Addendum to: The integrability conditions in the inverse problem of the calculus of variations for second-order ordinary differential equations

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Abstract

An omission in the outline of the general approach to the inverse problem in [5] is clarified.

While working on a comprehensive application of the theory established in [5], it has occurred to one of us that a rather subtle point about ordering the Helmholtz conditions has been overlooked in the paper's general theoretical sections 2 and 3. Although the effect on the validity of that general part is rather minor, it is not unimportant to clarify this matter because one might be led to erronous applications of the theory otherwise. The claim made in [5] (and also in the review text [4]) was that the conclusions about the full set of passivity conditions were valid irrespective of the ordering of the equations in the Riquier approach, as long as the second-order passivity conditions in the list, referred to as the $\mathcal{A}$-*conditions*, would not degenerate into conditions of order lower than two. Now in Riquier theory derivatives are divided into two types, principal and parametric. In deriving passivity conditions one should always substitute parametric for principal derivatives, and not the other way round; we were insufficiently careful about this point in [5]. As we shall explain below, our results in [5] are nevertheless correct, but with one proviso: the ordering referred to above must satisfy some minimal requirements which we shall capture here in the definition of a *proper ordering*.

With reference to the concepts, notations and numbered formulas in [5], consider the (vertical) closure conditions,

$$\mathrm{D}^V g\,(Z,X,Y) = \mathrm{D}^V g\,(Y,X,Z),$$

which are part of the original set of Helmholtz conditions. Here and below, X,Y,Z etc. are taken from a vector field basis adapted to the problem in hand. It is important to realize that for each fixed triple of different vector arguments X,Y,Z, the corresponding closure conditions more exactly read

$$\mathrm{D}^V g\,(Z,X,Y) = \mathrm{D}^V g\,(Y,X,Z) = \mathrm{D}^V g\,(X,Z,Y),$$

where the order of the last two arguments in each term does not matter in view of the symmetry of the tensor field g. In essence, two of these derivatives will be *principal derivatives* whereas the third one is *parametric*. This depends on the ordering which is chosen for the dependent variables, namely the components of g. If, for example, the term in the middle is the parametric derivative, the two equations which equivalently represent this triplet will be written as

$$\begin{aligned} \mathrm{D}^V g\,(Z,X,Y) &= \mathrm{D}^V g\,(Y,X,Z), \\ \mathrm{D}^V g\,(X,Z,Y) &= \mathrm{D}^V g\,(Y,X,Z). \end{aligned}$$

One further remark is in order here. In calling a term like $\mathrm{D}^V g\,(Z,X,Y)$ a principal derivative (as we will often do for brevity), we in fact tacitly transfer properties of derivatives of components of g to components of derivatives of g. That is to say, the true principal derivative is the 'leading term' $\mathrm{D}^V_Z(g(X,Y)) = Z^V(g(X,Y))$. But it may happen, for example, that external algebraic requirements force certain components of g to be zero, in which case the corresponding component of a derivative of g cannot be formally treated as a principal derivative. Of course, there must then be a shift in the list of principal derivatives; but this need not necessarily imply that the ordering rules for the remaining components of g have to be changed. Incidentally, it may even happen that the leading terms on both sides disappear due to algebraic restrictions, the corresponding closure condition thus becoming an algebraic equation itself. Such a situation will be integrated in our present discussion within the category of improper orderings.

To arrive at the definition of proper orderings and motivate it, let us reconsider the question of writing down D^V-prolongations of the closure conditions which are candidates to combine into passivity conditions in the sense of the Riquier theory, i.e. new relations between parametric derivatives (which then require one of the parametric derivatives to be promoted to the rank of principal derivative). Consider the following prolongations:

$$\begin{aligned} \mathrm{D}^V\mathrm{D}^V g\,(U,X,Y,Z) &= \mathrm{D}^V\mathrm{D}^V g\,(U,Z,Y,X), \\ \mathrm{D}^V\mathrm{D}^V g\,(X,U,Y,Z) &= \mathrm{D}^V\mathrm{D}^V g\,(X,Z,Y,U). \end{aligned}$$

