ON ALMOST-POISSON STRUCTURES IN NONHOLONOMIC MECHANICS. II. THE TIME-DEPENDENT FRAMEWORK

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Abstract

The structure of the equations of motion of a time-dependent mechanical system, subject to time-dependent nonholonomic constraints, is investigated in the Lagrangian as well as in the Hamiltonian setting. The treatment applies to systems with general nonlinear constraints, and the ambient space in which the constraint submanifold is embedded, is equipped with a cosymplectic structure. In analogy with the autonomous case, it is shown that one can define an almost-Poisson structure on the constraint submanifold, which plays a prominent role in the description of nonholonomic dynamics. Moreover, it is seen that the corresponding almost-Poisson bracket can also be interpreted as a Dirac-type bracket. Systems with a Lagrangian of mechanical type and affine nonholonomic constraints, are treated as a special case and two examples are discussed.

1 Introduction

In this paper, which should primarily be seen as a sequel to [5], we analyse, from a differential geometric point of view, the structure of the equations of motion of a time-dependent mechanical system subjected to general (i.e. possibly nonlinear) time-dependent nonholonomic constraints. Both the Lagrangian and the Hamiltonian formulation will be considered. The main purpose thereby is to show that, in analogy with the autonomous case (see [5, 15, 22, 26]), one can equip the constraint submanifold with an almost-Poisson structure which, in nonholonomic mechanics, takes over the role of the Poisson structure appearing in holonomic mechanics.

Given a holonomic, time-independent conservative mechanical system, described by a regular Lagrangian L, the time evolution of any observable F along the flow of the system can always be expressed in Poisson bracket form as $\dot{F} = \{F, E\}$, where E represents the energy. This form of the equations of motion is of course best known in the Hamiltonian setting, where $\{\ ,\ \}$ is the Poisson bracket induced by the canonical symplectic structure on the phase space of the system, and $E = H$ is the Hamiltonian. In canonical coordinates (q^i, p_i) we then simply recover the standard Hamilton equations of motion. However, also the Euler-Lagrange equations for L can be written in Poisson bracket form, with the (non-canonical) Poisson bracket on the velocity phase space being determined by the Poincaré-Cartan 2-form corresponding to L.

The Hamiltonian formulation of holonomic dynamics which, among others, has played a very important role in the development of the quantum mechanics of holonomic systems, has been further extended by Dirac to systems described by a singular Lagrangian (see e.g. [7]). Using the so-called second-class constraints, generated by the constraint algorithm for such systems, Dirac has constructed a noncanonical Poisson bracket on the final constraint submanifold in phase space, such that the constrained dynamics can be represented by a Hamiltonian vector field with respect to that bracket. This bracket is usually called the "Dirac bracket".

With a view on the quantisation of nonholonomic systems, the problem of whether the Hamiltonian formulation of dynamics can be further extended to systems with nonholonomic constraints, was addressed already in the early fifties in two papers by R.J. Eden (see [10, 11]). In [10], Eden succeeded in putting the phase space equations of motion of a nonholonomic system into Poisson bracket form by introducing some noncanonical variables, which he called "constrained variables". In [11], he then took this formulation of nonholonomic dynamics as a starting point to study the Heisenberg formalism for the corresponding quantum system.

In 1994, van der Schaft and Maschke presented a Hamiltonian formulation of autonomous mechanical systems with linear nonholonomic constraints in terms of a kind of Poisson bracket, defined on the constraint submanifold itself (cf. [26]). This bracket verifies all properties of a Poisson bracket, except for the Jacobi identity. In fact, it is proved in [26] that the latter holds iff the constraints are holonomic. In constructing their bracket, henceforth called an almost-Poisson bracket, van der Schaft and Maschke considered a noncanonical transformation to new momentum variables in phase space, similar to the "constrained momenta" introduced by Eden. Expressing the canonical Poisson structure in terms of these new variables and restricting everything to points of the constraint submanifold, the almost-Poisson structure is then obtained by 'truncating' the corresponding Poisson matrix in an appropriate way. This almost-Poisson structure has been further studied and used in the treatment of nonholonomic systems by several authors (see [2, 5, 15, 22]).

On the other hand, inspired by a geometrical construction of the Dirac bracket for singular Lagrangian systems, which is based on a symplectic decomposition of the tangent bundle of the ambient phase space along the (final) constraint submanifold, it was demonstrated in Ibort *et al* [13] that a similar construction can also be applied to nonholonomic systems. This idea was elaborated in more detail in [14], where the Dirac bracket is found to play a unifying role in the treatment of a large class of constrained dynamical systems: holonomic singular Lagrangian systems, and regular as well as singular Lagrangian systems with nonholonomic constraints. In the nonholonomic case, however, the Dirac bracket is no longer a Poisson bracket, but an almost-Poisson bracket. In [5] it was shown that, in spite of the difference in origin, along the constraint submanifold of a nonholonomic system the almost-Poisson bracket of van der Schaft and Maschke and the Dirac bracket actually coincide. In the same paper, the bracket formulation of nonholonomic dynamics was also extended to systems with general nonlinear constraints. In the latter case it turns out that, apart from the almost-Poisson bracket term, an additional term appears in the equations of motion, which only vanishes if the constraints are homogeneous (i.e. if they can be represented by functions that are homogeneous in the velocities).

Whereas most of the above quoted references are only concerned with autonomous systems, subject to time-independent constraints, the purpose of the present paper is to extend the bracket formulation of nonholonomic dynamics to the case of time-dependent systems with time-dependent constraints. In [10], Eden does consider time-dependent nonholonomic systems, but since he is interested in a relativistically invariant model, he immediately passes to the so-called homogeneous formalism, with time treated on the same footing as the other configuration variables. As such, he is turning the system into an "autonomous" one, with a singular Lagrangian which is homogeneous of degree one in the velocities. Here, however, we wish to stay within the realm of nonrelativistic classical mechanics, where time fulfills a priviliged role. More precisely, if Q represents the configuration space of a time-dependent nonholonomic system, we will use the space $\mathbb{R} \times TQ$ as the ambient space for the Lagrangian treatment, and $\mathbb{R} \times T^*Q$ for the Hamiltonian treatment. One of the main differences with the autonomous case is that the ambient space now carries a cosymplectic structure, rather than a symplectic structure. As a consequence, we will see that the (almost-Poisson) bracket does not suffice to express the equations of motion, but an additional vector field will make its appearance. This is typical for the situation encountered in the cosymplectic treatment of time-dependent systems (see e.g. [4, 16]).

From the point of view of the applications in particular to problems in mechanics and in control theory, the time-dependent case definitely deserves special attention. Systems with an explicit time-dependence of the Lagrangian and subject to time-dependent constraints are frequently encountered in nature. Time-dependent nonholonomic constraints occur, for instance, when considering the motion of a sphere that rolls without slipping on a surface moving in space with nonconstant angular velocity, or the motion of a rigid body subject to the constraint that the velocity of its center of mass should always oriented towards a point moving along a prescribed trajectory in space (i.e. a variant of the problem of the "curve of pursuit" (cf. [23], p. 14)).

The structure of the paper will be as follows. Sections 2, 3 and 4 are devoted to some mathematical preliminaries concerning almost-Poisson structures and a general setting for the construction of Dirac brackets, which can be seen as a special type of almost-Poisson brackets. Since the ambient space in which our treatment of time-dependent nonholonomic systems will take place, admits a cosymplectic structure, Section 5 will deal with the notion of Dirac brackets on general cosymplectic manifolds. In Section 6 we recall some aspects of the Lagrangian formalism for (unconstrained) time-dependent systems. The structure of the equations of motion of a time-dependent nonholonomic Lagrangian system with general (i.e. not necessarily linear or affine) time-dependent constraints, is studied in Section 7. First, an almost-Poisson structure is defined on the constraint submanifold, which is induced by 'projecting' the Poisson structure corresponding to the (unconstrained) Lagrangian according to a particular direct sum decomposition of the tangent bundle along the constraint submanifold. Next, it is shown that the equation describing the time evolution of any observable F along the flow of the constrained system essentially consists of three terms: the almost-Poisson bracket of F and the energy function, the derivative of F with respect to a vector field which identically vanishes in case the constraints are homogeneous with respect to the velocities, and the derivative of F with respect to a vector field obtained by projection of the Reeb vector field associated to the ambient cosymplectic manifold. It is this last term which mainly distinguishes the time-dependent from the time-independent setting (cf. [5]). In Section 8 we perform the Dirac bracket construction for time-dependent nonholonomic systems and show that, as in the autonomous case, on the constraint submanifold this bracket coincides with the almost-Poisson bracket considered before. Section 9 is devoted to the Hamiltonian formulation of time-dependent nonholonomic systems and in Section 10 we treat the special case of time-dependent Lagrangians of mechanical type with affine constraints. In that case we can construct coordinates adapted to the constraints in terms of which an explicit coordinate representation of the equations of motion on the constraint submanifold can be obtained, without interference of Lagrange multipliers. Finally, we illustrate some aspects of the theory on two examples.

Throughout this paper, all manifolds, mappings and tensor fields are assumed to be of class C^∞.

2 Almost-Poisson structures

We will say that an *almost-Poisson structure* on a differentiable manifold M is given by a bivector field Λ (i.e. a skew-symmetric 2-contravariant tensor field). The pair (M,Λ) is then called an almost-Poisson manifold. If an almost-Poisson structure is integrable, in the sense that $[\Lambda,\Lambda]=0$, where $[\ ,\ ]$ denotes the Schouten-Nijenhuis bracket, then (M,Λ) is a Poisson manifold and Λ a Poisson tensor (see [3, 27]). Several concepts and properties related to Poisson structures can be easily extended to the almost-Poisson setting. It should also be pointed out that almost-Poisson structures correspond to what are called (skew-symmetric) Leibniz structures in a recent paper by J. Grabowski *et al.*, in the context of general algebroid structures and differential calculi on vector bundles [12].

Given an almost-Poisson manifold (M,Λ), one can define a bracket on the ring of C^∞-functions on M, i.e. a $\mathbb{R}$-bilinear mapping $\{\ ,\ \} : C^\infty(M)\times C^\infty(M) \longrightarrow C^\infty(M)$ given by

$$\{F,G\} = \Lambda(dF,dG)$$

for arbitrary $F,G\in C^\infty(M)$. This bracket, called an *almost-Poisson bracket*, is skew-symmetric and verifies the Leibniz rule (i.e. it acts as a derivation on each of its arguments, with respect to the ordinary multiplication of functions). In general, however, it will not satisfy the Jacobi identity. The latter property only holds if we are dealing with a Poisson structure, in which case the bracket becomes a Poisson bracket.

The bivector field Λ determines a mapping $\sharp_\Lambda : T^*M \longrightarrow TM$, which is a vector bundle homomorphism over the identity given by

$$\langle \sharp_\Lambda(\alpha),\beta\rangle = \Lambda(x)(\alpha,\beta)$$

for all $x\in M$ and all α,β in T^*_xM. Given a function $F\in C^\infty(M)$, the vector field X_F, defined by

$$X_F = -\sharp_\Lambda \circ dF$$

is called the Hamiltonian vector field on (M, Λ) corresponding to F. Note that $X_F(G) = \{G, F\}$ for all $G \in C^\infty(M)$. The distribution C_Λ on M, defined by

$$x \in M \longrightarrow C_\Lambda(x) = \sharp_\Lambda(T_x^*M) \subseteq T_xM,$$

is called the characteristic distribution of the almost-Poisson structure. As in the case of a Poisson structure, it is not difficult to check that C_Λ is locally generated by the Hamiltonian vector fields. Unless the tensor field Λ has constant rank, C_Λ will be a generalised distribution, i.e. $C_\Lambda(M)$ need not be a genuine vector subbundle of TM.

Consider the smooth section $\hat{\omega}$ of the generalized vector bundle $\bigwedge^2(C_\Lambda^*(M)) \longrightarrow M$, given by

$$\hat{\omega}(x)(\sharp_\Lambda(\alpha), \sharp_\Lambda(\beta)) = \Lambda(x)(\alpha, \beta)$$

for all $x \in M$ and α and β in T_x^*M. It is clear that $\hat{\omega}(x)$ is well-defined and nondegenerate as a 2-form on $C_{\Lambda,x} (= C_\Lambda(x))$. In terms of this 2-form, the almost-Poisson bracket reads

$$\{F, G\} = \hat{\omega}(X_F, X_G).$$

If C_Λ is a completely integrable distribution, then $\hat{\omega}$ induces an almost-symplectic structure (i.e. a nondegenerate, but not necessarily closed differentiable 2-form) on each leaf of the corresponding foliation of M. In the special case where Λ is nondegenerate, $C_\Lambda(M) = TM$ and $\hat{\omega}$ defines an almost-symplectic form on M which is the "inverse" of Λ. If Λ is a Poisson tensor, then we know that C_Λ is completely integrable and the restriction of $\hat{\omega}$ to each leaf of C_Λ defines a symplectic form (see, for instance, [27]).

Remark 2.1 Note in passing that an almost-Poisson structure on a manifold M, with associated almost-Poisson tensor Λ, can also be seen as a special case of an almost-Dirac structure on M, in the sense of Courant [6], namely: graph $\sharp_\Lambda = \{(\sharp_\Lambda(\alpha), \alpha) |\ \alpha \in T^*M\}$ is a subbundle of $TM \oplus T^*M$ which is maximally isotropic with respect to the pairing $< (v_1, \alpha_1), (v_2, \alpha_2) >_+ = 1/2(< v_1, \alpha_2 > + < v_2, \alpha_1 >)$. Consequently, most of the results described in the present paper can also be approached from this point of view.

In the sequel, a regular distribution $\mathcal{U}$ on a manifold M will always refer to a smooth distribution of constant rank, and we will usually identify such a distribution with the vector subbundle $\mathcal{U}(M)$ of TM it determines.

