Bi-differential calculi and bi-Hamiltonian systems

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In a recent paper in this journal [1] A. Dimakis and F. Müller-Hoissen have shown how to generate conservation laws in completely integrable systems by using a bi-differential calculus. In the concluding section of their paper they ask how their approach 'is related to various other characterizations of completely integrable systems', and mention the bi-Hamiltonian formalism as one of these other approaches. We will briefly discuss aspects of the relationship between their work and the bi-Hamiltonian formalism in the finite-dimensional case.

We will be concerned with bi-differential calculi over the exterior algebra $\Omega(\mathcal{A}) = \bigwedge(M)$ on a manifold M, where $\mathcal{A} = C^\infty(M)$ is the algebra of real-valued C^∞ functions on M, and where one of the derivations is the exterior derivative d. (Actually Dimakis and Müller-Hoissen denote the derivation which plays the role of the exterior derivative here by δ; we have thought it better to stick to the standard notation of differential geometry.) The second derivation δ, which creates the bi-differential calculus, is required to be, like d, a derivation of degree 1 of the exterior algebra and to satisfy

$$\delta^2 = 0, \quad d\delta + \delta d = 0.$$

Our first observation is that, according to Frölicher-Nijenhuis theory, a derivation of degree 1 which (anti-)commutes with d (that is, a derivation of type d_* in the terminology of Frölicher and Nijenhuis) must be of the form $\delta = d_R$ for some type $(1,1)$ tensor field

R on M; and that the necessary and sufficient condition for d_R to satisfy ${d_R}^2 = 0$ is that the torsion, or Nijenhuis tensor, of R must be zero. Thus in this particular case, bi-differential calculi are in one-one correspondence with type $(1,1)$ tensor fields with vanishing torsion.

We will be concerned below mainly with the action of d_R on $C^\infty(M)$, for which we have the formula $d_R f = R^*(df)$, where we think of the tensor R as a homomorphism of the module of vector fields on M, and R^* as its adjoint acting on 1-forms. In fact a derivation δ of type d_* is determined by its action on functions — the condition $d\delta + \delta d = 0$ defines its action on s-forms for $s \geq 1$ — and it is easy to see that if δ is of degree 1 its action on functions must be given by $\delta f = R^*(df)$ for some R.

The basic step in the construction of Dimakis and Müller-Hoissen is to define inductively a sequence of $(s-1)$-forms $\chi^{(m)}$, $m = 0, 1, 2, \ldots$, where s is an integer for which closed s-forms are exact, by the rule

$$d\chi^{(m+1)} = d_R \chi^{(m)}.$$

That this is possible follows from the commutation relation $dd_R + d_R d = 0$: we have, for $m \geq 1$,

$$dd_R\chi^{(m)} = -d_R d\chi^{(m)} = -{d_R}^2\chi^{(m-1)} = 0,$$

so the scheme is consistent provided that $dd_R\chi^{(0)} = -d_R d\chi^{(0)} = 0$.

To make the correspondence with bi-Hamiltonian systems we suppose that M is a Poisson manifold, whose Poisson structure comes from a symplectic form ω_0; that R and ω_0 are such that for every pair of vector fields X and Y on M,

$$\omega_0(R(X), Y) = \omega_0(X, R(Y)),$$

so that ω_1, defined by $\omega_1(X, Y) = \omega_0(R(X), Y)$, is a 2-form; and that $d\omega_1 = 0$. Then if we set, for $f, g \in C^\infty(M)$,

$$\{f, g\}_1 = \omega_1(X_f, X_g)$$

where X_f is the Hamiltonian vector field corresponding to f with respect to ω_0, then $\{\cdot,\cdot\}_1$ is bilinear over $\mathbb{R}$, skew-symmetric, and satisfies the derivation property

$$\{f, gh\}_1 = g\{f, h\}_1 + \{f, g\}_1 h.$$

Furthermore, it follows from the vanishing of the torsion of R, together with the closure of ω_1, that the Jacobi identity holds, so that $\{\cdot,\cdot\}_1$ is a second Poisson bracket on M, which is moreover compatible with $\{\cdot,\cdot\}_0$, the Poisson bracket coming from ω_0. Thus in such a case a bi-differential calculus endows M with a Poisson-Nijenhuis structure, that is, with a second Poisson bracket compatible with the first; R is the recursion tensor of the structure.

