonomic system with general (not necessarily homogeneous) constraints is given by

$$\begin{aligned} \dot{f} &= R_H(f) + \{f, H\}_{nh} \\ &= (-C_{ab}Z^a(H)P(X_{\Psi^b})(f)) \\ &\quad + \left(\{f, H\} + C_{ab}Z^a(H)\{f, \Psi^b\} - C_{ab}Z^a(f)\{H, \Psi^b\} + C_{ab}C_{cd}\{\Psi^b, \Psi^l\}Z^a(f)Z^c(H)\right). \end{aligned}$$

Since R_H is tangent to M, we obtain for functions $f \in C^\infty(M)$:

$$\dot{f} = R_H(f) + \{f, H_M\}_M$$

where H_M denotes the restriction of H to M. In the case of homogeneous (for instance, linear) constraints, this reduces to

$$\dot{f} = \{f, H_M\}_M\,.$$

In view of the skew-symmetry of the almost-Poisson bracket, it immediately follows that the "energy" H_M will be a conserved quantity of the constrained dynamics iff $R_H(H_M) = 0$. This condition will clearly be verified if $R_H = P(X_H) - \mathcal{P}(X_H) \equiv 0$. Now, taking into account Lemma 3.2, we see that $P(X_H) - \mathcal{P}^*(X_H)$ identically vanishes if $X_H \in F$ (in points of M) or, equivalently, if $dH_{|M} \in (F^\perp)^o$. This, in particular, includes the case of homogeneous constraints since, as already observed above, in the homogeneous case we have $\mu^a(X_H) = 0$ for all the 1-forms μ^a spanning F^o.

Another case where the vector field R_H vanishes, occurs when $X_{H|M} \in F^\perp$, for we then have that $P(X_H) = \mathcal{P}(X_H) = 0$. This situation may be excluded, however, since in that case there is no constrained dynamics. More precisely, the constrained dynamics is trivial in the sense that $X_{H,M} = P(X_H) = 0$.

There may also be cases, however, where the energy will be conserved without R_H being zero. Suppose, for instance, that the constraint submanifold M is an invariant submanifold of the free (unconstrained) hamiltonian system, i.e. X_H is tangent to M. We then have that $X_{H,M} = P(X_H) = X_{H|M}$ and, hence, $X_{H,M}(H_M) = (X_H(H))_{|M} = 0$ although, in general, R_H will not be zero. (This can also be deduced from the local coordinate expression for R_H.)

6 A particular case: affine constraints.

We now treat the case whereby the given constraints are affine functions in the velocities, that is, $\phi^a = \mu^a_i \dot{q}^i - h^a$, $1 \leq a \leq k$, where μ^a_i and h^a are functions on the configuration space Q. The constraint submanifold D therefore yields an affine subbundle of TQ. For each a, the μ^a_i may be regarded as the components of a 1-form μ^a on Q. If H is the hamiltonian of the system under consideration, the constraint functions on T^*Q read

$$\Psi^a = \mu^a_i \frac{\partial H}{\partial p_i} - h^a, \ 1 \leq a \leq k$$

Let H again be of the form

$$H = \frac{1}{2} g^{ij} p_i p_j + V(q)$$

where g^{ij} are the components of (the inverse of) a metric on Q. The components C^{ab} of the matrix (2) here become

$$C^{ab} = \mu^a_i g^{ij} \mu^b_j \,,$$

and, in view of the compatibility assumption, this matrix is invertible. The components of the inverse matrix will again be denoted by C_{ab}. In the spirit of the van der Schaft and

Maschke construction for linear constraints (see Section 4) we now consider on T^*Q the following change of coordinates: $(q,p) \to (q,\tilde{p})$ with

$$\tilde{p}_\alpha = X^i_\alpha p_i \ \text{ and } \ \tilde{p}_{n-k+a} = Y^i_a p_i - C_{ab}h^b$$

for $1 \leq \alpha \leq n-k$, $1 \leq a \leq k$. Here, $Y^i_a = C_{ac}\mu^c_j g^{ji}$ and X^i_α is any set of functions satisfying $X^i_\alpha \mu^a_i = 0$ for all indices α and a, and such that the X^i_α and Y^i_a form the components of a regular $n \times n$ matrix. One can assign the following geometrical interpretation to the functions X^i_α, Y^i_a. The X^i_α can be regarded as the components of $n-k$ independent vector fields X_α which (locally) span the vector subbundle of TQ on which the affine bundle D is modelled. The Y^i_a then are the components of k vector fields, spanning a complementary subbundle of TQ and such that $< Y_a, \mu^b >= \delta^b_a$. The inverse of the above transformation then reads

