The Berwald-type connection associated to time-dependent second-order differential equations

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Abstract. We investigate the notions of a connection of Finsler type and of Berwald type on the first jet bundle $J^1\pi$ of a manifold E which is fibred over $\mathbb{R}$. Such connections are associated to a given horizontal distribution on the bundle $\pi_1^0 : J^1\pi \to E$, which in particular may come from a time-dependent system of second-order ordinary differential equations. In order to accomodate three existing constructions of a Berwald-type connection for a second-order system, we first introduce equivalence classes of connections of Finsler and Berwald type. By exploring the differences between the existing models in more depth, we come to a new construction which in many respects can be regarded as giving an optimal representative of the class of Berwald-type connections. We briefly enter into two related matters: one is the definition of connections of the type of Cartan, Chern-Rund and Hashiguchi when a metric tensor field is given; the other one is the potential effect of the newly acquired insights on the theory of derivations on forms along the projection π_1^0.

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1 Introduction

The Berwald connection is a well-known concept in Finsler geometry. It is in fact one of many related linear connections which have been studied in this field of research, other often discussed connections being for example those attributed to Chern-Rund, Cartan and Hashiguchi (see e.g. [2, 4, 10, 16, 17]). Although a paper by Vilms [22] uses the term Berwald connection also for a kind of linearization of a general non-linear connection on a vector bundle, this concept has not received widespread attention outside the 'Finsler community'. Quite recently, however, some applications have been developed in the study of second-order ordinary differential equations (SODE's), see e.g. [14, 8, 19], which make use of covariant derivative operators; these may be seen to come essentially from the Berwald-type connection associated to the non-linear connection of the given SODE. This by itself may be a sufficient reason for having a closer look at the relationship between various versions of such a connection which have been discovered independently in the literature.

A recent illuminating discussion of the relationship between different linear connections used in Finsler geometry has been given by Szilasi [21]. All such connections in Szilasi's account live on the tangent bundle $T(TM) \to TM$ of a tangent bundle $\tau : TM \to M$. Crampin [6] has pushed our understanding of this matter further ahead by explaining the more concise picture where all connections are constructed on the pullback bundle $\tau^*TM \to TM$ and by putting thereby the Berwald-type connection in the spotlight as the one to which all others can be related. Similar observations were also made by Anastasiei [1].

Given a horizontal distribution on TM (i.e. a possibly non-linear connection on $\tau : TM \to M$), with corresponding projection operators P_H and P_V, every vector field ξ on TM uniquely decomposes into a horizontal and vertical lift of vector fields along τ, which, as in [8], will be called ξ_H and ξ_V, respectively:

$$\xi = {\xi_H}^H + {\xi_V}^V. \tag{1}$$

The key point in Crampin's analysis is the following. The covariant derivative operator $\mathrm{D} : \mathcal{X}(TM) \times \mathcal{X}(\tau) \to \mathcal{X}(\tau)$, defined by

$$\mathrm{D}_\xi X = [P_H(\xi), X^V]_V + [P_V(\xi), X^H]_H, \tag{2}$$

determines the unique linear connection on $\tau^*TM \to TM$ which has the properties: (i) the restriction to fibres T_xM is the canonical complete parallelism; (ii) parallel translation along a horizontal curve is given by a rule of Lie transport. It is called the *Berwald-type connection on τ^*TM* determined by the given horizontal distribution.

If in particular the non-linear connection is the one canonically associated to a given SODE on TM, then the associated Berwald-type connection has all of its torsion tensor fields equal to zero, except for the one whose vanishing would require that the non-linear connection is flat. In fact, to recognize the (five) tensor fields referred to here as related to the concept of torsion, it is helpful to move to a bigger space, i.e. to consider another linear connection, this time defined on $T(TM) \to TM$, which is obtained from the one on τ^*TM by "doubling the formulas", as follows:

$$\nabla_\xi X^H = (\mathrm{D}_\xi X)^H, \qquad \nabla_\xi X^V = (\mathrm{D}_\xi X)^V. \tag{3}$$

The standard torsion tensor of this connection, $\nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]$, when decomposed into its horizontal and vertical part for various combinations of the arguments, gives rise to six tensor fields of type (1,2) in principle, but one of them is identically zero.

This brings us to the framework where Berwald and related connections are usually considered and gives us an opportunity to mention also the equivalent characterizing properties by which Szilasi singles out the Berwald-type connection in his overview. If J is the almost complex structure provided by a horizontal distribution, then a *Finsler connection* on $T(TM) \to TM$ is characterized by $\nabla P_H = \nabla J = 0$. It then follows that also

$\nabla S = 0$, where S is the canonical almost tangent structure on TM and that a Finsler connection is completely determined if you know the covariant derivative of vertical vectors. The Finsler connection is said to be of Berwald type if the vertical lifts of basic vector fields (i.e. vector fields on M) are parallel with respect to vertical vector fields and $\nabla_{X^H} Y^V = P_V([X^H, Y^V])$.

The other type of linear connections referred to at the beginning, although they were originally introduced merely in the framework of Finsler spaces, can be given a quite more general meaning also (cf. [1, 6]). All they require is one extra tool, namely a metric tensor field g along τ. For example, in the case that the horizontal distribution comes from a spray Γ, Crampin defines *vertical and horizontal Cartan tensors* by the following relations: $\forall X, Y, Z \in \mathcal{X}(\tau)$,

$$
\begin{aligned}
g(C_V(X,Y), Z) &= \mathrm{D}_{X^V} g(Y,Z), &\quad (4)\\
g(C_H(X,Y), Z) &= \mathrm{D}_{X^H} g(Y,Z). &\quad (5)
\end{aligned}
$$

For completeness, we should remark here that it is the type (1,2) tensor field C_V which is related to the Cartan tensor known in Finsler geometry. The main motivation behind the connections associated to Chern-Rund, Hashiguchi and Cartan comes from various degrees of trying to obtain a metrical connection. Since the difference between two linear connections is tensorial and the decomposition (1) of ξ will, in the present context, split this tensor field also in a vertical and horizontal component, denoted by δ^V and δ^H in [6], new connections can be derived from the Berwald connection by making assignments for δ^V and δ^H. In [6], for the further special case that Γ is the geodesic spray of a Finsler space, the other connections of interest are characterised as follows:

$$
\begin{array}{lll}
\delta^V = 0, & \delta^H = \frac{1}{2} C_H, & \text{(Chern-Rund)} \\
\delta^V = \frac{1}{2} C_V, & \delta^H = 0, & \text{(Hashiguchi)} \\
\delta^V = \frac{1}{2} C_V, & \delta^H = \frac{1}{2} C_H. & \text{(Cartan)}
\end{array}
\quad (6)
$$

The price to pay with these modifications of the Berwald-type connection is that every step towards the ideal of a metrical connection (the Cartan connection is fully metrical) introduces more torsion, i.e. a larger deviation from the ideal of a maximally torsion-free connection.

We can now come to the purpose of the present paper. Our interest comes in the first place from the study of time-dependent second-order differential equations. The manifold of interest for carrying such systems is the first jet bundle $J^1\pi$ of a bundle $\pi : E \to \mathbb{R}$. We are in particular interested in the fibration $\pi_1^0 : J^1\pi \to E$ and in the pullback bundle $\pi_1^{0*}(\tau_E)$ (to replace $\tau^* TM$). As in the framework of autonomous SODE's, time-dependent ones define a non-linear connection on the bundle $\pi_1^0 : J^1\pi \to E$. There are at least three known constructions in the literature of an associated linear connection. These were independently derived, from different perspectives, and do not make use of any other tool than the horizontal distribution coming from the given SODE. Therefore, they should somehow correspond to a generalized version of the concept of Berwald-type connection. Two of these linear connections were constructed on the full space $J^1\pi$ (i.e. on the bundle $T(J^1\pi) \to J^1\pi$), respectively by Massa and Pagani [15] and by

Byrnes [5]. The third construction by Crampin *et al* [8] is a more direct one on the bundle $\pi_1^{0*}(\tau_E) \to J^1\pi$ which generalizes the definition (2). Our primary objective is to explore a general scheme within which these three constructions can be compared and related to each other (see Sections 2 and 3): as expressed above, they should in some sense be equivalent and represent the 'time-dependent generalization' of the connection of Berwald type associated to the given horizontal distribution on $J^1\pi$. Their difference no doubt will come from a certain freedom or undeterminacy in 'fixing the time-component' of the connection. In order to identify natural procedures for deciding upon the best possible selection criteria for this matter, we shall approach the issue from the two different characterizations of Berwald-type connections described by Crampin [6] and Szilasi [21], respectively (Section 4). A second objective therefore is to come to an optimal definition or characterization of the Berwald-type connection in this framework. This will lead us in Section 5 to a new direct construction formula for a linear connection on $\pi_1^{0*}(\tau_E) \to J^1\pi$, which is more in line with the natural decomposition of $\mathcal{X}(\pi_1^0)$ and, when lifted to the bigger space $J^1\pi$ itself, is related to the construction of Massa and Pagani. We shall also briefly enter into the discussion of constructing Chern-Rund, Hashiguchi and Cartan type connections in the time-dependent case.

It is somehow intriguing that the construction of horizontal and vertical covariant derivative operators, the way they were derived from the classification theory of derivations of forms along π_1^0 in [20], gave rise exactly to the same, less optimal, construction of the linear connection of Crampin *et al*. The branching point in the theory of derivations in [20] was a freedom in selecting a natural vertical exterior derivative. By way of application of the newly acquired insights, therefore, we will discuss the reverse process in Section 6, namely the way different choices for the linear connection on $\pi_1^{0*}(\tau_E) \to J^1\pi$ affect the classification theory of derivations of forms along π_1^0.

For completeness, we should mention that a construction of certain connections for a time-dependent framework, in particular a Cartan-type connection, can also be found in [18]. We shall, however, not go into the details of comparing our analysis with this work because the general setting is different. Indeed, the carrier space in [18] is $\mathbb{R} \times TM$ (to which $J^1\pi$ is diffeomorphic, but not in a canonical way) and, unlike it was the case e.g. in [20], the constructions carried out in [18] have an intrinsic meaning only for a strict product bundle interpretation of $\mathbb{R} \times TM$. In other words, it is as though one specific trivialization of $J^1\pi$ is singled out and from then on coordinate transformations are not allowed to depend on time.

2 A general scheme for lifting linear connections from $\pi_1^{0*}(\tau_E)$ to $J^1\pi$ and vice versa

We begin by recalling that $\pi_1^{0*}(\tau_E)$ has a canonically defined section, the 'total time derivative operator' $\mathbf{T}$, and that, as a result, the module of sections of $\pi_1^{0*}(\tau_E)$ has the following natural direct sum decomposition:

$$\mathcal{X}(\pi_1^0) \equiv \overline{\mathcal{X}}(\pi_1^0) \oplus \langle \mathbf{T} \rangle, \qquad \mathbf{T} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} \tag{7}$$

where sections in $\overline{\mathcal{X}}(\pi_1^0)$ are annihilated by dt. Throughout this paper we shall write $X = \overline{X} + \langle X, dt\rangle \mathbf{T}$ for vectorfields in $\mathcal{X}(\pi_1^0)$. Whenever we have a (non-linear) connection or horizontal distribution at our disposal on $J^1\pi$, there is a corresponding decomposition of $\mathcal{X}(J^1\pi)$:

$$\mathcal{X}(J^1\pi) \equiv \overline{\mathcal{X}}(\pi_1^0)^H \oplus \overline{\mathcal{X}}(\pi_1^0)^V \oplus \langle \mathbf{T}^H \rangle, \tag{8}$$

where $\mathbf{T}^H$ is a SODE vector field on $J^1\pi$, often called the 'associated semi-spray' of the given connection. Needless to say, horizontal vector fields on $J^1\pi$ may have components both in the first and third set of the decomposition (8): for a general $\xi \in \mathcal{X}(J^1\pi)$, we may write

$$\xi = \xi_H{}^H + \overline{\xi}_V{}^V \;=\; \overline{\xi}_H{}^H + \overline{\xi}_V{}^V + \langle \xi, dt\rangle \mathbf{T}^H, \tag{9}$$

with $\overline{\xi}_H, \overline{\xi}_V \in \overline{\mathcal{X}}(\pi_1^0)$. An important remark in this respect is the following: the generalization from an autonomous framework to a time-dependent one in a way has two faces; some formulas tend to carry over in a natural way by thinking of the first decomposition in (9), as though one would formally copy the decomposition (1) with one extra dimension in the horizontal component; other features, however, tend to be better understood if one thinks of $\overline{\mathcal{X}}(\pi_1^0)$ as the analogue of $\mathcal{X}(\tau)$ and thus assigns a separate role to the one-dimensional distribution spanned by $\mathbf{T}^H$. Most of the technicalities in what follows (if not all) are related to this dichotomy.

We are in particular interested in the case where the connection is the one canonically associated to a given SODE Γ. Note that an advantage of the time-dependent set-up is that $\mathbf{T}^H$ then coincides with the given Γ, a feature which is not in general true for the autonomous framework. Beware, however, that if one starts from a general horizontal distribution and looks at the SODE $\Gamma_0 = \mathbf{T}^H$, the original connection need not coincide with the SODE connection of Γ_0. Non-linear connections which come from a SODE are characterized by the property that their torsion $[P_H, S]$ is zero, where S is the canonical vertical endomorphism on $J^1\pi$, given by

$$S = \theta^i \otimes \frac{\partial}{\partial v^i}, \qquad \theta^i = dx^i - v^i dt. \tag{10}$$

The computation of the Nijenhuis bracket $[P_H, S]$ is easy to carry out in a basis of local vector fields adapted to the decomposition (8). In fact, one finds that only two types of components are not trivially zero. We list them here for later use:

$$[P_H, S](\overline{X}^H, \overline{Y}^H) \;=\; [\overline{X}^H, \overline{Y}^V]_V{}^V - [\overline{Y}^H, \overline{X}^V]_V{}^V - [\overline{X}^H, \overline{Y}^H]_H{}^V, \tag{11}$$

$$[P_H, S](\mathbf{T}^H, \overline{X}^H) \;=\; [\mathbf{T}^H, \overline{X}^V]_V{}^V - [\mathbf{T}^H, \overline{X}^H]_H{}^V. \tag{12}$$

There are no points in our analysis which could exclusively be dealt with in the case of a SODE connection, so we will generally not specify the horizontal distribution even though that will require sometimes reformulating previous work of other authors in such a more general context.