The construction of the $\chi^{(m)}$, in the case $s = 1$, translates into the terminology of Poisson brackets as follows. We assume that M is such that closed 1-forms are exact.

From the definition,

$$\{f,g\}_1 = \omega_1(X_f, X_g) = \omega_0(X_f, R(X_g)) = -R(X_g)f = -d_R f(X_g);$$

thus the inductive definition of the functions $\chi^{(m)}$ can be expressed as follows:

$$\{\chi^{(m+1)}, \cdot\}_0 = \{\chi^{(m)}, \cdot\}_1.$$

It is easy to show that functions $\chi^{(m)}$ so defined are in involution with respect to both Poisson brackets — we shall outline the proof of a more general result below.

Dimakis and Müller-Hoissen usually impose the initial condition that $d\chi^{(0)} = 0$. However, the scheme will also work with the less restrictive initial condition that $dd_R\chi^{(0)} = -d_R d\chi^{(0)} = 0$, as they remark and as we remarked above. We will show that, under sufficiently generic conditions, the sum of the eigenfunctions of R satisfies this condition.

Note first that if X is an eigenvectorfield of R with eigenfunction λ, and if X' is an eigenvectorfield of R with eigenfunction λ', then from the symmetry condition on R

$$(\lambda - \lambda')\omega_0(X, X') = 0.$$

It follows that R can have at most n functionally independent eigenfunctions, where $\dim M = 2n$. We consider the case in which R has n functionally independent eigenfunctions, the maximum number, such that where the eigenvalues are distinct each is doubly degenerate. It follows from the vanishing of the torsion of R that if the eigenfunctions are λ_a, $a = 1, 2, \ldots, n$, and X_a is any eigenvectorfield corresponding to λ_a, then

$$X_a(\lambda_b) = 0, \quad b \neq a.$$

It is clear from dimensional considerations that the 2-dimensional eigendistribution corresponding to λ_a must contain a 1-dimensional subspace $\langle Y_a\rangle$ such that $Y_a(\lambda_a) = 0$; and we may therefore choose a (local) basis of vector fields $\{Y_a, Z_a \,|\, a = 1, 2, \ldots, n\}$ such that for each a, $\langle Y_a, Z_a\rangle$ is the eigendistribution corresponding to λ_a, $Y_a(\lambda_a) = 0$, and $Z_a(\lambda_a) = 1$. Now set

$$\chi^{(0)} = \sum_{a=1}^{n} \lambda_a.$$

Then for any eigenvectorfield X_a,

$$d_R\chi^{(0)}(X_a) = \sum_{b=1}^{n} d\lambda_b(R(X_a)) = \lambda_a X_a(\lambda_a) = \begin{cases} 0 & \text{if } X_a = Y_a \\ \lambda_a & \text{if } X_a = Z_a. \end{cases}$$

It follows that

$$d_R\chi^{(0)} = \sum_{a=1}^{n} \lambda_a d\lambda_a = \tfrac{1}{2} d\left(\sum_{a=1}^{n} {\lambda_a}^2\right).$$

The sequence of functions generated in this case can, without essential loss of generality, be taken to be the sums of the powers of the eigenfunctions of R, or equivalently the traces of the powers of R. We therefore recover the result that the traces of the powers of the recursion tensor of a Poisson-Nijenhuis structure are in involution with respect to both Poisson brackets. (The sequence of functions generated by the scheme of Dimakis and Müller-Hoissen is in principle infinite, but of course only the first n elements of the sequence are functionally independent.)

