A class of non-conservative Lagrangian systems on Riemannian manifolds

M. Crampin
Department of Applied Mathematics, The Open University,
Walton Hall, Milton Keynes MK7 6AA, UK

W. Sarlet
Department of Mathematical Physics and Astronomy, Ghent University,
Krijgslaan 281, B–9000 Gent, Belgium

November 30, 2000

1 Introduction

In a number of recent publications [8, 9, 10, 12], Rauch-Wojciechowski, Marciniak and Lundmark have discussed an interesting class of systems of second-order ordinary differential equations, whose members, when viewed as classical mechanical systems, are in a sense completely integrable. These systems originally generated interest because they are derived from the stationary flows of soliton-type evolution equations; but they have more recently been studied in their own right because they include well-known cases of integrable bi-Hamiltonian systems and cases where the Hamilton-Jacobi equation separates. Of these papers we will refer most often to [9], which contains the most general exposition of the theory which we seek to develop further here. In particular, [9] deals with systems with n of degrees of freedom, and therefore subsumes (at least so far as the issues we intend to discuss are concerned) [12], which is largely restricted to systems with two degrees of freedom.

The systems of second-order equations under consideration take the general form of Lagrange's equations in mechanics,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^i}\right) - \frac{\partial T}{\partial q^i} = Q_i,$$

where T is the kinetic energy function and the 'generalized forces' Q_i need not be derivable from a potential energy function. In all the publications mentioned above the kinetic energy is taken to have the Euclidean form, $T = \frac{1}{2}\sum(\dot{q}^i)^2$; we however will deal with the more general situation in which T is derived from a Riemannian metric, $T = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j$. In addition, the systems are required to possess quadratic integrals of the motion, of a special kind, called by Lundmark integrals of cofactor type. Of particular interest are those systems which possess two independent quadratic integrals of cofactor type: such a system can be regarded as the restriction of a bi-Hamiltonian flow on a phase space of one more dimension, and has a further hierarchy of integrals in involution.

As we have just pointed out, the primary aim of our paper is to derive in the Riemannian case results which parallel those that Lundmark *et al.* have obtained in the Euclidean one. In doing so we extend the range of application of their theory, of course. However, in justifying our efforts we would put greater emphasis on the increased level of geometrical insight we have achieved into the results of the group in Linköping. In particular, we claim to have considerably clarified, by generalizing them,

1. the concept and properties of a cofactor system;
2. the origin of the so-called 'fundamental equation' involved in the definition of a cofactor pair system; and
3. the construction of the bi-Hamiltonian structure associated with a cofactor pair system.

Several of the methods we use here were developed in a recent paper on bi-Hamiltonian systems and conformal Killing tensors [3], which was concerned with a certain class of conservative systems whose Hamilton-Jacobi equations separate; we will briefly indicate how such systems can be regarded as a subset of the cofactor pair systems discussed here.

The Linköping group refer to the differential equations they consider as 'Newton equations of quasi-Lagrangian type', because in the Euclidean case it turns out that when there is a quadratic first integral E the equations can formally be cast into the form $d/dt(\partial E/\partial \dot{q}^i) + \partial E/\partial q^i = 0$, which resembles Lagrange's equations but has a wrong sign. This is, in our opinion, a complete artifact of the systems under consideration, which has nothing to do with the fundamental issues which are at stake. In fact, the more general systems on Riemannian spaces we will introduce simply do not have this very non-intrinsic property. We have therefore decided to describe them as 'non-conservative Lagrangian systems' instead. In doing so we are conscious that our work, together with that of Rauch-Wojciechowski, Marciniak, Lundmark and others in this field, is closely connected with the researches of Bertrand and Darboux in the second half of the nineteenth century, which are summarised by Whittaker in articles 151 and 152 of [14].

The structure of the paper is as follows. In Section 2 we recall some aspects of Poisson structures for later use. We concentrate in particular on the construction of a non-standard Poisson structure on T^*Q out of the complete lift of a type (1,1) tensor field on Q. In Section 3, we take a non-conservative Lagrangian system as the starting point and investigate under what circumstances it has a quasi-Hamiltonian representation with respect to such a non-standard Poisson structure. This leads us to an interesting class of special conformal Killing tensors J, which are discussed in more detail in the next section. The main result in that section is that the cofactor tensor of such a J is a Killing tensor. Coming back then to the idea we started from, and inspired by the work of Lundmark [9] in the Euclidean case, we more formally introduce the notion of a cofactor system in Section 5 and complete the discussion of its quasi-Hamiltonian representation. In Section 6, we show how a cofactor system can also be given a Hamiltonian representation on an extended manifold. Section 7 is about cofactor pair systems, that is, systems which have a double cofactor representation. We show how this leads to a gauged bi-differential calculus which provides an intrinsic generalization of the 'fundamental equations' referred to above. We further establish complete integrability by exploiting the double Poisson structure on the extended space. Finally, we briefly explain the relation between this work and recent work on the separability of the Hamilton-Jacobi equation.

2 Poisson structures

It will be convenient to recall some generalities here about Poisson structures, which will at the same time serve to fix the sign conventions which we will adopt.

A Poisson structure on a manifold M is a bivector field Π which satisfies $[\Pi, \Pi] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket. The associated Poisson bracket of functions f, g is given by $\{f, g\} = \Pi(df, dg)$; the vanishing of the Schouten bracket entails the Jacobi identity for the Poisson bracket. Also associated with such a bivector field is a map P of 1-forms to vector fields on M, given by $\langle P(\alpha), \beta\rangle = \Pi(\alpha, \beta)$ for any pair of 1-forms α, β. The vector field $P(dh)$ is the Hamiltonian vector field corresponding to the Hamiltonian function h. The Poisson structure is non-singular if its Poisson map is.

Two Poisson structures Π_1, Π_2 are compatible if $[\Pi_1, \Pi_2] = 0$. When this condition holds, $a_1\Pi_1 + a_2\Pi_2$ is Poisson for any constants a_1 and a_2. The collection of Poisson bivectors $a_1\Pi_1 + a_2\Pi_2$ is called a Poisson pencil.