Let now D be a covariant derivative operator (i.e. a linear connection) on $\pi_1^{0*}(\tau_E)$ which induces a linear connection on the subbundle $\pi_1^{0*}(V\pi)$, i.e. which satisfies the assumption

$$\mathrm{D}_\xi(\overline{\mathcal{X}}(\pi_1^0)) \subset \overline{\mathcal{X}}(\pi_1^0) \quad \forall \xi \in \mathcal{X}(J^1\pi). \tag{13}$$

This is the only restriction which is required if we want to think of situations coming from an analogue in the tangent bundle set-up. Using the given horizontal distribution on $J^1\pi$ and having the autonomous doubling model (3) in mind, we define an associated class of linear connections on $J^1\pi$ by putting

$$\nabla_\xi \overline{X}^{H} = (\mathrm{D}_\xi \overline{X})^{H}, \qquad \nabla_\xi \overline{X}^{V} = (\mathrm{D}_\xi \overline{X})^{V}, \qquad \nabla_\xi \mathbf{T}^{H} = K(\xi), \tag{14}$$

where K is any type (1,1) tensor field on $J^1\pi$. It is easy to verify that for $\eta \in \mathcal{X}(J^1\pi)$, the operation

$$\nabla_\xi \eta = (\mathrm{D}_\xi \overline{\eta}_{H})^{H} + (\mathrm{D}_\xi \overline{\eta}_{V})^{V} + \xi(\langle \eta, dt\rangle)\mathbf{T}^{H} + \langle \eta, dt\rangle K(\xi) \tag{15}$$

defines a linear connection indeed, for any choice of K. All elements of such a class have the following easy to establish properties. Firstly,

$$\nabla_\xi(\overline{\mathcal{X}}(\pi_1^0)^{H}) \subset \overline{\mathcal{X}}(\pi_1^0)^{H}, \qquad \nabla_\xi(\overline{\mathcal{X}}(\pi_1^0)^{V}) \subset \overline{\mathcal{X}}(\pi_1^0)^{V} \qquad \forall \xi \in \mathcal{X}(J^1\pi). \tag{16}$$

Secondly, if J is the (degenerate) almost complex structure on $J^1\pi$, determined by the horizontal distribution according to the following defining relations: $J(\overline{X}^{H}) = \overline{X}^{V}$, $J(\overline{X}^{V}) = -\overline{X}^{H}$, $J(\mathbf{T}^{H}) = 0$, then we have

$$J(\nabla_\xi \overline{X}^{H}) = \nabla_\xi \overline{X}^{V}, \quad J(\nabla_\xi \overline{X}^{V}) = -\nabla_\xi \overline{X}^{H}, \quad \text{or equivalently} \quad \nabla_\xi J|_{\overline{\mathcal{X}}(J^1\pi)} = 0, \tag{17}$$

where $\overline{\mathcal{X}}(J^1\pi) \equiv \overline{\mathcal{X}}(\pi_1^0)^{H} \oplus \overline{\mathcal{X}}(\pi_1^0)^{V}$.

Conversely, let ∇ be any linear connection on $J^1\pi$ having the properties (16,17), then we define an associated class of linear connections on $\pi_1^{0*}(\tau_E)$ by putting:

$$\mathrm{D}_\xi \overline{X} = (\nabla_\xi \overline{X}^{H})_{H} = (\nabla_\xi \overline{X}^{V})_{V}, \qquad \mathrm{D}_\xi \mathbf{T} = L(\xi), \tag{18}$$

where L is any $C^\infty(J^1\pi)$-linear map from $\mathcal{X}(J^1\pi)$ to $\mathcal{X}(\pi_1^0)$. Indeed, for any $X \in \mathcal{X}(\pi_1^0)$, the relation

$$\mathrm{D}_\xi X = (\nabla_\xi \overline{X}^{H})_{H} + \xi(\langle X, dt\rangle)\mathbf{T} + \langle X, dt\rangle L(\xi) \tag{19}$$

is a linear connection on $\pi_1^{0*}(\tau_E)$ for any tensorial L. Obviously, each element of the class will have the property (13). If we take such an element and raise it to the bigger space $J^1\pi$ again according to the first procedure, we will obtain for every choice of K an element of the same class as the ∇ we started from. The type of connections on $J^1\pi$ we encounter in this construction are the ones we wish to call *connections of Finsler type*.

Definition *A pair (P_H, ∇) consisting of a horizontal distribution on $J^1\pi$ (represented by its horizontal projector) and a linear connection is said to be of Finsler type if we have the properties (16,17).*

Essentially, connections of Finsler type come from a class of linear connections D on $\pi_1^{0*}(\tau_E)$ with property (13) and we will sometimes figuratively term the couple (P_H, D) as being of Finsler type as well.

In order to obtain some equivalent characterizations of Finsler-type connections, we first prove two simple lemmas. A preliminary notational convention is in order here: various types of identity tensors (operating on different vector fields) will play a role in the sequel. The identity operator for $\mathcal{X}(\pi_1^0)$ will be denoted by I. We write its natural decomposition as

$$I = \overline{I} + dt \otimes \mathbf{T}, \qquad \text{with} \quad \overline{I} = \theta^i \otimes \frac{\partial}{\partial x^i}. \tag{20}$$

Likewise, $I_{J^1\pi}$ is the identity on $\mathcal{X}(J^1\pi)$ and $\overline{I}_{J^1\pi}$ is that part of $I_{J^1\pi}$ which vanishes on $\mathbf{T}^H$.

Lemma 1 $\quad S \circ J + J \circ S = -\overline{I}_{J^1\pi}$.

Proof: From $S(X^H) = X^V$ and the defining relations of J, it follows that

$$S(J(X^V)) = -X^V = -\overline{I}_{J^1\pi}(X^V), \quad S(J(X^H)) = S(J(\mathbf{T}^H)) = 0,$$

and

$$J(S(X^V)) = J(S(\mathbf{T}^H)) = 0, \quad J(S(X^H)) = -X^H = -\overline{I}_{J^1\pi}(X^H).$$

The result then readily follows. $\square$

Let now $P_{\overline{H}}$ be the 'strong horizontal projector' defined by $P_{\overline{H}}(\overline{X}^H) = \overline{X}^H$, $P_{\overline{H}}(\overline{X}^V) = 0$, $P_{\overline{H}}(\mathbf{T}^H) = 0$, and let M be the degenerate almost product structure determined by $M(\overline{X}^H) = \overline{X}^V$, $M(\overline{X}^V) = \overline{X}^H$, $M(\mathbf{T}^H) = 0$.

Lemma 2 $\quad J \circ P_{\overline{H}} - P_{\overline{H}} \circ J = M$.

Proof: The simple proof is similar to the one of Lemma 1. $\square$

Note that one can also obtain the relation $J \circ P_{\overline{H}} + P_{\overline{H}} \circ J = J$.

Proposition 1 *The following are equivalent characterizations of connections of Finsler type:*

$$\text{(16) and (17)} \quad \Longleftrightarrow \quad \nabla_\xi P_{\overline{H}}|_{\overline{\mathcal{X}}(J^1\pi)} = 0 \;\; \text{and} \;\; \nabla_\xi J|_{\overline{\mathcal{X}}(J^1\pi)} = 0. \tag{21}$$

$$\left\{ \begin{array}{l} \nabla_\xi P_{\overline{H}}|_{\overline{\mathcal{X}}(J^1\pi)} = 0 \\ \nabla_\xi J|_{\overline{\mathcal{X}}(J^1\pi)} = 0 \end{array} \right. \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \nabla_\xi P_{\overline{H}}|_{\overline{\mathcal{X}}(J^1\pi)} = 0 \\ \nabla_\xi S|_{\overline{\mathcal{X}}(J^1\pi)} = 0 \end{array} \right. \tag{22}$$

Proof: Making use of the information in (16), one easily finds from taking a covariant derivative of the defining relations of $P_{\overline{H}}$ that (16) implies $\nabla_\xi P_{\overline{H}}|_{\overline{\mathcal{X}}(J^1\pi)} = 0$. Conversely, this invariance implies that $P_{\overline{H}}(\nabla_\xi \overline{X}^H) = \nabla_\xi \overline{X}^H$ and $P_{\overline{H}}(\nabla_\xi \overline{X}^V) = 0$. The first of these says that $\nabla_\xi \overline{X}^H \in \overline{\mathcal{X}}(\pi_1^0)^H$, whereas the second only ensures that $\nabla_\xi \overline{X}^V \in \overline{\mathcal{X}}(\pi_1^0)^V \oplus \langle \mathbf{T}^H \rangle$ in a direct way. Indirectly however, using also $\nabla_\xi \overline{X}^V = J(\nabla_\xi \overline{X}^H)$ and Lemma 2, we find that $0 = J(\nabla_\xi \overline{X}^H) - M(\nabla_\xi \overline{X}^H) = \nabla_\xi \overline{X}^V - M(\nabla_\xi \overline{X}^H)$, which ensures that $\nabla_\xi \overline{X}^V$ belongs to $\overline{\mathcal{X}}(\pi_1^0)^V$ anyway.

Secondly, from $S(\overline{X}^V) = 0$ and (16), it follows that $\nabla_\xi S(\overline{X}^V) = 0$. From $S(\overline{X}^H) = \overline{X}^V$ and the second relation in (17), it follows that $\nabla_\xi S(\overline{X}^H) - S(J(\nabla_\xi \overline{X}^V)) = \nabla_\xi \overline{X}^V$. Using Lemma 1 and the information that $\nabla_\xi \overline{X}^V$ is vertical (from (16) again), it also follows that $\nabla_\xi S(\overline{X}^H) = 0$. This means that (16) and (17) imply $\nabla_\xi S|_{\overline{\mathcal{X}}(J^1\pi)} = 0$. For the converse, note first that $\nabla_\xi P_{\overline{H}}|_{\overline{\mathcal{X}}(J^1\pi)} = 0$ implies that $\nabla_\xi \overline{X}^H \in \overline{\mathcal{X}}(\pi_1^0)^H$. Next, from $\nabla_\xi S|_{\overline{\mathcal{X}}(J^1\pi)} = 0$ we find with the help of Lemma 1 again that

$$J(\nabla_\xi \overline{X}^V) = J(\nabla_\xi(S(\overline{X}^H))) = J(S(\nabla_\xi \overline{X}^H)) = (-\overline{I}_{J^1\pi} - S \circ J)(\nabla_\xi \overline{X}^H) = -\nabla_\xi \overline{X}^H.$$

Applying J to this relation, we obtain that also $J(\nabla_\xi \overline{X}^H) = \nabla_\xi \overline{X}^V$, so that $\nabla_\xi J|_{\overline{\mathcal{X}}(J^1\pi)} = 0$ indeed. This completes the proof. $\square$

3 The class of Berwald-type connections

The motivation for introducing the equivalence classes of linear connections of the previous section is, as indicated in the introduction, that we want to frame three existing constructions in the literature of a linear connection associated to a given SODE within one common scheme: that of a class of Berwald-type connections. The philosophy here is that one has to understand first all aspects lying at the origin of the difference between these constructions, before one can decide upon an optimal selection. Now, the Berwald-type connection for the autonomous framework, at least in its appearance on the pullback bundle τ^*TM in [6], is defined by (2). Within the present temporary scheme of equivalence classes of connections, we thus arrive at the following definition of the *class of Berwald-type connections*.

Definition *A linear connection* D *on* $\pi_1^{0*}(\tau_E)$ *with the property (13) belongs to the class of Berwald-type connections with respect to a given horizontal distribution, if it satisfies*

$$\mathrm{D}_\xi \overline{X} = [P_H(\xi), \overline{X}^V]_V + [P_V(\xi), \overline{X}^H]_H, \tag{23}$$

for all $\overline{X} \in \overline{\mathcal{X}}(\pi_1^0)$. *A Finsler pair* (P_H, ∇) *on* $J^1\pi$ *is said to be of Berwald type if it is lifted via (14) from a connection on* $\pi_1^{0*}(\tau_E)$ *with the property (23).*

It is of some interest to look at the effect of the various assumptions so far discussed on the torsion $T(\xi,\eta) = \nabla_\xi\eta - \nabla_\eta\xi - [\xi,\eta]$ of a pair (P_H, ∇) on $J^1\pi$. With the aid of the decomposition in horizontal and vertical parts, all components of T can be traced back to tensor fields acting on $\mathcal{X}(\pi_1^0)$. We introduce notations similar to those in [6] for these tensor fields and list them in the table below. The effect of assuming we have a connection of Finsler-type is that (18) can be invoked to express some covariant derivatives in terms of a D on $\pi_1^{0*}(\tau_E)$ (see the middle column). If in addition we have a connection of Berwald type, further simplifications occur through the definition (23). For completeness: the component $T(\overline{X}^V, \overline{Y}^V)_H$ of the torsion becomes trivially zero as soon as the assumption (16) is satisfied and is therefore not listed.