$$p_i = \nu^\alpha_i \tilde{p}_\alpha + \mu^a_i(\tilde{p}_{n-k+a} + C_{ab}h^b),$$

whereby the ν^α_i are the components of 1-forms ν^α on Q which, together with the μ^a, constitute a local basis of 1-forms, dual to the basis of vector fields (X_α, Y_a).

In the new coordinates we now have

$$C_{ab}\Psi^b(q, p(q,\tilde{p})) = C_{ab}(\mu^b_i g^{ij} p_j - h^b) = \tilde{p}_{n-k+a}.$$

In view of the regularity of (C_{ab}), the constraint submanifold M can equivalently be described by the equations $\tilde{p}_{n-k+a} = 0$. Putting $\tilde{H}(q,\tilde{p}) = H(q, p(q,\tilde{p}))$, the equations of motion for the constrained system become:

$$\begin{aligned}
\dot{q}^i &= \{q^i, q^j\}\frac{\partial \tilde{H}}{\partial q^j} + \{q^i, \tilde{p}_\beta\}\frac{\partial \tilde{H}}{\partial \tilde{p}_\beta} + \{q^i, \tilde{p}_{n-k+b}\}\frac{\partial \tilde{H}}{\partial \tilde{p}_{n-k+b}} \\
\dot{\tilde{p}}_\alpha &= \{\tilde{p}_\alpha, q^j\}\frac{\partial \tilde{H}}{\partial q^j} + \{\tilde{p}_\alpha, \tilde{p}_\beta\}\frac{\partial \tilde{H}}{\partial \tilde{p}_\beta} + \{\tilde{p}_\alpha, \tilde{p}_{n-k+b}\}\frac{\partial \tilde{H}}{\partial \tilde{p}_{n-k+b}} \\
\dot{\tilde{p}}_{n-k+a} &= \{\tilde{p}_{n-k+a}, q^j\}\frac{\partial \tilde{H}}{\partial q^j} + \{\tilde{p}_{n-k+a}, \tilde{p}_\beta\}\frac{\partial \tilde{H}}{\partial \tilde{p}_\beta} + \{\tilde{p}_{n-k+a}, \tilde{p}_{n-k+b}\}\frac{\partial \tilde{H}}{\partial \tilde{p}_{n-k+b}} + \bar{\lambda}_a
\end{aligned}$$

which, of course, have to be considered together with the constraint equations

$$\tilde{p}_{n-k+a} = 0 \ \text{ for } \ a = 1, \ldots, k$$

and where the right-hand sides are expressed in terms of the coordinates $(q,\tilde{p})$. For reasons of "symmetry" we have included the identically vanishing brackets $\{q^i, q^j\}$. A straightforward computation reveals that, in points of M,

$$\frac{\partial \tilde{H}}{\partial \tilde{p}_{n-k+b}}(q, \tilde{p}_\alpha, 0) = h^b(q).$$

Since the coordinates $(q^i, \tilde{p}_\alpha)$ are induced local coordinates on M, we find that the equations of motion on the constraint submanifold are given by

$$\begin{pmatrix} \dot{q}^i \\ \dot{\tilde{p}}_\alpha \end{pmatrix} = \begin{pmatrix} \{q^i, q^j\}_M & \{q^i, \tilde{p}_\beta\}_M \\ \{\tilde{p}_\alpha, q^j\}_M & \{\tilde{p}_\alpha, \tilde{p}_\beta\}_M \end{pmatrix} \begin{pmatrix} \frac{\partial H_M}{\partial q^i} \\ \frac{\partial H_M}{\partial \tilde{p}_\beta} \end{pmatrix} + \begin{pmatrix} \{q^i, \tilde{p}_{n-k+b}\} h^b \\ \{\tilde{p}_\alpha, \tilde{p}_{n-k+b}\} h^b \end{pmatrix}_{|M}$$