Let $\phi: M \to M$ be a diffeomorphism. For any bivector field Π on M we can use ϕ^*, the map of forms induced by ϕ, to transform Π into a new bivector field Π_ϕ by

$$(\Pi_\phi)(\alpha, \beta) = \phi^{-1*}\Pi(\phi^*\alpha, \phi^*\beta).$$

This is the natural extension to bivector fields of the map of vector fields induced by a diffeomorphism. The corresponding transform P_ϕ of the map P is given as a linear map $T_x^*M \to T_xM$, $x \in M$, by

$$P_\phi|_x = \phi_{*x} \circ P|_{\phi^{-1}(x)} \circ \phi_x^*.$$

There is no guarantee that Π_ϕ will be Poisson even when Π is: we will deal with a case in which it is below. For the present, merely note that if ϕ and ψ are two diffeomorphisms then $\Pi_{\phi\circ\psi} = (\Pi_\psi)_\phi$ and likewise $P_{\phi\circ\psi} = (P_\psi)_\phi$.

The cotangent bundle T^*Q of a manifold Q has a standard Poisson bivector Π_0 whose expression in terms of standard coordinates (q^i, p_i) is

$$\Pi_0 = \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}.$$

The corresponding Poisson map P_0 is given in terms of the canonical symplectic form ω by $P_0(\alpha) \lrcorner \omega = -\alpha$.

Let J be a non-singular type $(1,1)$ tensor field on Q. It defines a diffeomorphism $\hat{J}$ of T^*Q which is fibre preserving and linear on fibres, given by $\hat{J}(q^i, p_i) = (q^i, J_i^j p_j)$ (note that J acts here on covectors, that is to say, it is its adjoint that is involved). The bivector field $(\Pi_0)_{\hat{J}} = \Pi_J$ is a Poisson bivector if and only if the torsion, or Nijenhuis tensor, N_J of J is zero. When this is the case Π_J is compatible with Π_0, and we obtain an example of a Poisson-Nijenhuis structure. The corresponding Poisson map P_J is given by $P_J = \tilde{J} \circ P_0 = P_0 \circ \tilde{J}^*$, where $\tilde{J}$ is the complete lift of J, a type $(1,1)$ tensor field on T^*Q, and $\tilde{J}^*$ is its adjoint (acting on 1-forms).

In the sequel we will carry out several coordinate calculations involving these constructs, in situations where we have a symmetric connection at our disposal. We therefore give coordinate representations of them using bases of local vector fields and 1-forms adapted to the connection, given by

$$\frac{\partial}{\partial q^i} + \Gamma_{ij}^k p_k \frac{\partial}{\partial p_j} = X_i, \quad \frac{\partial}{\partial p_i}$$

for vector fields, where $\Gamma_{ij}^k = \Gamma_{ji}^k$ are the connection coefficients, and

$$dq^i, \quad dp_i - \Gamma_{ij}^k p_k dq^j = \pi_i$$

for 1-forms; these are dual bases. The indices i, j, k etc. range over $1, 2, \ldots, n = \dim M$, and the Einstein summation convention is in force. Then

$$\tilde{J} = J_j^i \left(X_i \otimes dq^j + \frac{\partial}{\partial p_j} \otimes \pi_i \right) + \left(J_{i|j}^k - J_{j|i}^k \right) p_k \frac{\partial}{\partial p_i} \otimes dq^j.$$

The vertical bar divides off the differentiation index in a covariant differential from the other indices. The condition $N_J = 0$ can be written

$$J_l^k \left(J_{i|j}^l - J_{j|i}^l \right) = J_j^l J_{i|l}^k - J_i^l J_{j|l}^k.$$

In order to calculate Hamiltonian vector fields with respect to P_J it is enough to know $\tilde{J}$ since one can use either of the formulae $P_J = \tilde{J} \circ P_0$ and $P_J = P_0 \circ \tilde{J}^*$; it is useful to remember that for a symmetric connection P_0 can be written

$$P_0 = \frac{\partial}{\partial p_i} \wedge X_i.$$

However, to facilitate comparison with [9, 10, 12] we give the formula for P_J:

$$P_J = J^i_j \frac{\partial}{\partial p_j} \wedge X_i - \tfrac{1}{2}\left(J^k_{i|j} - J^k_{j|i}\right) p_k \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j}.$$

Finally, we will have occasion to discuss situations where we have more than one type $(1,1)$ tensor field at our disposal. In the first place, suppose that J has vanishing torsion, and that A is another type $(1,1)$ tensor field such that JA also has vanishing torsion (where JA is the type $(1,1)$ tensor field whose components are $J^i_k A^k_j$, that is, JA is the composition $J \circ A$ acting on vector fields). Then Π_{JA} is a Poisson bivector. It can be expressed in terms of Π_J by means of the formula $\Pi_{\phi\circ\psi} = (\Pi_\psi)_\phi$ with $\phi = \hat{A}$, $\psi = \hat{J}$: note that since A and J act on T^*Q by their adjoints, $\widehat{JA} = \hat{A} \circ \hat{J}$. Thus $\Pi_{JA} = (\Pi_J)_{\hat{A}}$. It follows that for any Hamiltonian function H,

$$P_{JA}(d(\hat{A}^{-1*}H)) = \hat{A}_* P_J(dH).$$

Secondly, suppose that J and K both have vanishing torsion and that $[J,K] = 0$ where $[\cdot,\cdot]$ here is the Nijenhuis bracket. Then $aJ + bK$ has vanishing torsion for all constants a and b, so that P_{aJ+bK} is a Poisson map for all a and b; and from the formula for P_J we see that $P_{aJ+bK} = aP_J + bP_K$. Thus $aP_J + bP_K$ is a Poisson pencil in this case.

3 Non-conservative Lagrangian systems

A geometrical description of the kind of general Lagrange equations mentioned in the introduction can be obtained as follows. Let S denote the canonical vertical endomorphism on a tangent bundle TQ and Γ a second-order differential equation field. As was described in [13], Γ represents a non-conservative Lagrangian system, if there exists a 1-form $\phi = dL - \mu$ on TQ, where L is a regular Lagrangian and μ is semi-basic, such that $\mathcal{L}_\Gamma(S^*(\phi)) = \phi$. It is easy to verify that, in coordinates (q^i, u^i) on TQ, this requirement means that

$$\Gamma\left(\frac{\partial L}{\partial u^i}\right) - \frac{\partial L}{\partial q^i} = -M_i,$$

where the M_i are the components of μ (the minus sign here is a matter of convention). We shall consider the particular case in which the non-conservative forces $-M_i$ do not depend on the velocities, so that μ is a 1-form on Q; and L is a pure kinetic energy Lagrangian. The latter means that the base manifold Q is assumed to be Riemannian (or pseudo-Riemannian) with metric tensor $g = (g_{ij})$, and that $L = T = \frac{1}{2} g_{ij} u^i u^j$. If

Γ^i_{jk} are the Christoffel symbols for the corresponding Levi-Civita connection, and if we put $M^i = g^{ij}M_j$ as is usual, the resulting second-order differential equation field is of the form

$$\Gamma = u^i\frac{\partial}{\partial q^i} - \left(\Gamma^i_{jk}u^j u^k + M^i\right)\frac{\partial}{\partial u^i}.$$

Another way of characterizing such vector fields is to say that $\Gamma = \Gamma_0 - M^V$, where Γ_0 is the geodesic field for the connection and M^V is the vertical lift of a vector field M on Q.