Definition	Finsler	Berwald
$\mathcal{A}(\overline{X},\overline{Y}) = T(\overline{X}^H,\overline{Y}^H)_H$	$\mathrm{D}_{\overline{X}^H}\overline{Y} - \mathrm{D}_{\overline{Y}^H}\overline{X} - [\overline{X}^H,\overline{Y}^H]_H$	$[\overline{X}^H,\overline{Y}^V]_V - [\overline{Y}^H,\overline{X}^V]_V - [\overline{X}^H,\overline{Y}^H]_H$
$\mathcal{R}(\overline{X},\overline{Y}) = T(\overline{X}^H,\overline{Y}^H)_V$	$-[\overline{X}^H,\overline{Y}^H]_V$	$-[\overline{X}^H,\overline{Y}^H]_V$
$\mathcal{B}(\overline{X},\overline{Y}) = T(\overline{X}^H,\overline{Y}^V)_H$	$-\mathrm{D}_{\overline{Y}^V}\overline{X} - [\overline{X}^H,\overline{Y}^V]_H$	0
$\mathcal{P}(\overline{X},\overline{Y}) = T(\overline{X}^H,\overline{Y}^V)_V$	$\mathrm{D}_{\overline{X}^H}\overline{Y} - [\overline{X}^H,\overline{Y}^V]_V$	0
$\mathcal{S}(\overline{X},\overline{Y}) = T(\overline{X}^V,\overline{Y}^V)_V$	$\mathrm{D}_{\overline{X}^V}\overline{Y} - \mathrm{D}_{\overline{Y}^V}\overline{X} - [\overline{X}^V,\overline{Y}^V]_V$	0
$\mathcal{A}_{\mathbf{T}}(\overline{X}) = T(\mathbf{T}^H,\overline{X}^H)_H$	$\mathrm{D}_{\mathbf{T}^H}\overline{X} - (\nabla_{\overline{X}^H}\mathbf{T}^H)_H - [\mathbf{T}^H,\overline{X}^H]_H$	$[\mathbf{T}^H,\overline{X}^V]_V - (\nabla_{\overline{X}^H}\mathbf{T}^H)_H - [\mathbf{T}^H,\overline{X}^H]_H$
$\mathcal{R}_{\mathbf{T}}(\overline{X}) = T(\mathbf{T}^H,\overline{X}^H)_V$	$-(\nabla_{\overline{X}^H}\mathbf{T}^H)_V - [\mathbf{T}^H,\overline{X}^H]_V$	$-(\nabla_{\overline{X}^H}\mathbf{T}^H)_V - [\mathbf{T}^H,\overline{X}^H]_V$
$\mathcal{B}_{\mathbf{T}}(\overline{X}) = T(\mathbf{T}^H,\overline{X}^V)_H$	$-(\nabla_{\overline{X}^V}\mathbf{T}^H)_H - [\mathbf{T}^H,\overline{X}^V]_H$	$-(\nabla_{\overline{X}^V}\mathbf{T}^H)_H - [\mathbf{T}^H,\overline{X}^V]_H$
$\mathcal{P}_{\mathbf{T}}(\overline{X}) = T(\mathbf{T}^H,\overline{X}^V)_V$	$\mathrm{D}_{\mathbf{T}^H}\overline{X} - (\nabla_{\overline{X}^V}\mathbf{T}^H)_V - [\mathbf{T}^H,\overline{X}^V]_V$	$-(\nabla_{\overline{X}^V}\mathbf{T}^H)_V$

Perhaps one of the lines in the table requires an extra word. The $\mathcal{S}$-tensor, in the case of a Berwald-type connection, gives rise to the expression $\mathcal{S}(\overline{X},\overline{Y}) = [\overline{X}^V,\overline{Y}^H]_H - [\overline{Y}^V,\overline{X}^H]_H - [\overline{X}^V,\overline{Y}^V]_V$. This is manifestly zero when the arguments are basic vector fields, because a bracket such as $[\overline{X}^V,\overline{Y}^H]$ then is vertical, and therefore $\mathcal{S}$ is zero for all arguments. Note further that if the Berwald-type connection is given as a D on $\pi_1^{0*}(\tau_E)$, the covariant derivatives of $\mathbf{T}^H$ in the right column are determined by our choice of the tensor K in (14).

A case of particular interest is the one where the given horizontal distribution comes from a SODE connection. Indeed, in such a case we see from (11,12) that there are two further simplifications in the torsion for a connection of Berwald type: $\mathcal{A}(\overline{X},\overline{Y}) = 0$ and $\mathcal{A}_{\mathbf{T}}(\overline{X})$ reduces to $-(\nabla_{\overline{X}^H}\mathbf{T}^H)_H$.

Let us now discuss the three available constructions referred to above and verify whether they are of Berwald type indeed. As a preliminary remark, we should say that all of them were originally constructed with respect to the horizontal distribution associated to a given SODE Γ, but Γ sometimes only enters the picture by the fact that it is $\mathbf{T}^H$. We will try to make our presentation somewhat more general by adapting the original construction to allow for any non-linear connection (or horizontal distribution) on $J^1\pi$ as the starting point, although that will not be equally successful in all three cases.

The simplest construction to explain is the one by Crampin *et al* [8]. Essentially, it takes the direct construction formula (2), as first introduced in [11] for autonomous SODE's, as the model and tries to carry it over to the time-dependent framework to construct a linear connection on $\pi_1^{0*}(\tau_E)$. One then immediately observes that a correction term is needed for having D_ξ satisfying the derivation property. The defining relation of a linear connection, valid with respect to any given horizontal distribution, thus becomes

$$\mathrm{D}_\xi X = [P_H(\xi), X^V]_V + [P_V(\xi), X^H]_H + P_H(\xi)(\langle X, dt\rangle)\mathbf{T}. \tag{24}$$

Obviously, the requirement (23) is satisfied, so we are in the class of Berwald connections. It further follows that $\mathrm{D}_\xi\mathbf{T} = \overline{\xi}_V$. To say something about torsion in this case, we need to make a choice for the tensor K in (14). It looks natural here to maintain the spirit in which the defining relation (24) was conceived by simply taking over the formula which raises the connection to one on $J^1\pi$ from the autonomous framework. That is to say, we put (as in [8])

$$\nabla_\xi\eta = (\mathrm{D}_\xi\eta_H)^H + (\mathrm{D}_\xi\overline{\eta}_V)^V. \tag{25}$$

It is obvious then that the first two relations in (14) are satisfied and that the tensor K is defined by

$$K(\xi) = \nabla_\xi\mathbf{T}^H = (\mathrm{D}_\xi\mathbf{T})^H = \overline{\xi}_V{}^H. \tag{26}$$

As a result, we have $\mathcal{P}_{\mathbf{T}} = 0$ and also $\mathcal{B}_{\mathbf{T}} = 0$ (since $(\nabla_{\overline{X}^V}\mathbf{T}^H)_H = \overline{X} = -[\mathbf{T}^H,\overline{X}^V]_H$), while $\mathcal{A}_{\mathbf{T}}$ and $\mathcal{R}_{\mathbf{T}}$ reduce to $\mathcal{A}_{\mathbf{T}} = [\mathbf{T}^H,\overline{X}^V]_V - [\mathbf{T}^H,\overline{X}^H]_H$, $\mathcal{R}_{\mathbf{T}} = -[\mathbf{T}^H,\overline{X}^H]_V$. If, in addition, the non-linear connection comes from a SODE Γ, all torsion tensors which can vanish (without restrictions on the horizontal distribution) become zero, except for $\mathcal{R}_{\mathbf{T}}$.

Massa and Pagani [15] have constructed a linear connection on $J^1\pi$. Their way of building up the theory is somewhat harder to fit within our present approach, because a full horizontal distribution only becomes part of the data at the final stage of the argumentation, where a given SODE Γ is singled out. Briefly, their construction starts as follows. First of all, among all possibly existing linear connections on $J^1\pi$, Massa and Pagani consider only those which preserve the 1-form dt, the canonical vertical endomorphism S and (constant) parallel transport along the fibres. Explicitly, the as yet undetermined covariant derivative will have the properties: $\nabla_\xi dt = 0$, $\nabla_\xi S = 0$ and $\nabla_{\overline{X}^V}\overline{Y}^V = 0$, for every $\overline{X} \in \overline{\mathcal{X}}(\pi_1^0)$ and every *basic* $\overline{Y} \in \overline{\mathcal{X}}(\pi_1^0)$. With these assumptions, it is possible to construct two projection operators, which are assumed to be completely complementary at the subsequent stage, and are given by

$$P_{\overline{H}}(\eta) := T(\Gamma, S(\eta)), \qquad Q(\eta) := S(T(\Gamma,\eta)) + \langle \eta, dt\rangle \Gamma, \tag{27}$$

where T is the torsion tensor of the linear connection to be constructed and Γ is an arbitary SODE. It is shown that $P_{\overline{H}}(\eta)$ and $Q(\eta)$ do not depend on the choice of Γ. Note further that the image of Q contains all vertical vectors and all possible SODE's (which makes sense because the difference between two SODE's is vertical). After adopting some further restrictions (to which we come back later), a theorem is proved concerning existence and uniqueness of a linear connection which leaves a pre-selected SODE invariant. That SODE in fact, when added to the image of $P_{\overline{H}}$ completes the horizontal distribution to which the constructed linear connection can be thought of as being associated.

We explain now how this scheme can be slightly modified when an arbitrary horizontal distribution is given from the outset. Then, in particular, we have the SODE $\mathbf{T}^H$ at our disposal, which we can use to define the operators $P_{\overline{H}}$ and Q. In other words, we put

$$P_{\overline{H}}(\eta) := T(\mathbf{T}^H, S(\eta)), \qquad Q(\eta) := S(T(\mathbf{T}^H,\eta)) + \langle \eta, dt\rangle \mathbf{T}^H. \tag{28}$$

With $P_{\overline{H}} + Q = I_{J^1\pi}$ as part of the assumptions, we then have $P_H(\eta) = P_{\overline{H}}(\eta) + \langle \eta, dt\rangle \mathbf{T}^H$, and $P_V(\eta) = Q(\eta) - \langle \eta, dt\rangle \mathbf{T}^H$. The somewhat delicate point hereby is that, since the horizontal distribution is given, the defining relation for $P_{\overline{H}}$ here has to be regarded as an implicit restriction, via the torsion, on the class of amissable linear connections we want to consider.

Continuing now, in this modified picture, the line of reasoning of Massa and Pagani, assume that the class of potential ∇'s is further restricted by requiring that they satisfy $\nabla_\xi P_{\overline{H}} = 0$ and have a curvature tensor *curv* which vanishes on any pair of SODE's, or equivalently satisfies $curv(\Gamma, \overline{X}^V) = 0$ for each SODE Γ and all $\overline{X}$. One can prove as a minor modification of Theorem 2.2 in [15] that, with all hypotheses so far imposed, any admissible linear connection ∇ is now completely determined if we know $\nabla_\Gamma \mathbf{T}^H$ for an arbitrary SODE Γ. The final point is to agree to fix this remaining freedom by requiring that $\nabla_{\mathbf{T}^H}\mathbf{T}^H = 0$ and $\nabla_{\partial/\partial v^i}\mathbf{T}^H = 0$ from which it follows as in [15] that actually $\nabla_\xi \mathbf{T}^H = 0$, $\forall \xi$. We thus have arrived (in a perhaps rather roundabout way) at a prescription for a uniquely defined linear connection on $J^1\pi$, corresponding to any pre-assigned horizontal distribution.

A point to be observed, however, is that this construction contains a hidden restriction which comes from the fact that the two explicit formulas for $P_{\overline{H}}$ and Q in (28) are assumed to yield complementary projectors. Indeed, from the defining relation of $P_{\overline{H}}$, taking the later requirement $\nabla_\xi \mathbf{T}^H = 0$ into account, we have

$$\overline{X}^H = T(\mathbf{T}^H, \overline{X}^V) = \nabla_{\mathbf{T}^H}\overline{X}^V - [\mathbf{T}^H, \overline{X}^V],$$

from which it follows that

$$\nabla_{\mathbf{T}^H}\overline{X}^V = [\mathbf{T}^H, \overline{X}^V]_V{}^V.$$

On the other hand, we have

$$0 = Q(\overline{X}^H) = S(T(\mathbf{T}^H, \overline{X}^H)) = S(\nabla_{\mathbf{T}^H}\overline{X}^H) - S([\mathbf{T}^H, \overline{X}^H]).$$

Using the invariance of S, this implies that

$$\nabla_{\mathbf{T}^H}\overline{X}^V = [\mathbf{T}^H, \overline{X}^H]_H{}^V.$$

Compatibility of the two expressions for $\nabla_{\mathbf{T}^H}\overline{X}^V$ thus requires that $[P_H, S](\mathbf{T}^H, \overline{X}^H) = 0$, which is one of the conditions for having a SODE connection. In coordinates, if Γ^l_0, Γ^l_k denote the connection coefficients of the given horizontal distribution, this condition reads,

$$v^j V_k(\Gamma^i_j) - \Gamma^i_k + V_k(\Gamma^i_0) = 0, \tag{29}$$

where V_i is shorthand for $\partial/\partial v^i$. It can be verified that this is the only compatibility requirement coming from (28).