Let (M, Λ) be an almost-Poisson manifold and suppose there are two complementary regular distributions $\mathcal{U}$ and $\mathcal{W}$ on M such that we have a direct sum (or Whitney sum) decomposition $TM = \mathcal{U} \oplus \mathcal{W}$. The annihilator of $\mathcal{U}$ (resp. $\mathcal{W}$) in T^*M will be denoted by $\mathcal{U}^o$ (resp. $\mathcal{W}^o$). Let $\mathcal{P}$ denote the projector onto $\mathcal{U}$, i.e. $\mathcal{P} : TM \longrightarrow TM$ is a vector bundle homomorphism over the identity, satisfying $\mathcal{P}^2 = \mathcal{P}$, with $\ker(\mathcal{P}) = \mathcal{W}$, $\mathrm{Im}\,(\mathcal{P}) = \mathcal{U}$. The adjoint projector $\mathcal{P}^*$ is the vector bundle homomorphism $\mathcal{P}^* : T^*M \longrightarrow T^*M$, defined pointwise by the relation

$$< v, \mathcal{P}_x^*(\alpha) >=< \mathcal{P}_x(v), \alpha >,$$

for all $v \in T_xM$ and $\alpha \in T_x^*M$, such that $\ker(\mathcal{P}^*) = \mathcal{U}^o$ and $\mathrm{Im}\,(\mathcal{P}^*) = \mathcal{W}^o$. This corresponds to the direct sum decomposition of the cotangent bundle $T^*M = \mathcal{U}^o \oplus \mathcal{W}^o$. We can then construct a new bivector field $\Lambda_{\mathcal{P}}$ on M according to the prescription

$$\Lambda_{\mathcal{P}}(x)(\alpha, \beta) = \Lambda(x)(\mathcal{P}_x^*(\alpha), \mathcal{P}_x^*(\beta))$$

for all $\alpha, \beta \in T_x^*M$. It is easily seen that $\Lambda_{\mathcal{P}}$ is indeed a well-defined skew-symmetric 2-contravariant tensor field, and from its definition one can deduce that

$$\sharp_{\Lambda_{\mathcal{P}}} = \mathcal{P} \circ \sharp_\Lambda \circ \mathcal{P}^*.$$

This implies, in particular, that $\mathrm{Im}\,(\sharp_{\Lambda_{\mathcal{P}}}) \subset \mathcal{U}$. It is not difficult to verify that in case the distribution $\mathcal{U}$ is completely integrable, $\Lambda_{\mathcal{P}}$ induces an almost-Poisson structure on each leaf of the corresponding foliation.

Note that if Λ is a Poisson tensor field, it will not be true, in general, that $\Lambda_{\mathcal{P}}$ is Poisson. Without any further specification concerning the nature of the direct sum decomposition of TM, there is not much more to be said about the almost-Poisson structure $\Lambda_{\mathcal{P}}$. In the next section we will focuss our attention on a special type of splitting of TM which will lead us to an important class of almost-Poisson tensor fields, namely the so-called Dirac tensor fields.

3 A general construction of Dirac tensor fields

Consider an almost-Poisson manifold (M, Λ) and let $\mathcal{D}$ be a regular distribution on M for which the following property holds:

$$TM = \mathcal{D} \oplus \sharp_\Lambda(\mathcal{D}^o)\ , \tag{1}$$

where, as before, $\mathcal{D}^o$ denotes the annihilator of $\mathcal{D}$ in T^*M. Observe that condition (1) is equivalent to the following two conditions:

(i) the mapping $(\sharp_\Lambda)_{|\mathcal{D}_x^o} : \mathcal{D}_x^o \longrightarrow \sharp_\Lambda(\mathcal{D}_x^o)$ is an isomorphism for all $x \in M$,

(ii) $\mathcal{D}_x \cap \sharp_\Lambda(\mathcal{D}_x^o) = \{0\}$, for all $x \in M$.

The direct sum decomposition (1) gives rise to two complementary projectors

$$\mathcal{P} : TM \longrightarrow \mathcal{D}, \qquad \mathcal{Q} : TM \longrightarrow \sharp_\Lambda(\mathcal{D}^o),$$

with $\mathcal{P} + \mathcal{Q} = \mathrm{id}$. The adjoint projectors $\mathcal{P}^*$ and $\mathcal{Q}^*$ are defined by

$$< v, \mathcal{P}^*_x(\alpha) >=< \mathcal{P}_x(v), \alpha > \quad \text{and} \quad < v, \mathcal{Q}^*_x(\alpha) >=< \mathcal{Q}_x(v), \alpha >$$

for all $x \in M$ and all $v \in T_xM$, $\alpha \in T^*_xM$, and they correspond to the following splitting of T^*M:

$$T^*M = \mathcal{D}^o \oplus (\sharp_\Lambda(\mathcal{D}^o))^o\,. \tag{2}$$

For each $x \in M$, we have that $\operatorname{Im} \mathcal{P}^*_x = (\sharp_\Lambda(\mathcal{D}^o_x))^o$ and $\operatorname{Im} \mathcal{Q}^*_x = \mathcal{D}^o_x$. Moreover, from the definition of both projectors we deduce that

$$\Lambda(x)(\mathcal{P}^*_x(\alpha), \mathcal{Q}^*_x(\beta)) = 0, \tag{3}$$

for all $x \in M$ and arbitrary $\alpha, \beta \in T^*_xM$.

Using (1) it is easy to see that

$$C_{\Lambda,x} = (\mathcal{D}_x \cap C_{\Lambda,x}) \oplus (\mathcal{D}_x \cap C_{\Lambda,x})^\perp, \text{ for all } x \in M, \tag{4}$$

where $\perp$ denotes the symplectic complement of a subspace of $C_\Lambda(x)$ with respect to $\hat{\omega}(x)$. In fact, we have that

$$\mathcal{P}(C_{\Lambda,x}) = \mathcal{D}_x \cap C_{\Lambda,x}, \quad \mathcal{Q}(C_{\Lambda,x}) = (\mathcal{D}_x \cap C_{\Lambda,x})^\perp. \tag{5}$$

Thus, $\mathcal{D}_x \cap C_{\Lambda,x}$ is a symplectic subspace of the symplectic vector space $(C_{\Lambda,x}, \hat{\omega}(x))$.

We can now apply the procedure described in the previous section to construct a new almost-Poisson tensor field on M in terms of the projector $\mathcal{P}$.

Definition 3.1 *The bivector field $\Lambda_\mathcal{D}$ defined by*

$$\Lambda_\mathcal{D}(x)(\alpha, \beta) = \Lambda(x)(\mathcal{P}^*_x(\alpha), \mathcal{P}^*_x(\beta))$$

for $x \in M$ and $\alpha, \beta \in T^*_xM$, is called the Dirac tensor field determined by the almost-Poisson structure Λ and the distribution $\mathcal{D}$ verifying (1).

The tensor field $\Lambda_\mathcal{D}$ induces a bracket on the ring of smooth functions on M by putting

$$\{F, G\}_\mathcal{D} = \Lambda_\mathcal{D}(dF, dG), \tag{6}$$

for arbitrary $F, G \in C^\infty(M)$. This bracket is called the *Dirac bracket* associated to $\mathcal{D}$. The reason for using the terminology "Dirac tensor field" and "Dirac bracket" in this context, stems from the fact that the Dirac brackets encountered in the study of constrained Lagrangian and Hamiltonian systems are of this type. (For a unified geometric approach to various types of Dirac brackets in constrained dynamics, see [14]).

Since each 1-form β on M can be decomposed as $\beta = \mathcal{P}^*(\beta) + \mathcal{Q}^*(\beta)$, it follows from (3) that $\Lambda_\mathcal{D}$ is also determined by $\Lambda_\mathcal{D}(\alpha, \beta) = \Lambda(\mathcal{P}^*(\alpha), \beta)$, from which one easily deduces that

$$\sharp_{\Lambda_\mathcal{D}} = \sharp_\Lambda \circ \mathcal{P}^* = \mathcal{P} \circ \sharp_\Lambda. \tag{7}$$

In the following proposition we describe some further properties of the Dirac tensor field $\Lambda_\mathcal{D}$.

Proposition 3.2 *(i) If $\alpha \in \mathcal{D}^o$, then $\Lambda_\mathcal{D}(\alpha, \beta) = 0$, for all $\beta \in T^*M$.*

(ii) The characteristic distribution of $\Lambda_\mathcal{D}$ is given by

$$C_{\Lambda_\mathcal{D}} = \mathcal{P}(C_\Lambda) = \mathcal{D} \cap C_\Lambda.$$

(iii) If the 1-forms $\{\alpha_i\}$, $1 \leq i \leq m$, determine a local basis of $\mathcal{D}^o$, then

$$\Lambda_\mathcal{D}(\beta_1, \beta_2) = \Lambda(\beta_1, \beta_2) - \Lambda(\alpha_i, \beta_2) C^{ij} \Lambda(\alpha_j, \beta_1),$$

for all 1-forms β_1 and β_2, where C^{ij} are the entries of the inverse matrix of the regular skew-symmetric matrix $(C_{ij} = \Lambda(\alpha_i, \alpha_j))$.

Proof: Property (i) is a direct consequence of the definition of the adjoint projector $\mathcal{P}^*$, and (ii) also follows immediately, taking into account (5) and (7).

In order to prove (iii), let $\{\alpha_i\}$, $1 \leq i \leq m$ be a local basis of the annihilator of $\mathcal{D}$. Then, the vector fields $\sharp_\Lambda(\alpha_i)$ $(1 \leq i \leq m)$ locally span the distribution $\sharp_\Lambda(\mathcal{D}^o)$ and it follows from the given assumptions that the $m \times m$ matrix (C_{ij}), whose entries are

$$C_{ij} = \langle \sharp_\Lambda(\alpha_i), \alpha_j \rangle = \Lambda(\alpha_i, \alpha_j),$$

is nonsingular. Since (C_{ij}) is skew-symmetric, it follows that m necessarily has to be even. We denote the inverse matrix by (C^{ij}). Now, a direct computation shows that the projector $\mathcal{P}^*$ is locally given by

$$\mathcal{P}^* = \mathrm{id} \ - C^{ij} \alpha_i \otimes \sharp_\Lambda(\alpha_j).$$

Substituting this into the definition of the Dirac tensor field $\Lambda_\mathcal{D}$, we readily deduce that

$$\Lambda_\mathcal{D}(\beta_1, \beta_2) = \Lambda(\mathcal{P}^*(\beta_1), \mathcal{P}^*(\beta_2)) = \Lambda(\beta_1, \beta_2) - \Lambda(\alpha_i, \beta_2) C^{ij} \Lambda(\alpha_j, \beta_1)$$

for arbitrary 1-forms β_1 and β_2 on M. ∎

Using (3) again, one easily verifies that the 2-form induced by the Dirac tensor field $\Lambda_{\mathcal{D}}$ on its characteristic distribution, and which we will shortly denote by $\hat{\omega}_{\mathcal{D}}$, equals the restriction to $C_{\Lambda_{\mathcal{D}}}$ of the 2-form associated with the given almost-Poisson tensor Λ, i.e.

$$\hat{\omega}_{\mathcal{D}} = \hat{\omega}_{|C_{\Lambda_{\mathcal{D}}}}. \tag{8}$$

The following proposition gives some necessary and sufficient conditions for the Dirac tensor field $\Lambda_{\mathcal{D}}$ to be a Poisson tensor, such that the Dirac bracket satisfies the Jacobi-identity.

Proposition 3.3 *The Dirac tensor field $\Lambda_{\mathcal{D}}$ is a Poisson tensor field if and only if the distribution $\mathcal{D} \cap C_\Lambda$ is completely integrable and the restriction of $\hat{\omega}$ to each leaf of the corresponding foliation is a closed 2-form.*

Proof: First, assume that the distribution $\mathcal{D}\cap C_\Lambda$ is completely integrable and let $\mathcal{S}^{\mathcal{D}}$ denote the corresponding foliation. We further assume that for each $x \in M$, the restriction of $\hat{\omega}$ to the leaf $\mathcal{S}^{\mathcal{D}}_x$ passing through x is a closed 2-form. We already know from above that $T_x(\mathcal{S}^{\mathcal{D}}_x) = C_{\Lambda_{\mathcal{D}}}(x) = \mathcal{D}_x \cap C_\Lambda(x)$ is a symplectic subspace of $(C_\Lambda(x), \hat{\omega}(x))$. Taking into account (8), it then follows from the previous assumptions that $\hat{\omega}_{\mathcal{D}}$ induces a symplectic structure on $\mathcal{S}^{\mathcal{D}}_x$ and, hence, $\mathcal{S}^{\mathcal{D}}$ is a symplectic foliation of M. Denote by $\{\ ,\ \}_{\mathcal{S}^{\mathcal{D}}_x}$ the corresponding Poisson bracket on $\mathcal{S}^{\mathcal{D}}_x$. For any two smooth functions F and G on M we then find, using (4), (5) and (7), that

$$(\{F, G\}_{\mathcal{D}})_{|\mathcal{S}^{\mathcal{D}}_x} = \{F_{|\mathcal{S}^{\mathcal{D}}_x}, G_{|\mathcal{S}^{\mathcal{D}}_x}\}_{\mathcal{S}^{\mathcal{D}}_x}.$$

Since $\{\ ,\ \}_{\mathcal{S}^{\mathcal{D}}_x}$ satisfies the Jacobi identity, it follows that $\{\ ,\ \}_{\mathcal{D}}$ also satisfies the Jacobi identity, which proves that $\Lambda_{\mathcal{D}}$ is a Poisson tensor field.

Conversely, if $\Lambda_{\mathcal{D}}$ is a Poisson tensor, its characteristic distribution $C_{\Lambda_{\mathcal{D}}} = \mathcal{D} \cap C_\Lambda$ is completely integrable and every leaf of the induced foliation of M is a symplectic manifold (see [27, 28]). In fact, if $\mathcal{S}^{\mathcal{D}}_x$ again denotes the leaf containing $x \in M$, one easily verifies, using (7), that the symplectic form on $\mathcal{S}^{\mathcal{D}}_x$ is just the restriction of $\hat{\omega}$ to $\mathcal{S}^{\mathcal{D}}_x$. This restriction therefore is a closed 2-form. ∎

Suppose that Λ is a Poisson tensor on M, inducing a symplectic foliation $\mathcal{S}$, and that the distribution $\mathcal{D} \cap C_\Lambda$ is also completely integrable, with corresponding foliation $\mathcal{S}^{\mathcal{D}}$. It is then clear that, for each $x \in M$, $\mathcal{S}^{\mathcal{D}}_x$ is a symplectic submanifold of $\mathcal{S}_x$. In particular, this implies that the restriction of $\hat{\omega}$ to $S^{\mathcal{D}}_x$ is a closed 2-form. In view of Proposition 3.3 we may therefore conclude that:

Corollary 3.4 *If (M, Λ) is a Poisson manifold, then $\Lambda_{\mathcal{D}}$ is a Poisson tensor field if and only if the distribution $\mathcal{D} \cap C_\Lambda$ is completely integrable.*

We close this section with a result that will be of use later on, when dealing with almost-Poisson and Dirac brackets in nonholonomic mechanics.