We will show that it is possible, in certain interesting cases, to find a quasi-Hamiltonian representation for such a system; that is to say, to represent it as a scalar multiple of a Hamiltonian vector field. However, we will not assume that the Poisson structure with respect to which this vector field is Hamiltonian is the standard one; instead, we will look for a suitable Poisson structure of the form Π_J defined by some type $(1,1)$ tensor field J on Q whose torsion vanishes.

We use g to define a diffeomorphism $\hat{g}\colon TQ \to T^*Q$ by $p_i = g_{ij}u^j$. We will denote by $\hat{\Gamma}$ the transform of Γ by $\hat{g}$, that is, $\hat{\Gamma} = \hat{g}_*\Gamma$. We have

$$\hat{\Gamma} = g^{ij}p_jX_i - M_i\frac{\partial}{\partial p_i},$$

where the X_i are the vector fields on T^*Q adapted to the connection specified above.

Equip T^*Q with a Poisson structure Π_J and Poisson map P_J as described earlier. We wish to determine under what circumstances one can find a J with $N_J = 0$ such that the given system satisfies $F\hat{\Gamma} = P_J(dH)$ for some functions F and H. We will now solve this problem under the assumption that H is quadratic in the momenta, so that $H = \frac{1}{2}A^{ij}p_ip_j + V$ for some symmetric tensor A and function V on Q; we will further assume that A is non-singular.

For such H we find, after a little calculation, that

$$P_J(dH) = J^i_kA^{jk}p_jX_i - \left[\left(\tfrac{1}{2}J^l_iA^{jk}{}_{|l} - (J^j_{i|l} - J^j_{l|i})A^{kl}\right)p_jp_k + J^j_i\frac{\partial V}{\partial q^j}\right]\frac{\partial}{\partial p_i}.$$

For this to equal $F\hat{\Gamma}$ we must first have

$$J^i_kA^{jk} = Fg^{ij}.$$

Thus F must be a function on Q, so that the quadratic and zeroth-order terms in the other coefficients must be equated separately. As a result, we require that

$$\begin{aligned} J^l_iA^{jk}{}_{|l} &= (J^j_{i|l} - J^j_{l|i})A^{kl} + (J^k_{i|l} - J^k_{l|i})A^{jl} \\ J^j_i\frac{\partial V}{\partial q^j} &= FM_i. \end{aligned}$$

On lowering the index i in the first condition we see that J_{ij} is a scalar multiple of the inverse of A^{ij}, and so is symmetric because A^{ij} is. By differentiating this equation and multiplying by J twice we obtain

$$J_j^m J_k^n J_i^l A^{jk}{}_{|l} = \frac{\partial F}{\partial q^l} J^{mn} J_i^l - F J_i^l J^{mn}{}_{|l}.$$

The second condition therefore can equivalently be replaced by

$$\frac{\partial F}{\partial q^l} J^{mn} J_i^l = F\left(J_i^l J^{mn}{}_{|l} + g^{lm} J_k^n (J_{i|l}^k - J_{l|i}^k) + g^{ln} J_k^m (J_{i|l}^k - J_{l|i}^k)\right).$$

We now use the assumption that $N_J = 0$ to rearrange the last four terms; when this is done, the first term on the left-hand side cancels, and after some indices have been lowered we obtain

$$\frac{\partial F}{\partial q^l} J_{jk} J_i^l = F(J_k^l J_{ij|l} + J_j^l J_{ik|l} - J_i^l J_{jk|l}).$$

The part of this equation symmetric in i and j gives

$$J_{ij|k} = \tfrac{1}{2}(\alpha_i g_{jk} + \alpha_j g_{ik})$$

where we have written

$$\alpha_i = \frac{1}{F} J_i^j \frac{\partial F}{\partial q^j}.$$

The skew-symmetric part is then automatically satisfied. It further follows from the formula for $J_{ij|k}$ that $\alpha_k = (J_i^i)_{|k}$, or $\alpha = \alpha_i dq^i = d(\operatorname{tr} J)$. Hence, if $J_{ij|k}$ has the required structure, the α_i are actually determined, so that the relations $J_i^j \partial F/\partial q^j = F\alpha_i$ should be seen as equations for admissible functions F.

We next show that a particular solution for F is $\det J$. We have

$$\det J = \frac{1}{n!} \delta^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n} J_{i_1}^{j_1} J_{i_2}^{j_2} \cdots J_{i_n}^{j_n},$$

where $\delta^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n}$ is the generalized Kronecker delta (see for example [7]). Thus

$$\frac{\partial}{\partial q^k} \det J = \frac{1}{(n-1)!} \delta^{i i_2 \ldots i_n}_{j j_2 \ldots j_n} J_{i|k}^j J_{i_2}^{j_2} \cdots J_{i_n}^{j_n}.$$

Now

$$\frac{1}{(n-1)!} \delta^{i i_2 \ldots i_n}_{j j_2 \ldots j_n} J_{i_2}^{j_2} \cdots J_{i_n}^{j_n} = C_j^i$$

is the cofactor tensor of J, which satisfies

$$J_k^i C_j^k = (\det J)\delta_j^i.$$

So we may write

$$
\begin{aligned}
J^k_i \frac{\partial}{\partial q^k} \det J &= J^k_i C^l_j J^j_{l|k} = \tfrac{1}{2} J^k_i C^l_j (\alpha_l \delta^j_k + \alpha^j g_{kl}) \\
&= \tfrac{1}{2}(J^j_i C^l_j \alpha_l + J^k_i C^l_k \alpha_l) = (\det J)\alpha_i,
\end{aligned}
$$

where we have used the fact that C_{ij} is symmetric, which follows from the symmetry of J_{ij}.