What remains to be verified now is whether such a connection belongs to the Berwald class. Obviously, from the assumptions $\nabla_\xi P_{\overline{H}} = 0$ and $\nabla_\xi S = 0$, we will have a connection of Finsler type and the question is whether (23) holds. It easily follows from the defining relation for $P_{\overline{H}}$ in (28) and the property $\nabla_\xi \mathbf{T}^H = 0$ that (23) holds for $\xi = \mathbf{T}^H$. But it turns out that there is an obstruction for the rest of the property to hold true. To see this, let $\overline{X}, \overline{Y}$ be basic vector fields in $\overline{\mathcal{X}}(\pi^0_1)$. From one of the first assumptions, we have $\nabla_{\overline{X}^V}\overline{Y}^V = 0$, from which it follows via (18) that also $\nabla_{\overline{X}^V}\overline{Y}^H = 0$. Next, using the ∇-invariance of $\mathbf{T}^H$, one of the defining relations of the torsion tensor gives

$$\begin{aligned} [\mathbf{T}^H, \overline{X}^V] &= \nabla_{\mathbf{T}^H}\overline{X}^V - T(\mathbf{T}^H, \overline{X}^V), \\ &= \nabla_{\mathbf{T}^H}\overline{X}^V - \overline{X}^H, \end{aligned} \tag{30}$$

where we have used the definition (28) for $P_{\overline{H}}$ in the last line. The curvature requirement $curv\,(\mathbf{T}^H, \overline{X}^V) = 0$ subsequently learns that

$$\nabla_{\overline{X}^H} = \nabla_{\overline{X}^V}\nabla_{\mathbf{T}^H} - \nabla_{\mathbf{T}^H}\nabla_{\overline{X}^V} + \nabla_{\nabla_{\mathbf{T}^H}\overline{X}^V}. \tag{31}$$

Applying (31) to $\overline{Y}^V$, the last two terms vanish because $\overline{Y}$ is basic and $\nabla_{\mathbf{T}^H}\overline{X}^V$ is vertical in view of the properties (16). To compute the remaining vector field $\nabla_{\overline{X}^V}\nabla_{\mathbf{T}^H}\overline{Y}^V$ we proceed in coordinates. Using (30), one easily verifies that

$$\nabla_{\mathbf{T}^H}V_j = (V_j(\Gamma^l_0) + v^k V_j(\Gamma^l_k))V_l.$$

With $\overline{X}^V = X^i V_i$, $\overline{Y}^V = Y^i V_i$ (X^i, Y^i basic), we further have

$$\nabla_{\overline{X}^V} \nabla_{\mathbf{T}^H} \overline{Y}^V = \overline{X}^H(Y^i) V_i + Y^i X^j \nabla_{V_j} \nabla_{\mathbf{T}^H} V^i.$$

It then easily follows that

$$\nabla_{\overline{X}^H} \overline{Y}^V = [\overline{X}^H, \overline{Y}^V] + X^j Y^i \left(V_i V_j(\Gamma^l_0) + v^k V_i V_j(\Gamma^l_k) \right) V_l.$$

As we have seen in the table of torsion components, however, $T(\overline{X}^H, \overline{Y}^V) = 0$ is a necessary requirement for a connection to be of Berwald type and this would require here that

$$V_i V_j(\Gamma^l_0) + v^k V_i V_j(\Gamma^l_k) = 0. \tag{32}$$

It is easy to see through its two components $\mathcal{B}$ and $\mathcal{P}$ that the vanishing of this torsion is also sufficient for having the Berwald condition (23). The final point to observe is that (29) and (32) imply that

$$V_k(\Gamma^i_l) - V_l(\Gamma^i_k) = 0, \tag{33}$$

which is the coordinate expression for having $[P_H, S](\overline{X}^H, \overline{Y}^H) = 0$. We reach the rather striking conclusion that our attempt to generalize the construction of Massa and Pagani to arbitrary horizontal distributions only gives rise to a connection of Berwald type if that distribution is actually a SODE connection (which then brings us back to the actual construction in [15]).

Limiting ourselves then to the SODE case, the main difference between this connection and the one of Crampin *et al* comes from the fact that here $\nabla_\xi \mathbf{T}^H = 0$ for all ξ. The effect on the torsion is merely that $\mathcal{B}_{\mathbf{T}}$ is no longer zero. Instead we have $\mathcal{B}_{\mathbf{T}} = -\overline{I}$.

Let us come now to the third construction, which was independently set up by Byrnes [5]. Again, the original construction was carried out starting from a SODE connection, but we can easily generalize it here to the case of an arbitrary horizontal distribution. Indeed, the main idea of the construction of Byrnes was simply the following: (i) define the covariant derivatives of vector fields in $\overline{\mathcal{X}}(J^1\pi)$ by looking at the formula (2) for $\mathrm{D}_\xi X$ on the pullback bundle in the autonomous framework and taking horizontal and vertical lifts as appropriate; (ii) put $\nabla_\Gamma \Gamma = 0$; (iii) select the remaining derivatives of Γ in such a way that all torsion components which can be zero effectively vanish. By the nature of the construction, therefore, this is bound to give a connection belonging to the Berwald class. Transferred to the context of a general horizontal distribution, this idea becomes: (i) define $\nabla_\xi \overline{X}^H$ and $\nabla_\xi \overline{X}^V$ via (14) with $\mathrm{D}_\xi X$ given by (23); (ii) put $\nabla_{\mathbf{T}^H} \mathbf{T}^H = 0$; (iii) define $\nabla_{\overline{X}^V} \mathbf{T}^H$ and $\nabla_{\overline{X}^H} \mathbf{T}^H$ in such a way that the last four torsion components in the above table all vanish. This of course means then that the tensor K in (14) is constructed in a rather ad hoc manner.

Going back to the special case of a SODE connection, the only difference with our analysis of the first construction is that now also $\mathcal{R}_{\mathbf{T}} = 0$. Since the vertical part of the bracket $[\Gamma, \overline{X}^H]$ is determined by the so-called Jacobi endomorphism Φ, which is essentially the time-component of the curvature of the non-linear connection (see e.g. [20]), we could say here that the construction of Byrnes boils down to choosing the tensor K in (14) as:

$$K(\xi) = \overline{\xi}_V{}^H - \Phi(\overline{\xi}_H)^V. \tag{34}$$

Observe that from this point of view, i.e. if one regards the ∇ under consideration as being constructed from a D on $\pi_1^{0*}(\tau_E)$, the selection of K that was made in the construction of Massa and Pagani was simply $K = 0$.

We have now completed our programme of defining the class of Berwald-type connections in a sufficiently general way to be able to accomodate the constructions of Crampin *et al*, Massa and Pagani and Byrnes, and we have discovered the features which distinguish these constructions in that process. Can we, on the basis of these features, find reasons why one of these constructions should have preference over the others? If the ideal for a Berwald-type connection would be, as in the autonomous case, to have as much torsion zero as possible, then obviously the last construction would prevail. But it looks a lot less natural than the first one, for example, which is based on two direct formulas: (24) for the linear connection on $\pi_1^{0*}(\tau_E)$ and (25) for its lift to a connection on $J^1\pi$. The construction of Massa and Pagani deviates even further from the idea of maximally vanishing torsion, but we will now argue that it has a different interesting feature which the others fail to produce. At the start of Section 2, we have emphasized the importance of the natural decompositions (7) and (8) of the sections under consideration. Yet, when introducing Finsler-type connections, we required only part of that decomposition to be preserved by the covariant derivatives: see (13) for D and (16) for ∇. It would seem to be a natural assumption also to expect that these operators in addition would have the property $\mathrm{D}_\xi(\langle \mathbf{T}\rangle) \subset \langle \mathbf{T}\rangle$, respectively $\nabla_\xi(\langle \mathbf{T}^H\rangle) \subset \langle \mathbf{T}^H\rangle$. In this respect, only the construction of Massa and Pagani would be satisfactory in view of the property $\nabla_\xi \mathbf{T}^H = 0$.

Going back to our definition of the class of Berwald-type connections, it is obvious that the selection of a particular representative of the class is a matter of making a choice for $\mathrm{D}_\xi \mathbf{T}$ (when it concerns a connection on $\pi_1^{0*}(\tau_E)$) or for $\nabla_\xi \mathbf{T}^H$ (for a connection on $J^1\pi$). Clearly, there is much to say for giving preference to the simplest possible choice where these vector fields would both be zero. Note, however, that this would indirectly impose a restriction also on the freedom in lifting the connection (the choice of K in (14)) or lowering it (the choice of L in (18)). In the next section, therefore, we will explore some other interesting features of the theory, with an eye on discovering additional elements which can tell us whether there is a certain degree of optimality in choosing the simplest possible representative.

4 Further aspects of connections of Finsler and Berwald type

Recall that the only restriction so far considered for connections on $\pi_1^{0*}(\tau_E)$ was the requirement (13). It can equivalently be expressed as $\mathrm{D}_\xi \overline{I}|_{\overline{\mathcal{X}}(\pi_1^0)} = 0$. If a horizontal distribution is given and we lift the connection to one on $J^1\pi$ via (14), we have seen from (21, 22) that an immediate consequence is: $\nabla_\xi P_{\overline{H}}|_{\overline{\mathcal{X}}(J^1\pi)} = \nabla_\xi J|_{\overline{\mathcal{X}}(J^1\pi)} = \nabla_\xi S|_{\overline{\mathcal{X}}(J^1\pi)} = 0$. This should not come as a surprise as all the tensor fields under consideration here can in fact be constructed out of $\overline{I}$ via appropriate lifting operations. To be precise, we have $P_{\overline{H}} = \overline{I}^{H;H}$, $J = \overline{I}^{H;V} - \overline{I}^{V;H}$ and $S = \overline{I}^{H;V}$. These lifts, introduced in [20], are defined as follows for a general type (1,1) tensor field U along π_1^0:

$$\begin{array}{ll} U^{H;H}(X^H) = U(X)^H, & U^{H;H}(\overline{X}^V) = 0, \\ U^{H;V}(X^H) = U(X)^V, & U^{H;V}(\overline{X}^V) = 0, \\ U^{V;H}(X^H) = 0, & U^{V;H}(\overline{X}^V) = U(\overline{X})^H, \\ U^{V;V}(X^H) = 0, & U^{V;V}(\overline{X}^V) = U(\overline{X})^V. \end{array} \tag{35}$$

The interest of these operations is, as with the horizontal and vertical lifts of vector fields along π_1^0, that every type (1,1) tensor field $\mathcal{U}$ on $J^1\pi$ has a unique decomposition in the form

$$\mathcal{U} = U_1^{H;H} + U_2^{H;V} + U_3^{V;H} + U_4^{V;V}, \tag{36}$$

where the U_i are tensor fields along π_1^0 which have the following characteristics: U_1 is general, $U_2(\mathcal{X}(\pi_1^0)) \subset \overline{\mathcal{X}}(\pi_1^0)$, $U_3(\mathbf{T}) = 0$ and U_4 has the properties of U_2 and U_3. For Finsler-type connections, covariant derivatives of a $\mathcal{U}$ on $J^1\pi$ should to some extent be computable from the covariant derivatives of the U_i along π_1^0 which generate it. Ideally, of course, the latter should preserve the characteristic properties of each of the U_i.

If no further restrictions are imposed on the freedom in the procedures for raising or lowering the connection (see (14) and (18)), one can prove that for a $\mathcal{U}$ which maps $\overline{\mathcal{X}}(J^1\pi)$ into itself (the corresponding U_i in (36) then map $\overline{\mathcal{X}}(\pi_1^0)$ into itself), $\nabla_\xi \mathcal{U}|_{\overline{\mathcal{X}}(J^1\pi)} = 0$ if and only if $\mathrm{D}_\xi U_i|_{\overline{\mathcal{X}}(\pi_1^0)} = 0$, $i = 1, \ldots, 4$. But we may hope to discover natural additional restrictions as soon as we attempt to extend the scope of such a statement beyond the action on $\overline{\mathcal{X}}(\pi_1^0)$. It turns out that the very first restriction which imposes itself in this respect is to have the following direct link between the covariant derivatives of $\mathbf{T}$ and $\mathbf{T}^H$:

$$\nabla_\xi \mathbf{T}^H = (\mathrm{D}_\xi \mathbf{T})^H. \tag{37}$$

Indeed, we can state the following result which is proved in an appendix.

Proposition 2 *Under the assumption (37), we have for an arbitrary type (1,1) tensor field $\mathcal{U}$ on $J^1\pi$*

$$\nabla_\xi \mathcal{U}(\eta) = \mathrm{D}_\xi U_1(\eta_H)^H + \mathrm{D}_\xi U_2(\eta_H)^V + \mathrm{D}_\xi U_3(\overline{\eta}_V)^H + \mathrm{D}_\xi U_4(\overline{\eta}_V)^V. \tag{38}$$

The meaning of the extra condition (37) is the following. If D is the connection we start from, then the raising procedure (14) with $K(\xi) = \mathrm{D}_\xi \mathbf{T}^H$ corresponds exactly to the quite natural expression (25). If ∇ is the starting point, then the tensor L in (18) must be chosen in such a way that $L(\xi)^H = \nabla_\xi \mathbf{T}^H$, which is possible only if $\nabla_\xi \mathbf{T}^H$ is horizontal. Clearly, this is not the case for (our generalized version of) the construction of Byrnes, which means that it is rather unnatural to pursue maximally zero torsion in the time-dependent set-up: one should not insist on having $\mathcal{R}_{\mathbf{T}} = 0$. This is hardly surprising as $\mathcal{R}_{\mathbf{T}}$, just as $\mathcal{R}$ itself is related to the curvature of the non-linear connection one starts from.

