The inverse problem of the calculus of variations: separable systems

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Abstract

This paper deals with the inverse problem of the calculus of variations for systems of second-order ordinary differential equations. The case of the problem which Douglas, in his classification of pairs of such equations, called the "separated case" is generalized to arbitrary dimension. After identifying the conditions which should specify such a case for n equations in a coordinate-free way, two proofs of its variationality are presented. The first one follows the line of approach introduced by some of the authors in previous work, and is close in spirit, though being coordinate independent, to the Riquier analysis applied by Douglas for $n = 2$. The second proof is more direct and leads to the discovery that belonging to the "separated case" has an intrinsic meaning for the given second-order differential equations: the system is separable in the sense that it can be decoupled into n pairs of first-order equations.

1 Introduction

This paper is the third in a series dealing with the inverse problem of the calculus of variations for systems of second-order ordinary differential equations in the spirit of, and using methods developed from, Douglas's classic paper of 1941 [6]. The first of the series, [5], gave a geometrical interpretation of the key parts of Douglas's paper using a geometrical formalism developed over a number of years for the study of second-order differential equation fields, Lagrangian dynamical systems and related topics by Martínez and Sarlet and their co-workers: see [3] and references therein for an account of the relevant results. The second, [9], contained a complete general analysis of the integrability conditions, or more properly in the context the passivity conditions, for the Helmholtz equations for the inverse problem with arbitrarily many – say n – degrees of freedom (recall that Douglas dealt only with the two degree of freedom case). It is the availability of the same geometrical formalism that makes such a general account possible. The results were put to use to show that a particular class of systems with n degrees of freedom, corresponding to Class I of Douglas's classification in the two degrees of freedom case, is always variational.

In the present paper we extend another of Douglas's results from two to n degrees of freedom, namely the variationality of the separated subcase of Case IIa, which is Case IIa1 in Douglas's notation. Whereas other authors have tackled Case I by other methods [1, 7], to the best of our knowledge the result that Case IIa1 with arbitrarily many degrees of freedom is variational has not been proved before.

The first task is to generalize the specification of Case IIa1 so that it applies with arbitrarily many degrees of freedom. We suggested how this by no means obvious generalization should be made in [5]; we now give the details, in Section 2 below. We turn to the question of variationality in Section 3. There are three distinct ways known to us of proving that Case IIa1 is variational; two of them are explained in detail in this paper. Let us dispose of the remaining method. The differential Helmholtz conditions in the case in question consist of a system of systems of equations of the form $Z_\alpha(\rho) = \zeta_\alpha \rho$ for an unknown function ρ, where the Z_α are (in principle) known vector fields and the ζ_α known functions. There is a straightforward theory of the integrability of such a system of equations, based on Frobenius's theorem, whose main result is that when the distribution spanned by the Z_α is involutive the system is integrable if and only if the functions ζ_α satisfy the integrability conditions $Z_\beta(\zeta_\gamma) - Z_\gamma(\zeta_\beta) - \sum c^{\alpha}_{\beta\gamma}\zeta_\alpha = 0$, where $[Z_\beta, Z_\gamma] = \sum c^{\alpha}_{\beta\gamma} Z_\alpha$. That the necessary integrability conditions are satisfied in the case in question can be shown, but only by long and laborious calculations. We have therefore avoided this method in this paper.

We have done so for another reason as well. One of the other methods of proof is to use the Riquier theory of partial differential equations to show that the differential Helmholtz conditions are solvable. Now the Riquier theory forms the backbone of Douglas's paper; and our treatment of integrability and passivity conditions in the previous paper of this series, [9], is designed to be used in conjunction with the Riquier theory. By drawing on the results given in [9], therefore, we are able to give a proof of variationality which is not only relatively painless but also fits in with the line of attack of the previous papers.

The other proof arises as a by-product of an investigation into just what it means to say, as Douglas does, that systems of the kind that fall under Case IIa1 are separated. Douglas uses the term to mean that the differential Helmholtz conditions separate in a certain way; however, the example that he gives to illustrate his results can be separated in a stronger sense: it consists of a system of second-order equations in two unknowns which is simply the concatenation of two independent second-order equations each in one unknown. We show, in Section 4, that any system of second-order equations in Case IIa1 with n degrees of freedom can be separated into n separate systems of two first-order equations. However, in principle separability into independent pairs of first-order equations need not imply separability into single independent second-order equations. The question therefore arises as to whether there are non-trivial systems in Case IIa1, that is, systems which are not separable in the stronger sense. We resolve this question by presenting an example of a non-trivial Case IIa1 system, in Section 5.

In the course of the discussion of separability in Section 4, we introduce special coordinates, in terms of which it is easy to construct explicitly a Cartan 2-form for the system. Since the existence of a Cartan 2-form is sufficient for there to be a Lagrangian for the system, as is shown in [4], this provides another proof of variationality.

Rather than repeat the general background to the methods used in the present paper, we refer the reader to [9] for the requisite information; in fact the present paper is probably best regarded as a continuation of the previous one. However, it does seem helpful to explain some of our notation, and also to repeat below the main result of the integrability and passivity analysis in [9], which will be needed in Section 3.

The summation convention is *not* generally in force in this paper. However, we will have to write down some expressions involving sums, and in order to minimize our use of the summation symbol we will adopt the convention that whenever the multiply occurring index s appears summation is implied – and it is implied for that choice of index only. Indices i and j, and on occasion others from the same part of the alphabet, will range from 1 to n; the index s will sum over the same range.

In our approach to the inverse problem a system of second-order ordinary differential equations $\ddot{x}^i = f^i(t, x^j, \dot{x}^j)$ is represented by the vector field

$$\Gamma = \frac{\partial}{\partial t} + v^s \frac{\partial}{\partial x^s} + f^s \frac{\partial}{\partial v^s}$$

on the first jet bundle $J^1\pi$ of a bundle $\pi : E \to \mathbb{R}$.

Many of the geometrical objects of interest for our analysis are tensor fields along the projection $\pi_1^0 : J^1\pi \to E$. To be more precise, we will be concerned with objects which are sections of tensor product bundles formed from the vector bundle ${\pi_1^0}^*(V\pi)$, the pull-back by π_1^0 to $J^1\pi$ of the vertical subbundle of TE with respect to the projection π.