We can also give a simple example of what Dimakis and Müller-Hoissen call a gauged bi-differential calculus. In a gauged bi-differential calculus the derivations d and δ are replaced by operators

$$D_d = d + A, \quad D_\delta = \delta + B,$$

where in general A and B are square matrices of 1-forms and the operators act on square matrices of forms. The operators have to satisfy the conditions

$$D_d{}^2 = D_\delta{}^2 = 0, \quad D_d D_\delta + D_\delta D_d = 0.$$

In our example the operators act on functions and $D_d = d$; however,

$$D_\delta = d_R + df,$$

where d_R is the derivation of type d and degree 1 associated with the type $(1,1)$ tensor R as before, and f is a function whose properties are to be specified. It is easy to see that $D_d D_\delta + D_\delta D_d = 0$ follows from the fact that $dd_R + d_R d = 0$. If we assume that R has zero torsion, so that $d_R{}^2 = 0$, then the condition that $D_\delta{}^2 = 0$ reduces to $d_R df = 0$. If f satisfies this condition then we have a graded bi-differential calculus.

Following Dimakis and Müller-Hoissen we now have a new scheme for inductively generating a sequence of functions $\chi^{(m)}$, $m = 0, 1, 2, \ldots$:

$$d\chi^{(m+1)} = d_R \chi^{(m)} + \chi^{(m)} df.$$

(The original scheme is of course obtained by setting $f = 0$.) The consistency of this scheme follows from the general theory in [1], but can easily be demonstrated directly; we require that

$$d(d_R \chi^{(m)} + \chi^{(m)} df) = -d_R d\chi^{(m)} + d\chi^{(m)} \wedge df = 0;$$

for $m > 1$ we have

$$\begin{aligned} &-d_R(d_R\chi^{(m-1)} + \chi^{(m-1)} df) + d\chi^{(m)} \wedge df \\ &\quad = (d\chi^{(m)} - d_R\chi^{(m-1)}) \wedge df - \chi^{(m-1)} d_R df = -\chi^{(m-1)} d_R df = 0. \end{aligned}$$

We shall take for initial function $\chi^{(0)} = 1$: then $\chi^{(1)} = f$, apart from a constant which will be ignored. We now show that the functions so generated are in involution with respect to both Poisson brackets. The rule for generating $\chi^{(m+1)}$, when expressed in terms of Poisson brackets, and with f replaced by $\chi^{(1)}$, is

$$\{\chi^{(m+1)}, \cdot\}_0 = \{\chi^{(m)}, \cdot\}_1 + \chi^{(m)}\{\chi^{(1)}, \cdot\}_0.$$

Assume that $\{\chi^{(i)}, \chi^{(j)}\}_0 = \{\chi^{(i)}, \chi^{(j)}\}_1 = 0$ for all i, j with $1 \leq i, j \leq m$: we show that the same is true with $m+1$ in place of m. First, for $1 \leq i \leq m$

$$\{\chi^{(m+1)}, \chi^{(i)}\}_0 = \{\chi^{(m)}, \chi^{(i)}\}_1 + \chi^{(m)}\{\chi^{(1)}, \chi^{(i)}\}_0 = 0.$$

Then

$$0 = \{\chi^{(i+1)}, \chi^{(m+1)}\}_0 = \{\chi^{(i)}, \chi^{(m+1)}\}_1 + \chi^{(i)}\{\chi^{(1)}, \chi^{(m+1)}\}_0,$$

whence $\{\chi^{(i)}, \chi^{(m+1)}\}_1 = 0$.

Suppose that we take f to be the sum of the eigenfunctions of R, as before. It can then be shown that the functions so generated are the elementary symmetric polynomials in the eigenfunctions of R; and these functions are again in involution with respect to both Poisson brackets.

The construction just described appears in a recent paper by Ibort, Magri and Marmo [2], which is concerned with the so-called Gelfand-Zakharevich bi-Hamiltonian systems and their application to the problem of the separation of variables in the Hamilton-Jacobi equation for Hamiltonians of mechanical type. The proof that the functions $\chi^{(m)}$ are the elementary symmetric polynomials in the eigenfunctions of R may be found there.

A paper containing, among other things, a more detailed discussion of the issues raised above is being prepared by the present authors.

References

[1] A. Dimakis and F. Müller-Hoissen. Bi-differential calculi and integrable models. *J. Phys. A: Math. Gen.* **33** (2000), 957–974.

[2] A. Ibort, F. Magri and G. Marmo. Bihamiltonian structures and Stäckel separability. Preprint: University Carlos III, Madrid (1999).