Corollary 2.1 *Under the assumption of Proposition 2, we have*

$$\nabla_\xi \mathcal{U} = 0 \quad \Leftrightarrow \quad \left\{ \begin{array}{ll} \mathrm{D}_\xi U_1(\eta_H) = 0, & \mathrm{D}_\xi U_2(\eta_H) = 0, \\ \mathrm{D}_\xi U_3(\overline{\eta}_V) = 0, & \mathrm{D}_\xi U_4(\overline{\eta}_V) = 0, \end{array} \right. \quad \forall \eta_H, \overline{\eta}_V. \tag{39}$$

The proof is almost immediate. The only point to be careful about is that for the vertical parts in (38) the immediate conclusion is that the component in $\overline{\mathcal{X}}(\pi_1^0)$ of the corresponding vector field along π_1^0 must be zero. But U_2 and U_4 take their values in $\overline{\mathcal{X}}(\pi_1^0)$ and the property (13) of D then ensures that the same is true for their covariant derivatives. $\square$

The final point to observe is that the above results do not necessarily imply that the special features of the tensor fields U_i are preserved under covariant differentiation. One of the consequences then is that (39) in general is not sufficient to conclude that $\mathrm{D}_\xi U_i = 0,\ \forall i$. As a matter of fact, knowing that $U_3(\mathbf{T}) = 0$, we have $\mathrm{D}_\xi U_3(\mathbf{T}) = -U_3(\mathrm{D}_\xi \mathbf{T})$. It then follows that $\mathrm{D}_\xi U_3(\overline{\eta}_V) = 0,\ \forall \overline{\eta}_V$ implies $\mathrm{D}_\xi U_3 = 0$ if and only if

$$\mathrm{D}_\xi \mathbf{T} \in \langle \mathbf{T} \rangle. \tag{40}$$

The same is true for U_4. We thus have proved the following result.

Corollary 2.2 *If (37) holds together with (40), we have $\nabla_\xi \mathcal{U} = 0$ if and only if* $\mathrm{D}_\xi U_i = 0,\ i = 1, \ldots, 4$. $\square$

The linear connection (24) on $\pi_1^{0*}(\tau_E)$ as constructed in [8] does not have the property (40). The above considerations will prompt us to an improvement of the construction (24) in the next section.

Before doing that, however, we want to explore to what extent Crampin's characterization of Berwald-type connections has an analogue here and could perhaps also shed some light on ways to select a representative of the equivalence class we introduced with (23). The next considerations closely follow those in [6].

Let σ be a curve in E which has the interval $[a,b]$ in its domain, so $\sigma : [a,b] \subset \mathbb{R} \to E,\ u \mapsto \sigma(u)$. Consider the pullback bundle $\sigma^*(J^1\pi)$ (over some open interval containing $[a,b]$) and let $\tilde{\sigma}$ denote the corresponding map from $\sigma^*(J^1\pi)$ to $J^1\pi$. One easily verifies that tangent vectors to $\sigma^*(J^1\pi)$ whose image under the tangent map $T\tilde{\sigma}$ is horizontal in $T(J^1\pi)$ constitute a 1-dimensional vector space at each point. Expressed differently, the horizontal distribution on $J^1\pi$ pulls back to a 1-dimensional distribution on $\sigma^*(J^1\pi)$; this distribution contains a unique vector field which projects onto the coordinate vector field on $\mathbb{R}$. In coordinates (u, v^i) on $\sigma^*(J^1\pi)$, this vector field reads:

$$\dot{\sigma}^H = \frac{\partial}{\partial u} - \Big(\Gamma^i_0(\sigma(u), v)\, \sigma^{0\prime}(u) + \Gamma^i_j(\sigma(u), v)\, \sigma^{j\prime}(u)\Big)\, \frac{\partial}{\partial v^i}, \tag{41}$$

where the prime denotes differentiation with respect to u. The notation reflects the fact that at each point the value of this vector field is the horizontal lift of the tangent vector to σ (pulled back to $\sigma^*(J^1\pi)$). The integral curve of $\dot{\sigma}^H$ through a point (a, w) in $\sigma^*(J^1\pi)$ defines a section $\sigma^H_w : u \mapsto (u, \overline{\sigma}^H_w(u))$ of the pullback bundle, with $\overline{\sigma}^H_w(a) = w \in J^1_{\sigma(a)}\pi$. In coordinates, writing the section as $u \mapsto (u, X^i(u))$, the X^i are the solutions of the differential equations

$$X^{i\prime} = -\Gamma^i_0(\sigma(u), X)\, \sigma^{0\prime}(u) - \Gamma^i_j(\sigma(u), X)\, \sigma^{j\prime}(u), \qquad \text{with} \quad X^i(a) = w^i. \tag{42}$$

By the process of Lie dragging vertical tangent vectors to $J^1\pi$ along the flow of $\dot{\sigma}^H$, it is possible to define a (partial) rule of parallel transport as will now be explained. Recall first that $\pi_1^0 : J^1\pi \to E$ is an affine bundle modelled on $V\pi$, the sub-bundle of TE consisting of vertical tangent vectors to E, and that there is a natural vertical lift from $V_x\pi$ to each $V_w\pi_1^0$, the set of vertical tangent vectors to $J^1\pi$ at $w \in (\pi_1^0)^{-1}(x)$. Consider an element $\mathsf{v}_a \in V_{\sigma(a)}\pi$ and its vertical lift ${\mathsf{v}_a}_w^V$ to the starting point w of the curve σ_w^H. Let now Y be a vertical vector field along σ_w^H which takes the initial value ${\mathsf{v}_a}_w^V$ at the point w. Representing Y in coordinates as

$$Y|_{\sigma_w^H(u)} = Y^i(u) \frac{\partial}{\partial v^i}\bigg|_{\sigma_w^H(u)}, \tag{43}$$

The requirement $\mathcal{L}_{\dot\sigma^H} Y = 0$ uniquely determines Y: its components must be the solutions of the linear differential equations

$$Y^{i\prime} = -Y^k \left(\frac{\partial \Gamma_0^i}{\partial v^k}(\sigma(u), X(u))\, \sigma^{0\prime}(u) + \frac{\partial \Gamma_j^i}{\partial v^k}(\sigma(u), X(u))\, \sigma^{j\prime}(u) \right), \qquad \text{with} \quad Y^i(a) = \mathsf{v}_a^i. \tag{44}$$

The value of Y for $u = b$ defines a vertical vector at $\sigma_w^H(b)$ which can be thought of as being the vertical lift of a vector $\mathsf{v}_b \in V_{\sigma(b)}\pi$. So explicitly we put

$$Y|_{\sigma_w^H(b)} = {\mathsf{v}_b}_{\sigma_w^H(b)}^V, \tag{45}$$

and we can call this vector the parallel translate of ${\mathsf{v}_a}_w^V$ along the horizontal curve σ_w^H. It does not seem to make much sense to call v_b the parallel translate of v_a in the case of a non-linear connection, as the former depends on the choice of the point w through the initial value for the differential equations (42).

When the connection is linear, the equations (44) do not depend on $X(u)$ and thus Y becomes independent of the choice of w. We then have a rule of parallel transport from v_a to v_b. Observe, however, that we recover in this way only the rule of parallel transport for elements in the vector spaces on which the affine fibres of $J^1\pi$ are modelled. One then still has to define parallel transport in the usual way for one specific point of $J^1_{\sigma(a)}\pi$ to complete the construction for the fibre as an affine space.

Coming back to the general case of non-linear connections, we can complete the picture of parallel translation of vertical tangent vectors to $J^1\pi$ by calling two vertical vectors in points of the same fibre parallel if they are vertical lifts of the same element of $V\pi$. This is what will be understood here as having *complete parallelism in the fibres*. Translated to sections of the bundles under consideration, the criterion for a linear connection D on $\pi_1^{0*}(\tau_E)$ to have this property is that $\mathrm{D}_{\overline{X}^V}\overline{Y} = 0$ for basic $\overline{Y} \in \overline{\mathcal{X}}(\pi_1^0)$. Still following closely the analysis in [6], we can now prove the following result.

Proposition 3 *Given a general horizontal distribution on $J^1\pi$, every linear connection on $\pi_1^{0*}(\tau_E)$ with the properties that*

(i) *parallel translation along a horizontal curve in $J^1\pi$ is given by Lie transport, in the way explained above,*

(ii) *parallel translation along vertical curves is given by complete parallelism,* *belongs to the equivalence class of Berwald-type connections associated to that distribution.*

PROOF: We first show that any linear D with the property (23) has the properties (i) and (ii). That $\mathrm{D}_{\overline{X}^V}\overline{Y} = [\overline{X}^V, \overline{Y}^H]_H = 0$ for a basic $\overline{Y}$ has been argued repeatedly before. We further have $\mathrm{D}_{\dot\sigma^H}\overline{X} = (\mathcal{L}_{\dot\sigma^H}\overline{X}^V)_V$ and since $\mathcal{L}_{\dot\sigma^H}\overline{X}^V$ is vertical, it follows that $\mathrm{D}_{\dot\sigma^H}\overline{X} = 0 \Leftrightarrow \mathcal{L}_{\dot\sigma^H}\overline{X}^V = 0$.

Let us denote by δ the tensor which determines the difference between two connections D and $\tilde{\mathrm{D}}$: $\delta(\xi, X) = \mathrm{D}_\xi X - \tilde{\mathrm{D}}_\xi X$. We will prove next that if D and $\tilde{\mathrm{D}}$ both have the required properties, they can only differ in their action on $\mathbf{T}$, i.e. $\delta(\xi, \overline{X}) = 0$. Let w be any point of $J^1\pi$, in the fibre over x say, and consider first $\delta_w(\zeta_w, v_x)$ with an arbitrary v_x and a horizontal ζ_w. Take any curve σ in E with $x = \sigma(a)$ and $\dot\sigma^H(a) = \zeta_w$. Let Y be the vertical vector field along σ_w^H defined by Lie dragging $v_x{}_w^V$ in the manner described before. Then, by assumption, $\mathrm{D}_{\dot\sigma^H}Y = \tilde{\mathrm{D}}_{\dot\sigma^H}Y = 0$ and thus $\delta_w(\zeta_w, v_x) = 0$. Now take any vertical vector η_w and let $\overline{Y}$ be any basic vectorfield in $\overline{\mathcal{X}}(\pi_1^0)$ such that $\overline{Y}(x) = v_x$. Then $\mathrm{D}_{\eta_w}\overline{Y} = \tilde{\mathrm{D}}_{\eta_w}\overline{Y} = 0$ and hence $\delta_w(\eta_w, v_x) = 0$. The conclusion now readily follows. $\square$

This analysis confirms in the first place that it is acceptable to treat linear connections on $\pi_1^{0*}(\tau_E)$ as equivalent if the only distinction between them comes from a different action on $\mathbf{T}$. Contrary to the first part of this section, however, there are seemingly no indications in the above characterization of Berwald-type connections, which would point towards an optimal selection of the action on $\mathbf{T}$.

5 The optimal Berwald-type connection and derived constructions when a metric tensor field along π_1^0 is available

We will now attempt to come to an optimal choice of a representative of the class of Berwald-type connections associated to an arbitrary horizontal distribution on $J^1\pi$. Obviously, such a choice should combine all the good features we have encountered in discussing the different faces of the theory in the preceding sections. As we have seen, the essence of all such connections (as soon as they are of Finsler type) lies in a connection on $\pi_1^{0*}(\tau_E)$. So, in the first place, we want an explicit construction formula for a connection on $\pi_1^{0*}(\tau_E)$ which, unlike the explicit formula (24) of [8], does have the additional property (40) for preserving the natural decomposition (7). Secondly, we want to decide about an explicit rule for raising the connection to $J^1\pi$ which will then determine the optimal Berwald-type connection there. Preferably, there should also be an explicit expression for the inverse of this rule.

As explained in Section 3, the idea of the direct construction formula (24) was simply to copy the known formula (2) from the autonomous framework and see what correction terms are needed to have the right derivation properties for a connection on $\pi_1^{0*}(\tau_E)$. This way, one is guaranteed to arrive at a generalization which will give back the original theory when restricting to objects which are time-independent. There is, however, another way in which such an idea can be carried out: it consists in "copying the formula from the autonomous theory" with $\overline{X}$ in the place of X and then see what correction is needed to have a connection on $\pi_1^{0*}(\tau_E)$ again. This way, one arrives at the following explicit formula:

$$\mathrm{D}_\xi X = [P_{\scriptscriptstyle H}(\xi), \overline{X}^{\scriptscriptstyle V}]_{\scriptscriptstyle V} + [P_{\scriptscriptstyle V}(\xi), \overline{X}^{\scriptscriptstyle H}]_{\scriptscriptstyle H} + \xi(\langle X, dt\rangle)\mathbf{T}. \tag{46}$$

It is immediately clear that this connection has the property (40) since it is in fact the simplest representative for which $\mathrm{D}_\xi\mathbf{T} = 0$ for all ξ.

There is little doubt about the choice of an optimal lifting procedure now. Indeed, the further aspects of Finsler-type connections explored in the preceding section have revealed that it is advantageous to have the property (37), which will imply here that also $\nabla_\xi \mathbf{T}^{\scriptscriptstyle H} = 0$. The raising procedure then is just the natural one (25). Looking at the table of torsion components of Section 3, our optimal Berwald-type connection on $J^1\pi$ will have $\mathcal{B} = \mathcal{P} = \mathcal{S} = \mathcal{P}_{\mathbf{T}} = 0$ and $\mathcal{B}_{\mathbf{T}} = -\overline{I}$. If in particular the horizontal distribution comes from a SODE, we know that in addition $\mathcal{A} = 0$ and we will also have here $\mathcal{A}_{\mathbf{T}} = 0$. In the case of a SODE connection therefore, our optimal Berwald-type connection on $J^1\pi$ is just the linear connection constructed in [15].