Proposition 3.5 *Let (M, Λ) be an almost-Poisson manifold and suppose there exist two regular distributions $\mathcal{U}$ and $\mathcal{W}$ on M, such that:*
(i) rank $\mathcal{U}$ = rank $\mathcal{W}$ (= const.),
(ii) $(\mathcal{U} \cap \mathcal{W})^o \cap \ker \sharp_\Lambda = 0$,
(iii) $\sharp_\Lambda(\mathcal{W}^o) \cap \mathcal{U} = 0$,
then

$$TM = \mathcal{U} \oplus \sharp_\Lambda(\mathcal{W}^o). \tag{9}$$

If, in addition, we assume that $\sharp_\Lambda(\mathcal{W}^o) \subseteq \mathcal{W}$, then we also have that

$$TM = (\mathcal{U} \cap \mathcal{W}) \oplus \sharp_\Lambda((\mathcal{U} \cap \mathcal{W})^o), \tag{10}$$

and, moreover, the following equality holds

$$\Lambda_{\mathcal{P}} = \Lambda_{\mathcal{U} \cap \mathcal{W}},$$

(with $\mathcal{P}$ denoting the projector onto $\mathcal{U}$, induced by (9)), i.e. the almost-Poisson bracket associated to the distribution $\mathcal{U}$ equals the Dirac bracket determined by Λ and $\mathcal{U} \cap \mathcal{W}$.

Proof: First of all, since $\mathcal{W}^o \subset (\mathcal{U} \cap \mathcal{W})^o$, it follows from (ii) that $\mathcal{W}^o \cap \ker \sharp_\Lambda = 0$ which, taking into account the assumptions (i) and (iii), immediately leads to the direct sum decomposition (9).

If we now suppose that $\sharp_\Lambda(\mathcal{W}^o) \subseteq \mathcal{W}$, it follows from (9) that

$$\mathcal{W} = \sharp_\Lambda(\mathcal{W}^o) \oplus (\mathcal{U} \cap \mathcal{W}). \tag{11}$$

In view of the given assumptions it is clear that for the proof of (10) it only remains to be shown that $(\mathcal{U} \cap \mathcal{W}) \cap \sharp_\Lambda((\mathcal{U} \cap \mathcal{W})^o) = 0$. For that purpose, consider any point $x \in M$ and an arbitrary vector $v \in (\mathcal{U}_x \cap \mathcal{W}_x) \cap \sharp_\Lambda((\mathcal{U}_x \cap \mathcal{W}_x)^o)$. In particular, there exists a $\beta \in (\mathcal{U}_x \cap \mathcal{W}_x)^o$ such that $v = \sharp_\Lambda(\beta)$. For any $w \in \mathcal{W}_x$ we have, in view of (11), that $w = w_1 + w_2$ for some $w_1 \in \sharp_\Lambda(\mathcal{W}_x^o)$ and $w_2 \in \mathcal{U}_x \cap \mathcal{W}_x$, from which we immediately obtain $< w, \beta >=< w_1, \beta > + < w_2, \beta >=< w_1, \beta >$. Observing that w_1 can be written as $w_1 = \sharp_\Lambda(\alpha)$ for some $\alpha \in \mathcal{W}_x^o$, we find that

$$< w_1, \beta >= - < \sharp_\Lambda(\beta), \alpha >= 0,$$

since $\sharp_\Lambda(\beta) = v$ belongs to $\mathcal{W}_x$. Summarizing, we have shown that $< w, \beta >= 0$ for all $w \in \mathcal{W}_x$ and so, $\beta \in \mathcal{W}_x^o$. We therefore have that $v \in \mathcal{U}_x$ and $v = \sharp_\Lambda(\beta) \in \sharp_\Lambda(\mathcal{W}_x^o)$ which, in view of assumption (iii), necessarily implies that $v = 0$ and, hence, $(\mathcal{U}\cap\mathcal{W})\cap\sharp_\Lambda((\mathcal{U}\cap\mathcal{W})^o) = 0$. A simple dimensional argument now completes the proof of (10).

Let us denote by $\mathcal{P}$ (resp. $\tilde{\mathcal{P}}$) the projector of TM onto $\mathcal{U}$ (resp. $\mathcal{U}\cap\mathcal{W}$), corresponding to the decompostion (9) (resp. (10)). Before turning to the last part of the proposition, we first derive two useful intermediate results. First of all, note that for any $\alpha \in T^*M$

$$\mathcal{P}^*(\alpha) - \tilde{\mathcal{P}}^*(\alpha) \in \mathcal{W}^o. \tag{12}$$

This can be easily verified, using the definition of the two projectors (and their adjoints). Secondly, recalling that $\operatorname{Im}\mathcal{P}^* = (\sharp_\Lambda(\mathcal{W}^o))^o$ we immediately find that for any $\alpha \in T_x^*M$ and $\beta \in \mathcal{W}_x^o$ $(x \in M)$

$$\Lambda(\mathcal{P}_x^*(\alpha), \beta) = 0. \tag{13}$$

Returning now to the proof of the proposition, we first observe that, taking into account (12), for any given $\alpha_1, \alpha_2 \in T_x^*M$ there exist $\beta_1, \beta_2 \in \mathcal{W}_x^o$, such that

$$\tilde{\mathcal{P}}_x^*(\alpha_1) = \mathcal{P}_x^*(\alpha_1) + \beta_1 \qquad \tilde{\mathcal{P}}_x^*(\alpha_2) = \mathcal{P}_x^*(\alpha_2) + \beta_2.$$

Herewith we obtain

$$\begin{aligned}\Lambda_{\mathcal{U}\cap\mathcal{W}}(\alpha_1,\alpha_2) &= \Lambda(\tilde{\mathcal{P}}_x^*(\alpha_1), \tilde{\mathcal{P}}_x^*(\alpha_2))\\ &= \Lambda(\mathcal{P}_x^*(\alpha_1), \mathcal{P}_x^*(\alpha_2)) + \Lambda(\mathcal{P}_x^*(\alpha_1), \beta_2) + \Lambda(\beta_1, \mathcal{P}_x^*(\alpha_2)) + \Lambda(\beta_1, \beta_2).\end{aligned}$$

The second and third term on the right-hand side vanish in view of (13) and the previous relation becomes

$$\Lambda_{\mathcal{U}\cap\mathcal{W}}(\alpha_1,\alpha_2) = \Lambda_{\mathcal{P}}(\alpha_1,\alpha_2) + < \sharp_\Lambda(\beta_1), \beta_2 > .$$

Using the assumption that $\sharp_\Lambda(\mathcal{W}^o) \subset \mathcal{W}$, this relation further reduces to $\Lambda_{\mathcal{U}\cap\mathcal{W}}(\alpha_1,\alpha_2) = \Lambda_{\mathcal{P}}(\alpha_1,\alpha_2)$. Since this holds for all $\alpha_1, \alpha_2 \in T_x^*M$, at each point $x \in M$, this completes the proof of the proposition. ∎

4 Dirac tensor fields along submanifolds

Let (M, Λ) be an almost-Poisson manifold and consider a submanifold N of M and a regular distribution $\mathcal{D}$ on N, i.e. $\mathcal{D}(N)$ is a vector subbundle of TN which, in turn, can be regarded as a vector subbundle of T_NM. Assume that the following condition holds:

$$T_xM = \mathcal{D}_x \oplus \sharp_\Lambda(\mathcal{D}_x^o) \text{ ,for all } x \in N,$$

where the annihilator $\mathcal{D}_x^o$ is taken in T_x^*M. Note that in case Λ is a Poisson tensor and $\mathcal{D} = TN$, this condition is precisely the one needed for N to become a Poisson submanifold of M, with Poisson structure induced by Λ (see [28], Proposition 1.4). The construction of a Dirac tensor field described in the previous section being a pointwise construction, it can be carried over step by step to the present situation. More precisely, we can define a skew-symmetric 2-contravariant tensor field $\Lambda_{\mathcal{D}}$ along N according to the prescription given in Definition 3.1. Moreover, it is not difficult to see that $\Lambda_{\mathcal{D}}$ can be smoothly extended to a bivector field defined on an open neighbourhood of N, which coincides with $\Lambda_{\mathcal{D}}$ in points of N. Obviously, such an extension will not be unique.

The Dirac tensor field $\Lambda_{\mathcal{D}}$ restricts in a natural way to a skew-symmetric 2-contravariant tensor field $\Lambda_{N,\mathcal{D}}$ on the submanifold N. Indeed, at each point $x \in N$ and for all $\alpha, \beta \in T_x^*N$, we put

$$\Lambda_{N,\mathcal{D}}(x)(\alpha, \beta) = \Lambda_{\mathcal{D}}(x)(\tilde{\alpha}, \tilde{\beta}) \tag{14}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are arbitrary extensions to T_xM of α and β , respectively, i.e. $\tilde{\alpha}, \tilde{\beta} \in T_x^*M$ such that $\tilde{\alpha}_{|T_xN} = \alpha$ and $\tilde{\beta}_{|T_xN} = \beta$. Herewith, $\Lambda_{N,\mathcal{D}}$ is well-defined since $TN^o \subseteq \mathcal{D}^o$ (with the annihilators taken in T_N^*M).

The tensor field $\Lambda_{N,\mathcal{D}}$ now induces a bracket on the ring of smooth functions on N by putting

$$\{f, g\}_{N,\mathcal{D}} = \Lambda_{\mathcal{D}}((dF)_{|N}, (dG)_{|N}) = \Lambda(\mathcal{P}^*(dF), \mathcal{P}^*(dG))_{|N}, \tag{15}$$

where f and g are smooth functions on N, and F (resp. G) is an arbitrary smooth extension to M of f (resp. g) and $\mathcal{P}$ is the projector of T_NM onto $\mathcal{D}$. In a similar way as in Section 3 it can be shown that $\{\ ,\ \}_{N,\mathcal{D}}$ is a Poisson bracket on N if and only if the distribution $\mathcal{D} \cap C_\Lambda$ is completely integrable and the restriction of $\hat{\omega}$ to each leaf of $\mathcal{D} \cap C_\Lambda$ is a closed 2-form.

With a view on applications to nonholonomic mechanics, it is interesting to consider the following situation. Let N again be a submanifold of an almost-Poisson manifold (M, Λ) and let $\mathcal{F}$ be a regular distribution on M along N, i.e. a subbundle of T_NM, such that

$$T_NM = TN \oplus \sharp_\Lambda(\mathcal{F}^o),$$

with associated projector $\mathcal{P} : T_NM \longrightarrow TN$. We then have that

$$(a)\ TN \cap \sharp_\Lambda(\mathcal{F}^o) = 0.$$

Let us assume that, in addition, the following conditions are also satisfied

$$(b)\ (TN \cap \mathcal{F})^o \cap \ker \sharp_\Lambda = 0, \quad (c)\ \sharp_\Lambda(\mathcal{F}^o) \subseteq \mathcal{F}.$$

It is not difficult to verify that we are in a situation where Proposition 3.5 can be applied, restricted to the submanifold N. In particular, one can construct, along N, an almost-Poisson tensor field $\Lambda_{\mathcal{P}}$ and a Dirac tensor field $\Lambda_{TN\cap\mathcal{F}}$. Proposition 3.5 then tells us that both tensor fields are equal, which further implies that both induce the same almost-Poisson structure on N.

In the case of nonholonomic dynamics, (M, Λ) is a Poisson manifold (for instance, the phase space of a mechanical system with its canonical Poisson structure), N denotes the constraint submanifold and $\mathcal{F}^o$ represents the co-distribution in which the constraint forces take their values (see e.g. [5, 15, 21, 22]). In this framework, the above condition (a) corresponds to what was previously called the "regularity" or "compatibility condition" (see e.g. [5, 19]), whereas condition (b) is automatically verified if Λ is nondegenerate (which is obviously the case if it is the canonical Poisson tensor on the phase space of a mechanical system). Condition (c) also holds in nonholonomic mechanics if we are dealing with linear or affine constraints or, more generally, with nonlinear constraints, provided in the latter case we adopt the so-called Chetaev rule for determining the bundle $\mathcal{F}$ (cf. Section 7). The almost-Poisson structure induced on the constraint submanifold N is precisely the one considered by van der Schaft and Maschke [26] in the framework of the Hamiltonian description of autonomous mechanical systems with linear nonholonomic constraints. The associated almost-Poisson bracket on N is then also called the "nonholonomic bracket" (see also [2, 5, 22, 15]). The main point of [5] precisely consisted in showing that this nonholonomic bracket can be related to two different splittings of the tangent bundle of the phase space of the system along the constraint submanifold.

In what follows, we wish to extend the bracket formulation of nonholonomic mechanics to the time-dependent setting. For reasons to be explained later on, we will follow a common practice in classical treatments of mechanics by describing a time-dependent system on the direct product of the (velocity or momentum) phase space of the system with the real time axis, rather than using a more general jet bundle formulation. The underlying geometric structure will be that of a cosymplectic manifold. The next section will be devoted to the construction of Dirac brackets on general cosymplectic manifolds.

5 Cosymplectic manifolds and Dirac tensor fields

First, we recall some general facts about cosymplectic manifolds (see [1, 4]).

Let V be a real vector space of dimension $2n+1$, η an exterior 1-form and Ω an exterior 2-form on V. Then the triple (V, η, Ω) is called a cosymplectic vector space if $\eta \wedge \Omega^n \neq 0$.

If (V, η, Ω) is a cosymplectic vector space and V^* is the dual space of V then the linear map

$$b_{\eta,\Omega} : V \longrightarrow V^*, \qquad v \in V \longrightarrow b_{\eta,\Omega}(v) = i_v\Omega + (\eta(v))\eta \in V^* , \tag{16}$$

is a linear isomorphism and the inverse of $-\flat_{\eta,\Omega}$ will be denoted by $\sharp_{\eta,\Omega}$. (Note the minus sign, which is merely introduced for conventional reasons.) The vector $\mathcal{R} = -\sharp_{\eta,\Omega}(\eta)$ is called the Reeb vector of the cosymplectic vector space (V, η, Ω). It is characterized by the relations $i_{\mathcal{R}}\eta = 1$ and $i_{\mathcal{R}}\Omega = 0$.

Now, suppose that M is a smooth $(2n+1)$-dimensional manifold. M is said to be cosymplectic if a closed 1-form η and a closed 2-form Ω on M exist such that the triple (T_xM, η_x, Ω_x) is a cosymplectic vector space for all $x \in M$. It is possible to select coordinates (t, q^A, p_A) on M (called Darboux coordinates) such that

$$\eta = dt \quad \text{and} \quad \Omega = dq^A \wedge dp_A.$$

Denote by $\mathcal{R}$ the Reeb vector field of the cosymplectic structure (η, Ω) which in Darboux coordinates reads:

$$\mathcal{R} = \frac{\partial}{\partial t}.$$

We further denote by

$$\flat_{\eta,\Omega} : \mathfrak{X}(M) \longrightarrow \Lambda^1(M), X \longmapsto \flat_{\eta,\Omega}(X) = i_X\Omega + (\eta(X))\eta\ ,$$

the $C^\infty(M)$-linear isomorphism between vector fields and 1-forms on M, induced by the cosymplectic structure and, as in the linear case, we let $\sharp_{\eta,\Omega}$ be the inverse of $-\flat_{\eta,\Omega}$.