When the second is subtracted from the first, the left-hand side is zero in view of the commutator identity Eqn. (34) in [5], whereas the right-hand side, using the same identity, becomes

$$\mathrm{D}^V\mathrm{D}^V g\,(Z,U,Y,X) = \mathrm{D}^V\mathrm{D}^V g\,(Z,X,Y,U),$$

or written differently

$$\mathrm{D}^V_Z\Big(\mathrm{D}^V g\,(U,Y,X) - \mathrm{D}^V g\,(X,Y,U)\Big) + \Big(\mathrm{D}^V g\,(\mathrm{D}^V_Z U,Y,X) - \mathrm{D}^V g\,(X,Y,\mathrm{D}^V_Z U)\Big) + \ldots = 0,$$

where the dots represent two more terms similar to the second. After expanding vector fields such as $\mathrm{D}^V_Z U$ in terms of the selected local basis, this whole expression consists of pairs of terms and for each pair, irrespective of the ordering which has been chosen, we have one of the following situations: either one of the terms is a principal derivative and the other one is the parametric derivative in the corresponding closure condition, or both terms are principal derivatives. Riquier's method requires that principal derivatives are systematically replaced by parametric ones in order to see whether a new passivity condition can be obtained. In the first situation the cancellation of terms is immediate; in the second, both terms have to be substituted by the same parametric derivative after which they will cancel each other as well. This is a typical example of the legitimate use of previously obtained equations to show that no new conditions arise. A key point in the above calculation, however, the one which escaped our attention in [5], is that a derivative with respect to a common vector field Z can be brought outside the brackets in the first term. A rather subtle point about ordering slips into the procedure here. There is no guarantee, for an arbitrary ordering, that when we set up the two prolongations whose left-hand sides involve comparable second-order derivatives of the same component of g ($g(Y,Z)$ in the example), the right-hand sides will both have Z in the same position. It may happen, for example, that $\mathrm{D}^V g\,(Z,Y,X)$ is a principal derivative and therefore must be replaced by $\mathrm{D}^V g\,(Y,X,Z)$. In such a case, the subsequent calculations cannot be traced back to closure conditions and their prolongations, and thus may give rise to passivity conditions. We therefore introduce the following concept.

Definition. *An ordering of the components of g is said to be proper when the following requirements are met:*

(i) *if the list of vertical closure conditions contains two items in which the principal derivatives are of the form $\mathrm{D}^V g\,(Z_1,Y,X)$ and $\mathrm{D}^V g\,(Z_2,Y,X)$, then the corresponding parametric derivatives are derivatives with respect to the same vector argument (X or Y as the case may be);*

(ii) *none of the vertical closure conditions reduces to an algebraic relation.*

Some comments are in order here. The last item in the above definition was recognized already in [5] as requiring a new start of the algorithmic process: it has always been understood that whenever algebraic relations are encountered which restrict the number of independent components of g, such information has to be exhausted before the systematic search for passivity conditions is resumed. The Riquier process also requires ordering independent variables. It is always tacitly understood that vertical and horizontal derivatives are ordered in the same way. Since the horizontal closure conditions are obtained via the commutator identity $[\nabla, \mathrm{D}^V] = -\mathrm{D}^H$, it is then clear that these will inherit the ordering structure of the vertical closure conditions.

We have now to verify that the assumption of having a proper ordering makes the rest of the general considerations in [5] still work, even though we do not allow ourselves to replace terms in the calculations by substitutions from earlier conditions, unless we are sure that this involves substituting parametric derivatives for principal ones.

Before proceeding, however, it is of interest to give some examples of proper and improper orderings. In the two columns below, we give the vertical closure conditions following from two possible orderings for the case $n = 2$, one which starts with the diagonal elements of g, the other one doing the opposite ($g_{ij|k}$ is shorthand for $\mathrm{D}^V g\,(X_k, X_i, X_j)$):

$$
\begin{array}{ll}
g_{11|2} = g_{12|1} & \qquad g_{12|1} = g_{11|2} \\
g_{22|1} = g_{12|2} & \qquad g_{12|2} = g_{22|1}
\end{array}
$$

Assuming no diagonal elements are zero, the first ordering is proper, simply because the two principal derivatives do not involve the same component of g. The second one is improper, however, because the right-hand sides of the two equations for g_{12} have derivatives with respect to different variables. It is obvious that cross differentiation of these two equations will lead to a new relation between parametric derivatives. Next, we look at three different orderings for $n = 3$:

$$
\begin{array}{lll}
g_{11|2} = g_{12|1} & \qquad g_{12|1} = g_{11|2} & \qquad g_{11|2} = g_{12|1} \\
g_{11|3} = g_{13|1} & \qquad g_{13|1} = g_{11|3} & \qquad g_{11|3} = g_{13|1} \\
g_{22|1} = g_{12|2} & \qquad g_{22|1} = g_{12|2} & \qquad g_{12|2} = g_{22|1} \\
g_{22|3} = g_{23|2} & \qquad g_{22|3} = g_{23|2} & \qquad g_{12|3} = g_{23|1} \\
g_{33|1} = g_{13|3} & \qquad g_{33|1} = g_{13|3} & \qquad g_{13|2} = g_{23|1} \\
g_{33|2} = g_{23|3} & \qquad g_{33|2} = g_{23|3} & \qquad g_{13|3} = g_{33|1} \\
g_{12|3} = g_{13|2} & \qquad g_{12|3} = g_{13|2} & \qquad g_{23|2} = g_{22|3} \\
g_{23|1} = g_{13|2} & \qquad g_{23|1} = g_{13|2} & \qquad g_{23|3} = g_{33|2}
\end{array}
$$

The first of these, in which the dependent variables can be ordered $g_{11} > g_{22} > g_{33} > g_{12} > g_{23} > g_{13}$, is proper, assuming that none of the g_{ij} is identically zero (we remind the reader that a principal derivative must be a derivative of a dependent variable of higher order than the order of the dependent variable occurring in any parametric derivative of the same degree in the same equation). The second ordering, with $g_{22} > g_{33} > g_{12} > g_{23} > g_{13} > g_{11}$, for example, is likewise also proper. Observe, however, that the first ordering looses its validity if $g_{11} = 0$, since then neither $g_{11|2}$ nor $g_{11|3}$ can be principal; note that these two covariant derivatives need not be zero, it is the vanishing of their leading terms which causes the problem. With $g_{11} = 0$, one is forced to pass to the second column, where the ordering induced on the remaining dependent variables is still proper. If, however, also g_{22} were zero, a similar transition to an induced ordering on the remaining variables would result in an improper ordering. The third ordering above is not proper from the outset: things go wrong in the last two equations, resulting in a D^V-D^V passivity condition.

Assuming the ordering is proper, we now go through the whole procedure of completing the set of passivity conditions again and indicate where amendments have to be made.

The first equation encountered in Section 3 of [5], coming from ∇-D^H compatibility, reads

$$(\Phi\lrcorner\,\mathrm{D}^V g + i_\Psi g)(Z,X,Y) = (\Phi\lrcorner\,\mathrm{D}^V g + i_\Psi g)(Y,X,Z).$$

Whereas before we used closure conditions to substitute for the first term on both sides, we must face the fact now that it is impossible to know whether such terms will be principal or parametric. Instead, therefore, we look at the second term on both sides, and more particularly at the parts involving $\mathrm{D}^V\Phi$ (see the definition of Ψ in (30)). Such terms can be substituted for by using D^V-derivatives of the first algebraic requirement (13). This is permitted because the idea is always that algebraic conditions on g are imposed first (to determine the set of independent components of g which will then become the unknowns in the differential conditions). In other words, algebraic conditions (and their derivatives) are to be regarded as identities in our approach and thus can be used in the process of simplifying expressions much in the same way as the curvature relations (34-36). In carrying out the indicated substitutions in the present case, we will create for example a term $\mathrm{D}^V g\,(Z,X,\Phi(Y))$ in the right-hand side. This term, together with the first one on the left precisely make up a pair of type (12) which cannot be both parametric and will cancel each other in the Riquier procedure as explained above. What we are left with is, as before, the algebraic curvature condition (20) which is assumed to be already taken care of.

Concerning D^H-D^H compatibility as discussed in [5], the crucial point is again that for a proper ordering, the right-hand sides of the two prolongations we start from will have a common argument Z in second position. The procedure which leads to the 'intermediate relation' of the bottom of the page is fine. From there on, the following modified arguments must be invoked. When the first term is replaced via the identity (35), the second-order terms combine to closure conditions and their D^H-prolongations and thus cancel out. There remains:

$$\begin{aligned}0 &= \mathrm{D}^V g\,(R(U,Z),Y,X) + \mathrm{D}^V g\,(R(X,U),Y,Z) + \mathrm{D}^V g\,(R(Z,X),Y,U)\\ &\quad + g(\mathrm{D}^V R\,(Y,U,Z),X) + g(\mathrm{D}^V R\,(Y,X,U),Z) + g(\mathrm{D}^V R\,(Y,Z,X),U),\end{aligned}$$

where use has been made of the property $g\,(Y,\sum \mathrm{D}^V R(X,U,Z)) = 0$ following from (24). Again, we don't touch the first-order terms now as it is undecidable at the moment which of them is principal and which is parametric. Instead, we make a substitution for the algebraic terms, coming from the D^V_Y prolongation of the curvature condition $\sum g(R(U,Z),X) = 0$. This creates first-order terms in g which are exactly the ones we need to cancel out, by making proper use of the closure conditions, those we already had.