There remains the question about an explicit formula for the inverse procedure of lowering a connection on $J^1\pi$ to one on $\pi_1^{0*}(\tau_E)$. Such a formula of course must have the properties (18) and can simply be taken to be

$$\mathrm{D}_\xi X = (\nabla_\xi X^{\scriptscriptstyle H})_{\scriptscriptstyle H}, \qquad \forall X \in \mathcal{X}(\pi_1^0). \tag{47}$$

As an aside, note that there is another explicit formula by which a ∇ on $J^1\pi$ can be lowered to a D on $\pi_1^{0*}(\tau_E)$, namely

$$\mathrm{D}_\xi X = (\nabla_\xi X^{\scriptscriptstyle V})_{\scriptscriptstyle V} + \xi(\langle X, dt\rangle)\mathbf{T}, \qquad \forall X \in \mathcal{X}(\pi_1^0). \tag{48}$$

In the case of our optimal Berwald-type connection on $J^1\pi$, these two procedures give rise to the same D, thanks to the property $\nabla_\xi \mathbf{T}^{\scriptscriptstyle H} = 0$. By contrast, for example, if we were to start from the connection (24), raise it to $J^1\pi$ via (25) and subsequently come back to a connection on $\pi_1^{0*}(\tau_E)$ via the procedure (48), we would not end up with the connection we started from, but rather with the connection (46).

Summarizing what preceeds, we come to the following formal definition.

Definition *The optimal Berwald-type connection on $\pi_1^{0*}(\tau_E)$, associated to a given horizontal distribution on $J^1\pi$, is defined explicitly by (46). The corresponding Berwald-type connection on $J^1\pi$ is produced by (25).*

Suppose now that we have an additional tool at our disposal, namely a symmetric type (0,2) tensor field g along π_1^0, having the property $g(\mathbf{T}, .) = 0$ and being non-singular when restricted to $\overline{\mathcal{X}}(\pi_1^0)$. We would like then to generalize the concepts (4-5) of the autonomous framework to arrive in the end at suitable generalizations of connections of the type of Cartan, Chern-Rund and Hashiguchi. It should be emphasized at this point that the context in which we wish to achieve this is far more general than the case of geodesic sprays on a Finsler manifold: both the horizontal distribution we start from and the tensor field g along π_1^0 are completely arbitrary and need not have anything to do with each other.

Let us agree that the main point about a Cartan-type connection is that it should be fully metrical and that the other two should be horizontically or vertically metrical only. There is, however, not a unique way of achieving such properties, even though from now on we agree that the Berwald-type connection we start from is fixed by (46). As we learn for example from [18] (Chapter X, Theorem 2.4), there is a lot of freedom still in pursuing the idea of constructing a metrical connection. One way to proceed here, for example, would be to define Cartan-type tensor fields $C_{\scriptscriptstyle V}$ and $C_{\scriptscriptstyle H}$ exactly as in equations (4,5), at least when all arguments are elements of $\overline{\mathcal{X}}(\pi_1^0)$. This may seem to be the most direct way to proceed. We prefer, however, to define $C_{\scriptscriptstyle V}$ and $C_{\scriptscriptstyle H}$ in this general context in a different way; it will lead to a metrical connection which is more closely related to the work of the Miron school on what they call "generalized Lagrange spaces" (cf. [1, 18]).

Definition *The vertical and horizontal Cartan tensor fields associated to the Berwald-type connection (46) and the metric tensor field g along π_1^0, are type (1,2) tensor fields $C_{\scriptscriptstyle V}$ and $C_{\scriptscriptstyle H}$ along π_1^0, determined by the relations*

$$\begin{aligned} g(C_{\scriptscriptstyle V}(\overline{X},\overline{Y}),\overline{Z}) &= \mathrm{D}_{\overline{X}^V} g\,(\overline{Y},\overline{Z}) + \mathrm{D}_{\overline{Y}^V} g\,(\overline{X},\overline{Z}) - \mathrm{D}_{\overline{Z}^V} g\,(\overline{X},\overline{Y}), & (49)\\ g(C_{\scriptscriptstyle H}(X,\overline{Y}),\overline{Z}) &= \mathrm{D}_{X^H} g\,(\overline{Y},\overline{Z}) + \mathrm{D}_{\overline{Y}^H} g\,(X,\overline{Z}) - \mathrm{D}_{\overline{Z}^H} g\,(X,\overline{Z}), & (50)\end{aligned}$$

and by the following restrictions for fixing the remaining time-components: $C_{\scriptscriptstyle V}(\,.\,,\mathbf{T}) = C_{\scriptscriptstyle V}(\mathbf{T},\,.\,) = 0,\ C_{\scriptscriptstyle H}(\,.\,,\mathbf{T}) = 0$.

Thinking then of another linear connection on $\pi_1^{0*}(\tau_E)$, $\hat{\mathrm{D}}$ say, which differs from the Berwald-type connection by the tensor field δ, i.e. such that $\hat{\mathrm{D}}_\xi X - \mathrm{D}_\xi X = \delta(\xi, X)$, we introduce type (1,2) tensor fields δ^V and δ^H along π_1^0, defined by

$$\delta^V(\overline{Z},X) = \delta(\overline{Z}^V, X), \qquad \delta^V(\mathbf{T},X) = 0, \tag{51}$$

$$\delta^H(Z,X) = \delta(Z^H, X). \tag{52}$$

Having optimalized the freedom in the class of Berwald-type connections by making $\mathrm{D}_\xi\mathbf{T} = 0$, we will do the same for the derived connections related to g which we will now discuss. That is to say, we choose to have also $\hat{\mathrm{D}}_\xi\mathbf{T} = 0$, which implies that $\delta^V(Z,\mathbf{T}) = 0$ and $\delta^H(Z,\mathbf{T}) = 0$ (but $\delta^H(\mathbf{T},X)$ need not be zero). This selection makes the following definitions perfectly compatible with the properties of the tensor fields $C_{\scriptscriptstyle V}$ and $C_{\scriptscriptstyle H}$ introduced above.

Definition *The Cartan-type connection on $\pi_1^{0*}(\tau_E)$, associated to the given metric tensor field g along π_1^0, deviates from the Berwald-type connection by*

$$\delta^V = \tfrac{1}{2}C_{\scriptscriptstyle V}, \qquad \delta^H = \tfrac{1}{2}C_{\scriptscriptstyle H}. \tag{53}$$

The Hashiguchi-type connection is likewise defined by

$$\delta^V = \tfrac{1}{2}C_{\scriptscriptstyle V}, \qquad \delta^H = 0. \tag{54}$$

Finally, the connection of Chern-Rund type is determined by

$$\delta^V = 0, \qquad \delta^H = \tfrac{1}{2}C_{\scriptscriptstyle H}. \tag{55}$$

Proposition 4 *The Cartan-type connection is metrical in the sense that* $\hat{\mathrm{D}}_\xi g = 0,\ \forall \xi$. *For the connection of Hashiguchi type we have* $\hat{\mathrm{D}}_{\overline{X}^V} g = 0$, *while for the connection of Chern-Rund type:* $\hat{\mathrm{D}}_{X^H} g = 0$.

PROOF: Let us see what the meaning is of, for example, the assumption $\delta^H = \frac{1}{2} C_H$. We have

$$\begin{aligned}
\hat{\mathrm{D}}_{X^H} g\,(\overline{Y},\overline{Z}) &= \hat{\mathrm{D}}_{X^H}(g(\overline{Y},\overline{Z})) - g(\hat{\mathrm{D}}_{X^H}\overline{Y},\overline{Z}) - g(\overline{Y},\hat{\mathrm{D}}_{X^H}\overline{Z}) \\
&= \mathrm{D}_{X^H}(g(\overline{Y},\overline{Z})) - g(\mathrm{D}_{X^H}\overline{Y},\overline{Z}) - g(\overline{Y},\mathrm{D}_{X^H}\overline{Z}) \\
&\quad - g(\delta^H(X,\overline{Y}),\overline{Z}) - g(\overline{Y},\delta^H(X,\overline{Z})) \\
&= \mathrm{D}_{X^H} g\,(\overline{Y},\overline{Z}) - \tfrac{1}{2}\left(g(C_H(X,\overline{Y}),\overline{Z}) + g(\overline{Y},C_H(X,\overline{Z}))\right),
\end{aligned}$$

and all terms on the right cancel out when the defining relation (50) is used to replace the terms involving C_H. $\hat{\mathrm{D}}_{X^H} g$ further inherits the property of g of vanishing whenever one of the arguments is $\mathbf{T}$, therefore $\hat{\mathrm{D}}_{X^H} g = 0$. The meaning of the assumption $\delta^V = \frac{1}{2} C_V$ is similar. The statements of the proposition now immediately follow. □

Needless to say, as in the autonomous case, making the connection more metrical has the effect of having less of the torsion components equal to zero. Without going into the details here, it is worth mentioning that the advantage of taking (49,50) as defining relations for the tensors C_V and C_H (rather than a direct transcription of (4,5) which would have only the first term in the right-hand side) is that more of the torsion components are still zero. This is similar to the result for autonomous systems stated as Theorem 2.1 in Chapter X of [18].

An interesting special case occurs when there is a direct link between the horizontal distribution and the tensor field g, in the following sense. Consider a general (regular) time-dependent Lagrangian system; let the horizontal distribution be the one coming from the Euler-Lagrange equations and take $(g_{ij}(t,x,v))$ to be the Hessian of the Lagrangian L. Then the second and third term in the right-hand side of the defining relations (49,50) cancel each other in view of the Helmholtz conditions satisfied by the tensor field g. Moreover, since in such a case also $\mathrm{D}_\Gamma g = 0$, we will have $C_H(\mathbf{T},\,.\,) = 0$. The effect of this last property is that all four connections (Berwald, Cartan, Hashiguchi and Chern-Rund) then share the same "dynamical covariant derivative operator" D_Γ. This feature was emphasized (for autonomous systems) in Crampin's recent discussion of the second variation formula [7], because the dynamical covariant derivative and the Jacobi endomorphism is all one needs in such an analysis. Note, however, that if we are not in the Lagrangian case, D_Γ and $\hat{\mathrm{D}}_\Gamma$ may be different; in fact, the necessary and sufficient condition for them to be identical is that $\mathrm{D}_{\mathbf{T}^H} g = 0$.

6 Coordinate expressions

We wish to make the different levels of generality and the diferent types of connections which have been considered in the previous section a bit more perceptible by presenting a survey now of the relevant coordinate expressions in each case. This will make it easier for the reader to compare our results with related features in, for example, the books of Miron and Anastasiei [18] and Antonelli et al [3], where the theory is often developed through coordinate calculations.

At the first level, all that is given is an arbitrary horizontal distribution and we can simply express the corresponding Berwald-type connection from (46). If in addition a metric tensor field g along π_1^0 is given, we list the coordinate expressions for the tensor fields $C_{\scriptscriptstyle V}$ and $C_{\scriptscriptstyle H}$ defined by (49,50) and the connection coefficients for the resulting Cartan-type connection. For a second stage, we look at the special interest case where the horizontal distribution comes from an arbitrary SODE Γ on $J^1\pi$. Finally, we have a closer look at the particular case when both the SODE Γ and the tensor field g are determined by a regular Lagrangian function L.

So, to begin with, consider an arbitrary horizontal distribution, locally spanned by vector fields

$$H_0 = \frac{\partial}{\partial t} - \Gamma^j_0(t,x,v)\frac{\partial}{\partial v^j}, \quad H_i = \frac{\partial}{\partial x^i} - \Gamma^j_i(t,x,v)\frac{\partial}{\partial v^j}. \tag{56}$$

We have

$$\mathbf{T}^{\scriptscriptstyle H} = \frac{\partial}{\partial t} + v^i\frac{\partial}{\partial x^i} - \left(\Gamma^j_0 + v^i\Gamma^j_i\right)\frac{\partial}{\partial v^j}. \tag{57}$$

Using shorthand notations already introduced in Section 3, a straightforward application of the defining relation (46) shows that the Berwald-type connection on $\pi_1^{0*}(\tau_E)$ is determined by

$$\mathrm{D}_{\mathbf{T}^H}\frac{\partial}{\partial x^j} = \kappa^k_j\frac{\partial}{\partial x^k}, \quad \mathrm{D}_{H_i}\frac{\partial}{\partial x^j} = V_j(\Gamma^k_i)\frac{\partial}{\partial x^k}, \quad \mathrm{D}_{V_i}\frac{\partial}{\partial x^j} = 0, \tag{58}$$

where

$$\kappa^k_j = V_j(\Gamma^k_0) + v^l V_j(\Gamma^l_k),$$

and of course $\mathrm{D}_{\mathbf{T}^H}\mathbf{T} = \mathrm{D}_{H_i}\mathbf{T} = \mathrm{D}_{V_i}\mathbf{T} = 0$. Since we will have, by construction, $\mathrm{D}_\xi\mathbf{T} = 0$ for all connections which follow, we will not repeat these zero-components below.