Given a 1-form on M, we define the vector fields $\operatorname{grad}\alpha$, X_α and $\mathcal{E}_\alpha$ as follows:

$$\begin{aligned}
\operatorname{grad}\alpha &= -\sharp_{\eta,\Omega}(\alpha),\\
X_\alpha &= -\sharp_{\eta,\Omega}(\alpha) - \alpha(\mathcal{R})\mathcal{R},\\
\mathcal{E}_\alpha &= \mathcal{R} + X_\alpha.
\end{aligned}$$

These vector fields are called the gradient, Hamiltonian and evolution vector field, respectively, corresponding to α. For what follows it is useful to recall that X_α is also completely determined by the relations $i_{X_\alpha}\Omega = \alpha - \langle\mathcal{R}, \alpha\rangle\eta$ and $i_{X_\alpha}\eta = 0$.

In Darboux coordinates, with $\alpha = \alpha_o dt + \alpha_A dq^A + \bar{\alpha}^A dp_A$, we obtain

$$\begin{aligned}
\operatorname{grad}\alpha &= \alpha_0\frac{\partial}{\partial t} + \bar{\alpha}^A\frac{\partial}{\partial q^A} - \alpha_A\frac{\partial}{\partial p_A},\\
X_\alpha &= \bar{\alpha}^A\frac{\partial}{\partial q^A} - \alpha_A\frac{\partial}{\partial p_A},\\
\mathcal{E}_\alpha &= \frac{\partial}{\partial t} + \bar{\alpha}^A\frac{\partial}{\partial q^A} - \alpha_A\frac{\partial}{\partial p_A}.
\end{aligned}$$

For $F \in C^\infty(M)$ we will use the shorthand notations $\operatorname{grad}F = \operatorname{grad}dF$, $X_F = X_{dF}$ and $\mathcal{E}_F = \mathcal{E}_{dF}$.

If M is a $2n+1$-dimensional cosymplectic manifold, with cosymplectic structure (η, Ω), we can define a skew-symmetric 2-contravariant tensor field $\Lambda_{\eta,\Omega}$ on M as follows:

$$\Lambda_{\eta,\Omega}(\alpha, \beta) = \Omega(X_\alpha, X_\beta) = \Omega(\mathcal{E}_\alpha, \mathcal{E}_\beta) = \Omega(\mathrm{grad}\,\alpha, \mathrm{grad}\,\beta).$$

One can then easily verify that $(M, \Lambda_{\eta,\Omega})$ is a regular Poisson manifold, with Poisson structure of constant rank $2n$. The corresponding Poisson bracket is given by

$$\{F, G\}_{\eta,\Omega} = \Lambda_{\eta,\Omega}(dF, dG)$$

for $F, G \in C^\infty(M)$. The symplectic foliation of this Poisson structure is determined by the equation $\eta = 0$. The Hamiltonian vector fields X_α, defined above, coincide with the vector fields $-\sharp_{\Lambda_{\eta,\Omega}}(\alpha)$, defined by the Poisson structure, which (at least in the case of exact 1-forms) explains the terminology. Note that the maps $\sharp_{\eta,\Omega}$ and $\sharp_{\Lambda_{\eta,\Omega}}$ are different; indeed, $\sharp_{\eta,\Omega}(\alpha) = -\mathrm{grad}\,\alpha$, whereas $\sharp_{\Lambda_{\eta,\Omega}}(\alpha) = -X_\alpha$ (for more details, see [1, 4]).

Consider now a distribution $\mathcal{D}$ on the cosymplectic manifold M and denote the restrictions of η and Ω to $\mathcal{D}$ by $\eta_\mathcal{D}$ and $\Omega_\mathcal{D}$, respectively. The distribution $\mathcal{D}$ will then be called a *cosymplectic distribution* if $(\mathcal{D}_x, \eta_\mathcal{D}(x), \Omega_\mathcal{D}(x))$ is a cosymplectic vector space, for all $x \in M$. In such a case, we have that

$$b_{\eta_\mathcal{D},\Omega_\mathcal{D}} : \mathcal{D} \longrightarrow \mathcal{D}^*, v \in \mathcal{D}_x \longmapsto b_{\eta_\mathcal{D},\Omega_\mathcal{D}}(v) = i_v\Omega_\mathcal{D}(x) + < v, \eta_\mathcal{D}(x) > \eta_\mathcal{D}(x) \in \mathcal{D}^*_x$$

is a linear isomorphism.

Proposition 5.1 *Let $\mathcal{D}$ be a regular cosymplectic distribution on a cosymplectic manifold (M, η, Ω). Then,*

$$TM = \mathcal{D} \oplus \sharp_{\Lambda_{\eta,\Omega}}(\mathcal{D}^o).$$

Proof: Let $\{\alpha_i; 1 \leq i \leq m = 2n+1 - \mathrm{rank}\,\mathcal{D}\}$, be a local basis of the annihilator of $\mathcal{D}$ in a neighbourhood of a given point $x \in M$. Then, in that neighbourhood, $\sharp_{\Lambda_{\eta,\Omega}}(\mathcal{D}^o)$ is generated by the vector fields $\{X_{\alpha_i} = -\sharp_{\Lambda_{\eta,\Omega}}(\alpha_i); 1 \leq i \leq m\}$.

First we prove that the tangent vectors $X_{\alpha_i}(x) \in T_xM$ are linearly independent. Suppose that there exists a linear combination $\lambda^i X_{\alpha_i}(x) = 0$ such that not all the coefficients $\lambda_i \in \mathbb{R}$ are zero. Then, we have

$$0 = \lambda^i(i_{X_{\alpha_i}}\Omega)(x) = \lambda^i\alpha_i(x) - \lambda^i\langle\mathcal{R}(x), \alpha_i(x)\rangle\eta(x)$$

which implies $\lambda^i\langle\mathcal{R}(x), \alpha_i(x)\rangle\eta(x) = \lambda^i\alpha_i(x)$. Since $\lambda^i\alpha_i(x) \neq 0$, it follows that $\eta(x)$ belongs to $\mathcal{D}^o_x$, which is in contradiction with the assumption that $\mathcal{D}$ is a cosymplectic distribution. Thus, we have

$$\dim \mathcal{D}_x + \dim \sharp_{\Lambda_{\eta,\Omega}}(\mathcal{D}^o_x) = \dim T_xM.$$

Next, we prove that $\mathcal{D}_x \cap \sharp_{\Lambda_{\eta,\Omega}}(\mathcal{D}_x^o) = \{0\}$, for all $x \in M$. Indeed, take $\alpha \in \mathcal{D}^o$, defined in a neighbourhood of the point x, then

$$X_\alpha = -\sharp_{\Lambda_{\eta,\Omega}}(\alpha) \in \sharp_{\Lambda_{\eta,\Omega}}(\mathcal{D}^o).$$

Suppose now that $X_\alpha(x) \in \mathcal{D}_x$. If $\mathcal{R}_\mathcal{D}(x)$ is the Reeb vector of the cosymplectic vector space $(\mathcal{D}_x, \eta_\mathcal{D}(x), \Omega_\mathcal{D}(x))$ then

$$\begin{array}{rcl} (i_{X_\alpha}\Omega_\mathcal{D})(x) & = & \alpha_\mathcal{D}(x) - \langle R_\mathcal{D}(x), \alpha(x)\rangle \eta_D(x), \\ (i_{X_\alpha}\eta_\mathcal{D})(x) & = & 0, \end{array}$$

with $\alpha_\mathcal{D}$ the restriction of the selected 1-form α to $\mathcal{D}$. Since $\alpha \in \mathcal{D}^o$, it follows that the right-hand side of the first equation is zero and, hence, $X_\alpha(x) = 0$, since $(\mathcal{D}_x, \eta_\mathcal{D}(x), \Omega_\mathcal{D}(x))$ is a cosymplectic vector space.

Summing up the above results we have that $T_xM = \mathcal{D}_x \oplus \sharp_{\Lambda_{\eta,\Omega}}(\mathcal{D}_x^o)$, for all $x \in M$. ∎

It follows from the above proposition that, given a cosymplectic distribution $\mathcal{D}$ on a cosymplectic manifold (M, η, Ω), there exist two complementary projectors:

$$\mathcal{P} : TM \longrightarrow \mathcal{D}, \qquad \mathcal{Q} : TM \longrightarrow \sharp_{\Lambda_{\eta,\Omega}}(\mathcal{D}^o),$$

with corresponding adjoint projectors:

$$\mathcal{P}^* : T^*M \longrightarrow \left(\sharp_{\Lambda_{\eta,\Omega}}(\mathcal{D}^o)\right)^o, \qquad \mathcal{Q}^* : T^*M \longrightarrow \mathcal{D}^o.$$

According to the discussion in Section 3, we then have an induced Dirac tensor field $(\Lambda_{\eta,\Omega})_\mathcal{D}$, defined by

$$(\Lambda_{\eta,\Omega})_\mathcal{D}(\beta_1, \beta_2) = \Lambda_{\eta,\Omega}(\mathcal{P}^*(\beta_1), \mathcal{P}^*(\beta_2))$$

for all 1-forms β_1 and β_2 on M. In the next proposition we give an alternative definition for this Dirac tensor field in terms of the 2-form Ω.

Proposition 5.2 *We have*

$$(\Lambda_{\eta,\Omega})_\mathcal{D}(\beta_1, \beta_2) = \Omega(\mathcal{P}(X_{\beta_1}), \mathcal{P}(X_{\beta_2})).$$

Proof: In fact, we obtain

$$\begin{array}{rcl} (\Lambda_{\eta,\Omega})_\mathcal{D}(\beta_1, \beta_2) & = & \Lambda_{\eta,\Omega}(\mathcal{P}^*(\beta_1), \mathcal{P}^*(\beta_2)) \\ & = & \Omega(X_{\mathcal{P}^*(\beta_1)}, X_{\mathcal{P}^*(\beta_2)}). \end{array}$$

Since $\sharp_{\Lambda_{\eta,\Omega}} \circ \mathcal{P}^* = \mathcal{P} \circ \sharp_{\Lambda_{\eta,\Omega}}$ (see (7)), we deduce that

$$X_{\mathcal{P}^*(\beta)} = \mathcal{P}(X_\beta)$$

for each 1-form β, which completes the proof of the proposition. ∎

Using Corollary 3.4, we also immediately have:

Proposition 5.3 *$(\Lambda_{\eta,\Omega})_{\mathcal{D}}$ is a Poisson tensor if and only if the distribution $\bar{\mathcal{D}}$ on M given by*

$$x \in M \longrightarrow \bar{\mathcal{D}}_x = \{v \in \mathcal{D}_x | \; \eta_x(v) = 0\} \subseteq T_x M$$

is completely integrable.

Remark 5.4 That integrability of $\bar{\mathcal{D}}$ does not necessarily imply integrability of $\mathcal{D}$ is illustrated by the following simple example. Consider $\mathbb{R}^5$, with coordinates denoted by (t, q^1, q^2, p_1, p_2), and with the natural cosymplectic structure (η, Ω) defined by

$$\eta = dt, \qquad \Omega = dq^1 \wedge dp_1 + dq^2 \wedge dp_2.$$

Then, the distribution $\mathcal{D}$ generated by the vector fields

$$\frac{\partial}{\partial q^1}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial t} + p_1 \frac{\partial}{\partial q^2}$$

is cosymplectic and the distribution $\bar{\mathcal{D}}$, which is spanned by $\{\frac{\partial}{\partial q^1}, \frac{\partial}{\partial p_1}\}$, is completely integrable. However, it is clear that the distribution $\mathcal{D}$ itself is not completely integrable.

The following result will be useful in the treatment of time-dependent Lagrangian and Hamiltonian systems.

Lemma 5.5 *Let (M, η, Ω) be a cosymplectic manifold and let H be an arbitrary function on M. If we put $\Omega_H = \Omega + dH \wedge \eta$, then (η, Ω_H) also determines a cosymplectic structure on M, with Reeb vector field $\mathcal{E}_H$ (the evolution vector field corresponding to H with respect to (η, Ω)). Moreover, for any given 1-form α on M, the corresponding Hamiltonian vector fields X_α and $\mathbb{X}_\alpha$ defined with respect to the cosymplectic structures (η, Ω) and (η, Ω_H), respectively, coincide, i.e.*

$$X_\alpha = \mathbb{X}_\alpha.$$

Proof: It is obvious that Ω_H is closed and $\eta \wedge \Omega_H^n \neq 0$. Hence, (η, Ω_H) is a cosymplectic structure. Moreover, we have

$$i_{\mathcal{E}_H}\Omega_H = 0, \quad i_{\mathcal{E}_H}\eta = 1,$$

which already proves the first part of the lemma.

Now, for a given 1-form α on M we find:

$$\begin{aligned} i_{X_\alpha}\Omega_H &= i_{X_\alpha}\Omega + X_\alpha(H)\eta \\ &= i_{X_\alpha}\Omega - \alpha(X_H)\eta \\ &= \alpha - \alpha(\mathcal{R})\eta - \alpha(X_H)\eta \\ &= \alpha - \langle \mathcal{E}_H, \alpha \rangle \eta, \end{aligned}$$

from which we deduce that $X_\alpha = \mathbb{X}_\alpha$. ∎

Since the Hamiltonian vector fields with respect to a cosymplectic structure coincide with the Hamiltonian vector fields with respect to the induced Poisson structure, we immediately deduce from the previous lemma that the cosymplectic structures (η, Ω) and (η, Ω_H) determine the same Poisson structure on M, i.e.

$$\Lambda_{\eta,\Omega} = \Lambda_{\eta,\Omega_H}.$$

Indeed, by definition of these Poisson tensor fields we have that, for any 1-form α on M,

$$\Lambda_{\eta,\Omega}(\alpha, \ .\ ) = -X_\alpha = -\mathbb{X}_\alpha = \Lambda_{\eta,\Omega_H}(\alpha, \ .\ ).$$

In the next section we briefly recall some elements from the geometric treatment of time-dependent Lagrangian mechanics in a cosymplectic framework.

6 Lagrangian formalism for time-dependent systems

Consider a time-dependent mechanical system with configuration space given by an n-dimensional differentiable manifold Q, and which is described by a Lagrangian $L : \mathbb{R} \times TQ \longrightarrow \mathbb{R}$. If local coordinates on Q are given by (q^A), $1 \leq A \leq n$, then the corresponding bundle coordinates on $\mathbb{R} \times TQ$ will be denoted by (t, q^A, v^A) , where t stands for the standard ('time') coordinate on $\mathbb{R}$. We further denote by $\tau_Q : \mathbb{R} \times TQ \longrightarrow \mathbb{R} \times Q$ the canonical projection, i.e. $\tau_Q(t, q^A, v^A) = (t, q^A)$.