Now suppose that F is any solution: we show that F is a constant multiple of $\det J$. We have

$$
\frac{1}{F} J^j_i \frac{\partial F}{\partial q^j} = \alpha_i = \frac{1}{\det J} J^j_i \frac{\partial}{\partial q^j} \det J,
$$

from which it follows that

$$
J^j_i \frac{\partial}{\partial q^j}\left(\frac{F}{\det J}\right) = 0,
$$

and so $F = k \det J$.

Once we have fixed a J in a quasi-Hamiltonian representation $F\hat{\Gamma} = P_J(dH)$ for the given system, multiplying F by a constant factor is a quite irrelevant degree of freedom, since it can be compensated for by adapting the Hamiltonian. So without loss of generality we can take $F = \det J$, whence the first condition on J becomes $J^i_k A^{jk} = (\det J) g^{ij}$, and identifies A^i_j as the cofactor tensor of J^i_j. Finally, there is a restriction on the non-conservative forces, which must have the form

$$
M_i = (\det J)^{-1} J^j_i \frac{\partial V}{\partial q^j}
$$

for some function V on Q.

We shall come back to the formulation of the conclusions of this analysis in Section 5, after looking in more detail at the special kind of tensor fields J it has revealed.

4 Special conformal Killing tensors

A tensor J which satisfies the condition $J_{ij|k} = \frac{1}{2}(\alpha_i g_{jk} + \alpha_j g_{ik})$ for some α_i has very interesting properties. In the first place, $J_{(ij|k)} = \alpha_{(i} g_{jk)}$ (brackets denote symmetrization), which says that J is a conformal Killing tensor of g; and furthermore $\alpha = \alpha_i dq^i$ is exact, so it is a conformal Killing tensor of gradient type. In the course of the argument in the previous section it was assumed that the torsion of J vanishes (this was necessary to ensure that Π_J is Poisson): but in fact the vanishing of the torsion is an easy consequence of the defining condition. Moreover, as we showed in [3], a conformal Killing tensor whose torsion vanishes and which has functionally independent eigenfunctions must necessarily satisfy this condition. A symmetric type $(0,2)$ tensor J on Q such that

$$J_{ij|k} = \tfrac{1}{2}(\alpha_i g_{jk} + \alpha_j g_{ik})$$

will therefore be called a *special conformal Killing tensor*. In the Euclidean case a tensor is a special conformal Killing tensor if and only if it is an elliptic coordinates matrix in Lundmark's terminology [9], or a planar inertia tensor in Benenti's [1].

We will deal only with special conformal Killing tensors which are non-singular. The inverse of a type $(1,1)$ tensor will be denoted by an overbar when we need to use indices.

The determinant of a type $(1,1)$ tensor is a scalar (this is not so for a type $(2,0)$ or $(0,2)$ tensor), so whenever we use determinants it is to be assumed that the corresponding tensor is in type $(1,1)$ form. This applies also to the formula $A = (\det J)J^{-1}$, which may be used to define the cofactor tensor of J when it is non-singular. Elsewhere, the usual rules for raising and lowering indices apply. Thus for example $A_{ij} = (\det J)\bar{J}_{ij}$: it is symmetric if J is.

When J is special conformal Killing, by taking the covariant derivative of the equation $A_{ij}J^j_l = (\det J)g_{il}$ and using the defining condition one can deduce that

$$A_{ij|k} = (\det J)\left(\bar{J}_{ij}\bar{J}_{kl} - \tfrac{1}{2}\bar{J}_{ik}\bar{J}_{jl} - \tfrac{1}{2}\bar{J}_{il}\bar{J}_{jk}\right)\alpha^l,$$

from which one easily derives the following remarkable property of any special conformal Killing tensor.

Proposition 1 The cofactor tensor of a non-singular special conformal Killing tensor is a Killing tensor.

Proof It follows immediately from the formula above that $A_{(ij|k)} = 0$. □

Note further that A has the same eigenvectors as J.

A special conformal Killing tensor J may be used to define a couple of differential operators with nice properties. In the first place, we can form the operator d_J in the sense of Frölicher-Nijenhuis theory [5]: this is the derivation of degree 1 of the exterior algebra $\bigwedge(Q)$ of forms on Q, over the algebra $C^\infty(Q)$ of real-valued C^∞ functions on Q, which anti-commutes with the exterior derivative d (i.e. is a derivation of type d_*), and whose action on $C^\infty(Q)$ is given by $d_J f = J^*(df)$. Furthermore, d_J has the coboundary property ${d_J}^2 = 0$ because the torsion N_J is zero. What is more, since by assumption J is non-singular, d_J satisfies a Poincaré lemma: that is to say, for a k-form θ, the condition $d_J\theta = 0$ is sufficient as well as necessary for the local existence of a $(k-1)$-form φ such that $\theta = d_J\varphi$. This result can be found in a paper of Willmore [15].

In the previous section we came across an interesting property in which d_J is involved. We showed there that if J is a special conformal Killing tensor, $F = \det J$ satisfies

$J_i^j \partial F/\partial q^j = F\alpha_i$ where $\alpha = d(\operatorname{tr} J)$. Hence,

$$d_J(\det J) = (\det J)\alpha = (\det J)d(\operatorname{tr} J).$$

(In fact this holds for any tensor J whose torsion vanishes.) By acting with d_J on both sides, it further easily follows that $d_J\alpha = 0$, that is, $d_Jd(\operatorname{tr} J) = 0$. (In fact for any tensor J whose torsion vanishes, $d_J(\operatorname{tr} J) = \frac{1}{2}d(\operatorname{tr} J^2)$.)

These properties enable us to define the following differential operator D_J, which also acts on forms θ on Q; D_J will turn out to have an important role in relation to the fundamental equation mentioned in the introduction.

$$D_J\theta = (\det J)^{-1}d_J((\det J)\theta) = d_J\theta + \alpha \wedge \theta.$$

Note that D_J is not a derivation (in the sense of Frölicher-Nijenhuis), but it is clear from the first expression that D_J satisfies ${D_J}^2 = 0$, so it is an example of a (scalar) gauged differential operator, in the terminology of [4]. Moreover, we see that once again $D_J\theta = 0$ is a sufficient condition for there to be a form φ (locally) such that $\theta = D_J\varphi$: we have $d_J((\det J)\theta) = 0$, so there is a φ' such that $(\det J)\theta = d_J\varphi'$, whence $\varphi = (\det J)^{-1}\varphi'$ satisfies $D_J\varphi = \theta$.