Assume next that a symmetric tensor field of the form $g = g_{ij}(t,x,v)\,\theta^i\otimes\theta^j$ is given. Then, it follows from (49) that the vertical Cartan tensor $C_{\scriptscriptstyle V}$ is of the form $C_{\scriptscriptstyle V} = C_{\scriptscriptstyle V}{}^k_{ij}\,\theta^i\otimes\theta^j\otimes(\partial/\partial x^k)$, with

$$C_{\scriptscriptstyle V}{}^k_{ij} = g^{kl}\Big(V_i(g_{lj}) + V_j(g_{li}) - V_l(g_{ij})\Big). \tag{59}$$

The horizontal Cartan tensor, on the other hand, has a non-zero dt-component; it is of the form

$$C_{\scriptscriptstyle H} = C_{\scriptscriptstyle H}{}^k_{ij}\,\theta^i\otimes\theta^j\otimes\frac{\partial}{\partial x^k} + C_{\scriptscriptstyle H}{}^k_{0i}\,dt\otimes\theta^i\otimes\frac{\partial}{\partial x^k},$$

with

$$\begin{aligned} C_{\scriptscriptstyle H}{}^k_{0i} &= -\kappa^k_i + g^{kl}\Big(\mathbf{T}^{\scriptscriptstyle H}(g_{li}) - \kappa^m_l g_{mi}\Big), && (60)\\ C_{\scriptscriptstyle H}{}^k_{ij} &= -\Big(V_i(\Gamma^k_j) + V_j(\Gamma^k_i)\Big) + g^{kl}\Big(H_i(g_{lj}) + H_j(g_{li}) - H_l(g_{ij})\Big) \\ &\quad + g^{kl}\Big(g_{im}(V_j(\Gamma^m_l) - V_l(\Gamma^m_j)) + g_{jm}(V_i(\Gamma^m_l) - V_l(\Gamma^m_i))\Big). && (61)\end{aligned}$$

As a result, the Cartan-type connection along π_1^0, the way it is intrinsically defined by (53), is determined locally by the following relations:

$$\begin{aligned}
\hat{\mathrm{D}}_{\mathbf{T}^H}\frac{\partial}{\partial x^j} &= \left[\tfrac{1}{2}\kappa_j^k + \tfrac{1}{2}g^{kl}\left(\mathbf{T}^H(g_{lj}) - \kappa_l^m g_{mj}\right)\right]\frac{\partial}{\partial x^k}, &(62)\\
\hat{\mathrm{D}}_{H_i}\frac{\partial}{\partial x^j} &= \Big[\tfrac{1}{2}\Big(V_j(\Gamma_i^k) - V_i(\Gamma_j^k)\Big) + \tfrac{1}{2}g^{kl}\Big(H_i(g_{lj}) + H_j(g_{li}) - H_l(g_{ij})\Big)\\
&\quad + g^{kl}\Big(g_{im}(V_j(\Gamma_l^m) - V_l(\Gamma_j^m)) + g_{jm}(V_i(\Gamma_l^m) - V_l(\Gamma_i^m))\Big)\Big]\frac{\partial}{\partial x^k}, &(63)\\
\hat{\mathrm{D}}_{V_i}\frac{\partial}{\partial x^j} &= \tfrac{1}{2}g^{kl}\Big(V_i(g_{lj}) + V_j(g_{li}) - V_l(g_{ij})\Big)\frac{\partial}{\partial x^k}. &(64)
\end{aligned}$$

We leave it as an exercise for the reader to write down in the same way the local determining equations for the connections of Hashiguchi and of Chern-Rund type, as defined by (54) and (55) respectively.

Coming now to the second stage, let the horizontal distribution be the one canonically associated to a given SODE

$$\Gamma = \frac{\partial}{\partial t} + v^i\frac{\partial}{\partial x^i} + f^i(t,x,v)\frac{\partial}{\partial v^i}. \tag{65}$$

This means that the coefficients in (56) are given by

$$\Gamma_i^j = -\tfrac{1}{2}\frac{\partial f^j}{\partial v^i}, \quad \Gamma_0^j = -f^j - v^k\Gamma_k^j, \tag{66}$$

and that $\mathbf{T}^H = \Gamma$. The two conditions which essentially determine whether a non-linear connection is a SODE-connection, have already been mentioned in coordinates (see (29) and (33)). They read: $\kappa_j^i = \Gamma_j^i$ and $V_k(\Gamma_j^i) = V_j(\Gamma_k^i)$. The first of these has an immediate effect on the coefficients in the equations for the associated Berwald-type connection, which now become:

$$\mathrm{D}_\Gamma\frac{\partial}{\partial x^j} = \Gamma_j^k\frac{\partial}{\partial x^k}, \quad \mathrm{D}_{H_i}\frac{\partial}{\partial x^j} = V_j(\Gamma_i^k)\frac{\partial}{\partial x^k}, \quad \mathrm{D}_{V_i}\frac{\partial}{\partial x^j} = 0. \tag{67}$$

The second results in obvious cancellations in the horizontal covariant derivative of the Cartan connection (still for an arbitrary metric tensor field g along π_1^0). We get:

$$\begin{aligned}
\hat{\mathrm{D}}_\Gamma\frac{\partial}{\partial x^j} &= \left[\tfrac{1}{2}\Gamma_j^k + \tfrac{1}{2}g^{kl}\left(\Gamma(g_{lj}) - \Gamma_l^m g_{mj}\right)\right]\frac{\partial}{\partial x^k}, &(68)\\
\hat{\mathrm{D}}_{H_i}\frac{\partial}{\partial x^j} &= \tfrac{1}{2}g^{kl}\Big(H_i(g_{lj}) + H_j(g_{li}) - H_l(g_{ij})\Big)\frac{\partial}{\partial x^k}, &(69)\\
\hat{\mathrm{D}}_{V_i}\frac{\partial}{\partial x^j} &= \tfrac{1}{2}g^{kl}\Big(V_i(g_{lj}) + V_j(g_{li}) - V_l(g_{ij})\Big)\frac{\partial}{\partial x^k}. &(70)
\end{aligned}$$

Obviously, the elegance of this result is that both the horizontal and vertical covariant derivative resemble the classical formula for the Levi-Civita connection.

Consider now finally the particular case of a Lagrangian system. That is to say, let $L(t,x,v)$ be a given regular Lagrangian function on $J^1\pi$; then, there is an intrinsically defined metric tensor field g along π_1^0, whose coefficients are $g_{ij} = V_iV_j(L)$. Let further Γ denote the SODE field governing the Euler-Lagrange equations, i.e. take the f^i in (65) to be

$$f^i = g^{ij}\left(\frac{\partial L}{\partial x^j} - v^k\frac{\partial^2 L}{\partial x^k\partial v^j} - \frac{\partial^2 L}{\partial t\partial v^j}\right). \tag{71}$$

The Berwald-type connection remains determined by (67), but we can express the relevant coefficients Γ_j^k and $V_j(\Gamma_i^k)$ here in terms of the Lagrangian L. One can verify that:

$$\begin{aligned}\Gamma_j^k &= \tfrac{1}{2}g^{kl}\left(\Gamma(g_{lj}) + \frac{\partial^2 L}{\partial v^l\partial x^j} - \frac{\partial^2 L}{\partial x^l\partial v^j}\right), &(72)\\ V_j(\Gamma_i^k) &= \tfrac{1}{2}g^{kl}\Big(\Gamma V_j(g_{li}) + H_j(g_{li}) + H_i(g_{lj}) - H_l(g_{ij})\\ &\qquad - \Gamma_l^m V_m(g_{ij}) - \Gamma_j^m V_m(g_{li}) - \Gamma_i^m V_m(g_{lj})\Big). &(73)\end{aligned}$$

Turning then to the Cartan-type connection for this case, the following simplifications of the previous situation can be verified. First of all, we obviously have $V_j(g_{li}) - V_l(g_{ij}) = 0$. Furthermore, the property $\mathrm{D}_\Gamma g = 0$ means in coordinates that $\Gamma(g_{lj}) = \Gamma_l^m g_{mj} + \Gamma_j^m g_{ml}$, from which it easily follows that the right-hand side in (68) is equal to Γ_j^k (i.e. is the same as for the Berwald connection, as argued already in the previous section). As a result, the Cartan-type connection for the Lagrangian case is determined by

$$\begin{aligned}\hat{\mathrm{D}}_\Gamma\frac{\partial}{\partial x^j} &= \Gamma_j^k\frac{\partial}{\partial x^k}, &(74)\\ \hat{\mathrm{D}}_{H_i}\frac{\partial}{\partial x^j} &= \tfrac{1}{2}g^{kl}\Big(H_i(g_{lj}) + H_j(g_{li}) - H_l(g_{ij})\Big)\frac{\partial}{\partial x^k}, &(75)\\ \hat{\mathrm{D}}_{V_i}\frac{\partial}{\partial x^j} &= \tfrac{1}{2}g^{kl}V_i(g_{lj})\frac{\partial}{\partial x^k}. &(76)\end{aligned}$$

To finish this summary of coordinate expressions, let us repeat that one should be a little cautious in comparing our expressions with those in [18] for time-dependent Lagrangians. The point is that the set-up is different: due to a strict separation between time and space variables in [18], some of the concepts developed in that work loose there intrinsic meaning within the jet bundle approach which we have adopted.

7 From covariant derivatives to exterior derivatives and the classification of derivations

In [20] a systematic study was made of the theory and classification of derivations of scalar and vector-valued forms along π_1^0. A classification of such derivations, in the line of the standard work of Frölicher and Nijenhuis [9], makes use of a vertical and horizontal exterior derivative. For the horizontal derivative one needs a horizontal distribution, while the vertical derivative is canonically available from the intrinsic structure of $J^1\pi$. Yet, not surprisingly, there is not just one canonically defined vertical exterior derivative: one encounters a certain freedom in fixing the time-component. Scalar differential forms along π_1^0 can be identified with semi-basic forms on $J^1\pi$ and there is a natural derivation of degree 1 on $J^1\pi$ which preserves semi-basic forms, namely (in the notations of [9]) $d_S = [i_S, d]$. To maintain the analogy with the autonomous theory, the authors in [20] decided to model their vertical exterior derivative d^V on d_S, even though this derivation does not have the coboundary property ${d^V}^2 = 0$. The authors were well aware of the availability of another vertical derivation which does have that property. But from the point of view of setting up the theory of derivations, this other one comes somehow in the second place as it can be derived from d^V: it is the derivation $d^V_{\overline{I}} = [i_{\overline{I}}, d^V]$. Much later in the story of classifying derivations, one encounters vertical and horizontal covariant derivatives which appear to coincide with the ones coming from the linear connection (24) in [8].

The purpose of this final section is to approach this matter from the other end. That is to say, by way of application of the newly acquired insights, we wish to explore to what extent the optimal choice of a Berwald-type connection adds something to the debate about the best possible choice of a vertical exterior derivative.

Let us first discuss some generalities about the way to construct an exterior derivative from a covariant derivative. Suppose a covariant derivative $\mathrm{D}^\star$ on $\pi_1^{0*}(\tau_E)$ is given, which has been extended by duality to a (self-dual) degree 0 derivation on tensor fields along π_1^0 (the present discussion, by the way, applies just as well to covariant derivatives on a general manifold). Putting $[X,Y]_\star = \mathrm{D}^\star_X Y - \mathrm{D}^\star_Y X$, we have a bilinear (over $\mathbb{R}$) skew-symmetric operator on $\mathcal{X}(\pi_1^0)$ which satisfies a Leibniz rule, namely $[FX,Y]_\star = F\,[X,Y]_\star - (\mathrm{D}^\star_Y F)X$, but which need not have the Jacobi identity property. Any other bracket operator with these properties differs from the first one by a vector-valued 2-form (torsion form) along π_1^0. In other words, given $\mathrm{D}^\star$, the most general skew-symmetric bracket operator with the above Leibniz property is of the form

$$[X,Y]_\star = \mathrm{D}^\star_X Y - \mathrm{D}^\star_Y X + T^\star(X,Y), \tag{77}$$

where $T^\star$ is any element of $V^2(\pi_1^0)$ (the $C^\infty(J^1\pi)$-module of vector-valued 2-forms along π_1^0). Let now ω be a scalar k-form along π_1^0 (notation: $\omega \in \bigwedge^k(\pi_1^0)$) or a vector-valued k-form (then $\omega \in V^k(\pi_1^0)$).

Proposition 5 *The operator $d^\star$, defined by*

$$\begin{aligned} d^\star\omega(X_0,\ldots,X_k) &= \sum_{i=0}^{k}(-1)^i \mathrm{D}^\star_{X_i}(\omega(X_0,\ldots,\hat{X}_i,\ldots,X_k)) \\ &\quad + \sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j]_\star, X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k), \end{aligned} \tag{78}$$

is a derivation of degree 1 on $\bigwedge(\pi_1^0)$ and $V(\pi_1^0)$.

Proof: From the defining relation, it follows that the action of $d^\star$ on functions F on $J^1\pi$, 1-forms α and vector fields X along π_1^0 is given by: $d^\star F(X) = \mathrm{D}^\star_X F$, $d^\star\alpha(X,Y) = \mathrm{D}^\star_X(\alpha(Y)) - \mathrm{D}^\star_Y(\alpha(X)) - \alpha([X,Y]_\star)$, $d^\star X(Y) = \mathrm{D}^\star_Y X$. It is easy to verify that this restricted action has the necessary properties for a derivation, i.e. we have $d^\star(FG) = Fd^\star G + Gd^\star F$, $d^\star(F\alpha) = d^\star F \wedge \alpha + Fd^\star\alpha$ and $d^\star(FX) = F\,d^\star X + d^\star F \wedge X$. It follows that there is a unique derivation $\hat{d}^\star$ wich coincides with $d^\star$ when restricted to functions, 1-forms and vector fields. Defining $\hat{d}^\star_X = [i_X, \hat{d}^\star]$ as usual, one can create another self-dual degree zero derivation $(\hat{d}^\star_X)^*$ which is obtained from $\hat{d}^\star_X|_{\bigwedge^1(\pi^0_1)}$ by imposing the duality rule

$$\langle(\hat{d}^\star_X)^*Y, \alpha\rangle = \hat{d}^\star_X(\langle Y, \alpha\rangle) - \langle Y, \hat{d}^\star_X\alpha\rangle$$

$\forall X, Y \in \mathcal{X}(\pi^0_1)$ and $\alpha \in \bigwedge^1(\pi^0_1)$. It was proved in [13] (see Prop. 3.3) that $\hat{d}^\star$ then has the property

$$\begin{aligned}\hat{d}^\star\omega(X_0,\ldots,X_k) &= \sum_{i=0}^{k}(-1)^i\hat{d}^\star_{X_i}(\omega(X_0,\ldots,\hat{X}_i,\ldots,X_k)) \\ &+ \sum_{0\leq i<j\leq k}(-1)^{i+j}\omega((\hat{d}^\star_{X_i})^*(X_j), X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots X_k).\end{aligned}$$

One easily computes, however, that $(\hat{d}^\star_X)^*Y = [X, Y]_\star$. Comparison of the above result with the defining relation (78) then shows that $\hat{d}^\star \equiv d^\star$ and the result follows. □

We come back to the actual situation now, where we have two explicitly defined covariant derivatives on $\pi^{0*}_1(\tau_E)$ which are of Berwald type. We write the original one (24) for this section as D and the newly introduced one (46) as $\tilde{\mathrm{D}}$. We make a further notational convention which has the advantage of focussing entirely on operations which involve only tensorial objects along the projection: if $\xi \in \mathcal{X}(J^1\pi)$ is itself the horizontal or vertical lift of some vector field $Y \in \mathcal{X}(\pi^0_1)$, we shall write

$$\mathrm{D}^H_Y X \text{ for } \mathrm{D}_{Y^H}X, \text{ and likewise } \mathrm{D}^V_Y X \text{ for } \mathrm{D}_{Y^V}X. \tag{79}$$

Such notations will make it easier to relate the discussion to the calculus of derivations developed in [12, 13, 20]. Similar notations are used for $\tilde{\mathrm{D}}$.