The two intrinsic geometrical objects, characterizing the tangent bundle TQ, are the Liouville vector field Δ and the vertical endomorphism S. Both can be regarded in an obvious way as living on $\mathbb{R} \times TQ$, and we shall denote these extensions by the same symbols. So, we have

$$\Delta = v^A \frac{\partial}{\partial v^A}, \quad S = \frac{\partial}{\partial v^A} \otimes dq^A.$$

We will also consider the type $(1,1)$-tensor field $\tilde{S}$ defined by

$$\tilde{S} = S - \Delta \otimes dt = \frac{\partial}{\partial v^A} \otimes (dq^A - v^A dt).$$

The energy function associated to the Lagrangian L is given by

$$E_L = \Delta(L) - L = v^A \frac{\partial L}{\partial v^A} - L.$$

Herewith, we can define the Poincaré-Cartan 1-form corresponding to L as

$$\begin{aligned}\Theta_L &= S^*(dL) - E_L dt = \frac{\partial L}{\partial v^A} dq^A - E_L dt \\ &= \frac{\partial L}{\partial v^A}(dq^A - v^A dt) + L dt,\end{aligned}$$

and the Poincaré-Cartan 2-form is then given by

$$\Omega_L = -d\,\Theta_L.$$

Note that Θ_L can also be written as $\Theta_L = \tilde{S}^*(dL) + Ldt$. In the subsequent discussion we will also need the following 2-form

$$\omega_L = -d(S^*(dL)) = \Omega_L - dE_L \wedge dt.$$

Remark 6.1 As will become apparent later on, the energy function E_L plays an important role in the "bracket" formulation of time-dependent mechanics. It should be pointed out, therefore, that in the more general jet bundle framework, where the configuration space-time of a mechanical system is modelled by a fibred manifold E over $\mathbb{R}$, with projection $\pi : E \longrightarrow \mathbb{R}$, and where the Lagrangian is a function defined on the first jet bundle $J^1\pi$, one is no longer assured of the existence of a globally defined "energy function" associated to a given L, unless one fixes a trivialisation of π (which always exists since $\mathbb{R}$ is contractible) [9, 20]. In the latter case, however, we put $E \cong \mathbb{R} \times Q$ for some manifold Q and $J^1\pi$ can then be identified with $\mathbb{R} \times TQ$, which brings us back to the framework we are considering here.

We say that L is regular if the Hessian matrix of L with respect to the velocities is regular. A direct computation shows that L is regular if and only if the pair (dt, ω_L) determines a cosymplectic structure on $\mathbb{R} \times TQ$. The corresponding Reeb vector field $\mathcal{R}_L$ is locally given by

$$\mathcal{R}_L = \frac{\partial}{\partial t} - \mathcal{W}^{AB} \frac{\partial^2 L}{\partial v^A \partial t} \frac{\partial}{\partial v^B}\,,$$

where $\mathcal{W}^{AB}$ are the components of the inverse of the Hessian matrix of L.

If (dt, ω_L) is a cosymplectic structure, then (dt, Ω_L) is also a cosymplectic structure, and conversely (see Lemma 5.5). Moreover, from Lemma 5.5 we deduce that in the case of a regular Lagrangian the following equations of motion are equivalent:

$$\left\{ \begin{array}{l} i_X \omega_L = dE_L - \mathcal{R}_L(E_L)dt \\ i_X dt = 1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} i_X \Omega_L = 0 \\ i_X dt = 1 \end{array} \right. \tag{17}$$

In the regular case, equations (17) have a unique solution $X = \Gamma_L$, which is precisely the evolution vector field corresponding to the energy function E_L, relative to the cosymplectic structure (dt, ω_L), as well as the Reeb vector field of the cosymplectic structure (dt, Ω_L). The vector field Γ_L is a time-dependent second-order differential equation field, i.e. $\tilde{S}(\Gamma_L) = 0$ and $dt(\Gamma_L) = 1$. Its integral curves are determined by the solutions of the Euler-Lagrange equations for the Lagrangian L:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^A}\right) - \frac{\partial L}{\partial q^A} = 0, \quad 1 \leq A \leq n.$$

From the discussion at the end of the previous section, we further deduce that the cosymplectic structures (dt, ω_L) and (dt, Ω_L) determine the same Poisson tensor field on $\mathbb{R} \times TQ$ which we shall denote henceforth by Λ_L. In particular, we have

$$\Lambda_L(\alpha, \beta) = \omega_L(X_\alpha, X_\beta).$$

The corresponding Poisson bracket of functions on $\mathbb{R} \times TQ$ reads

$$\{F, G\}_L = \Lambda_L(dF, dG) = \omega_L(X_F, X_G).$$

It then follows that the evolution of any observable $F \in C^\infty(\mathbb{R} \times TQ)$ along the integral curves of Γ_L can be described in terms of this Poisson bracket by the equation

$$\dot{F} = \Gamma_L(F) = \mathcal{R}_L(F) + \{F, E_L\}_L. \tag{18}$$

7 Nonholonomic Lagrangian systems

We now consider a Lagrangian system described by a regular Lagrangian $L \in C^\infty(\mathbb{R} \times TQ)$ and subjected to (time-dependent) nonholonomic constraints. The constraints are modelled by a closed embedded submanifold M of $\mathbb{R} \times TQ$, which is locally determined by the vanishing of k independent smooth functions $\phi_i(t, q^A, v^A)$, $1 \leq i \leq k$, where k is the codimension of M. For simplicity we will always assume that $\tau_Q(M) = \mathbb{R} \times Q$, such that the given constraints do not generate restrictions on the admissible positions of the system in the configuration space-time $\mathbb{R} \times Q$ (i.e. there are no holonomic constraints). We do not require, however, that M is a vector or affine subbundle of $\mathbb{R} \times TQ$ (over $\mathbb{R} \times Q$), i.e. we do not confine the discussion to the case of constraints that are linear or affine in the velocities.

As is well-known, the standard derivation of the equations of motion of a nonholonomic system is based on a generalisation of d'Alembert's principle for holonomic constraints, which states that the infinitesimal work of the reaction (or constraint) force vanishes for any infinitesimal "virtual displacement" of the system. In the case of nonholonomic constraints, however, the notion of virtual displacement does not have the same clear-cut meaning it has in the case of holonomic constraints (see e.g. [22, 24]). Whereas for linear or affine nonholonomic constraints there is still a natural choice available for the bundle of admissible infinitesimal virtual displacements (or virtual velocities), things are much less transparent when dealing with nonlinear constraints. In that case, a common procedure to determine the bundle of virtual velocities relies on the so-called Chetaev rule. Using the above notations, Chetaev's rule states that the virtual velocities (w^A) at a given point (t, q^A, v^A) of the constraint submanifold M, are determined by the relations

$$\frac{\partial \phi_i}{\partial v^A} w^A = 0.$$

It should be emphasised, however, that application of Chetaev's rule does not always lead to a physically acceptable model for the dynamics of a system with nonlinear nonholonomic constraints. A critical discussion of this matter, as well as some counterexamples to the use of Chetaev's rule, can be found in a recent paper by Marle [22].

When adopting Chetaev's rule, the dynamics of the nonholonomic system introduced above is determined by the relations (see e.g. [17, 18])

$$\left\{ \begin{array}{l} (i_X \Omega_L)_{|M} \in \tilde{S}^*(TM)^o \\ i_X \, dt = 1 \\ X_{|M} \in TM. \end{array} \right. \tag{19}$$

Here, $\tilde{S}^*(TM)^o$ is the co-distribution along M in which the constraint forces take their values; it is the annihilator of the bundle of admissible virtual velocities determined by Chetaev's rule. Let $\mathcal{F}$ be the distribution on $\mathbb{R} \times TQ$ along M, defined by $\mathcal{F} = \sharp_{dt,\Omega_L}(\tilde{S}^*(TM)^o)$. It can be shown that the system (19) admits a unique solution $X = \Gamma_{L,M}$ provided the following two conditions are satisfied (cf. [17, 18]):

- **admissibility condition:**

$$\dim (T_x M)^o = \dim (\tilde{S}^*(T_x M)^o), \quad \text{for all } x \in M.$$

- **compatibility condition:**

$$\mathcal{F} \cap TM = 0.$$

The distribution $\mathcal{F}$ is locally generated by the restriction to M of the k vector fields Z_i, defined by

$$i_{Z_i}\Omega_L + dt(Z_i)\, dt = \tilde{S}^*(d\phi_i).$$

Observe that the Z_i are precisely the gradient vector fields corresponding to the 1-forms $\tilde{S}^*(d\phi_i)$ with respect to the cosymplectic structure (dt, Ω_L). It is not difficult to prove that

$$Z_i = X_{\tilde{S}^*(d\phi_i)}, \tag{20}$$

since $\tilde{S}^*(d\phi_i)(\Gamma_L) = 0$, with Γ_L the Euler-Lagrange vector field corresponding to the unconstrained Lagrangian system, which is also the Reeb vector field of the cosymplectic structure (dt, Ω_L) (cf. (17)). Recall that, in view of Lemma 5.5, for any 1-form (or any function) the corresponding Hamiltonian vector field, computed with respect to the cosymplectic structure (dt, Ω_L), coincides with the one computed with respect to the alternative cosymplectic structure (dt, ω_L).

If the constrained system verifies both the compatibility and admissibility condition, we conclude that

$$T_x(\mathbb{R} \times TQ) = T_xM \oplus \mathcal{F}_x\ ,$$

for all $x \in M$, which shows that we have a direct sum decomposition $T_M(\mathbb{R}\times TQ) = TM \oplus \mathcal{F}$. Herewith we can again associate two complementary projectors:

$$\mathcal{P} : T_M(\mathbb{R} \times TQ) \longrightarrow TM \qquad \mathcal{Q} : T_M(\mathbb{R} \times TQ) \longrightarrow \mathcal{F}.$$

One can then prove that the solution $\Gamma_{L,M}$ of the constrained dynamics is precisely obtained by projecting the (unconstrained) Euler-Lagrange vector field Γ_L onto M, i.e.,

$$\mathcal{P}(\Gamma_{L|M}) = \Gamma_{L,M}.$$

In local coordinates, the projector $\mathcal{P}$ is given by

$$\mathcal{P} = \mathrm{id} - \mathcal{C}^{ij} Z_i \otimes d\phi_j,$$

where $\mathcal{C}^{ij}$ are the entries of the inverse of the regular symmetric matrix $(\mathcal{C}_{ij}) = (Z_i(\phi_j))$ (for more details, see [18]).

Let us now consider the Poisson tensor field Λ_L on $\mathbb{R} \times TQ$, induced by the cosymplectic structure (dt, ω_L). In view of the above direct sum decomposition of $T_M(\mathbb{R} \times TQ)$ we can define the following tensor field along the constraint submanifold M:

$$(\Lambda_L)_{\mathcal{P}}(\alpha, \beta) = \Lambda_L(\mathcal{P}^*\alpha, \mathcal{P}^*\beta),$$

for arbitrary sections α, β of $T^*_M(\mathbb{R} \times TQ)$. Because of the regularity of the matrix $(\mathcal{C}_{ij})$, it is not difficult to see that this tensor field can be smoothly extended to a bivector field, denoted by Λ_{nh}, defined on a neigbourhood of M in $\mathbb{R} \times TQ$. The index 'nh' refers to 'nonholonomic'. This extension is clearly not unique, but we are only interested in its evaluation in points of M. The corresponding almost-Poisson bracket is denoted by $\{\ ,\ \}_{nh}$. For any two functions $F, G \in C^\infty(\mathbb{R} \times TQ)$ we then have

$$(\{F, G\}_{nh})_{|M} = (\Lambda_L)_{\mathcal{P}}((dF)_{|M}, (dG)_{|M}),$$

and this is a well-defined function on M, independent of the chosen extension Λ_{nh} of $(\Lambda_L)_{\mathcal{P}}$. The restriction of $\Lambda_{nh|M}(= (\Lambda_L)_{\mathcal{P}})$ to sections of T^*M induces an almost-Poisson structure on M which we shall denote by Λ_M. More precisely, we have at each point $x \in M$ and for arbitrary covectors $\alpha_x, \beta_x \in T^*_x M$ that

$$\Lambda_M(x)(\alpha_x, \beta_x) = \Lambda_{nh}(x)(\tilde{\alpha}_x, \tilde{\beta}_x),$$

where $\tilde{\alpha}_x$ and $\tilde{\beta}_x$ are arbitrary extensions of α_x and β_x, respectively, to $T^*_x(\mathbb{R} \times TQ)$. The almost-Poisson bracket of two functions $f, g \in C^\infty(M)$ then reads:

$$\{f, g\}_M = \Lambda_M(df, dg) = (\{F, G\}_{nh})_{|M}$$

where F and G are arbitrary smooth extensions of f and g, respectively, to $\mathbb{R} \times TQ$. This bracket is the analogue of the almost-Poisson bracket for time-independent nonholonomic systems introduced, within the Hamiltonian framework, by van der Schaft and Maschke [26] (see also [2, 5, 15, 22]).

Next, we define the following section of $T_M(\mathbb{R} \times TQ)$ over M:

$$R_M = \mathcal{P}(X_{\mathcal{Q}^*((dE_L)_{|M})}).$$

It is readily seen that R_M is in fact a section of TM, i.e. a vector field on M, and its expression in local coordinates is given by

$$R_M = -\left(\mathcal{C}^{ij}\frac{\partial \phi_i}{\partial v^A} v^A\right)_{|M} \left(X_{\phi_j} + \mathcal{C}^{lk}\{\phi_l, \phi_j\}_L Z_k\right)_{|M}.$$

From this expression it can be seen that if all the constraint functions ϕ_i are homogeneous functions in the velocities, the vector field R_M vanishes identically. This will be the case if the dilation vector field Δ is tangent to the constraint submanifold. In particular, R_M vanishes if we are dealing with linear nonholonomic constraints.

We can now derive an expression for the time evolution of an arbitrary function on the constraint submanifold along the integral curves of the constrained dynamics, in terms of the bracket $\{\ ,\ \}_M$.