where, in the second term on the right-hand side, $_{|M}$ indicates that we have to put all $\tilde{p}_{n-k+a} = 0$ after computing the brackets. In the first term on the right-hand side, $\{\ ,\ \}_M$ denotes the almost-Poisson bracket on M and H_M denotes the restriction of H to M, i.e. $H_M(q^i, \tilde{p}_\alpha) = \tilde{H}(q^i, \tilde{p}_\alpha, 0)$. Observe now that the projector P (cf. Section 2) is here given by

$$P = \mathrm{id} - \frac{\partial}{\partial \tilde{p}_{n-k+a}} \otimes d\tilde{p}_{n-k+a}$$

and the vector field R_H, restricted to M, reads

$$R_H = \left(\{q^i, \tilde{p}_{n-k+b}\} h^b\right) \frac{\partial}{\partial q^i} + \left(\{\tilde{p}_\alpha, \tilde{p}_{n-k+b}\} h^b\right) \frac{\partial}{\partial \tilde{p}_\alpha}\,.$$

For any smooth function f on M we then have

$$\dot{f} = R_H(f) + \{f, H_M\}_M\,.$$

Example 6.1 A rolling ball on a rotating table *([3, 16, 17])*
A (homogeneous) sphere of radius r and unit mass ($m = 1$) rolls without sliding on a horizontal table which rotates with constant angular velocity Ω about a vertical axis through one of its points. The configuration space of this system can be identified with $\mathbb{R}^2 \times \mathrm{SO}(3)$. Choose a cartesian reference frame with origin at the center of rotation of the table and z-axis along the rotation axis. Let (x, y) denote the position of the point of contact of the sphere with the table. Since we do not consider external forces, the lagrangian is given by the kinetic energy of the sphere, i.e.

$$L(=T) = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + k^2(\omega_x^2 + \omega_y^2 + \omega_z^2)\right) \tag{6}$$

where $(\omega_x, \omega_y, \omega_z)$ are the components of the angular velocity of the sphere. In terms of Euler's angles we have that:

$$\omega_x = \dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi, \quad \omega_y = \dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi, \quad \omega_z = \dot{\psi} + \dot{\varphi}\cos\theta.$$

It is well-known that the the problem of the rolling ball on a rotating table can be most elegantly treated by using the formalism of "quasi-coordinates", i.e. using the above expression for the lagrangian, with the components of the angular velocity then playing the role of "quasi-velocities" (see e.g. [16, 17]). We will briefly return to this approach at the end. However, in order to illustrate the procedure outlined above for the general case of affine constraints, we will pass to the natural bundle coordinates on $T(\mathbb{R}^2 \times \mathrm{SO}(3))$. The lagrangian for the sphere then becomes

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + k^2(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta)\right)$$

and the constraint equations, expressing the condition of rolling without slipping, are:

$$\begin{aligned}
\dot{x} - r\dot{\theta}\sin\psi + r\dot{\varphi}\sin\theta\cos\psi(= \dot{x} - r\omega_y) &= -\Omega y \\
\dot{y} + r\dot{\theta}\cos\psi + r\dot{\varphi}\sin\theta\sin\psi(= \dot{y} + r\omega_x) &= \Omega x
\end{aligned}$$

or, after some algebraic manipulations,

$$\begin{aligned}
\dot{x}\cos\psi + \dot{y}\sin\psi + r\dot{\varphi}\sin\theta &= \Omega x\sin\psi - \Omega y\cos\psi \\
\dot{x}\sin\psi - \dot{y}\cos\psi - r\dot{\theta} &= -\Omega y\sin\psi - \Omega x\cos\psi\,.
\end{aligned}$$

We will now construct the associated almost-Poisson bracket for this nonholonomic system, as well as the vector field R_H, where, in the present example, $H = T$.