We now come to the most delicate part of this note: the confirmation that the so-called $\mathcal{A}$-conditions remain valid.

Since the structure of the D^H-closure conditions is the same as that of the D^V-equations, we will always have to compare two mixed second-order derivatives of g with respect to the same vector argument, but these will certainly have the same vector argument Z in second position in the right-hand sides. If on the other hand we have to consider a prolongation such as

$$\mathrm{D}^H\mathrm{D}^V g\,(U,X,Y,Z) = \mathrm{D}^H\mathrm{D}^V g\,(U,Z,Y,X)$$

with $U \neq X$, this necessarily means that both $\mathrm{D}^V g\,(X,Y,Z)$ and $\mathrm{D}^V g\,(U,Y,Z)$ figure in the list of principal derivatives so that, assuming that the ordering is proper, the corresponding right-hand sides have the same vector argument Z (or Y) in first position. Hence, the prolongation to which the above one has to be compared will necessarily be of the form

$$\mathrm{D}^V\mathrm{D}^H g\,(X,U,Y,Z) = \mathrm{D}^V\mathrm{D}^H\, g(X,Z,Y,U),$$

with the same Z in second position again. By subtracting the first from the second and using the identity (36) which introduces a component called θ of the curvature of the linear connection, and by replacing also the right-hand side of the second equation by using (36), we arrive at an equation of the following form:

$$\mathrm{D}^H\mathrm{D}^V g\,(U,Z,Y,X) - \mathrm{D}^H\mathrm{D}^V g\,(Z,X,Y,U) + g(\theta(Y,X)Z,U) - g(\theta(Y,X)U,Z) = 0.$$

We now carefully investigate the nature of the second-order terms. Observe first that in view of the preceding ordering assumptions, $\mathrm{D}^V g\,(Z,Y,X)$ and $\mathrm{D}^V g\,(Z,Y,U)$ are both parametric (and the same is true for the corresponding D^H-derivatives). Hence, the only way in which one or both second-order terms can be parametric is when $\mathrm{D}^V g\,(U,Y,X)$ and/or $\mathrm{D}^V g\,(X,Y,U)$ are parametric as well. Clearly, however, these cannot both be parametric, so we have to discuss separately the case that one of them is principal and the case that they are both principal. For the first case, assume to fix ideas that $\mathrm{D}^V g\,(X,Y,U)$ is principal and the other one parametric. Then we can substitute the latter for the former and the second-order equation becomes:

$$\begin{aligned}&(\mathcal{A}(U,Z)g)(Y,X) :=\\ &\quad \mathrm{D}^H\mathrm{D}^V g\,(U,Z,Y,X) - \mathrm{D}^H\mathrm{D}^V g\,(Z,U,Y,X) + g(\theta(Y,X)Z,U) - g(\theta(Y,X)U,Z) = 0.\end{aligned}$$

We thus obtain an '$\mathcal{A}$-condition' in which both second-order terms are parametric. In the second case, when both $\mathrm{D}^V g\,(U,Y,X)$ and $\mathrm{D}^V g\,(X,Y,U)$ are principal (and thus also $\mathrm{D}^H g\,(U,Y,X)$ is principal), we know that $\mathrm{D}^V g\,(Y,X,U)$ is parametric. In the second term of our intermediate relation, we can do the substitution of parametric for principal immediately, whereas in the first term we first have to swap the arguments (U, Z) by using the curvature identity (36), then do the substitution and then swap the first two arguments again to obtain the term $\mathrm{D}^H\mathrm{D}^V g\,(Y, Z, U, X)$. It is easy to verify that the algebraic terms in this procedure get rearranged in such a way that the final expression is exactly the tensor $\mathcal{A}(Y, Z)g$ evaluated on the arguments (U, X) and we have again obtained an '$\mathcal{A}$-condition' in which both second-order terms are parametric.

Note in passing that the interesting Proposition 2 of [5] about $\mathcal{A}(X, Y)g$-tensors remains valid; although we did not use this property in deriving the above passivity conditions now, it will still be useful in subsequent considerations.