Theorem 7.1 *For any function $F \in C^\infty(M)$ we have*

$$\Gamma_{L,M}(F) = \{F, E_{L|M}\}_M + R_M(F) + \mathcal{P}(\mathcal{R}_{L|M})(F), \tag{21}$$

or, equivalently,

$$\Gamma_{L,M}(F) = \{F, E_{L|M}\}_M + \mathcal{P}(\mathcal{E}_{\mathcal{Q}^*(dE_{L|M})})(F).$$

Proof: Let $\tilde{F}$ be an extension of F to $\mathbb{R} \times TQ$. Since all objects appearing in the subsequent computation are restricted to points of M, we will systematically omit, for ease of writing, the restriction sign $(|M)$. We have that

$$\begin{aligned}
\Gamma_{L,M}(F) &= [\mathcal{P}(\Gamma_L)](F) \\
&= \mathcal{P}^*(d\tilde{F})(\Gamma_L) \\
&= \omega_L(X_{\mathcal{P}^*(d\tilde{F})}, \Gamma_L) + \langle \mathcal{R}_L, \mathcal{P}^*(d\tilde{F}) \rangle \\
&= \mathcal{P}(\mathcal{R}_L)(\tilde{F}) + \omega_L(X_{\mathcal{P}^*(d\tilde{F})}, \Gamma_L).
\end{aligned}$$

Observe now that, along M,

$$\Gamma_L = \mathcal{E}_{E_L} = \mathcal{E}_{\mathcal{P}^*(dE_L)} + X_{\mathcal{Q}^*(dE_L)},$$

and thus

$$\begin{aligned}
\omega_L(X_{\mathcal{P}^*(d\tilde{F})}, \Gamma_L) &= \omega_L(X_{\mathcal{P}^*(d\tilde{F})}, \mathcal{E}_{\mathcal{P}^*(dE_L)}) + \omega_L(X_{\mathcal{P}^*(d\tilde{F})}, X_{\mathcal{Q}^*(dE_L)}) \\
&= \{F, E_L\}_M + R_M(F).
\end{aligned}$$

Therefore, we deduce that

$$\Gamma_{L,M}(F) = \{F, E_L\}_M + R_M(F) + \mathcal{P}(\mathcal{R}_L)(F).$$

∎

Remark 7.2 The right-hand side of the evolution equation (21) for an observable F consists of three distinguished parts: (i) the bracket term $\{F, E_{L|M}\}_M$, (ii) the term $R_M(F)$, which vanishes if the constraints are homogeneous, and (iii) the derivative of F with respect to the projection of the Reeb vector field $\mathcal{R}_L$. Note that the first two of these also appear in the case of time-independent nonholonomic systems (see [5]), whereas the appearance of the last term is characteristic for the time-dependent formalism.

In the next section, we will show that the bracket $\{\ ,\ \}_M$ on M is also induced by a Dirac tensor field corresponding to a cosymplectic distribution along M.

8 The Dirac tensor field for nonholonomic lagrangian systems

First we note that from the definitions of the Poisson tensor field Λ_L and the 2-form ω_L, one may deduce that

$$\Lambda_L(\tilde{S}^*(\alpha), \tilde{S}^*(\beta)) = \omega_L(X_{\tilde{S}^*(\alpha)}, X_{\tilde{S}^*(\beta)}) = 0, \tag{22}$$

for any two 1-forms α and β on $\mathbb{R} \times TQ$. This can be most easily verified using coordinate expressions, for instance. Consider now again the distribution $\mathcal{F}$ along M, introduced in the previous section, and recall that $\mathcal{F}$ is locally spanned by the vector fields $Z_i = X_{\tilde{S}^*(d\phi_i)} = -\sharp_{\Lambda_L}(\tilde{S}^*(d\phi_i))$. In particular, it follows that $\mathcal{F} = \sharp_{\Lambda_L}(\tilde{S}^*(TM)^o)$. If $\mathcal{W}$ denotes the distribution along M whose annihilator, $\mathcal{W}^o$, is given by $\tilde{S}^*((TM)^o)$, one can prove that

$$\mathcal{F} \subseteq \mathcal{W}.$$

Indeed, for any $\alpha \in \tilde{S}^*((TM)^o)(= \mathcal{W}^o)$ and for all Z_i $(1 \leq i \leq k)$, we have, taking into account (22),

$$< Z_i, \alpha >= \Lambda_L(\tilde{S}^*(d\phi_i), \alpha) = 0,$$

from which it readily follows that $\mathcal{F} \subseteq \mathcal{W}$.

In [18] it was proved that the compatibility condition for the nonholonomic constraints implies that

$$\mathcal{D} = \mathcal{W} \cap TM$$

is a cosymplectic distribution along M for the cosymplectic structure (dt, Ω_L). It is then easy to verify that $\mathcal{D}$ is also a cosymplectic distribution for the cosymplectic structure (dt, ω_L). Therefore, in view of Proposition 5.1, we have the decomposition

$$T_M(\mathbb{R} \times TQ) = \mathcal{D} \oplus \sharp_{\Lambda_L}(\mathcal{D}^o).$$

Let $\tilde{\mathcal{P}}$ denote the corresponding projector onto $\mathcal{D}$. Observe that the 1-forms $\{\tilde{S}^*(d\phi_i), d\phi_i; 1 \leq i \leq k\}$, restricted to M, define a local basis of $\mathcal{D}^o$. Since $\mathcal{D}^o \cap \ker(\sharp_{\Lambda_L}) = 0$, it follows that the vector fields $\{Z_i, X_{\phi_i}; 1 \leq i \leq k\}$ determine a local basis of the distribution $\sharp_{\Lambda_L}(\mathcal{D}^o)$ (see (20)). The projector $\tilde{\mathcal{P}}$ is then locally given by

$$\tilde{\mathcal{P}} = \mathrm{id} - \mathcal{C}^{ij}\mathcal{C}^{kl}\{\phi_l, \phi_j\}_L Z_i \otimes \tilde{S}^*(d\phi_k) + \mathcal{C}^{ij} X_{\phi_i} \otimes \tilde{S}^*(d\phi_j) - \mathcal{C}^{ij} Z_i \otimes d\phi_j.$$

It should be emphasised that, in general, the dynamics of the given nonholonomic system will not be recovered by projection of the unconstrained Euler-Lagrange vector field under $\tilde{\mathcal{P}}$, i.e. $\Gamma_{L,M} \neq \tilde{\mathcal{P}}(\Gamma_{L|M})$. As in the autonomous case (cf. [5]), however, one can prove that the equality will hold if the constraint submanifold M can be (locally) described as the common zero level set of functions ϕ_i that are homogeneous in the velocities (for instance, in the case of purely linear constraints).

We know from the discussion in Section 4 that the Poisson tensor Λ_L and the projector $\tilde{\mathcal{P}}$ induce a Dirac tensor field $(\Lambda_L)_{\mathcal{D}}$ along the constraint submanifold M, which restricts to an almost-Poisson tensor field $(\Lambda_L)_{M,\mathcal{D}}$ on M. We will now show that the latter coincides with the almost-Poisson tensor Λ_M introduced in the previous section. Indeed, from the above we know that the vector bundle $T_M(\mathbb{R} \times TQ)$ admits the following two direct sum decompositions

$$T_M(\mathbb{R} \times TQ) = TM \oplus \mathcal{F} \quad \text{and} \quad T_M(\mathbb{R} \times TQ) = \mathcal{D} \oplus \sharp_{\Lambda_L}(\mathcal{D}^o),$$

with $\mathcal{F} = \sharp_{\Lambda_L}(\mathcal{W}^o)$, $\mathcal{D} = TM \cap \mathcal{W}$ and whereby $\sharp_{\Lambda_L}(\mathcal{W}^o) \subseteq \mathcal{W}$ and $(TM \cap \mathcal{W})^o \cap \ker(\sharp_{\Lambda_L}) = 0$. From Proposition 3.5, and from the related discussion in Section 4, we can then immediately derive the following result (with $(\Lambda_L)_{\mathcal{P}}$ the almost-Poisson structure constructed in the previous section).

Proposition 8.1 *Along the constraint submanifold M, the almost-Poisson tensor field $(\Lambda_L)_{\mathcal{P}}$ and the Dirac tensor field $(\Lambda_L)_{\mathcal{D}}$ coincide. In particular, they induce the same almost-Poisson structure (and, hence, the same almost-Poisson bracket) on M, i.e.*

$$\Lambda_M \equiv (\Lambda_L)_{M,\mathcal{D}}.$$

Remark 8.2 From the results of Section 4 and Section 5, we also deduce that $(\Lambda_L)_{M,\mathcal{D}}$ is a Poisson tensor if and only if the distribution $\bar{\mathcal{D}}$ on M, given by

$$x \in M \longrightarrow \bar{\mathcal{D}}_x = \{X \in \mathcal{D}_x / dt_x(X) = 0\} \subseteq T_x M,$$

is completely integrable.

In what follows we will be concerned with the conversion of the preceeding analysis to the Hamiltonian framework.

9 Hamiltonian formulation

The connection between the Lagrangian and the Hamiltonian description of a system with Lagrangian $L \in C^\infty(\mathbb{R} \times TQ)$ is established by the Legendre transformation

$$leg : \mathbb{R} \times TQ \longrightarrow \mathbb{R} \times T^*Q, (t, q^A, v^A) \longmapsto (t, q^A, p_A = \frac{\partial L}{\partial v^A})$$

(see, for instance, [8] for a global (intrinsic) definition). One can easily check that L is regular if and only if leg is a local diffeomorphism. In that case, the Lagrangian and Hamiltonian formulations are locally equivalent.

For simplicity we shall always assume here that L is hyperregular, that is, leg is a global diffeomorphism (which is the usual case in mechanics). The Hamiltonian function $H : \mathbb{R} \times T^*Q \longrightarrow \mathbb{R}$ is then defined by $H = E_L \circ leg^{-1}$. Denote by ω_Q the canonical symplectic form on T^*Q: $\omega_Q = dq^A \wedge dp_A$. Its pull-back to $\mathbb{R} \times T^*Q$ by means of the canonical projection $\mathbb{R} \times T^*Q \longrightarrow T^*Q$, will also be denoted by ω_Q. The pair (dt, ω_Q) then defines a cosymplectic structure on $\mathbb{R} \times T^*Q$ with Reeb vector field $\partial/\partial t$. Next, consider the 2-form $\Omega_H = \omega_Q + dH \wedge dt$. By Lemma 5.5 we know that (dt, Ω_H) is also a cosymplectic structure. Via the Legendre map, we have that

$$leg^*\omega_Q = \omega_L, \quad \text{and} \quad leg^*\Omega_H = \Omega_L.$$

It follows that the evolution vector field $\mathcal{E}_H$, with respect to the cosymplectic structure (dt, ω_Q), is leg-related with Γ_L, the Euler-Lagrange vector field corresponding to L. The equations of motion corresponding to $\mathcal{E}_H$ are precisely the Hamilton equations.

We now consider a nonholonomic system, described by a hyperregular Lagrangian L and a constraint submanifold $M \subset \mathbb{R} \times TQ$, such that both the admissibility and compatibility condition are satisfied. Denote by $\bar{M}$ the image of M under the Legendre transformation. If M is again locally defined by k independent functions ϕ_i, then the constraint functions on $\mathbb{R} \times T^*Q$, describing $\bar{M}$, read $\Psi_i = \phi_i \circ leg^{-1}$, i.e.

$$\Psi_i(t, q^A, p_A) = \phi_i(t, q^A, \frac{\partial H}{\partial p_A}).$$

If $\mathcal{H}_{AB}$ are the components of the inverse matrix of the regular matrix $(\mathcal{H}^{AB} = \dfrac{\partial^2 H}{\partial p_A \partial p_B})$ then, in terms of the canonical coordinates, the equations of motion of the nonholonomic system on $\mathbb{R} \times T^*Q$ can be written as follows:

$$\begin{aligned} \dot{q}^A &= \frac{\partial H}{\partial p_A} \\ \dot{p}_A &= -\frac{\partial H}{\partial q^A} - \bar{\lambda}^i \mathcal{H}_{AB} \frac{\partial \Psi_i}{\partial p_B}, \end{aligned}$$

together with the constraint equations

$$\Psi_i(t, q^A, p_A) = 0,$$

and where the $\bar{\lambda}^i$ are Lagrange multipliers. An intrinsic representation of the equations of motion is obtained as follows (see e.g. [18]). Consider the codistribution $(leg^{-1})^*(\tilde{S}^*((TM)^o))$, which is locally generated by the independent 1-forms

$$\mu_i = \frac{\partial \Psi_i}{\partial p_A} \mathcal{H}_{AB}(dq^B - \frac{\partial H}{\partial p_B} dt) \ , \ 1 \leq i \leq k,$$

and let $\bar{\mathcal{W}}$ be the distribution along $\bar{M}$ such that $\bar{\mathcal{W}}^o = (leg^{-1})^*(\tilde{S}^*((TM)^o))$. The Hamilton equations of motion of the nonholonomic system can then be rewritten in intrinsic form as

$$\left\{ \begin{array}{l} (i_Y \Omega_H)_{|\bar{M}} \in \bar{\mathcal{W}}^o \\ i_Y dt = 1 \\ Y_{|\bar{M}} \in T\bar{M} \end{array} \right.$$

or, equivalently,

$$\left\{ \begin{array}{l} (i_Y(\omega_Q + dH \wedge dt))_{|\bar{M}} \in \bar{\mathcal{W}}^o \\ i_Y dt = 1 \\ Y_{|\bar{M}} \in T\bar{M} \end{array} \right. \tag{23}$$

Remark 9.1 An alternative derivation of the Hamilton equations of motion in the case of time-dependent nonholonomic systems, which immediately starts at the level of the constraint submanifold and does not necessarily require the given Lagrangian to be regular, can be found in [25].