We will always use the term 'tensor field along π_1^0' in this special sense. In particular, for us the multiplier in the inverse problem is a symmetric type $(0,2)$ tensor field along π_1^0. Vector fields along π_1^0, in other words sections of $\pi_1^{0*}(V\pi)$, will be denoted by such symbols as X, Y and Z.

We will also make extensive use of certain differential operators on tensor fields along π_1^0. It was shown in [3] that a second-order differential equation field Γ determines a linear connection on $\pi_1^{0*}(V\pi)$, whose covariant differentiation operator is denoted by D. Any vector field ξ on $J^1\pi$ determines unique vector fields X and Y along π_1^0 by means of its decomposition into vertical and horizontal components: $\xi = X^V + Y^H + \langle \xi, dt \rangle \Gamma$, where X^V and $Y^H + \langle \xi, dt \rangle \Gamma$ are respectively the vertical and horizontal components of ξ with respect to the horizontal distribution on $J^1\pi$ determined by the second-order differential equation field Γ. Then the covariant differentiation operator D_ξ associated with the linear connection on $\pi_1^{0*}(\tau_E)$ can be written as

$$\mathrm{D}_\xi = \mathrm{D}_X^V + \mathrm{D}_Y^H + \langle \xi, dt \rangle \nabla.$$

The component operators D^V, D^H and ∇ have important roles to play in our analysis. Since they are built out of a covariant derivative, they extend to operate on tensor fields in the usual way. In our formulation the Helmholtz equations for the multiplier g are written

$$\begin{aligned}
&\nabla g = 0 \\
&\mathrm{D}^V g(Z, X, Y) = \mathrm{D}^V g(Y, X, Z) \\
&g(\Phi(X), Y) = g(\Phi(Y), X),
\end{aligned}$$

where Φ is the Jacobi endomorphism, a type $(1,1)$ tensor field along π_1^0 which is essentially defined by the formula $[\Gamma, X^H] = (\nabla X)^H + (\Phi(X))^V$.

The main result of the integrability and passivity analysis in [9] is that the complete set of integrability and/or passivity conditions associated with the Helmholtz conditions is:

- the additional first-order condition

$$\mathrm{D}^H g(Z, X, Y) = \mathrm{D}^H g(Y, X, Z);$$

- the two sets of algebraic requirements

$$g(\nabla^r \Phi(X), Y) = g(\nabla^r \Phi(Y), X) \qquad r = 1, 2, 3, \ldots$$

and

$$\sum g(\nabla^r R(X,Y),Z)=0 \qquad r=0,1,2,\dots,$$

where R is a type $(1,2)$ tensor-field along π_1^0, which is a component of the curvature of the connection D, and is related to Φ by $\mathrm{D}^V\Phi(X,Y)-\mathrm{D}^V\Phi(Y,X)=3R(X,Y)$;

- the second-order conditions

$$\begin{aligned}&\mathrm{D}^V\mathrm{D}^H g(U,Z,Y,X)+g(\theta(Y,X)Z,U)\\&\quad=\mathrm{D}^V\mathrm{D}^H g(Z,U,Y,X)+g(Z,\theta(Y,X)U),\end{aligned}$$

where θ is a type $(1,3)$ tensor field along π_1^0 (here written as a type $(1,1)$ tensor valued twice covariant tensor), which is another part of the curvature of the connection D, and which is symmetric in all of its arguments as a consequence of the first Bianchi identities.

The latter conditions were written in the shorthand form $\mathcal{A}(U,Z)g(Y,X)=0$ in [9]; we will use the same notation here, and refer to the conditions as the $\mathcal{A}$ conditions.

2 Case IIa1

We will follow, and adapt, the classification of second-order systems in relation to the inverse problem given by Douglas. For $n=2$, this classification can be interpreted as being based on the linear dependence or independence of the tensors I, Φ, $\nabla\Phi$, ... The simplest case is that in which Φ is a multiple of the identity tensor; this, which like Douglas we call Case I, was generalized to arbitrary n in [9]. The next simplest is the case in which $\nabla\Phi$ is a linear combination of Φ and the identity; this is Douglas's Case II. One can easily see, however, that such a linear dependence leads to conditions which become more restrictive with increasing n. Note that, for $n=2$, a condition like $\nabla\Phi=\alpha\Phi+\beta I$ is equivalent to the condition that $\nabla\Phi$ and Φ commute, but the latter becomes much less restrictive than the former for large n. Since $[\nabla\Phi,\Phi]=0$ will be a sufficient condition for the situation we want to describe in this paper for arbitrary n, we will accept this (and not $\nabla\Phi=\alpha\Phi+\beta I$) as the defining assumption for what we will call Case II.

Within Case II we distinguish those systems for which Φ has n distinct eigenvalues (which we take to be real); systems with this property comprise Case IIa.

Consider such a system. Let $\{X_i\}$ be an eigenvector basis for Φ, with

$$\Phi(X_i)=\lambda_i X_i \qquad \text{(no summation)}.$$

¿From the Helmholtz condition $g(\Phi(X), Y) = g(\Phi(Y), X)$ it follows that

$$0 = g(\Phi(X_i), X_j) - g(\Phi(X_j), X_i) = (\lambda_i - \lambda_j) g(X_i, X_j),$$

whence

$$g(X_i, X_j) = 0 \quad \text{for } i \neq j.$$

That is to say, in order to be a multiplier, g must be diagonal with respect to any eigenvector basis.

¿From the condition $[\nabla\Phi, \Phi] = 0$, it follows that $(\Phi - \lambda_i I)(\nabla\Phi(X_i)) = 0$, which implies that $\nabla\Phi(X_i)$ is proportional to X_i. This in turn implies that $(\Phi - \lambda_i I)(\nabla X_i)$ is proportional to X_i, from which it follows that also $\nabla X_i \propto X_i$. But X_i is determined only up to a scalar factor, and this freedom may be used to rescale X_i so that

$$\nabla X_i = 0.$$

Let us set

$$g(X_i, X_i) = \rho_i.$$

Then with $\nabla X_i = 0$ in force, the condition $\nabla g = 0$ is equivalent to

$$\Gamma(\rho_i) = 0,$$

that is, the functions ρ_i must be first integrals of Γ.

We require that $\rho_i \neq 0$ for all $i = 1, 2, \ldots, n$, for the multiplier to be regular.