On the phase space we consider coordinates

$$(x, y, \theta, \varphi, \psi, \tilde{p}_\alpha, \tilde{p}_a)\ ,\ 1 \leq \alpha \leq 3,\ 4 \leq a \leq 5$$

where

$$\begin{aligned}
\tilde{p}_1 &= rp_x\sin\psi - rp_y\cos\psi + p_\theta \\
\tilde{p}_2 &= rp_x\sin\theta\cos\psi + rp_y\sin\theta\sin\psi - p_\varphi \\
\tilde{p}_3 &= p_\psi
\end{aligned}$$

and

$$\begin{aligned}
\tilde{p}_4 &= \frac{k^2}{k^2 + r^2}\left(p_x\cos\psi + p_y\sin\psi + \frac{rp_\varphi}{k^2\sin\theta} - \frac{rp_\psi\cos\theta}{k^2\sin\theta} - \Omega(x\sin\psi - y\cos\psi)\right) \\
\tilde{p}_5 &= \frac{k^2}{k^2 + r^2}\left(p_x\sin\psi - p_y\cos\psi - \frac{r}{k^2}p_\theta + \Omega(x\cos\psi + y\sin\psi)\right)\ .
\end{aligned}$$

In these coordinates the constraints are:

$$\begin{aligned}
\tilde{p}_4 &= 0 \\
\tilde{p}_5 &= 0\,.
\end{aligned}$$

Therefore, the non zero terms of the nonholonomic bracket $\{\ ,\ \}_{nh}$ on the phase space are:

$$\begin{array}{ll}
\{x,\tilde p_1\}_{nh} = r\sin\theta & \{y,\tilde p_1\}_{nh} = -r\cos\psi \\
\{\theta,\tilde p_1\}_{nh} = 1 & \{x,\tilde p_2\}_{nh} = r\sin\theta\cos\psi \\
\{y,\tilde p_2\}_{nh} = r\sin\theta\sin\psi & \{\varphi,\tilde p_2\}_{nh} = -1 \\
\{\psi,\tilde p_3\}_{nh} = 1 & \{\tilde p_1,\tilde p_2\}_{nh} = -r\cos\theta(p_x\cos\psi + p_y\sin\psi) \\
\{\tilde p_1,\tilde p_3\}_{nh} = r(p_x\cos\psi + p_y\sin\psi) & \{\tilde p_2,\tilde p_3\}_{nh} = r\sin\theta(p_y\cos\psi - p_x\sin\psi)\,.
\end{array}$$

The almost-Poisson bracket $\{\ ,\ \}_M$ is obtained by expressing the right-hand sides in terms of $(x,y,\theta,\varphi,\psi,\tilde p_1,\ldots,\tilde p_5)$ by means of the inverse of the above transformation, and then putting $\tilde p_4 = \tilde p_5 = 0$.

The projector P is

$$P = \mathrm{id} - \frac{\partial}{\partial\tilde p_4}\otimes d\tilde p_4 - \frac{\partial}{\partial\tilde p_5}\otimes d\tilde p_5$$

and the vector field R_H then becomes

$$\begin{aligned}
R_H &= h^a P(X_{\tilde p_a}) \\
&= h^a\left[\{q^i,\tilde p_a\}\frac{\partial}{\partial q^i} + \{\tilde p_\alpha,\tilde p_a\}\frac{\partial}{\partial\tilde p_\alpha}\right] \\
&= \frac{k^2}{k^2+r^2}\left[-\Omega y\frac{\partial}{\partial x} + \Omega x\frac{\partial}{\partial y} + \Omega(x\cos\psi + y\sin\psi)\frac{r}{k^2}\frac{\partial}{\partial\theta}\right. \\
&\quad -\Omega(y\cos\psi - x\sin\psi)\frac{r}{k^2\sin\theta}\frac{\partial}{\partial\varphi} + \Omega(y\cos\psi - x\sin\psi)\frac{r\cos\theta}{k^2\sin\theta}\frac{\partial}{\partial\psi} \\
&\quad +\Omega(x\sin\psi - y\cos\psi)\left(r\Omega + \frac{r}{k^2\sin^2\theta}(p_\varphi\cos\theta - p_\psi)\right. \\
&\quad \left. -\frac{r^2\cos\theta}{k^2\sin\theta}(p_x\cos\psi + p_y\sin\psi)\right)\frac{\partial}{\partial\tilde p_1} \\
&\quad + \left(r\Omega^2(x\cos\psi + y\sin\psi)\sin\theta + \Omega\frac{r^2}{k^2}\cos\theta(xp_x + yp_y)\right)\frac{\partial}{\partial\tilde p_2} \\
&\quad \left. + \Omega(xp_x + yp_y)\frac{\partial}{\partial\tilde p_3}\right],
\end{aligned}$$

where the components on the right-hand side still have to be expressed in terms of the $\tilde p$ and then put $\tilde p_4 = \tilde p_5 = 0$.