As for the remainder of the general search for passivity conditions, specifically the rather complicated computations reported in Appendix B of [5], one can verify that the conclusions remain unaltered. Essentially, the difference between the inaccurate arguments used in [5] and the proper ones we appeal to now is of the same type as discussed above for ∇-D^H and D^H-D^H compatibility: instead of making substitutions in derivative terms where we cannot know whether they are principal or parametric, we have to appeal at an earlier stage to prolongations of algebraic conditions. We limit ourselves to a sketch of the way this works in each case.

Consider first ∇-$\mathcal{A}$ compatibility. The beginning of the computation as explained on p. 267 of [5] remains unaltered, up to and including the intermediate result at the bottom of the page. We now proceed to work out first the explicit form for the terms involving the tensors Ψ and Rie, using the defining relation (30) for Ψ and the property (53) for Rie. In addition, we use the Bianchi identity (46) to substitute for the $\nabla\theta$ terms which gives rise to more terms involving a second D^V-derivative of Φ. The resulting expression contains a number of $\mathrm{D}^V g$ terms which involve the curvature R in one of the arguments and we first work on eliminating those terms. Two of them are $\mathrm{D}^V g\,(Y, R(X, U), Z) - \mathrm{D}^V g\,(X, R(Y, U), Z)$; we replace them by using a D^V_Y-prolongation of the algebraic curvature condition $\sum g(R(X, U), Z) = 0$ and the similar one with Y and X interchanged. In doing so, we further take into account the relation

$$\mathrm{D}^V R\,(Y, X, U) = \tfrac{1}{3}\mathrm{D}^V\mathrm{D}^V\Phi(Y, X, U) - \tfrac{1}{3}\mathrm{D}^V\mathrm{D}^V\Phi(Y, U, X),$$

which follows from (19). The result is that a number of terms now cancel out via legitimate use of closure conditions, or because the first two arguments in a $\mathrm{D}^V\mathrm{D}^V\Phi$ tensor can always be interchanged. Remarkably, however, quite a number of terms also add up, and we obtain as the next intermediate result the equation

$$\begin{aligned}
&\mathrm{D}^V\mathrm{D}^V g\,(X, \Phi Y, U, Z) - \mathrm{D}^V\mathrm{D}^V g\,(Y, \Phi X, U, Z) = \\
&\quad - 2\,\mathrm{D}^V g\,(R(X, Y), U, Z) - \mathrm{D}^V g\,(Y, R(Z, X), U) + \mathrm{D}^V g\,(Y, U, R(Z, X)) \\
&\quad + \mathrm{D}^V g\,(X, R(Z, Y), U) - \mathrm{D}^V g\,(X, U, R(Y, Z)) + \mathrm{D}^V g\,(Y, \mathrm{D}^V\Phi(U, X), Z) \\
&\quad + \mathrm{D}^V g\,(Y, U, \mathrm{D}^V\Phi(Z, X)) - \mathrm{D}^V g\,(X, \mathrm{D}^V\Phi(U, Y), Z) - \mathrm{D}^V g\,(X, U, \mathrm{D}^V\Phi(Z, Y)) \\
&\quad + \tfrac{2}{3}\,g(\mathrm{D}^V\mathrm{D}^V\Phi(Y, U, X), Z) - \tfrac{2}{3}\,g(\mathrm{D}^V\mathrm{D}^V\Phi(Y, U, Z), X) - \tfrac{2}{3}\,g(\mathrm{D}^V\mathrm{D}^V\Phi(X, U, Y), Z)
\end{aligned}$$

$$+\tfrac{2}{3}\,g(\mathrm{D}^V\mathrm{D}^V\Phi(X,U,Z),Y)+\tfrac{1}{3}\,g(\mathrm{D}^V\mathrm{D}^V\Phi(U,Z,X),Y)-\tfrac{1}{3}\,g(\mathrm{D}^V\mathrm{D}^V\Phi(U,Z,Y),X).$$

Invoking the D^V_U-prolongation of $\sum g(R(X,Y),Z)=0$, we next replace the four terms having a factor $\frac{2}{3}$ (taking again the expression of $\mathrm{D}^V R$ in terms of Φ into account). All $\mathrm{D}^V g$ terms having R acting in one of the arguments now cancel out and the second-order equation reduces to

$$\begin{aligned}
&\mathrm{D}^V\mathrm{D}^V g\,(X,\Phi Y,U,Z)-\mathrm{D}^V\mathrm{D}^V g\,(Y,\Phi X,U,Z)=\\
&\quad \mathrm{D}^V g\,(Y,\mathrm{D}^V\Phi(U,X),Z)+\mathrm{D}^V g\,(Y,U,\mathrm{D}^V\Phi(Z,X))-\mathrm{D}^V g\,(X,\mathrm{D}^V\Phi(U,Y),Z)\\
&\quad -\mathrm{D}^V g\,(X,U,\mathrm{D}^V\Phi(Z,Y))-g(\mathrm{D}^V\mathrm{D}^V\Phi(U,Z,Y),X)+g(\mathrm{D}^V\mathrm{D}^V\Phi(U,Z,X),Y)
\end{aligned}$$