We now introduce 'connection coefficients' τ_{ij}^k for the basis $\{X_i\}$:

$$\mathrm{D}^V_{X_i} X_j = \tau_{ij}^s X_s \qquad \text{(summation over } s \text{ intended)}.$$

The so-called closure conditions, $\mathrm{D}^V g(X, Y, Z) = \mathrm{D}^V g(Y, X, Z)$, when expressed in terms of the basis, lead to conditions which the ρ_i must satisfy, involving the τ_{ij}^k. Now

$$\begin{aligned}\mathrm{D}^V g(X_i, X_j, X_k) &= X_i{}^V(g(X_j, X_k)) - g(\tau_{ij}^s X_s, X_k) - g(X_j, \tau_{ik}^s X_s) \\ &= X_i{}^V(\rho_j \delta_{jk}) - \tau_{ij}^k \rho_k - \tau_{ik}^j \rho_j,\end{aligned}$$

so the closure conditions amount to

$$X_i{}^V(\rho_j)\delta_{jk} - \tau^k_{ij}\rho_k - \tau^j_{ik}\rho_j = X_j{}^V(\rho_i)\delta_{ik} - \tau^k_{ji}\rho_k - \tau^i_{jk}\rho_i.$$

When i, j and k are all different (possible only when $n \geq 3$) we obtain purely algebraic conditions

$$\tau^k_{ij}\rho_k + \tau^j_{ik}\rho_j = \tau^k_{ji}\rho_k + \tau^i_{jk}\rho_i.$$

When $j = k \neq i$ we obtain the differential conditions

$$X_i{}^V(\rho_j) = (2\tau^j_{ij} - \tau^j_{ji})\rho_j - \tau^i_{jj}\rho_i.$$

The inverse problem for Case IIa is to find non-vanishing functions ρ_i which are first integrals of Γ and which satisfy these conditions.

Following Douglas, we make a further subdivision of Case IIa, motivated as follows. In general the differential conditions $X_i{}^V(\rho_j) = (2\tau^j_{ij} - \tau^j_{ji})\rho_j - \tau^i_{jj}\rho_i$ involve both ρ_i and ρ_j; the separated case for Douglas is that for which the term involving ρ_i is absent. If this were the only consideration, however, we would be left with the algebraic conditions for $n > 2$. We therefore adopt the following definition: Case IIa1 is the subcase of Case IIa in which the algebraic conditions on ρ_i are satisfied trivially, and the differential equations are separated in Douglas's sense. That is to say, Case IIa1 is that for which $\tau^k_{ij} = 0$ whenever i, j and k are all different, and $\tau^i_{jj} = 0$ for $i \neq j$.

This may appear to be a somewhat arbitrary requirement; however, it can be conveniently specified in a covariant way, that is, in a way which shows that it is indeed geometrically natural. This specification involves a certain differential tensor concomitant of Φ which we denote by H_Φ. This object, which is a tensor field along π^0_1 of type $(1, 2)$, is defined as follows:

$$\begin{aligned} H_\Phi(X, Y) = {} & \mathrm{D}^V\Phi(\Phi(X), \Phi(Y)) - \Phi(\mathrm{D}^V\Phi(X, \Phi(Y))) - \Phi(\mathrm{D}^V\Phi(\Phi(X), Y)) \\ & + \Phi^2(\mathrm{D}^V\Phi(X, Y)). \end{aligned}$$

In Case IIa we can express H_Φ in terms of the eigenvector basis $\{X_i\}$, as follows:

$$\begin{aligned} H_\Phi(X_i, X_j) &= \lambda_i\lambda_j\mathrm{D}^V\Phi(X_i, X_j) - \lambda_j\Phi(\mathrm{D}^V\Phi(X_i, X_j)) - \lambda_i\Phi(\mathrm{D}^V\Phi(X_i, X_j)) \\ &\quad + \Phi^2(\mathrm{D}^V\Phi(X_i, X_j)) \\ &= (\lambda_i I - \Phi)(\lambda_j I - \Phi)\mathrm{D}^V\Phi(X_i, X_j). \end{aligned}$$

But

$$\mathrm{D}^V\Phi(X_i, X_j) = \mathrm{D}^V_{X_i}(\Phi(X_j)) - \Phi\left(\mathrm{D}^V_{X_i}X_j\right) = X_i{}^V(\lambda_j)X_j + (\lambda_j I - \Phi)\mathrm{D}^V_{X_i}X_j,$$

whence

$$H_\Phi(X_i, X_j) = (\lambda_i I - \Phi)(\lambda_j I - \Phi)^2\mathrm{D}^V_{X_i}X_j = (\lambda_i - \lambda_s)(\lambda_j - \lambda_s)^2\tau^s_{ij}X_s$$

(where summation is intended over *all* terms containing an occurrence of the index s). Thus $H_\Phi = 0$ if and only if $\tau^k_{ij} = 0$ unless $k = i$ or $k = j$. That is to say, Case IIa1, which we also call the separable case, is determined by the conditions

- $[\nabla\Phi, \Phi] = 0$
- Φ has n distinct real eigenvalues
- $H_\Phi = 0$.

This gives a covariant definition of Case IIa1. In practice, however, it is usually more convenient to use the formulation in terms of the vanishing of connection coefficients – simply because calculations are usually most conveniently carried out using an eigenvector basis.

In Case IIa1 the closure conditions reduce to

$$X_i{}^V(\rho_j) = (2\tau^j_{ij} - \tau^j_{ji})\rho_j \qquad i \neq j.$$

3 The variationality of Case IIa1

We will show that Case IIa1 is variational in arbitrary dimension. We will continue to use $\{X_i\}$ to denote an eigenvector basis of Φ such that $\nabla X_i = 0$.