Using these expressions for the bracket and for the vector field R_H one will find, after some tedious calculations, the following equations of motion on M:

$$\dot x = -\frac{k^2\Omega y}{k^2+r^2} + \frac{r}{k^2+r^2}\left[(\tilde p_2 + \tilde p_3\cos\theta)\cos\psi\csc\theta + \tilde p_1\sin\psi\right]$$

$$
\begin{aligned}
\dot{y} &= \frac{k^2\Omega x}{k^2+r^2} + \frac{r}{k^2+r^2}\left[(\tilde{p}_2 \csc\theta + \tilde{p}_3\cot\theta)\sin\psi - \tilde{p}_1\cos\psi\right] \\
\dot{\theta} &= \frac{r\Omega}{k^2+r^2}(x\cos\psi + y\sin\psi) + \frac{\tilde{p}_1}{k^2+r^2} \\
\dot{\phi} &= -\frac{1}{k^2+r^2}\left[\csc^2\theta(\tilde{p}_2 + \tilde{p}_3\cos\theta + r\Omega y\cos\psi\sin\theta - r\Omega x\sin\theta\sin\psi)\right] \\
\dot{\psi} &= \frac{\csc^2\theta}{k^2}\left[\tilde{p}_3 + \frac{\cos\theta}{k^2+r^2}\left(-\tilde{p}_3 r^2\cos\theta + k^2(\tilde{p}_2 + r\Omega y\cos\psi\sin\theta - r\Omega x\sin\theta\sin\psi)\right)\right] \\
\dot{\tilde{p}}_1 &= \frac{1}{4\ k^2(k^2+r^2)}[(\tilde{p}_2+\tilde{p}_3\cos\theta)\csc^3\theta(4k^2\tilde{p}_3 + 2\tilde{p}_3 r^2 \\
&\quad +4k^2\tilde{p}_2\cos\theta - 2\tilde{p}_3 r^2\cos(2\theta) - k^2 rx\Omega\cos(2\theta-\psi) + k^2 rx\Omega\cos(2\theta+\psi) \\
&\quad +k^2 ry\Omega\sin(2\theta-\psi) + k^2 ry\Omega\sin(2\theta+\psi)] \\
\dot{\tilde{p}}_2 &= \frac{r\csc\theta}{4k^2(k^2+r^2)}(-2\tilde{p}_1\tilde{p}_3 r + 2\tilde{p}_1\tilde{p}_3 r\cos 2\theta \\
&\quad +2k^2\tilde{p}_2 x\Omega\cos(\theta-\psi) + k^2\tilde{p}_1 x\Omega\cos(2\theta-\psi) \\
&\quad +4k^2\tilde{p}_3 x\Omega\cos\psi + 2k^2\tilde{p}_2 x\Omega\cos(\theta+\psi) \\
&\quad -k^2\tilde{p}_1 x\Omega\cos(2\theta+\psi) - 2k^2\tilde{p}_2 y\Omega\sin(\theta-\psi) \\
&\quad -k^2\tilde{p}_1 y\Omega\sin(2\theta-\psi) + 4k^2\tilde{p}_3 y\Omega\sin\psi \\
&\quad +2k^2\tilde{p}_2 y\Omega\sin(\theta+\psi) - k^2\tilde{p}_1 y\Omega\sin(2\theta+\psi)) \\
\dot{\tilde{p}}_3 &= 0\,.
\end{aligned}
$$

It should be emphasised again that these equations can be written in a much more compact form when making use of quasi-coordinates. For a classical treatment of the theory of quasi-coordinates, with application to the problem under consideration, we may refer to [16, 17]. A more recent discussion of the use of quasi-coordinates in the study of nonholonomic systems with symmetry, within a differential geometrical setting, can be found in a paper by Koiller [8].

Obviously, the treatment of the problem of the rolling ball in terms of an almost-Poisson stucture, can also be translated in terms of quasi-coordinates. Without going into the details, we just briefly sketch here the main steps of such an approach.