Finally, using a D^V-D^V-prolongation of the algebraic condition (13), we can substitute for the two algebraic terms in the above relation. In the resulting expression, the first order terms cancel out via closure conditions and we are left with

$$\begin{aligned}
&\mathrm{D}^V\mathrm{D}^V g\,(X,\Phi Y,U,Z)-\mathrm{D}^V\mathrm{D}^V g\,(U,Z,\Phi Y,X)=\\
&\qquad \mathrm{D}^V\mathrm{D}^V g\,(Y,\Phi X,U,Z)-\mathrm{D}^V\mathrm{D}^V g\,(U,Z,\Phi X,Y).
\end{aligned}$$

For a proper ordering, whenever a combination of terms of the form $\mathrm{D}^V\mathrm{D}^V g\,(X,Y,U,Z)-\mathrm{D}^V\mathrm{D}^V g\,(U,Z,X,Y)$ occurs, these terms can never be both parametric. For if they were, every selection of a triple of arguments from each term, referring to first-order derivatives which would then also be parametric, would still have two arguments in common (one from the first couple and one from the second). Say these common arguments are X and U. This means that there would be two corresponding equations in the list of closure conditions with left-hand sides of the form $\mathrm{D}^V g\,(\,\cdot\,,X,U)$ and that cross differentiation of these equations would already have produced the difference of two terms we started from at the level of D^V-D^V compatibility. But we have shown that this cannot occur for a proper ordering. Therefore, when parametric derivatives are substituted for principal ones the expressions on both sides of the above equation become separately zero.

We next turn to D^V-$\mathcal{A}$ compatibility, following the computation which starts on p. 268 of [5]. The intermediate condition mentioned there is not quite correct because improper use was made of closure conditions. The correct expression reads

$$\begin{aligned}
&\mathrm{D}^V\mathrm{D}^V\mathrm{D}^H g\,(Y,W,X,U,Z)-\mathrm{D}^V\mathrm{D}^V\mathrm{D}^H g\,(Z,X,Y,U,W)=\\
&\qquad \mathrm{D}^V g\,(W,\theta(U,Z)X,Y)-\mathrm{D}^V g\,(X,\theta(W,Y)U,Z)+g\,(\mathrm{D}^V\theta(Y,U,Z)X,W)\\
&\qquad -g\,(\mathrm{D}^V\theta(Y,U,Z)W,X)+g\,(\mathrm{D}^V\theta(Z,U,W)X,Y)-g\,(\mathrm{D}^V\theta(X,W,Y)U,Z).
\end{aligned}$$

Now, the information which is contained in the prolongations we started from to derive this expression is that $\mathrm{D}^V g\,(Z,U,W)$, $\mathrm{D}^V g\,(U,W,Z)$, $\mathrm{D}^V g\,(X,U,Z)$, $\mathrm{D}^V g\,(Y,U,Z)$ are all parametric, whereas $\mathrm{D}^V g\,(W,U,Z)$ and $\mathrm{D}^V\mathrm{D}^H g\,(X,Y,U,Z)$ are principal. It follows that certainly $\mathrm{D}^V\mathrm{D}^V\mathrm{D}^H g\,(Y,W,X,U,Z)$ is principal and thus must be replaced. To do this we must bring the argument W into third position first and therefore proceed as follows. Considering the term $\mathrm{D}^V\mathrm{D}^H g\,(W,X,U,Z)$, swap the first two arguments using the curvature identity (36), then make the substitution, subsequently swap the first two arguments again via (36) and finally take a D^V_Y derivative of the resulting expression. When the third-order principal derivative above is replaced in this way, cancellations occur in view of the closure conditions and also through the Bianchi identity (46). The leading terms of the remaining expression are those of the D^V_Z-derivative of $(\mathcal{A}(Y,X)g)(U,W)$. If $(\mathcal{A}(Y,X)g)(U,W) = 0$ is one of the passivity conditions which has been added to the original closure conditions, just one of its leading terms will be principal. So another substitution is required and one can verify that the remaining terms then all cancel out. If, on the other hand, $(\mathcal{A}(Y,X)g)(U,W) = 0$ is not one of the passivity conditions, the implication is that its leading terms are both principal derivatives so that again substitutions must be made. Using Proposition 2 we conclude, however, that modulo prolongations the computation will be formally the same as in the first case. Hence, no new passivity conditions can be obtained.