According to the general theory of [9], as summarized in the Introduction, we can show that Case IIa1 is variational by first identifying those g which satisfy the two sets of algebraic conditions

$$g(\nabla^r\Phi(X_i), X_j) = g(\nabla^r\Phi(X_j), X_i) \qquad r = 0, 1, 2, \ldots$$

and

$$\sum g(\nabla^r R(X_i, X_j), X_k) = 0 \qquad r = 0, 1, 2, \ldots;$$

and second, showing that for such g there is an orthonomic passive system of differential equations containing the differential Helmholtz conditions

$$\nabla g = 0 \qquad \mathrm{D}^{V} g(X_i, X_j, X_k) = \mathrm{D}^{V} g(X_j, X_i, X_k).$$

The algebraic conditions involving Φ are clearly satisfied in Case IIa in general by any g which is diagonal with respect to one, and thus any, eigenvector basis of Φ. On the other hand, in Case IIa1 $R(X_i, X_j)$ has components in the directions of X_i and X_j only. This follows from the formula

$$\mathrm{D}^{V}\Phi(X_i, X_j) = \tau^{i}_{ij}(\lambda_j - \lambda_i)X_i + {X_i}^{V}(\lambda_j)X_j,$$

which shows in particular that $\mathrm{D}^{V}\Phi(X_i, X_j)$ is a linear combination of X_i and X_j; given that $\mathrm{D}^{V}\Phi(X_i, X_j) - \mathrm{D}^{V}\Phi(X_j, X_i) = 3R(X_i, X_j)$ it is apparent that $R(X_i, X_j)$ has the same property. The fact that g is diagonal with respect to the basis $\{X_i\}$ then ensures that the algebraic conditions which involve the tensor R are satisfied in Case IIa1. Thus there are no algebraic conditions to be satisfied by g in addition to the requirement that it be diagonal with respect to the basis $\{X_i\}$.

We know from our general results that to the differential Helmholtz conditions $\nabla g = 0$ and $\mathrm{D}^{V} g(X_i, X_j, X_k) = \mathrm{D}^{V} g(X_j, X_i, X_k)$ there must be added the further first-order conditions $\mathrm{D}^{H} g(X_i, X_j, X_k) = \mathrm{D}^{H} g(X_j, X_i, X_k)$. We know further that if there are any integrability or passivity conditions for this augmented set of equations they are to be found amongst the $\mathcal{A}$ conditions $\mathcal{A}(X_i, X_j)g(X_k, X_l) = 0$. These conditions require further attention if they give rise to new first-order or algebraic conditions. However, in Case IIa1 the $\mathcal{A}$ conditions are automatically satisfied in virtue of the first-order equations and their prolongations, as we will show below. It follows that the first-order equations form a passive set.

As we have seen, in Case IIa1 the equations $\mathrm{D}^{V} g(X_i, X_j, X_k) = \mathrm{D}^{V} g(X_j, X_i, X_k)$ may be written ${X_j}^{V}(\rho_i) = \mu_{ji}\rho_i$, $i \neq j$, where for convenience we have set $2\tau^{i}_{ji} - \tau^{i}_{ij} = \mu_{ji}$. We will now express the additional equations $\mathrm{D}^{H} g(X_i, X_j, X_k) = \mathrm{D}^{H} g(X_j, X_i, X_k)$ in a similar form. First, we show that the connection coefficients for D^{H} are related in a simple way to those for D^{V} when $\nabla X_i = 0$. Since $[\nabla, \mathrm{D}^{V}] = -\mathrm{D}^{H}$ and $\nabla X_i = 0$ we have

$$\begin{aligned}\nabla \mathrm{D}^{V}_{X_i} X_j &= \Gamma(\tau^{s}_{ij})X_s \\ &= \mathrm{D}^{V}_{X_i}\nabla X_j - \mathrm{D}^{H}_{X_i} X_j = -\mathrm{D}^{H}_{X_i} X_j,\end{aligned}$$

whence, writing $\dot{\tau}^{s}_{ij}$ for $\Gamma(\tau^{s}_{ij})$ for convenience,

$$\mathrm{D}^{H}_{X_i} X_j = -\dot{\tau}^{s}_{ij} X_s.$$

In Case IIa1 we therefore have

$$\begin{aligned}
&\mathrm{D}^{H}_{X_i} X_j = -\dot\tau^i_{ij} X_i - \dot\tau^j_{ij} X_j \quad i \neq j\\
&\mathrm{D}^{H}_{X_i} X_i = -\dot\tau^i_{ii} X_i.
\end{aligned}$$

It is now easy to see that the additional conditions are

$$X_j{}^H(\rho_i) = -(2\dot\tau^i_{ji} - \dot\tau^i_{ij})\rho_i = -\dot\mu_{ji}\rho_i \qquad i \neq j.$$

We have therefore to show that the augmented set of first-order equations

$$\Gamma(\rho_i) = 0, \qquad X_j{}^H(\rho_i) = -\dot\mu_{ji}\rho_i, \qquad X_j{}^V(\rho_i) = \mu_{ji}\rho_i \qquad i \neq j,$$

where $\mu_{ji} = 2\tau^i_{ji} - \tau^i_{ij}$, is passive.

We will do so by examining the $\mathcal{A}$ conditions. However, it is not necessary to consider all of them, as we now explain. In computing the passivity conditions of the first-order equations we may freely make use of the prolongations of the equations to substitute for the second derivatives which occur. Thus in evaluating the $\mathcal{A}$ conditions we need to work only modulo prolongations. If we can show that any particular expressions $\mathcal{A}(X_i, X_j)g(X_k, X_l)$ vanish as a consequence of the first-order equations and their prolongations, so must all those which are equivalent to them modulo prolongations of those equations. Now a lemma proved in the previous paper [9] states that $\mathcal{A}(X_i, X_j)g(X_k, X_l) = \mathcal{A}(X_i, X_k)g(X_j, X_l)$ modulo prolongations. We take advantage of this result in the following way. The $\mathcal{A}$ operator is skew-symmetric, so that we may assume $i \neq j$. For $k = l$ at least one of the indices i or j (since they are different) will not be k. Using the lemma, therefore, we may shuffle the indices so that those occurring in the final two arguments are different. We will show that $\mathcal{A}(X_i, X_j)g(X_k, X_l)$ vanishes in virtue of the first-order equations whenever $k \neq l$; it then follows from the preceding remarks that $\mathcal{A}(X_i, X_j)g(X_k, X_l)$ vanishes in virtue of the first-order equations and their prolongations without any such restriction.