The further analysis now proceeds in full analogy to the one carried out on the Lagrangian side. Let $\bar{\mathcal{F}}$ be the distribution on $\mathbb{R} \times T^*Q$ along $\bar{M}$, defined by

$$\bar{\mathcal{F}} = leg_*(\mathcal{F}) = leg_*(\sharp_{\Lambda_L}(\tilde{S}^*(TM)^o)).$$

The assumed admissibility and compatibility of the constraints imply that we have a direct sum decomposition of $T_{\bar{M}}(\mathbb{R} \times T^*Q)$:

$$T_{\bar{M}}(\mathbb{R} \times T^*Q) = T\bar{M} \oplus \bar{\mathcal{F}}$$

and, hence, one can define two complementary projectors

$$\begin{array}{r} \bar{\mathcal{P}} : T_{\bar{M}}(\mathbb{R} \times T^*Q) \longrightarrow T\bar{M} \\ \bar{\mathcal{Q}} : T_{\bar{M}}(\mathbb{R} \times T^*Q) \longrightarrow \bar{\mathcal{F}}. \end{array}$$

In the sequel, all Hamiltonian and evolution vector fields are always defined with respect to the cosymplectic structure (dt, ω_Q). It can then be checked that a solution of the equations of motion (23) for the constrained system is given by the vector field $Y = \mathcal{E}_{H,\bar{M}}$, which is obtained by projection of the evolution vector field describing the unconstrained Hamiltonian system, i.e.

$$\mathcal{E}_{H,\bar{M}} = \bar{\mathcal{P}}((\mathcal{E}_H)_{|M}).$$

Next, the Poisson tensor field Λ_{dt,ω_Q} and the projector $\bar{\mathcal{P}}$ give rise to an almost-Poisson structure $(\Lambda_{dt,\omega_Q})_{\bar{\mathcal{P}}}$ along $\bar{M}$, which can be smoothly extended to a bivector field $\Lambda_{\bar{nh}}$ on an open neighbourhood of $\bar{M}$, with associated bracket $\{\ ,\ \}_{\bar{nh}}$. This extension is not unique, but for any two smooth functions $\tilde{F}, \tilde{G}$ on $\mathbb{R} \times T^*Q$ we have

$$(\{\tilde{F}, \tilde{G}\}_{\bar{nh}})_{|\bar{M}} = \Lambda_{dt,\omega_Q}(\bar{\mathcal{P}}^*(d\tilde{F}), \bar{\mathcal{P}}^*(d\tilde{G}))_{|\bar{M}},$$

which is a well-defined function on $\bar{M}$, independent of the chosen extension of $(\Lambda_{dt,\omega_Q})_{\bar{\mathcal{P}}}$. An almost-Poisson structure on $\bar{M}$, denoted by $\Lambda_{\bar{M}}$, can then be obtained by restricting the action of $\Lambda_{\bar{nh}}$ to 1-forms on $\bar{M}$. The corresponding almost-Poisson bracket is denoted by $\{\ ,\ \}_{\bar{M}}$ and for any two functions $F, G \in C^\infty(\bar{M})$ we then have

$$\{F, G\}_{\bar{M}} = (\{\tilde{F}, \tilde{G}\}_{\bar{nh}})_{|\bar{M}}$$

for arbitrary extensions $\tilde{F}$ and $\tilde{G}$ of F and G, respectively. As on the Lagrangian side, one can again show that this bracket can also be derived from a Dirac tensor field along $\bar{M}$, corresponding to the cosymplectic distribution $T\bar{M} \cap \bar{\mathcal{W}}$.

Introducing the vector field $R_{\bar{M}}$ on $\bar{M}$, given by $R_{\bar{M}} = \bar{\mathcal{P}}(X_{\bar{\mathcal{Q}}^*((dH)_{|\bar{M}})})$, the Hamiltonian counterpart of Theorem 7.1 reads as follows.

Theorem 9.2 *For any function $F \in C^\infty(\bar{M})$ we have that*

$$\dot{F} = \mathcal{E}_{H,\bar{M}}(F) = \{F, H_{|\bar{M}}\}_{\bar{M}} + \bar{\mathcal{P}}\left(\frac{\partial}{\partial t}\right)_{|\bar{M}}(F) + R_{\bar{M}}(F),$$

or, equivalently,

$$\dot{F} = \{F, H_{|\bar{M}}\}_{\bar{M}} + \bar{\mathcal{P}}(\mathcal{E}_{\bar{\mathcal{Q}}^*(dH)_{|\bar{M}}})(F).$$

Concerning the structure of the equation describing the evolution of an observable in the Hamiltonian setting, the same remark can be made as on the Lagrangian side (cf. the remark following Theorem 7.1).

Finally, from the above constructions one can easily infer that the *leg*-map induces an almost-Poisson morphism between (M, Λ_M) and $(\bar{M}, \Lambda_{\bar{M}})$, i.e.

$$(leg_{|M})^*\{F, G\}_{\bar{M}} = \{(leg_{|M})^*F, (leg_{|M})^*G\}_M$$

for arbitrary $F, G \in C^\infty(\bar{M})$. In addition, we have that $\Gamma_{L,M}$ and $\mathcal{E}_{H,\bar{M}}$ are *leg*-related, i.e.

$$(leg_{|M}) * (\Gamma_{L,M}(x)) = \mathcal{E}_{H,\bar{M}}(leg(x)), \qquad \text{for all } x \in M.$$

10 A particular case: affine constraints

In this section we will treat the case where the constraint submanifold M is an affine subbundle of $\mathbb{R}\times TQ$ over $\mathbb{R}\times Q$, and the Lagrangian L is polynomial of degree two in the velocities. This is the case usually encountered in mechanics. Under these conditions, and assuming (hyper-)regularity of L, it easily follows that the constraint submanifold $\bar{M} = leg(M)$ is also an affine subbundle of $\mathbb{R} \times T^*Q$. The purpose now is to show that, in this case, one can construct local coordinates adapted to the submanifold $\bar{M}$, in terms of which the constrained Hamilton equations of motion can be written without Lagrange multipliers, i.e. we obtain a direct representation of the equations of motion on $\bar{M}$. Up to some technicalities, the construction mimics the one applied in the autonomous case (see [5, 26]).

To fix the ideas, let g be a symmetric covariant tensor field of order 2 on $\mathbb{R}\times Q$ which restricts to a time-dependent Riemannian metric on Q. Consider then a system with Lagrangian $L : \mathbb{R} \times TQ \longrightarrow \mathbb{R}$ of the form

$$L(t, q, v) = \frac{1}{2} g\left((1, v), (1, v)\right) - V(t, q),$$

where $V : \mathbb{R} \times Q \longrightarrow \mathbb{R}$ represents the potential energy of the system and $(1, v)$ is a tangent vector at $(t, q) \in \mathbb{R} \times Q$ with time component 1. Assume that the system is subjected to affine nonholonomic constraints, described by the vanishing of k independent functions of the form

$$\phi_i(t, q, v) = \mu_{iA}(t, q)v^A - h_i(t, q), \quad i = 1, \dots, k.$$

This determines an affine subbundle M of $\tau_Q : \mathbb{R} \times TQ \longrightarrow \mathbb{R} \times Q$. If D is the vector subbundle of $\mathbb{R} \times TQ$ (over $\mathbb{R} \times Q$) on which M is modelled then, for any fixed i, the μ_{iA} may be regarded as the components of a time-dependent 1-form μ_i on Q in such a way that the set $\{\mu_i\}_{i=1,\dots,k}$ is a local basis of the annihilator D^o of D in $\mathbb{R} \times T^*Q$.

The nonholonomic system under consideration is both admissible and compatible. In fact, the components of the matrix $\mathcal{C}$ are here given by

$$\mathcal{C}_{ij} = -\mu_{iA} g^{AB} \mu_{jB},$$

where the g^{AB} are the components of the inverse of the positive definite matrix $(g_{AB} = g(\frac{\partial}{\partial q^A}, \frac{\partial}{\partial q^B}))$.

The Hamiltonian of the system is of the form

$$H = \frac{1}{2} g^{AB}(t, q) p_A p_B + k^A(t, q) p_A + \bar{V}(t, q).$$

The constraint functions on $\mathbb{R} \times T^*Q$ read

$$\Psi_i = \mu_{iA}(g^{AB} p_B + k^A) - h_i, \quad 1 \leq i \leq k$$

or

$$\Psi_i = \mu_{iA} g^{AB} p_B + \mathcal{A}_i, \quad 1 \leq i \leq k$$

where, for notational convenience, we have put $\mathcal{A}_i = \mu_{iA} k^A - h_i$.

In order to obtain an explicit coordinate representation of the equations of motion on the constraint submanifold, from which the Lagrange multipliers have been eliminated, we will consider a (non-canonical!) transformation $(t, q^A, p_A) \longrightarrow (t, q^A, \tilde{p}_A(t, q, p))$ with the following time-dependent change of momenta (see [5, 26] for the time-independent case)

$$\begin{aligned} \tilde{p}_\alpha &= X_\alpha^A p_A \\ \tilde{p}_{n-k+i} &= Y_i^A p_A - \mathcal{C}^{ij}(\mu_{jA} k^A - h_j) = Y_i^A p_A - \mathcal{C}^{ij} \mathcal{A}_j \end{aligned}$$

for $1 \leq \alpha \leq n - k$ and $1 \leq i \leq k$. Here, $Y_i^A = -\mathcal{C}^{ij} \mu_{jB} g^{BA}$ and X_α^A is any set of functions satisfying $X_\alpha^A \mu_{iA} = 0$ for all indices α and i, and such that the X_α^A and Y_i^A form the components of a regular $n \times n$ matrix. The functions X_α^A can be regarded as the components of $n - k$ independent, time-dependent vector fields X_α on Q, which (locally) span the vector subbundle D. The functions Y_i^A are then the components of k time-dependent vector fields on Q, spanning a complementary subbundle of D and such that $< Y_i, \mu_j >= \delta_{ij}$. The inverse of this transformation is

$$p_A = \nu_A^\alpha \tilde{p}_\alpha + \mu_{iA}(\delta^{ij} \tilde{p}_{n-k+j} + \mathcal{C}^{ij} \mathcal{A}_j)$$

where the ν_A^α are the components of time-dependent 1-forms ν^α on Q which, together with the μ_i, constitute a local basis of 1-forms, dual to the basis of vector fields (X_α, Y_i). From the definition of the new fibre coordinates one can readily see that

$$\tilde{p}_{n-k+i} = -\mathcal{C}^{ij} \Psi_j$$

and, hence, in view of the regulartity of $(\mathcal{C}^{ij})$, the constraint submanifold is described by the vanishing of the k 'momenta' $\tilde{p}_{n-k+i}$. In particular, it follows from this that the set $(t, q^A, \tilde{p}_\alpha)$ determines a local coordinate chart on the constraint submanifold $\bar{M}$.

Remark 10.1 On a direct product structure, such as $\mathbb{R} \times T^*Q$, transformations should, strictly speaking, not mix time and phase space coordinates. However, since we are dealing here with a purely local description of the constraint submanifold, there is no harm in considering time-dependent transformations on T^*M.

In terms of the new coordinates $(t, q^A, \tilde{p}_\alpha, \tilde{p}_{n-k+i})$, the matrix representation of the Poisson tensor field Λ_{dt,ω_Q} becomes:

$$\Lambda_{dt,\omega_Q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \{q^A, q^B\} & \{q^A, \tilde{p}_\beta\} & \{q^A, \tilde{p}_{n-k+j}\} \\ 0 & \{\tilde{p}_\alpha, q^B\} & \{\tilde{p}_\alpha, \tilde{p}_\beta\} & \{\tilde{p}_\alpha, \tilde{p}_{n-k+j}\} \\ 0 & \{\tilde{p}_{n-k+i}, q^B\} & \{\tilde{p}_{n-k+i}, \tilde{p}_\beta\} & \{\tilde{p}_{n-k+i}, \tilde{p}_{n-k+j}\} \end{pmatrix}$$

where, of course, the brackets $\{q^A, q^B\}$ are identically zero. As in the autonomous case (see [5, 26]), a matrix representation of the almost-Poisson structure $\Lambda_{\bar{M}}$ in terms of the coordinates $(t, q^A, \tilde{p}_\alpha)$ on the constraint submanifold, is obtained by deleting the last k rows and the last k columns from the above matrix, and then restricting the remaining components to points of $\bar{M}$. The latter is achieved by putting the variables $\tilde{p}_{n-k+i}$ everywhere equal to zero.

Putting $\tilde{H}(t,q,\tilde{p}) = H(t,q,p(t,q,\tilde{p}))$, the constrained equations of motion can first be written as

$$
\begin{aligned}
\dot{q}^A &= \{q^A, q^B\}\frac{\partial \tilde{H}}{\partial q^B} + \{q^A, \tilde{p}_\beta\}\frac{\partial \tilde{H}}{\partial \tilde{p}_\beta} + \{q^A, \tilde{p}_{n-k+j}\}\frac{\partial \tilde{H}}{\partial \tilde{p}_{n-k+j}} \\
\dot{\tilde{p}}_\alpha &= \{\tilde{p}_\alpha, q^B\}\frac{\partial \tilde{H}}{\partial q^B} + \{\tilde{p}_\alpha, \tilde{p}_\beta\}\frac{\partial \tilde{H}}{\partial \tilde{p}_\beta} + \{\tilde{p}_\alpha, \tilde{p}_{n-k+j}\}\frac{\partial \tilde{H}}{\partial \tilde{p}_{n-k+j}} \\
&\quad + \frac{\partial X^A_\alpha}{\partial t}\left(\nu^\beta_A \tilde{p}_\beta + \mu_{iA}(\delta^{ij}\tilde{p}_{n-k+j} + \mathcal{C}^{ij}\mathcal{A}_j)\right) \\
\dot{\tilde{p}}_{n-k+i} &= \{\tilde{p}_{n-k+i}, q^B\}\frac{\partial \tilde{H}}{\partial q^B} + \{\tilde{p}_{n-k+i}, \tilde{p}_\beta\}\frac{\partial \tilde{H}}{\partial \tilde{p}_\beta} + \{\tilde{p}_{n-k+i}, \tilde{p}_{n-k+j}\}\frac{\partial \tilde{H}}{\partial \tilde{p}_{n-k+j}} \\
&\quad + \frac{\partial Y^A_i}{\partial t}\left(\nu^\alpha_A \tilde{p}_\alpha + \mu_{jA}(\delta^{jl}\tilde{p}_{n-k+l} + \mathcal{C}^{jl}\mathcal{A}_l)\right) - \frac{\partial(\mathcal{C}^{ij}\mathcal{A}_j)}{\partial t} - \bar{\lambda}^i
\end{aligned}
$$

togehter with the constraint equations $\tilde{p}_{n-k+i} = 0$, $1 \leq i \leq k$. The brackets on the right-hand side are still the Poisson brackets induced by the Poisson tensor Λ_{dt,ω_Q}. If we denote by $H_{\bar{M}}$ the restriction of H to $\bar{M}$, then we have along the constraint submanifold:

$$
\left(\frac{\partial \tilde{H}}{\partial q^A}\right)_{|\bar{M}} = \frac{\partial H_{\bar{M}}}{\partial q^A}, \qquad \left(\frac{\partial \tilde{H}}{\partial \tilde{p}_\alpha}\right)_{|\bar{M}} = \frac{\partial H_{\bar{M}}}{\partial \tilde{p}_\alpha}, \qquad \left(\frac{\partial \tilde{H}}{\partial \tilde{p}_{n-k+i}}\right)_{|\bar{M}} = h_i \ .
$$

Taking into account the expressions for the almost-Poisson brackets $\{\ ,\ \}_{\bar{M}}$ of the coordinate functions $(q^A, \tilde{p}_\alpha)$ the equations of motion on $\bar{M}$ become

$$
\begin{aligned}
\begin{pmatrix} \dot{q}^A \\ \dot{\tilde{p}}_\alpha \end{pmatrix} &= \begin{pmatrix} \{q^A, q^B\}_{\bar{M}} & \{q^A, \tilde{p}_\beta\}_{\bar{M}} \\ \{\tilde{p}_\alpha, q^B\}_{\bar{M}} & \{\tilde{p}_\alpha, \tilde{p}_\beta\}_{\bar{M}} \end{pmatrix} \begin{pmatrix} \frac{\partial H_{\bar{M}}}{\partial q^B} \\ \frac{\partial H_{\bar{M}}}{\partial \tilde{p}_\beta} \end{pmatrix} \\
&\quad + \begin{pmatrix} \{q^A, \tilde{p}_{n-k+i}\}h_i \\ \{\tilde{p}_\alpha, \tilde{p}_{n-k+i}\}h_i + \frac{\partial X^A_\alpha}{\partial t}\left(\nu^\beta_A \tilde{p}_\beta + \mu_{iA}\mathcal{C}^{ij}\mathcal{A}_j\right) \end{pmatrix}_{|\bar{M}}
\end{aligned}
$$

where the restriction sign '$_{|\bar{M}}$' on the right-hand side means that, after computing the brackets, one has to put the $\tilde{p}_{n-k+i}$ everywhere equal to zero.