For the D^H-$\mathcal{A}$ compatibility, the few lines of indications in [5] are sufficient to lead the way through quite similar calculations. The only correction which has to be made is, as before, that the second D^V-derivative of the curvature condition (20), previously mentioned as the final outcome of the calculation, now has to be invoked at an earlier stage, to create the terms which are needed for a proper use of closure conditions.

Finally we must reconsider whether alternants formed out of '$\mathcal{A}$-conditions' among themselves could give rise to new relations between parametric derivatives. Why this does not happen will now be briefly explained. Suppose we have the following '$\mathcal{A}$-conditions' in the list of passivity conditions of order two:

$$\begin{aligned} \mathrm{D}^V\mathrm{D}^H g\,(X,Y,U,Z) &= \mathrm{D}^V\mathrm{D}^H g\,(Y,X,U,Z) + \text{l.o.} \\ \mathrm{D}^V\mathrm{D}^H g\,(V,W,U,Z) &= \mathrm{D}^V\mathrm{D}^H g\,(W,V,U,Z) + \text{l.o.} \end{aligned}$$

where the abbreviation l.o. refers to unspecified terms of lower order. The two left-hand sides are principal, the right-hand sides parametric, which implies that also all derivatives $\mathrm{D}g(\,.\,,U,Z)$, where D stands for either D^V or D^H and the dot stands for one of the arguments X,Y,V,W, are parametric. An alternant arises from the following two prolongations:

$$\begin{aligned} \mathrm{D}^V\mathrm{D}^H\mathrm{D}^V\mathrm{D}^H g\,(V,W,X,Y,U,Z) &= \mathrm{D}^V\mathrm{D}^H\mathrm{D}^V\mathrm{D}^H g\,(V,W,Y,X,U,Z) + \text{l.o.} \\ \mathrm{D}^V\mathrm{D}^H\mathrm{D}^V\mathrm{D}^H g\,(X,Y,V,W,U,Z) &= \mathrm{D}^V\mathrm{D}^H\mathrm{D}^V\mathrm{D}^H g\,(X,Y,W,V,U,Z) + \text{l.o.} \end{aligned}$$

The procedure to make the left-hand side of the second equation identical to that of the first consist of the following series of swappings of arguments by virtue of the curvature identities (34-36): interchange first the middle two derivative arguments which gives rise to $\mathrm{D}^V\mathrm{D}^V\mathrm{D}^H\mathrm{D}^H g\,(X,V,Y,W,U,Z)$; next, interchange X and V and Y and W; finally, interchange the middle derivatives X and W again. Carrying out the same steps simultaneously in the right-hand side of the equation, the second equation is replaced by one of the form

$$\mathrm{D}^V\mathrm{D}^H\mathrm{D}^V\mathrm{D}^H g\,(V,W,X,Y,U,Z) = \mathrm{D}^V\mathrm{D}^H\mathrm{D}^V\mathrm{D}^H g\,(W,V,X,Y,U,Z) + \text{l.o.}$$

When this transformed equation is subtracted from the first one, the left-hand sides cancel out and we obtain a relation of the form

$$\mathrm{D}^V\mathrm{D}^H\mathrm{D}^V\mathrm{D}^H g\,(V,W,Y,X,U,Z) = \mathrm{D}^V\mathrm{D}^H\mathrm{D}^V\mathrm{D}^H g\,(W,V,X,Y,U,Z) + \mathrm{l.o.}$$