Note finally that the condition on the non-conservative forces derived in the previous section can now be written in coordinate-free form with the aid of the 1-form $\mu = M_i dq^i$ of the beginning of that section. The condition reads

$$\mu = (\det J)^{-1}d_JV = D_J((\det J)^{-1}V).$$

Hence, in order for the non-conservative Lagrangian system Γ to have a quasi-Hamiltonian representation as described in the previous section, there must be a function $V' = (\det J)^{-1}V$ such that $\mu = D_JV'$. But so long as we are concerned only with local considerations, this is equivalent to the condition $D_J\mu = 0$.

5 Cofactor systems

We can now describe explicitly the class of non-conservative Lagrangian systems Γ we are analysing: they are those determined by a metric tensor g which admits a special conformal Killing tensor J, and a 1-form μ on the configuration manifold Q such that $D_J\mu = 0$. Systems of this type, in the Euclidean case, are what Lundmark calls cofactor systems, though he does not define them in quite the same way; we will use the same terminology even though it doesn't really match our definition.

Definition A non-conservative system Γ on TQ, generated by a metric tensor field g and a 1-form μ on Q, is said to be a *cofactor system*, if g admits a non-singular special conformal Killing tensor J and μ satisfies $D_J\mu = 0$.

The results of the preceding sections can now be summarized as follows.

Theorem 2 A non-conservative system Γ on TQ determined by the couple (g,μ) on Q, has a quasi-Hamiltonian representation $F\hat{\Gamma} = P_J(dH)$, where J is a type $(1,1)$ tensor field on Q and H is a function on T^*Q quadratic in momenta, if and only if it is a cofactor system.

Proof The argument developed in Section 3 proves the following assertion: the condition $F\hat{\Gamma} = P_J(dH)$ with H quadratic, assuming $N_J = 0$, is equivalent to the requirements for having a cofactor system. But as we observed in Section 4, a special conformal Killing tensor automatically has zero torsion. Therefore, conversely, every cofactor system has a quasi-Hamiltonian representation of the desired type. □

Notice that for a special conformal Killing tensor

$$P_J \;=\; J^i_j \frac{\partial}{\partial p_j} \wedge X_i - \tfrac{1}{4}(\alpha_i p_j - \alpha_j p_i)\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j}.$$

Lundmark *et al.* approach the analysis of non-conservative Lagrangian systems by discussing the conditions under which such a system has a quadratic first integral $E = \frac{1}{2}A_{ij}u^i u^j + V$. Any cofactor system has a quasi-Hamiltonian representation with a quadratic Hamiltonian, which then is necessarily conserved, but the cofactor systems are a subclass of the non-conservative Lagrangian systems with quadratic integrals. We will complete the picture by identifying exactly which of the properties or conditions we have encountered entail that the function $E = \frac{1}{2}A_{ij}u^i u^j + V$ is a constant of the motion. We have

$$\Gamma(E) = \tfrac{1}{2}A_{ij|k}u^i u^j u^k - \left(A_{ij}M^j - \frac{\partial V}{\partial q^i}\right)u^i.$$

Thus, in order that $\Gamma(E)$ be zero, A must satisfy $A_{(ij|k)} = 0$, which is to say that it must be a Killing tensor. Moreover, we must have $A^*\mu = dV$. As we have seen, the first condition is satisfied automatically when A is the cofactor tensor of a special conformal Killing tensor (but of course there may be Killing tensors which are not of this type), and the restriction on μ then takes the form $\mu = D_J((\det J)^{-1}V)$. These remarks are supposed to explain the origins of the name 'cofactor system'.

We will also take this opportunity to comment on the use of the term 'quasi-Lagrangian' to describe non-conservative Lagrangian systems with quadratic integrals in the Euclidean case. Note that the commutator of any second-order equation field $\Gamma = u^i\partial/\partial q^i + f^i\partial/\partial u^i$ on TQ with $\partial/\partial u^j$ is given by

$$\left[\Gamma, \frac{\partial}{\partial u^j}\right] = -\frac{\partial}{\partial q^j} - \frac{\partial f^i}{\partial q^j}\frac{\partial}{\partial u^i}.$$

It follows that for any first integral E of Γ we will have

$$\Gamma\left(\frac{\partial E}{\partial u^j}\right) + \frac{\partial E}{\partial q^j} = -\frac{\partial f^i}{\partial u^j}\frac{\partial E}{\partial u^i}.$$

Hence, if the right-hand sides of the given equations (i.e. the functions f^i) are velocity independent, it will trivially be the case that every first integral E leads to a relation which formally looks like Euler-Lagrange equations with the wrong sign. In the Euclidean case, in Cartesian coordinates, the right-hand sides are indeed velocity independent. However, if the space is not Euclidean the equations will certainly not have this feature; indeed, they will not even in the Euclidean case if curvilinear coordinates are used. On the other hand, as we have already seen and will see further in what follows, the systems we are considering do have all the intrinsic features which explain the essential properties of what were called quasi-Lagrangian systems in [9, 10, 12]. It seems to us, therefore, that the fact that the systems considered there have quasi-Lagrangian representations is not significant.

Given the prominent role played by the function $\frac{1}{2}A_{ij}u^iu^j+V$ in the concept of a cofactor system, one might naturally ask why one should not, to obtain a quasi-Hamiltonian representation, map TQ to T^*Q by $u^i \mapsto A_{ij}u^j$ rather than $u^i \mapsto g_{ij}u^j$. To do so is equivalent to carrying out the map $\hat{A}$ on T^*Q. Note that since $JA = (\det J)I$, and the torsion of a multiple of the identity vanishes, P_{JA} is certainly a Poisson map. The dynamics is transformed to $\hat{A}_*\hat{\Gamma}$. In Section 2 we showed that $P_{JA}(d(\hat{A}^{-1*}H)) = \hat{A}_*P_J(dH)$. It follows that

$$(\det J)\hat{A}_*\hat{\Gamma} = \hat{A}_*P_J(dH) = P_{JA}(d(\hat{A}^{-1*}H)),$$

which is to say that $\hat{A}_*\hat{\Gamma}$ is quasi-Hamiltonian with respect to $P_{JA} = P_{(\det J)I}$, with Hamiltonian

$$\hat{A}^{-1*}H = (\det J)^{-1}\hat{J}^*H = (\det J)^{-1}(\tfrac{1}{2}A^{ij}J_i^kJ_j^lp_kp_l + V) = \tfrac{1}{2}J^{ij}p_ip_j + (\det J)^{-1}V.$$

This formulation, in the Euclidean case, is essentially that given by the second of the two non-standard Poisson structures in [10].