The difference between the two Berwald-type connections is given by $\mathrm{D}_\xi X - \tilde{\mathrm{D}}_\xi X = \langle X, dt\rangle\overline{\xi}_V$. This translates into the following relations between horizontal and vertical derivatives:

$$\mathrm{D}^H_Y X = \tilde{\mathrm{D}}^H_Y X, \tag{80}$$
$$\mathrm{D}^V_Y X = \tilde{\mathrm{D}}^V_Y X + \langle X, dt\rangle\overline{Y}. \tag{81}$$

The idea is now to let the * of the above general considerations play the role of ^{H} and ^{V}. Clearly, if we make the same choice of torsion forms for the brackets coming from both connections and subsequently use (78) to construct exterior derivatives, we will obtain the same horizontal exterior derivatives d^H and $\tilde{d}^H$, but the vertical exterior derivatives will be different.

From the general classification results of self-dual derivations in [20], we know that the difference between D^V_Y and $\tilde{\mathrm{D}}^V_Y$ is a so-called derivation of type μ_*. Such a derivation is of algebraic type and consists of two parts. For a derivation of degree r, for example, we write $\mu_Q = a_Q - i_Q$, where Q is a type (1,1) tensor-valued r-form along π^0_1; a_Q vanishes on scalar forms, while i_A vanishes on vector fields. Specifically now, we derive from (81) that

$$\mathrm{D}^{V}_{Y} = \tilde{\mathrm{D}}^{V}_{Y} + \mu_A, \qquad \text{with} \quad A = dt \otimes \overline{Y}. \tag{82}$$

It follows that for the dual action on 1-forms

$$\mathrm{D}^{V}_{Y}\alpha = \tilde{\mathrm{D}}^{V}_{Y}\alpha - i_A\alpha = \tilde{\mathrm{D}}^{V}_{Y}\alpha - \langle \overline{Y}, \alpha\rangle dt.$$

Since the vertical bracket in [20] had no torsion, we take $\tilde{T}^{V} = 0$ as well. We then have

$$\begin{aligned} d^{V}\alpha(X,Y) &= \mathrm{D}^{V}_{X}\alpha(Y) - \mathrm{D}^{V}_{Y}\alpha(X) \\ &= \tilde{\mathrm{D}}^{V}_{X}\alpha(Y) - \tilde{\mathrm{D}}^{V}_{Y}\alpha(X) - \langle \overline{X}, \alpha\rangle\langle Y, dt\rangle + \langle \overline{Y}, \alpha\rangle\langle X, dt\rangle, \end{aligned}$$

so that

$$d^{V}\alpha = \tilde{d}^{V}\alpha + i_{dt\wedge I}\alpha.$$

Similarly, for the action on vector fields we find

$$d^{V}X = \tilde{d}^{V}X + a_{\overline{I}\otimes dt}X, \quad \text{where} \quad a_{\overline{I}\otimes dt}X = \langle X, dt\rangle\, \overline{I}.$$

In conclusion, the difference between the two vertical derivatives is expressed by

$$\tilde{d}^{V} = d^{V} - i_{dt\wedge I} - a_{\overline{I}\otimes dt} = d^{V}_{\overline{I}} - a_{\overline{I}\otimes dt}. \tag{83}$$

It may come a bit as a surprise that $\tilde{d}^{V}$ is not the same as $d^{V}_{\overline{I}}$. One can verify, however, that just like $d^{V}_{\overline{I}}$, $\tilde{d}^{V}$ has the coboundary property $\tilde{d}^{V\,2} = 0$. Indeed, on scalar forms this is obvious since the a_*-term then does not contribute. To complete the argument, since both terms in the right-hand side of (83) manifestly vanish on $\partial/\partial x^i$, it then suffices to check that $\tilde{d}^{V\,2}\mathbf{T} = 0$. This follows easily from the fact that $d^{V}_{\overline{I}}\mathbf{T} = a_{\overline{I}\otimes dt}\mathbf{T} = \overline{I}$ and thus $\tilde{d}^{V}\mathbf{T} = 0$. In coordinates, the action of the new $\tilde{d}^{V}$ on the local basis of 1-forms and vector fields is given by

$$\tilde{d}^{V}\theta^i = 0, \quad \tilde{d}^{V}dt = 0, \quad \tilde{d}^{V}\frac{\partial}{\partial x^i} = 0, \quad \tilde{d}^{V}\mathbf{T} = 0.$$

It would perhaps be worthwhile to enter more deeply into the question of the effect of selecting $\tilde{d}^{V}$ as the fundamental vertical exterior derivative on the classification theory of derivations along π^0_1. This, of course, is beyond the scope of the present paper. In a sense, one expects that the influence of such a change will be minor as long as one deals with forms acting on $\overline{\mathcal{X}}(\pi^0_1)$. We finish our discussion by deriving a couple of properties which express this expectation in more precise terms.

Let ω be an element of $\bigwedge^k(\pi^0_1)$. Applying the definition of derivations of type i_* (cf. [12]), we find that

$$\begin{aligned} &i_{dt\wedge I}\omega\,(\overline{X}_1, \ldots, \overline{X}_{k+1}) = \\ &\qquad \frac{1}{2!(k-1)!}\sum_{\sigma\in S_{k+1}} (\operatorname{sign}\sigma)\,\omega\Big((dt\wedge I)(\overline{X}_{\sigma(1)}, \overline{X}_{\sigma(2)}), \overline{X}_{\sigma(3)}, \ldots, \overline{X}_{\sigma(k+1)}\Big) = 0. \end{aligned}$$

Since derivations of type a_* act trivially on scalar forms, we can conclude from this that

$$\tilde{d}^V \omega\,(\overline{X}_1, \ldots, \overline{X}_{k+1}) = d^V \omega\,(\overline{X}_1, \ldots, \overline{X}_{k+1}).$$

Secondly, for $L \in V^k(\pi_1^0)$ we have

$$a_{\overline{I}\otimes dt} L\,(\overline{X}_1, \ldots, \overline{X}_{k+1}) = \frac{1}{k!} \sum_{\sigma \in S_{k+1}} (\operatorname{sign} \sigma)\, \overline{X}_{\sigma(1)} \left(\langle L(\overline{X}_{\sigma(2)}, \ldots, \overline{X}_{\sigma(k+1)}), dt \rangle \right),$$

from which it follows that if L takes values in $\overline{\mathcal{X}}(\pi_1^0)$, the actions of $\tilde{d}^V$ and d^V coincide when the resulting forms are restricted to $\overline{\mathcal{X}}(\pi_1^0)$ again.

Appendix: Proof of Proposition 2

The idea is to compute $\nabla_\xi \mathcal{U}(\eta)$ for arbitrary ξ and η and $\mathcal{U}$ in its decomposition (36). At the start, we only assume that ∇ comes via (14) from some D with property (13); we want to find out which further restrictions (if any) impose themselves in a natural way for obtaining a closed form expression such as (38).

Let us start by looking in detail at the term $\nabla_\xi U_3^{V;H}(\eta)$, knowing that U_3 vanishes on $\mathbf{T}$. We have

$$\begin{aligned}
\nabla_\xi U_3^{V;H}(\eta) &= \nabla_\xi \left(U_3(\overline{\eta}_V)^H \right) - U_3^{V;H} \left((\mathrm{D}_\xi \overline{\eta}_H)^H + (\mathrm{D}_\xi \overline{\eta}_V)^V + \xi(\langle \eta, dt \rangle) \mathbf{T}^H + \langle \eta, dt \rangle \nabla_\xi \mathbf{T}^H \right) \\
&= \nabla_\xi \left(\overline{U_3(\overline{\eta}_V)}^H + \langle U_3(\overline{\eta}_V), dt \rangle \mathbf{T}^H \right) - U_3(\mathrm{D}_\xi \overline{\eta}_V)^H - \langle \eta, dt \rangle U_3^{V;H}(\nabla_\xi \mathbf{T}^H) \\
&= \left(\mathrm{D}_\xi (\overline{U_3(\overline{\eta}_V)}) \right)^H + \xi\left(\langle U_3(\overline{\eta}_V), dt \rangle \right) \mathbf{T}^H + \langle U_3(\overline{\eta}_V), dt \rangle \nabla_\xi \mathbf{T}^H \\
&\quad - U_3(\mathrm{D}_\xi \overline{\eta}_V)^H - \langle \eta, dt \rangle U_3^{V;H}(\nabla_\xi \mathbf{T}^H) \\
&= (\mathrm{D}_\xi (U_3(\overline{\eta}_V)))^H - \langle U_3(\overline{\eta}_V), dt \rangle (\mathrm{D}_\xi \mathbf{T})^H + \langle U_3(\overline{\eta}_V), dt \rangle \nabla_\xi \mathbf{T}^H \\
&\quad - U_3(\mathrm{D}_\xi \overline{\eta}_V)^H - \langle \eta, dt \rangle U_3^{V;H}(\nabla_\xi \mathbf{T}^H) \\
&= (\mathrm{D}_\xi U_3(\overline{\eta}_V))^H + \langle U_3(\overline{\eta}_V), dt \rangle (\nabla_\xi \mathbf{T}^H - (\mathrm{D}_\xi \mathbf{T})^H) - \langle \eta, dt \rangle U_3^{V;H}(\nabla_\xi \mathbf{T}^H).
\end{aligned}$$

Under the condition (37), this reduces to

$$\nabla_\xi U_3^{V;H}(\eta) = (\mathrm{D}_\xi U_3(\overline{\eta}_V))^H.$$

The computation for U_4 is quite similar. Since U_4 takes values in $\overline{\mathcal{X}}(\pi_1^0)$, there is in fact a further simplification, we find:

$$\nabla_\xi U_4^{V;V}(\eta) = (\mathrm{D}_\xi U_4(\overline{\eta}_V))^V - \langle \eta, dt \rangle U_4^{V;V}(\nabla_\xi \mathbf{T}^H),$$

from which it follows under the same assumption (37) that

$$\nabla_\xi U_4^{V;V}(\eta) = (\mathrm{D}_\xi U_4(\overline{\eta}_V))^V.$$

For U_1 we have, taking this time (37) already into account,

$$\begin{aligned}
\nabla_\xi U_1^{H;H}(\eta) &= \nabla_\xi \left(U_1(\eta_H)^H\right) - U_1^{H;H}\left((\mathrm{D}_\xi\overline{\eta}_H)^H + \xi(\langle\eta, dt\rangle)\mathbf{T}^H + \langle\eta, dt\rangle\nabla_\xi\mathbf{T}^H\right) \\
&= \nabla_\xi \left(\overline{U_1(\eta_H)}^H + \langle U_1(\eta_H), dt\rangle\mathbf{T}^H\right) - U_1(\mathrm{D}_\xi\overline{\eta}_H)^H \\
&\quad - \xi(\langle\eta, dt\rangle)U_1(\mathbf{T})^H - \langle\eta, dt\rangle U_1(\mathrm{D}_\xi\mathbf{T})^H \\
&= \left(\mathrm{D}_\xi(\overline{U_1(\eta_H)})\right)^H + \xi(\langle U_1(\eta_H), dt\rangle)\mathbf{T}^H + \langle U_1(\eta_H), dt\rangle\nabla_\xi\mathbf{T}^H - U_1(\mathrm{D}_\xi\eta_H)^H \\
&= (\mathrm{D}_\xi(U_1(\eta_H)))^H - U_1(\mathrm{D}_\xi\eta_H)^H \\
&= \mathrm{D}_\xi U_1(\eta_H)^H.
\end{aligned}$$

The computation for U_2 is similar, with an extra simplification again because U_2 takes values in $\overline{\mathcal{X}}(\pi_1^0)$. We find

$$\nabla_\xi U_2^{H;V}(\eta) = \mathrm{D}_\xi U_2(\eta_H)^V,$$

which completes the proof of Proposition 2. □

Note finally that the statement preceding Proposition 2 can easily be proved from the above computations as well. Indeed, if each of the U_i maps $\overline{\mathcal{X}}(\pi_1^0)$ into itself and we restrict ourselves to such vector field arguments, none of the terms which prompted the assumption (37) will occur.

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