The result is obtained by a direct computation. The equation $\mathcal{A}(X_i, X_j)g(X_k, X_l) = 0$ when written out explicitly takes the form

$$\begin{aligned}
&\mathrm{D}^{V}\mathrm{D}^{H} g(X_i, X_j, X_k, X_l) + g(\theta(X_k, X_l)X_i, X_j)\\
&\qquad = \mathrm{D}^{V}\mathrm{D}^{H} g(X_j, X_i, X_k, X_l) + g(X_i, \theta(X_k, X_l)X_j).
\end{aligned}$$

We will express the various terms which occur as expressions in the connection coefficients τ^i_{jk} and $\dot\tau^i_{jk}$; the result will follow essentially from the fact that both of these coefficients vanish except when $i=j$ or $i=k$. Straightforward computations give, for $k \neq l$,

$$
\begin{aligned}
&\mathrm{D}^{V}\mathrm{D}^{H} g(X_i, X_j, X_k, X_l)\\
&\quad = \dot\tau^k_{jl} {X_i}^V(\rho_k) - \tau^k_{il} {X_j}^H(\rho_k) + ({X_i}^V(\dot\tau^k_{jl}) - \tau^s_{ij}\dot\tau^k_{sl} - \tau^s_{il}\dot\tau^k_{js})\rho_k\\
&\qquad + \dot\tau^l_{jk} {X_i}^V(\rho_l) - \tau^l_{ik} {X_j}^H(\rho_l) + ({X_i}^V(\dot\tau^l_{jk}) - \tau^s_{ij}\dot\tau^l_{sk} - \tau^s_{ik}\dot\tau^l_{js})\rho_l\\
&\qquad - (\tau^s_{ik}\dot\tau^s_{jl} + \tau^s_{il}\dot\tau^s_{jk})\rho_s
\end{aligned}
$$

and

$$
\begin{aligned}
&g(\theta(X_k, X_l)X_i, X_j)\\
&\quad = (-{X_k}^V(\dot\tau^j_{li}) - {X_l}^H(\tau^j_{ki}) + \tau^s_{ki}\dot\tau^j_{ls} + \tau^s_{kl}\dot\tau^j_{si} - \tau^j_{ks}\dot\tau^s_{li} - \tau^j_{si}\dot\tau^s_{lk})\rho_j.
\end{aligned}
$$

So far we have used only the fact that g is diagonal with respect to the eigenvector basis. It is immediately clear that when in addition $H_\Phi = 0$ both of these expressions vanish when i, j, k and l are all different, simply as a consequence of the properties then satisfied by the connection coefficients: note that in a term involving summation such as $\tau^s_{ij}\dot\tau^k_{sl}$ there is no choice of the summation index s for which both of the connection coefficients are non-zero. Thus $\mathcal{A}(X_i, X_j)g(X_k, X_l)$ vanishes identically when i, j, k and l are all different.

Bearing in mind that $\mathcal{A}(X_i, X_j)g(X_k, X_l)$ is skew-symmetric in i and j and symmetric in k and l, and that $i \neq j$ and by assumption $k \neq l$, it is enough to consider only the cases $i=k$, $j \neq l$ and $i=k$, $j=l$ to complete the argument.

¿From the above formulae, also using now the first-order equations to substitute for terms of the form ${X_j}^V(\rho_k)$ and ${X_j}^H(\rho_k)$, we find that when $j \neq l$

$$
\begin{aligned}
\mathrm{D}^{V}\mathrm{D}^{H} g(X_k, X_j, X_k, X_l) &= -(\tau^k_{kj}\dot\tau^k_{kl} + \tau^k_{kl}\dot\tau^k_{kj})\rho_k\\
g(\theta(X_k, X_l)X_k, X_j) &= 0\\
\mathrm{D}^{V}\mathrm{D}^{H} g(X_j, X_k, X_k, X_l) &= ({X_j}^V(\dot\tau^k_{kl}) - \tau^k_{kj}\dot\tau^k_{kl} - \tau^j_{jl}\dot\tau^k_{kj} - \tau^l_{jl}\dot\tau^k_{kl})\rho_k\\
g(X_k, \theta(X_k, X_l)X_j) &= g(X_k, \theta(X_j, X_k)X_l)\\
&= (-{X_j}^V(\dot\tau^k_{kl}) + \tau^j_{jl}\dot\tau^k_{kj} + \tau^l_{jl}\dot\tau^k_{kl} - \tau^k_{kl}\dot\tau^k_{kj})\rho_k,
\end{aligned}
$$

where in the final expression we have taken advantage of the fact that θ is symmetric in all of its arguments. By inspection of these formulae it follows that $\mathcal{A}(X_k, X_j)g(X_k, X_l) = 0$ when $j \neq l$, on account of the first-order equations.

For the remaining case we find that

$$\begin{aligned}
\mathrm{D}^{\mathrm{V}}\mathrm{D}^{H}g(X_k,X_l,X_k,X_l) &= -2\tau^k_{kl}\dot\tau^k_{kl}\rho_k + ({X_k}^{V}(\dot\tau^l_{lk}) - \tau^l_{lk}\dot\tau^l_{lk} - \tau^k_{kk}\dot\tau^l_{lk})\rho_l\\
g(\theta(X_k,X_l)X_k,X_l) &= (-{X_k}^{V}(\dot\tau^l_{lk}) + \tau^k_{kk}\dot\tau^l_{lk} - \tau^l_{lk}\dot\tau^l_{lk})\rho_l,
\end{aligned}$$

whence

$$\mathrm{D}^{\mathrm{V}}\mathrm{D}^{H}g(X_k,X_l,X_k,X_l) + g(\theta(X_k,X_l)X_k,X_l) = -2\tau^k_{kl}\dot\tau^k_{kl}\rho_k - 2\tau^l_{lk}\dot\tau^l_{lk}\rho_l.$$

Since this expression is invariant under the interchange of k and l we have

$$\begin{aligned}
&\mathrm{D}^{\mathrm{V}}\mathrm{D}^{H}g(X_k,X_l,X_k,X_l) + g(\theta(X_k,X_l)X_k,X_l)\\
&\quad = \mathrm{D}^{\mathrm{V}}\mathrm{D}^{H}g(X_l,X_k,X_l,X_k) + g(\theta(X_l,X_k)X_l,X_k)\\
&\quad = \mathrm{D}^{\mathrm{V}}\mathrm{D}^{H}g(X_l,X_k,X_k,X_l) + g(X_k,\theta(X_k,X_l)X_l),
\end{aligned}$$

where the symmetry of θ has been used again to obtain the last expression. But this says that $\mathcal{A}(X_k,X_l)g(X_k,X_l) = 0$.

The first-order equations are therefore passive. It remains to show that they can be put in orthonomic form.