Observe that the projector $\bar{\mathcal{P}}$ from Section 9, is here given by

$$
\bar{\mathcal{P}} = \mathrm{id} - \frac{\partial}{\partial \tilde{p}_{n-k+i}} \otimes d\tilde{p}_{n-k+i}
$$

and, in the coordinates $(t, q^A, \tilde{p}_\alpha)$, the vector field $R_{\bar{M}}$ then reads

$$R_{\bar{M}} = \bar{\mathcal{P}}(X_{\bar{\mathcal{Q}}^*(dH)}) = \left(\{q^A, \tilde{p}_{n-k+i}\}h_i\right)_{|\bar{M}} \frac{\partial}{\partial q^A} + \left(\{\tilde{p}_\alpha, \tilde{p}_{n-k+i}\}h_i\right)_{|M} \frac{\partial}{\partial \tilde{p}_\alpha}.$$

In these coordinates, the projection by $\bar{\mathcal{P}}$ of the Reeb vector field $\partial/\partial t$ of the cosymplectic structure (dt, ω_Q) becomes

$$\frac{\partial}{\partial t} + \frac{\partial X^A_\alpha}{\partial t}\left(\nu^\beta_A \tilde{p}_\beta + \mu_{iA}\mathcal{C}^{ij}\mathcal{A}_j\right)\frac{\partial}{\partial \tilde{p}_\alpha}.$$

With these expressions for $R_{\bar{M}}$ and $\mathcal{P}(\partial/\partial t)$, we can also recover the above equations of motion from Theorem 9.2.

In the last section we will treat two examples to illustrate some of the previous results.

11 Examples

Example 11.1 Consider a particle with mass $m = 1$ moving in a plane and subject to the constraint

$$\phi = \dot{x} - t\dot{y} - a = 0$$

for some $a \in \mathbb{R}$. The constraint is affine in the velocities and the 1-form

$$\mu = dx - tdy$$

defines a 3-dimensional time-dependent vector subbundle of $T\mathbb{R}^2 = \mathbb{R}^4$.

Suppose that there are no external forces acting on the particle, such that we may put $V = 0$. The kinetic energy of the unconstrained system is given by

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$$

and its Hamiltonian then reads

$$H = \frac{1}{2}(p_x^2 + p_y^2).$$

The constraint equation becomes $\Psi = p_x - tp_y - a = 0$, which defines a submanifold $\bar{M}$ of $\mathbb{R} \times T^*\mathbb{R}^2$. In this case, the matrix $\mathcal{C}$ is 1×1 and is given by the scalar function

$$\mathcal{C} = -(1 + t^2).$$

Proceeding as in Section 10, we consider the following time-dependent change of the momenta:

$$\tilde{p}_1 = tp_x + p_y, \qquad \tilde{p}_2 = \frac{1}{1 + t^2}(p_x - tp_y - a),$$

such that the constraint equation can be written as $\tilde{p}_2 = 0$ and $(t, x, y, \tilde{p}_1)$ will be adapted coordinates for the constraint submanifold $\bar{M}$. The restriction of the Hamiltonian to $\bar{M}$ here becomes

$$H_{\bar{M}} = \frac{1}{2(1+t^2)}\left(\tilde{p}_1^2 + a^2\right).$$

In terms of the adapted coordinates, the almost-Poisson structure on $\bar{M}$ is determined by

$$\{x, \tilde{p}_1\}_{\bar{M}} = t, \qquad \{y, \tilde{p}_1\}_{\bar{M}} = 1,$$

all the other brackets being zero.

In order to obtain the equations of motion on $\bar{M}$ we construct the following vector field:

$$\bar{\mathcal{P}}(\mathcal{E}_{\bar{Q}^*(dH)_{|\bar{M}}}) \;=\; \frac{\partial}{\partial t} + \frac{a}{1+t^2}\frac{\partial}{\partial x} - \frac{at}{1+t^2}\frac{\partial}{\partial y} + \frac{1}{1+t^2}(t\tilde{p}_1 + a)\frac{\partial}{\partial \tilde{p}_1}.$$

From Theorem 9.2 we then know that the equations of motion of the constrained system follow from:

$$\dot{F} = \bar{\mathcal{P}}(\mathcal{E}_{\bar{Q}^*(dH)_{|\bar{M}}})(F) + \{F, H_{\bar{M}}\}_{\bar{M}}.$$

For the system under consideration we therefore obtain:

$$\dot{x} = \frac{t\tilde{p}_1 + a}{1+t^2}, \quad \dot{y} = \frac{\tilde{p}_1 - at}{1+t^2}, \quad \dot{\tilde{p}}_1 = \frac{t\tilde{p}_1 + a}{1+t^2},$$

which can be easily integrated.

Example 11.2 A rolling ball on a rotating table with time-dependent angular velocity. A (homogeneous) sphere of radius r and unit mass ($m = 1$) rolls without sliding on a horizontal table which rotates with non constant angular velocity $\Omega(t)$ about a vertical axis through one of its points. Apart from the constant gravitational force, no other external forces are assumed to act on the sphere. The case of constant angular velocity has been treated in [5], to which we also refer for more references to this problem. Since the time-dependence of Ω only induces minor changes as far as the computations are concerned, we will not spell them out in full detail.

Choose a cartesian reference frame with origin at the center of rotation of the table and z-axis along the rotation axis. Let (x, y) denote the position of the point of contact of the sphere with the table. The configuration space for the sphere on the table is $Q = \mathbb{R}^2 \times \mathrm{SO}(3)$ where $\mathrm{SO}(3)$ may be parametrized by the Eulerian angles φ, θ, ψ. The kinetic energy of the sphere is then given by

$$T = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + k^2(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta)\right)$$

and the potential energy being a constant, we may put $V = 0$. The constraint equations are:

$$\dot{x} - r\dot{\theta}\sin\psi + r\dot{\varphi}\sin\theta\cos\psi = -\Omega(t)y$$
$$\dot{y} + r\dot{\theta}\cos\psi + r\dot{\varphi}\sin\theta\sin\psi = \Omega(t)x$$

or, after some algebraic manipulations,

$$\dot{x}\cos\psi + \dot{y}\sin\psi + r\dot{\varphi}\sin\theta = \Omega(t)x\sin\psi - \Omega(t)y\cos\psi$$
$$\dot{x}\sin\psi - \dot{y}\cos\psi - r\dot{\theta} = -\Omega(t)y\sin\psi - \Omega(t)x\cos\psi\,.$$

We can proceed from here to construct the equations of motion of the sphere, following the general theory but, as in the autonomous case, the use of the Eulerian angles as part of the coordinates will lead to very complicated expressions (cf. [5]). Instead, one may choose to exploit the symmetry of the problem, and one way to do this is by the use of appropriate 'quasi-coordinates'.

First of all, observe that the kinetic energy, when expressed in terms of angular velocities, reads:

$$T = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + k^2(\omega_x^2 + \omega_y^2 + \omega_z^2)\right)$$

where $(\omega_x, \omega_y, \omega_z)$ are the components of the angular velocity of the sphere. The constraint equations, expressing the rolling conditions can be rewritten as:

$$\dot{x} - r\omega_y = -\Omega(t)y$$
$$\dot{y} + r\omega_x = \Omega(t)x.$$

We now introduce quasi-coordinates (q_1, q_2, q_3), adapted to the symmetry of the problem, by putting

$$\dot{q}_1 = \omega_x,\ \dot{q}_2 = \omega_y,\ \dot{q}_3 = \omega_z.$$

As it is well known, the coordinates q^i only have a symbolic meaning (cf. [5]).

The Hamiltonian of the sphere can be written as:

$$H = \frac{1}{2k^2}\left(k^2p_x^2 + k^2p_y^2 + p_1^2 + p_2^2 + p_3^2\right)$$

where $p_x = \dot{x}, p_y = \dot{y}, p_1 = k^2\omega_x, p_2 = k^2\omega_y, p_3 = k^2\omega_z$. The variables p_1, p_2, p_3 are not canonical momenta, but they are linear combinations of the momenta conjugated to the Eulerian angles. On the (extended) phase space we introduce new fibre coordinates

$$\begin{aligned} \hat{p}_1 &= rp_x + p_2 \\ \hat{p}_2 &= rp_y - p_1 \\ \hat{p}_3 &= p_3 \end{aligned}$$

and

$$\begin{aligned}
\hat{p}_4 &= \frac{k^2}{k^2+r^2}\left(p_x - \frac{rp_2}{k^2} + \Omega(t)y\right)\\
\hat{p}_5 &= \frac{k^2}{k^2+r^2}\left(p_y + \frac{r}{k^2}p_1 - \Omega(t)x\right) .
\end{aligned}$$

In terms of these, the constraint equations, describing the constraint submanifold $\bar{M}$ of $\mathbb{R}\times T^*(\mathbb{R}^2\times \mathrm{SO}(3))$, become:

$$\begin{aligned}
\hat{p}_4 &= 0\\
\hat{p}_5 &= 0 .
\end{aligned}$$

When constructing the almost-Poisson structure on $\bar{M}$, we find that the only non-zero 'fundamental' brackets are:

$$\begin{array}{ll}
\{x,\hat{p}_1\}_{\bar{M}} = r & \{q_2,\hat{p}_1\}_{\bar{M}} = 1\\
\{y,\hat{p}_2\}_{\bar{M}} = r & \{q_1,\hat{p}_2\}_{\bar{M}} = -1\\
\{q_3,\hat{p}_3\}_{\bar{M}} = 1 &
\end{array}$$

in which the appropriate 'operational' meaning has to be attached to the quasi-coordinates q^i. The projector $\bar{\mathcal{P}}$ is

$$\bar{\mathcal{P}} = \mathrm{id} - \frac{\partial}{\partial \tilde{p}_4}\otimes d\tilde{p}_4 - \frac{\partial}{\partial \tilde{p}_5}\otimes d\tilde{p}_5$$

and the vector field $\bar{\mathcal{P}}(\mathcal{E}_{\bar{\mathcal{Q}}^*(dH)_{|\bar{M}}})$ then becomes

$$\begin{aligned}
\bar{\mathcal{P}}(\mathcal{E}_{\bar{\mathcal{Q}}^*(dH)_{|\bar{M}}}) &= \frac{\partial}{\partial t} - \frac{\Omega(t)k^2y}{k^2+r^2}\frac{\partial}{\partial x} + \frac{\Omega(t)k^2x}{k^2+r^2}\frac{\partial}{\partial y}\\
&\quad + \frac{r\Omega(t)x}{k^2+r^2}\frac{\partial}{\partial q_1} + \frac{r\Omega(t)y}{k^2+r^2}\frac{\partial}{\partial q_2}\\
&\quad + \frac{rk^2\Omega^2(t)x}{k^2+r^2}\frac{\partial}{\partial \hat{p}_1} + \frac{rk^2\Omega^2(t)y}{k^2+r^2}\frac{\partial}{\partial \hat{p}_2} .
\end{aligned}$$

The restriction of the Hamiltonian H to $\bar{M}$, expressed in the adapted coordinates, is given by

$$H_{\bar{M}} = \frac{1}{2(k^2+r^2)}\left[\hat{p}_1^2 + \hat{p}_2^2 + k^2\Omega^2(x^2+y^2)\right] + \frac{\hat{p}_3^2}{2k^2} .$$

For any function F on $\bar{M}$ we have that

$$\dot{F} = \bar{\mathcal{P}}(\mathcal{E}_{\bar{\mathcal{Q}}^*(dH)_{|\bar{M}}})(F) + \{F, H_{\bar{M}}\}_{\bar{M}}$$

(cf. Theorem 9.2) and, hence, using the above expressions for $\bar{\mathcal{P}}(\mathcal{E}_{\bar{\mathcal{Q}}^*(dH)_{\bar{M}}})$, $H_{\bar{M}}$ and for the brackets $\{\ ,\ \}_{\bar{M}}$, we finally obtain the following system of differential equations for the rolling sphere on a rotating table:

$$\begin{aligned}
\dot{x} &= -\frac{\Omega(t)k^2y}{k^2+r^2}+\frac{r\hat{p}_1}{k^2+r^2}\\
\dot{y} &= \frac{\Omega(t)k^2x}{k^2+r^2}+\frac{r\hat{p}_2}{k^2+r^2}\\
\dot{q}_1 &= \frac{r\Omega(t)x}{k^2+r^2}-\frac{\hat{p}_2}{k^2+r^2}\\
\dot{q}_2 &= \frac{r\Omega(t)y}{k^2+r^2}+\frac{\hat{p}_1}{k^2+r^2}\\
\dot{q}_3 &= \frac{\hat{p}_3}{k^2}\\
\dot{\hat{p}}_1 &= 0\\
\dot{\hat{p}}_2 &= 0\\
\dot{\hat{p}}_3 &= 0
\end{aligned}$$

or, recalling that $\dot{q}^i$ represent the components of the angular velocity of the sphere, the above equations lead to:

$$\begin{aligned}
\dot{x} &= -\frac{\Omega(t)k^2y}{k^2+r^2}+C_1\\
\dot{y} &= \frac{\Omega(t)k^2x}{k^2+r^2}+C_2\\
\omega_x &= \frac{r\Omega(t)x}{k^2+r^2}-\frac{C_2}{r}\\
\omega_y &= \frac{r\Omega(t)y}{k^2+r^2}-\frac{C_1}{r}\\
\omega_z &= C_3
\end{aligned}$$

where C_i, $1 \leq i \leq 3$ are constants depending on the initial data.

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