Let us take $(x, y, \theta, \varphi, \psi, \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z)$ as coordinates on the tangent bundle, with the components of the angular velocity now being regarded as the "velocities" associated to some quasi-coordinates. Following the classical treatments, such as [16, 17], we put $\omega_x = \dot{q}^1, \omega_y = \dot{q}^2, \omega_z = \dot{q}^3$, with q^1, q^2, q^3 denoting quasi-coordinates. The latter merely have a symbolic meaning in the sense that in the present example, for instance, the partial derivative operators $\partial/\partial q^i$ should be interpreted as linear combinations of the partial derivatives with respect to Euler's angles. Also to the differential forms dq^i one should attach the appropriate meaning, i.e. they do not represent exact differentials but, instead, we should read them as $dq^1 = \cos\psi\, d\theta + \sin\theta \sin\psi\, d\varphi$, *etc....* The lagrangian is now given by (6) and the nonholonomic constraints read

$$\dot{x} - r\omega_y = -\Omega y, \qquad \dot{y} + r\omega_x = \Omega x \ .$$

The hamiltonian for the system becomes

$$H = \frac{1}{2k^2}\left(k^2 p_x^2 + k^2 p_y^2 + p_1^2 + p_2^2 + p_3^2\right)$$

where $p_x = \dot{x}, p_y = \dot{y}, p_1 = k^2\omega_x, p_2 = k^2\omega_y, p_3 = k^2\omega_z$. Note that the "momenta" p_1, p_2, p_3, associated to the quasi-coordinates, are not canonical momenta: they are linear combinations of the canonical momenta conjugated to Euler's angles. On the phase space we then introduce new variables

$$\begin{aligned} \hat{p}_1 &= r p_x + p_2 \\ \hat{p}_2 &= r p_y - p_1 \\ \hat{p}_3 &= p_3 \end{aligned}$$

and

$$\begin{aligned} \hat{p}_4 &= \frac{k^2}{k^2 + r^2}\left(p_x - \frac{r p_2}{k^2} + \Omega y\right) \\ \hat{p}_5 &= \frac{k^2}{k^2 + r^2}\left(p_y + \frac{r}{k^2} p_1 - \Omega x\right) \ . \end{aligned}$$

In these coordinates the constraint equations become:

$$\begin{aligned} \hat{p}_4 &= 0 \\ \hat{p}_5 &= 0 \, . \end{aligned}$$

With the correct "operational" meaning attached to the quasi-coordinates q^i we may then again compute the almost-Poisson bracket $\{\ ,\ \}_M$ and the vector field R_H and, eventually, we will obtain the equations of motion on M in the following form:

$$\begin{aligned} \dot{x} &= -\frac{\Omega k^2 y}{k^2 + r^2} - \frac{r\hat{p}_1}{k^2 + r^2} \\ \dot{y} &= \frac{\Omega k^2 x}{k^2 + r^2} + \frac{r\hat{p}_2}{k^2 + r^2} \\ \dot{q}^1 &= \frac{r\Omega x}{k^2 + r^2} - \frac{\hat{p}_2}{k^2 + r^2} \\ \dot{q}^2 &= \frac{r\Omega y}{k^2 + r^2} + \frac{\hat{p}_1}{k^2 + r^2} \\ \dot{q}^3 &= \frac{\hat{p}_3}{k^2} \\ \dot{\hat{p}}_1 &= 0 \\ \dot{\hat{p}}_2 &= 0 \\ \dot{\hat{p}}_3 &= 0 \, . \end{aligned}$$

Recalling that the $\dot{q}^i$'s represent the components of the angular velocity of the sphere, the above equations lead to:

$$\begin{aligned}
\dot{x} &= -\frac{\Omega k^2 y}{k^2+r^2} + C_1 \\
\dot{y} &= \frac{\Omega k^2 x}{k^2+r^2} + C_2 \\
\omega_x &= \frac{r\Omega x}{k^2+r^2} - \frac{C_2}{r} \\
\omega_y &= \frac{r\Omega y}{k^2+r^2} - \frac{C_1}{r} \\
\omega_z &= C_3
\end{aligned}$$

where C_i, $1 \leq i \leq 3$ are constants depending on the initial data. From these we immediately extract the reduced equations of motion for the rolling sphere on a rotating plate as derived, for instance, in [16, 17]. Exploiting the rotational symmetry of the problem, an alternative way of deriving these reduced equations consists in applying a geometric reduction theory for nonholonomic systems with symmetry (see [3]).

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