For this purpose, given that the equations are separated, it is enough to consider those equations involving just one of the dependent variables, say ρ_n. These involve only the eigenvectors X_a for $a = 1,2,\ldots,n-1$. The full set of eigenvectors $\{X_i\}$, $i = 1,2,\ldots,n$, is of course linearly independent, so that if we write $X_i = \sum_j X_i^j\partial/\partial x^j$ the matrix of coefficients (X_i^j) is non-singular. The $n\times(n-1)$ sub-matrix (X_a^j) has rank $n-1$, and so by relabelling the coordinates if necessary we can find an $(n-1)\times(n-1)$ matrix (Y_a^b) such that $\sum_b Y_a^b X_b = \partial/\partial x^a$. The first-order equations for ρ_n may be replaced by the equivalent set

$$\Gamma(\rho_n) = 0, \qquad \left(\frac{\partial}{\partial x^a}\right)^H(\rho_n) = -\sum_b Y_a^b\dot\mu_{bn}\rho_n, \qquad \left(\frac{\partial}{\partial x^a}\right)^V(\rho_n) = \sum_b Y_a^b\mu_{bn}\rho_n.$$

We may write these in the form

$$\frac{\partial\rho_n}{\partial t} = \text{expression in } \frac{\partial\rho_n}{\partial x^i}, \frac{\partial\rho_n}{\partial v^i}$$

$$\frac{\partial \rho_n}{\partial x^a} = \text{expression in } \frac{\partial \rho_n}{\partial v^i},\ \rho_n$$

$$\frac{\partial \rho_n}{\partial v^a} = \text{expression in } \rho_n.$$

When written in this form these equations are orthonomic.

It follows that Case IIa1 is variational. The freedom in the solution of the multiplier problem is clearly n functions of 2 variables.

4 Varieties of separability

Let $\{X_i\}$ be a basis of eigenvectors of Φ which are ∇-invariant, as before. Let $\mathcal{N}$ be any subset of $\{0, 1, 2, \ldots, n\}$, and consider on $J^1\pi$ the distribution (vector field system) $\mathcal{D}_{\mathcal{N}}$ spanned by the following vector fields: ${X_i}^H$ and ${X_i}^V$ for all $i \in \mathcal{N}$, $i > 0$, and Γ if $0 \in \mathcal{N}$. Then $\mathcal{D}_{\mathcal{N}}$ is involutive for every $\mathcal{N}$ if $H_\Phi = 0$. This result is a consequence of the following formulae for the brackets of the vector fields ${X_j}^V$, ${X_k}^H$ and Γ when $H_\Phi = 0$:

$$\begin{aligned}
[{X_j}^V, {X_k}^V] &= (\mathrm{D}^V_{X_j} X_k)^V - (\mathrm{D}^V_{X_k} X_j)^V = (\tau^j_{jk} - \tau^j_{kj}){X_j}^V + (\tau^k_{jk} - \tau^k_{kj}){X_k}^V \\
[{X_j}^H, {X_k}^V] &= (\mathrm{D}^H_{X_j} X_k)^V - (\mathrm{D}^V_{X_k} X_j)^H = -\dot\tau^j_{jk}{X_j}^V - \dot\tau^k_{jk}{X_k}^V - \tau^k_{jk}{X_j}^H - \tau^k_{kj}{X_k}^H \\
[{X_j}^H, {X_j}^V] &= -\dot\tau^j_{jj}{X_j}^V - \tau^j_{jj}{X_j}^H \\
[{X_j}^H, {X_k}^H] &= (\mathrm{D}^H_{X_j} X_k)^H - (\mathrm{D}^H_{X_k} X_j)^H + R(X_j, X_k)^V \\
&= -(\dot\tau^j_{jk} - \dot\tau^j_{kj}){X_j}^H - (\dot\tau^k_{jk} - \dot\tau^k_{kj}){X_k}^H + {R_{jk}}^j {X_j}^V + {R_{jk}}^k {X_k}^V \\
[\Gamma, {X_j}^V] &= (\nabla X_j)^V - {X_j}^H = -{X_j}^H \\
[\Gamma, {X_j}^H] &= (\nabla X_j)^H + \Phi(X_j)^V = \lambda_j {X_j}^V,
\end{aligned}$$

where $R(X_j, X_k) = {R_{jk}}^j X_j + {R_{jk}}^k X_k$ (recall that $R(X_j, X_k)$ is a linear combination of X_j and X_k when $H_\Phi = 0$).

In fact this gives us another way of specifying Case IIa1: it is the subcase of Case IIa for which $\mathcal{D}_{\{i\}}$ and $\mathcal{D}_{\{i,j\}}$ are involutive for all $i, j \in \{1, 2, \ldots, n\}$. In Case IIa in general we have

$$[{X_i}^H, {X_j}^V] = (\mathrm{D}^H_{X_i} X_j)^V - (\mathrm{D}^V_{X_j} X_i)^H = -\dot\tau^s_{ij}{X_s}^V - \tau^s_{ji}{X_s}^H$$

(where the possibility that $i = j$ is allowed). This vector field will belong to $\mathcal{D}_{\mathcal{N}}$, $\mathcal{N} = \{i, j\}$ or $\mathcal{N} = \{i\}$, only if $\tau^s_{ji} = 0$ when s is not i or j, which is equivalent to $H_\Phi = 0$. To summarize: in Case IIa the following conditions are equivalent

- $H_\Phi = 0$
- $\mathcal{D}_{\{i\}}$ and $\mathcal{D}_{\{i,j\}}$ are involutive for all $i, j \in \{1, 2, \ldots, n\}$
- $\mathcal{D}_{\mathcal{N}}$ is involutive for all $\mathcal{N} \subset \{0, 1, 2, \ldots, n\}$.

These properties of the distributions $\mathcal{D}_{\mathcal{N}}$ will allow us to show, below, that the description of Case IIa1 as the separable case has much wider relevance than just to the differential equations arising from the Helmholtz conditions.

One way of stating the Frobenius integrability theorem is as follows. Let $\mathcal{D}$ be a distribution on a differentiable manifold M. We denote by $\mathcal{D}^\perp$ the annihilator of $\mathcal{D}$, that is, the $C^\infty(M)$-module of 1-forms which give zero when paired with any vector field in $\mathcal{D}$. Then if, and only if, $\mathcal{D}$ is involutive for each point $x \in M$ there is a neighbourhood $\mathcal{O}$ of x and functionally independent functions $u^1, u^2, \ldots, u^k$ defined on $\mathcal{O}$, where k is the co-dimension of $\mathcal{D}$, such that $\mathcal{D}^\perp\big|_{\mathcal{O}}$ is the span of $\{du^i \mid i = 1, 2, \ldots, k\}$.