Both fourth-order terms in this expression are manifestly principal, however, the first one via the $\mathrm{D}^V_V\mathrm{D}^H_W$-derivative, the second one because of the $\mathrm{D}^V_X\mathrm{D}^H_Y$-derivative. Hence, further substitutions are required. In the right-hand side the substitution can be done immediately, after which the interchanged arguments Y, X can be moved to the front positions again by the same three-step procedure as before. In the left-hand side, on the other hand, we have to transport the arguments V, W to the inner positions first by the three-step procedure, and we must next interchange them via the appropriate '$\mathcal{A}$-condition'. At the end of this process, it is clear that the fourth-order terms on both sides are identical. Needless to say, however, many lower order terms have been created throughout the use of (35-36) and by substitutions of parametric for principal derivatives via '$\mathcal{A}$-conditions'. One can verify, however, that all lower order terms in the end cancel out. This is unfortunately a very tedious calculation. We content ourselves therefore with giving some hints about the most important steps in the process. To begin with, the $\mathrm{D}^H g$ terms which show up in the overall lower order part immediately cancel out in view of the full symmetry of $\mathrm{D}^V\theta$. The terms in $\mathrm{D}^V\mathrm{D}^H g$ all occur in pairs which constitute the highest-order terms of an $\mathcal{A}$-type tensor. By the fact that Proposition 2 of [5] still applies, we know that the process of substituting parametric for principal derivatives will eventually give rise to purely algebraic terms here. The other algebraic terms in g (which do not cancel out immediately) have vector fields such as $\mathrm{D}^V\mathrm{D}^H\theta\,(X,Y,U,Z)W$ in their arguments. These can be turned into terms involving $\mathrm{D}^V\mathrm{D}^V\mathrm{D}^V R$ as follows. First interchange the derivatives acting on θ. This requires an extension of (36), because θ is not purely covariant: in fact

$$\begin{aligned}\mathrm{D}^V\mathrm{D}^H\theta\,(X,Y,U,Z)W - \mathrm{D}^H\mathrm{D}^V\theta\,(Y,X,U,Z)W &= \mu_{\theta(X,Y)}\theta\,(U,Z)W = \\ -\,\theta(\theta(X,Y)U,Z)W - \theta(U,\theta(X,Y)Z)W &- \theta(U,Z)\theta(X,Y)W + \theta(X,Y)\theta(U,Z)W.\end{aligned}$$

The symmetry of $\mathrm{D}^V\theta$ can now be used to interchange two of its arguments. The identity above can then be used again to restore the original order of differentiation. If the arguments to be interchanged in the middle step are chosen correctly, all the derivatives of θ which arise combine so as to give a third derivative of R via the Bianchi identity (47). All other algebraic terms which are created in this process precisely cancel out the ones obtained before. Using finally the third-order D^V-derivative of the cyclic curvature condition (20), one finds that all remaining terms cancel out by virtue of the closure conditions and their prolongations.

It is worth observing again that the generality of this search for further passivity conditions requires the '$\mathcal{A}$-conditions' to be effectively conditions of second order. Unfortunately, in many of the particular cases in a Douglas-type classification, the '$\mathcal{A}$-conditions' will degenerate into first-order or even algebraic conditions. Obviously, the last three compatibility investigations carried out above make no sense in such cases because they start with setting up second-order prolongations of the closure conditions or the '$\mathcal{A}$-conditions' themselves. In case the '$\mathcal{A}$-conditions' would degenerate into algebraic conditions, for example, the philosophy of our general approach would require these to be imposed first and then to see whether it is still possible to set up a proper ordering for the restricted set of unknowns.

The upshot of the corrections discussed in this note is that Theorem 1 of [5] remains valid, provided we replace the final sentence by: *The completeness of the scheme only applies when the ordering which is selected is proper and no degeneracy occurs in the second-order passivity conditions.*

In fact, this hardly restricts the range of applicability of the general conclusions as the number of particular cases in a Douglas-type classification where no degeneracy problem occurs is rather restricted anyway. This does not mean, on the other hand, that our approach is useless in the majority of cases. All it means is that for many cases in a classification study one will have to proceed in an ad hoc manner from the closure conditions on, but the same technique of using a more geometrical calculus, adapted to the Jordan normal form structure of the Jacobi endomorphism Φ, remains valid and will provide more structure and insight into the analysis than working in a coordinate basis does.

We have claimed so far to have generalized two interesting subcases of Douglas's scheme to an arbitrary number of degrees of freedom. To finish this note, we briefly argue why these two general cases are indeed correct. In the first case, treated in [5], where Φ is a multiple of the identity, there are no algebraic restrictions on g. The diagonalizewise ordering of the components of g we selected is easily seen to be proper and there is no degeneracy in the second-order passivity conditions, so all conclusions are correct. In the generalization of Douglas's Case IIa1 we reported in [1], g is diagonal and the ordering we chose is manifestly proper. The $\mathcal{A}$-conditions do degenerate here so we are in danger of having to proceed in an ad hoc manner. But we showed that these conditions are actually identically satisfied for this case, so no further action is needed.

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