Once one has noticed this trick one realises that there are other possible ways of obtaining a quasi-Hamiltonian representation of a cofactor system. Indeed, by applying the map $\widehat{J^{-1}}$ one sees that there is a quasi-Hamiltonian representation with respect to the standard Poisson structure. However, when we come to discuss cofactor pair systems these alternatives will not do, because they will associate different vector fields on T^*Q with the original vector field Γ on TQ: it is far better to stick with the single vector field $\hat{\Gamma} = \hat{g}_*\Gamma$ on T^*Q and represent it in quasi-Hamiltonian form with respect to two Poisson structures.

6 Hamiltonian structure for a cofactor system

We now show how to represent a cofactor system as a Hamiltonian vector field with respect to a Poisson structure defined on an extended manifold. This involves an application of what is in fact a general construction which applies to any quasi-Hamiltonian vector field. This construction is the subject of the following theorem.

Theorem 3 Let Π be a Poisson bivector on a manifold M, and Z a vector field on M with the property that there is a nowhere-vanishing function F such that FZ is a Hamiltonian vector field with respect to Π, with Hamiltonian function H. Then there is a Poisson bivector $\hat{\Pi}$ on $M \times \mathbb{R}$ which projects onto Π, and a vector field, Hamiltonian with respect to $\hat{\Pi}$, whose restriction to the zero section is Z. Furthermore, $H + zF$ is a Casimir of $\hat{\Pi}$ (where z is the coordinate on $\mathbb{R}$).

Proof Let π denote the projection $\pi\colon M \times \mathbb{R} \to M$. We can extend Π to $M \times \mathbb{R}$ simply by ignoring z: that is, for 1-forms on M we put $\Pi(\pi^*\alpha, \pi^*\beta) = \Pi(\alpha, \beta)$, while $\Pi(dz, \cdot) = 0$ (so that z is a Casimir for Π). We consider a bivector of the form

$$\hat{\Pi} = \Pi + (Z + zW) \wedge \frac{\partial}{\partial z},$$

where W is a vector field independent of z (that is, $\langle W, dz \rangle = 0$ and $\mathcal{L}_{\partial/\partial z} W = 0$), and seek a W for which $\hat{\Pi}$ is Poisson. If we find one then the Hamiltonian vector field corresponding to $-z$ will be $Z + zW$, agreeing with Z on $z = 0$; and the projection of $\hat{\Pi}$ to M will be Π (or in other words π will be a Poisson map). We require that $[\hat{\Pi}, \hat{\Pi}] = 0$. Now for any bivector field Ω and vector fields X, Y,

$$[\Omega + X \wedge Y, \Omega + X \wedge Y] = [\Omega, \Omega] + 2(\mathcal{L}_X \Omega \wedge Y - X \wedge \mathcal{L}_Y \Omega - X \wedge Y \wedge [X, Y]),$$

so we require that

$$\mathcal{L}_{Z+zW} \Pi \wedge \frac{\partial}{\partial z} = (Z + zW) \wedge \frac{\partial}{\partial z} \wedge (-W) = Z \wedge W \wedge \frac{\partial}{\partial z}.$$

For any bivector field Ω, function f and vector field V,

$$\mathcal{L}_{fV} \Omega = f \mathcal{L}_V \Omega - V \wedge S(df),$$

where S is the map of 1-forms corresponding to Ω. So $[\hat{\Pi}, \hat{\Pi}] = 0$ is equivalent to

$$(\mathcal{L}_Z \Pi + z \mathcal{L}_W \Pi) \wedge \frac{\partial}{\partial z} = Z \wedge W \wedge \frac{\partial}{\partial z}.$$

The conditions for $\hat{\Pi}$ to be Poisson are

$$\mathcal{L}_Z \Pi = Z \wedge W, \quad \mathcal{L}_W \Pi = 0.$$

Now $Z = F^{-1} P(dH)$, where P is the Poisson map corresponding to Π, so (since the Lie derivative of Π by a Hamiltonian vector field is zero)

$$\mathcal{L}_Z \Pi = -P(dH) \wedge P(dF^{-1}) = F^{-2} P(dH) \wedge P(dF),$$

so both requirements are satisfied if

$$W = F^{-1}P(dF).$$

That is to say,

$$\hat{\Pi} = \Pi + \Big(Z + zF^{-1}P(dF)\Big) \wedge \frac{\partial}{\partial z} = \Pi + F^{-1}P(dH + zdF) \wedge \frac{\partial}{\partial z}$$

is a Poisson tensor on $M \times \mathbb{R}$.

It follows by a direct calculation that $H + zF$ is a Casimir of $\hat{\Pi}$. $\square$

It is useful to note, for future reference, that in fact the bivector $\Pi + (kZ + zW) \wedge \partial/\partial z$ is Poisson for every constant k; or what amounts to the same thing, $\hat{\Pi}$ is compatible with the (highly degenerate) Poisson bivector $Z \wedge \partial/\partial z$. This is established by the following computation:

$$\begin{aligned}
\left[\hat{\Pi}, Z \wedge \frac{\partial}{\partial z}\right] &= \mathcal{L}_Z \hat{\Pi} \wedge \frac{\partial}{\partial z} - Z \wedge \mathcal{L}_{\partial/\partial z} \hat{\Pi} \\
&= \left(\mathcal{L}_Z \Pi + z[Z, W] \wedge \frac{\partial}{\partial z}\right) \wedge \frac{\partial}{\partial z} - Z \wedge \left(\left[\frac{\partial}{\partial z}, Z + zW\right] \wedge \frac{\partial}{\partial z}\right) \\
&= (Z \wedge W) \wedge \frac{\partial}{\partial z} - Z \wedge \left(W \wedge \frac{\partial}{\partial z}\right) = 0.
\end{aligned}$$

Of course the Hamiltonian vector field corresponding to $-z$ for this modified Poisson structure restricts on $z = 0$ not to Z but to a constant multiple of it.