We now apply this version of Frobenius's theorem to Case IIa1. Of the many involutive distributions at our disposal we choose to apply it to the distributions $\mathcal{D}_{\mathcal{N}}$ for which $\mathcal{N}$ is the subset of $\{0, 1, 2, \ldots, n\}$ obtained by omitting just one of the non-zero integers $1, 2, \ldots, n$: when the omitted integer is i we denote the corresponding distribution by $\mathcal{D}_i$. Each $\mathcal{D}_i$ has co-dimension 2. Let u^i_α, $\alpha = 1, 2$, i fixed, be independent (local) functions such that ${\mathcal{D}_i}^\perp$ is (locally) the span of $\{du^i_\alpha \mid \alpha = 1, 2\}$. Then

$$\Gamma(u^i_\alpha) = 0, \qquad {X_j}^H(u^i_\alpha) = {X_j}^V(u^i_\alpha) = 0 \qquad \text{for all } j \neq i.$$

The independence of the two functions u^i_α, for each i, amounts to the claim that the matrix

$$\begin{bmatrix} {X_i}^H(u^i_1) & {X_i}^H(u^i_2) \\ {X_i}^V(u^i_1) & {X_i}^V(u^i_2) \end{bmatrix}$$

is non-singular. Denote its determinant by Δ_i.

Consider now the $2n+1$ locally defined functions $\{t, u^i_\alpha \mid i = 1, 2, \ldots, n; \alpha = 1, 2\}$, where t is the "time". They are easily shown to be independent, for example by considering the $(2n+1)$-form

$$\chi = dt \wedge du^1_1 \wedge du^1_2 \wedge du^1_1 \wedge du^1_2 \wedge \cdots \wedge du^n_1 \wedge du^n_2 :$$

the result of evaluating χ on $\{\Gamma, X_1{}^H, X_1{}^V, X_2{}^H, X_2{}^V, \ldots, X_n{}^H, X_n{}^V\}$ – a basis of vector fields on $J^1\pi$ – is easily seen to be

$$\chi(\Gamma, X_1{}^H, X_1{}^V, X_2{}^H, X_2{}^V, \ldots, X_n{}^H, X_n{}^V) = \prod_{i=1}^{n} \Delta_i \neq 0,$$

and so $\chi \neq 0$. Here we have made use of the facts that $\Gamma(t) = 1$ while $X_i{}^H(t) = X_i{}^V(t) = 0$ for $i = 1, 2, \ldots, n$.

One immediate consequence is that we can write down a Cartan 2-form for Γ in terms of the u^i_α. Let

$$\Omega = du^1_1 \wedge du^1_2 + du^2_1 \wedge du^2_2 + \cdots + du^n_1 \wedge du^n_2.$$

Then $d\Omega = 0$, trivially. Since $\Gamma(u^i_\alpha) = 0$, $\Gamma \lrcorner \Omega = 0$, and since $X_j{}^V(u^i_\alpha) = 0$ for $j \neq i$, $\Omega(X_i{}^V, X_j{}^V) = 0$. Finally, $dt \wedge (\wedge^n \Omega) \neq 0$ as we have shown above, and so a fortiori $\wedge^n \Omega \neq 0$. These properties characterize the Cartan 2-form(s) of a second-order differential equation field Γ, as given in [4]. In effect, this gives us a second proof of the variationality of Case IIa1. More generally, for each $i = 1, 2, \ldots, n$ let f_i be a nowhere-vanishing function of two variables; then $\sum_{i=1}^n f_i(u^i_\alpha)\, du^i_1 \wedge du^i_2$ is a Cartan 2-form for Γ. It is interesting to note that the freedom of choice of Cartan 2-forms constructed in this way is again n functions of two variables.

We can regard (t, u^i_α) as a local coordinate system on $J^1\pi$. Since for each i, du^i_α annihilates all vector fields in $\mathcal{D}_\mathcal{N}$ for each $\mathcal{N}$ not containing i, u^i_α is constant on all integral submanifolds of each $\mathcal{D}_\mathcal{N}$ for which $\mathcal{N}$ does not contain i. Thus with respect to these coordinates the coordinate 2-planes over which the u^i_α (fixed i) vary are the integral submanifolds of $\mathcal{D}_{\{i\}}$, and the coordinate 3-planes over which t and the u^i_α vary are the integral submanifolds of $\mathcal{D}_{\{0,i\}}$. About any point O in $J^1\pi$ we can find a neighbourhood in which $J^1\pi$ is a fibre product over $\mathbb{R}$ of n 3-dimensional manifolds each fibred over $\mathbb{R}$, each such manifold being (the appropriate part of) the integral submanifold of $\mathcal{D}_{\{0,i\}}$ through O, and its standard fibre being (the appropriate part of) the integral submanifold of $\mathcal{D}_{\{i\}}$ through O. The coordinate system described above is adapted to this local product structure. With respect to these coordinates the differential equations for the integral curves of Γ are just $\dot{u}^i_\alpha = 0$. Note that, by leaving the given vector field Γ out of the discussion of simultaneously integrable distributions above, one will arrive more generally, at a similar local product structure and, with respect to correspondingly adapted coordinates, the equations for the integral curves of Γ will decouple into n separate systems of two first-order equations. To summarize: if for a given second-order system Γ, Φ is diagonalizable with distinct (real) eigenvalues, and further satisfies $H_\Phi = 0$ and $[\nabla\Phi, \Phi] = 0$, then Γ decouples into n separate systems of two first-order equations.

In general, these n separate systems need not be second-order equations. Indeed, a coordinate transformation which will transform the given system into its decoupled form will, in general, not be a point transformation and therefore will not preserve the second-order character of the system.

Since we are in a situation where Φ has only one-dimensional eigenspaces, a characterization of complete separability in the sense of second-order equations is easily expressed, on the basis of results in [10] and [2], as follows. Suppose that Φ is diagonalizable with distinct eigenvalues. Then, for Γ to be completely separable into n decoupled second-order equations, it is necessary and sufficient that $C^V_\Phi = 0$ and $[\nabla\Phi, \Phi] = 0$. Here C^V_Φ is the type (1,2) tensor field along π^0_1 defined by

$$C^V_\Phi(X, Y) = [\mathrm{D}^V_X\Phi, \Phi](Y) = \mathrm{D}^V\Phi(X, \Phi(Y)) - \Phi(\mathrm{D}^V\Phi(X, Y)).$$

The above result is explicitly stated in [10] for autonomous equations and easily follows from Theorem 3.11 in [2] in the case of time-dependent systems.

For clarity we give a sketch of the proof. It is trivial to verify that any system of n decoupled second-order equations in one dependent variable, with different Φ^i_i, satisfies the given conditions. To see that the conditions are also sufficient, observe first that $C^V_\Phi = 0$ means that the eigendistributions of Φ are D^V-invariant and are therefore spanned by basic vector fields, more precisely vector fields on E which are vertical with respect to the fibration over $\mathbb{R}$. Since the eigendistributions are also ∇-invariant in view of $[\nabla\Phi, \Phi] = 0$, and $[\nabla, \mathrm{D}^V] = -\mathrm{D}^H$, one easily deduces that they are also D^H-invariant. This can be shown to imply that all the basic eigendistributions are simultaneously integrable. Hence, there exist coordinates on $\pi : E \to \mathbb{R}$, with respect to which each of the X_i is a multiple of $\partial/\partial x^i$. The ∇-invariance then implies that $\Gamma^i_j = 0$ for $i \neq j$ and the coordinate expression of Φ^i_j subsequently reveals that also $\partial f^i/\partial x^j = 0$ for $i \neq j$. The complete separation follows.

To see how this fits as a special case within the situation of Case IIa1, it suffices to observe that the tensor field H_Φ can be expressed as follows

$$H_\Phi(X, Y) = C^V_\Phi(\Phi(X), Y) - \Phi\left(C^V_\Phi(X, Y)\right),$$

so that $C^V_\Phi = 0$ implies $H_\Phi = 0$, but not the other way around.

5 Illustrative example

It is very easy to construct examples of systems of second-order differential equations satisfying the requirements of Case IIa1. Indeed, with reference to the discussion in the previous section, examples could be manufactured in the following way: take any system of n separate second-order equations with the property that each equation contributes a distinct component of the overall Φ; apply an arbitrary, possibly time-dependent transformation to the base space coordinates, which will make the transformed equations look hopelessly coupled. But we know that the conditions $C^V_\Phi = 0$ and $[\nabla\Phi, \Phi] = 0$ will be satisfied by construction. Thus, we are bound to have an example of a system of differential equations belonging to Case IIa1.

The true challenge, therefore, is to find an example which does not decouple in the sense of references [8] and [2]. The following two-dimensional geodesic flow was considered in [1] (for convenience, we label the coordinates here with a subscript):

$$\begin{aligned} \ddot{x}_1 &= -2\dot{x}_1\dot{x}_2 \\ \ddot{x}_2 &= (\dot{x}_2)^2. \end{aligned}$$

The matrix representation of Φ, with respect to the coordinate fields, is given by

$$\begin{pmatrix} -2(v_2)^2 & 2v_1v_2 \\ 0 & 0 \end{pmatrix}$$

and is seen to have distinct eigenvalues. One easily verifies that $\nabla\Phi$ and Φ commute.

Eigenvectors of Φ are given by

$$X_1 = \frac{\partial}{\partial x_1}, \qquad X_2 = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2}.$$

It is clear that the second eigenspace cannot be spanned by a basic vector field, which is sufficient to conclude that $C^V_\Phi \neq 0$. Yet H_Φ will be zero, as can be seen for example from the relations

$$\mathrm{D}^V_{X_i} X_1 = 0, \qquad \mathrm{D}^V_{X_1} X_2 = X_1, \qquad \mathrm{D}^V_{X_2} X_2 = X_2,$$

which show that only τ^1_{12} and τ^2_{22} are non-zero. As a result, the equations are not separable by a point transformation, but should have the weaker separability property of Case IIa1, as we will show explicitly now.

The distributions $\mathcal{D}_{\{1\}}$ and $\mathcal{D}_{\{2\}}$ are spanned respectively by

$$X_1{}^H = \frac{\partial}{\partial x_1} - v_2\frac{\partial}{\partial v_1}, \qquad X_1{}^V = \frac{\partial}{\partial v_1},$$

and

$$X_2{}^H = \Gamma - \frac{\partial}{\partial t}, \qquad X_2{}^V = v_1\frac{\partial}{\partial v_1} + v_2\frac{\partial}{\partial v_2}.$$

The given coordinates are already adapted to the distribution $\mathcal{D}_{\{1\}}$ which has x_2 and v_2 as invariants. Integrals for the distribution $\mathcal{D}_{\{2\}}$ are found to be $3x_1 + v_1(v_2)^{-1}$ and $e^{3x_2}v_1(v_2)^{-1}$. If we introduce the non-point transformation

$$\bar{x}_1 = x_1 + \frac{v_1}{3v_2}, \qquad \bar{x}_2 = x_2,$$

$$\bar{v}_1 = \frac{e^{3x_2}v_1}{v_2}, \qquad \bar{v}_2 = v_2$$

then $\mathcal{D}_{\{1\}}$ is spanned by $\partial/\partial\bar{x}_1$, $\partial/\partial\bar{v}_1$, whereas the two other coordinate fields now span $\mathcal{D}_{\{2\}}$. Indeed, we have

$$X_2{}^H = \bar{v}_2\frac{\partial}{\partial\bar{x}_2} + (\bar{v}_2)^2\frac{\partial}{\partial\bar{v}_2}, \qquad X_2{}^V = \bar{v}_2\frac{\partial}{\partial\bar{v}_2}.$$

Since Γ is actually $X_2{}^H + \partial/\partial t$, we see that the transformed differential equations are indeed decoupled into two pairs of first-order equations:

$$\dot{\bar{x}}_1 = 0, \qquad \dot{\bar{v}}_1 = 0, \quad \text{and} \quad \dot{\bar{x}}_2 = \bar{v}_2, \qquad \dot{\bar{v}}_2 = (\bar{v}_2)^2.$$

This illustrates that any system of coordinates adapted to both distributions will give rise to this form of decoupling. The analysis of the previous section has shown that one can in principle even construct such coordinates which at the same time straighten out the given vector field Γ. In the present example, this would be achieved by choosing $\exp(-\bar{x}_2)$ as a new coordinate.

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