Returning to the case of a given cofactor system, we may use the construction in the theorem above to represent $\hat{\Gamma}$ as the restriction to $z = 0$ of a Hamiltonian vector field on $T^*Q \times \mathbb{R}$. The function F in this case is $\det J$, which is a function on Q. The Hamiltonian vector field associated with $\det J$ by Π_J is $-(d_J(\det J))^V = -(\det J)\alpha^V$. Thus

$$\hat{\Pi}_J = \Pi_J + (\hat{\Gamma} - z\alpha^V) \wedge \frac{\partial}{\partial z},$$

or in other words

$$\begin{aligned}
\hat{P}_J &= J^i_j \frac{\partial}{\partial p_j} \wedge X_i - \tfrac{1}{4}(\alpha_i p_j - \alpha_j p_i) \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j} \\
&\quad + g^{ij} p_j X_i \wedge \frac{\partial}{\partial z} - (M_i + z\alpha_i) \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial z}.
\end{aligned}$$

This agrees with the Poisson structure for a cofactor system in the n-dimensional Euclidean case given in [9] and in 2 dimensions in [12], and is the obvious generalization once one has realised that introducing the metric involves using the basis adapted to the connection.

Note that the Hamiltonian representation of the given cofactor system on $T^*Q \times \mathbb{R}$ is

$$\hat{\Gamma} - z\alpha^V = \hat{\Gamma}_0 - (\mu + z\alpha)^V;$$

so it is obtained by a kind of deformation of the non-conservative forces.

7 Cofactor pair systems

We now consider non-conservative Lagrangian systems which are of cofactor type in two ways — what Lundmark calls cofactor pair systems.

Suppose that the metric g admits two independent special conformal Killing tensors J and K. The condition for a given system $\Gamma = \Gamma_0 - M^V$ to be of cofactor type (at least locally) with respect to both J and K is that $D_J\mu = D_K\mu = 0$. We must examine the relation between the operators D_J and D_K.

If J and K are two special conformal Killing tensors then clearly $aJ+bK$ is also a special conformal Killing tensor for any constants a, b, with corresponding 1-form $a\alpha+b\beta$, where $\alpha = d(\operatorname{tr} J)$, $\beta = d(\operatorname{tr} K)$. Using the representation $D_J\theta = d_J\theta + \alpha \wedge \theta$ we see that

$$D_{aJ+bK} = aD_J + bD_K;$$

since ${D_J}^2 = {D_K}^2 = {D_{aJ+bK}}^2 = 0$, it follows that

$$D_J D_K + D_K D_J = 0.$$

It is worth pointing out that it also follows that the Nijenhuis bracket $[J, K]$ vanishes, and that $d_J\beta + d_K\alpha = 0$. (Incidentally, for any pair of tensors such that $[J, K] = 0$, $d_J(\operatorname{tr} K) + d_K(\operatorname{tr} J) = d(\operatorname{tr} JK)$.)

There are functions V' and W' such that $\mu = D_J V' = D_K W'$, where $V = (\det J)V'$ and $W = (\det K)W'$ are the 'potentials' in the quadratic integrals of Γ. These functions satisfy

$$D_J D_K W' = 0 \text{ and } D_K D_J V' = 0;$$

since $D_J D_K + D_K D_J = 0$, they are both solutions of the same equation,

$$D_J D_K \phi = 0.$$

This is the generalization of the so-called 'fundamental equation' of [8, 9, 12]. In view of the anti-commutativity of the operators D_J and D_K, the 2-form $D_J D_K\phi$ is obviously skew-symmetric in J and K. It is given in terms of the components of J and K by

$$D_J D_K \phi = \left(J_i^k K_j^l \phi_{|kl} + \tfrac{3}{2}(J_i^k \beta_j - K_i^k \alpha_j)\phi_{|k} + \left((J_i^k \beta_k)_{|j} + \alpha_i \beta_j \right) \right) dq^i \wedge dq^j.$$

The term involving a covariant derivative of $J^*\beta$ may not seem to have the required skew-symmetry property at first sight, but is essentially $d_J\beta$, which is equal to $-d_K\alpha$.

We leave it to the reader to verify that this covariant fundamental equation reduces exactly to the one Lundmark put forward in coordinates for the Euclidean case. It suffices to take for the matrix J an expression of the form $J_{ij} = aq_iq_j + b_iq_j + b_jq_i + c_{ij}$ (the position of the indices is rather irrelevant in the Euclidean case, so we write them here as lower indices); accordingly, $\alpha_i = 2(aq_i + b_i)$.

We now establish the complete integrability of a cofactor pair system. Since $D_{aJ+bK}\mu = aD_J\mu + bD_K\mu = 0$, the vector field $\hat{\Gamma}$ corresponding to the given cofactor pair system is the restriction to $z = 0$ of the Hamiltonian vector field of $-z$ for the Poisson structure

$$\hat{\Pi}(a, b) = \Pi_{aJ+bK} + (\hat{\Gamma} - z(a\alpha^V + b\beta^V)) \wedge \frac{\partial}{\partial z}$$

for every a and b. Note that this is not the same as

$$a\left(\Pi_J + (\hat{\Gamma} - z\alpha^V) \wedge \frac{\partial}{\partial z}\right) + b\left(\Pi_K + (\hat{\Gamma} - z\beta^V) \wedge \frac{\partial}{\partial z}\right)$$

except when $a+b = 1$, so this does not define a Poisson pencil as written. This is a minor difficulty, which can be dealt with by adding $(a + b - 1)\hat{\Gamma} \wedge \partial/\partial z$ to $\hat{\Pi}(a, b)$, and using the standard results about Poisson pencils to prove complete integrability; however, it seems interesting, and more pleasing, to establish complete integrability directly.

We write $A(a, b)$ for the cofactor tensor of $aJ + bK$. It is a homogeneous polynomial of degree $n - 1$ in a, b (where $n = \dim Q$). Let $V(a, b)$ be a solution of $D_{aJ+bK}((\det(aJ + bK)^{-1}V(a, b)) = \mu$ or equivalently $dV(a, b) = A(a, b)^*\mu$; it is again a homogeneous polynomial in a and b of degree $n - 1$. We set

$$A(a, b) = \sum_{m=1}^{n} A_{(m)}a^{n-m}b^{m-1}, \quad V(a, b) = \sum_{m=1}^{n} V_{(m)}a^{n-m}b^{m-1}$$

and thereby define n functions $H_{(m)}$ on T^*Q, $m = 1, 2, \ldots, n$ by

$$H_{(m)} = \tfrac{1}{2}A^{ij}_{(m)}p_ip_j + V_{(m)}.$$

Note that $H_{(1)}$ is the Hamiltonian function for $(\det J)\hat{\Gamma}$ with respect to Π_J, and $H_{(n)}$ the Hamiltonian function for $(\det K)\hat{\Gamma}$ with respect to Π_K.

Theorem 4 The functions $H_{(m)}$ are first integrals of $\hat{\Gamma}$ which are in involution with respect to the Poisson brackets associated with Π_J and Π_K.

Proof We know that

$$C(a, b) = H(a, b) + z\det(aJ + bK) = \tfrac{1}{2}A(a, b)^{ij}p_ip_j + V(a, b) + z\det(aJ + bK)$$

is a Casimir of $\hat{\Pi}(a, b)$, and $\hat{\Gamma} - z(a\alpha^V + b\beta^V)$ is a Hamiltonian vector field, for every a and b. So in particular $(\hat{\Gamma} - z(a\alpha^V + b\beta^V))C(a, b) \equiv 0$. Now $H(a, b) = \sum_{m=1}^n H_{(m)}$. On setting $z = 0$ we find that $H_{(m)}$ is a first integral of $\hat{\Gamma}$ for $m = 1, 2, \ldots, n$. Now set

$$\det(aJ + bK) = \sum_{l=0}^{n} \Delta_{(l)} a^{n-l} b^l.$$

It then follows from the fact that $C(a, b)$ is a Casimir that (on T^*Q)

$$\{\cdot, H_{(m)}\}_J + \{\cdot, H_{(m-1)}\}_K = \Delta_{(m)}\hat{\Gamma}$$

for $2 \leq m \leq n$, while $\{\cdot, H_{(1)}\}_J = (\det J)\hat{\Gamma}$ and $\{\cdot, H_{(n)}\}_K = (\det K)\hat{\Gamma}$ (which just confirms that $\hat{\Gamma}$ is Hamiltonian up to a scalar factor for the Poisson brackets determined by both J and K). It follows that

$$\{H_{(r)}, H_{(s)}\}_J + \{H_{(r)}, H_{(s-1)}\}_K = \Delta_{(m)}\hat{\Gamma}(H_{(r)}) = 0$$

for $1 \leq r \leq n$ and $2 \leq s \leq n$, from which the usual kind of induction argument leads to the $H_{(m)}$ being in involution with respect to both Poisson brackets. $\square$

We make some final observations now, which will establish a link between our new results and related work in Hamilton-Jacobi theory. The cofactor tensor of $aJ + bK$ is a Killing tensor for every a, b (or at least those for which $aJ + bK$ is non-singular). It follows that $A_{(m)}$ is a Killing tensor for each m. In particular, if $K = I$, and if J has functionally independent eigenfunctions, we generate from the one special conformal Killing tensor n independent Killing tensors, one of which is g and another of which is the cofactor tensor of J; and since the eigenvectors of $aJ + bI$ are the same as the eigenvectors of J, these Killing tensors have the same eigenvectors. Furthermore, they commute pairwise in the sense of their corresponding quadratic functions having vanishing Poisson bracket (in this case, $\{\cdot, \cdot\}_K$ is the standard Poisson bracket). It follows that these Killing tensors form a Stäckel system (for full details see [2]). Moreover, $\hat{\Gamma}$ is the Hamiltonian flow of the Hamiltonian $\frac{1}{2}g^{ij}p_ip_j + V$, where $D_J dV = 0$; it then follows by results of [3] and [6] that the Hamilton-Jacobi equation for this Hamiltonian is separable in orthogonal coordinates.

The fact that the existence of a special conformal Killing tensor leads to the orthogonal separability of the Hamilton-Jacobi equation for the geodesic flow was first pointed out by Benenti in [1]. The more general case, in which there is a potential, has been discussed from points of view close to that of the present paper in [3] and [6]. In [6] a bi-Hamiltonian structure was introduced, essentially equivalent to the Poisson pencil $\hat{\Pi}_{aJ+bI}$. In [3] the special nature of the conformal Killing tensor which plays such a central role in the theory was investigated.

8 Conclusions

As is often the case, the generalization of the results of [8, 9, 10, 12] has led to the clarification of several of the concepts and methods used in these papers. In particular, the Poisson structures introduced there have been shown to be examples of two general constructions: first, the construction of a Poisson-Nijenhuis structure on a cotangent bundle via the complete lift of a type $(1,1)$ tensor with vanishing torsion, which is well-known; and second, the 'lifting' of a Poisson structure and a quasi-Hamiltonian vector field to an extended space which is the subject of Theorem 3 above. This latter result appears to be new. It should be noted that it is not dependent on the existence of a bi-Hamiltonian or quasi-bi-Hamiltonian structure, but only on a single quasi-Hamiltonian vector field. Nevertheless, it clearly has potential application in the field of quasi-bi-Hamiltonian systems (as defined for example in [11]), of which the case $K = I$ discussed at the end of the last section is an example. Indeed, the results obtained here for a particular class of what might be called bi-quasi-Hamiltonian systems should be capable of generalization to provide a theory of such systems. We are currently investigating this possibility. It is noteworthy that the differential operators D_J and D_K play an important role here, so that the theory of cofactor pairs provides an example of a gauged bi-differential calculus in the sense of [4]; this seems to us to be likely to be a feature of the general theory of bi-quasi-Hamiltonian systems we have in mind. Finally, from the opposite point of view as one might say, the properties of cofactor tensors of special conformal Killing tensors have not been noticed before in published accounts of Benenti's theory of inertia tensors and the orthogonal separability of the Hamilton-Jacobi equation, so far as we are aware. A paper on this subject, [2], is in preparation. In [6], Ibort *et al.* speculate that there is 'a deep relation, still to be worked out, between the geometry of Killing tensors on a Riemannian manifold and the geometry of a particular class of Poisson manifolds'. It seems to us that this remarkable property of special conformal Killing tensors is part of this deep geometrical structure.

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