Complex second-order differential equations and separability

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Abstract. A general theory is developed about a form of maximal decoupling of systems of second-order ordinary differential equations. Such a decoupling amounts to the construction of new variables with respect to which all equations in the system are either single equations, or pairs of equations (not coupled with the rest) which constitute the real and imaginary part of a single complex equation. The theory originates from a natural extension of earlier results by allowing the Jacobian endomorphism of the system, which is assumed to be diagonalizable, to have both real and complex eigenvalues. An important tool in the analysis is the characterization of complex second-order equations on the tangent bundle TM of a manifold, in terms of properties of an integrable almost complex structure living on the base manifold M.

1 Introduction

Martínez *et al* [8] developed a constructive geometrical characterization of systems of second-order ordinary differential equations (SODE) which can be completely decoupled into a number of single second-order equations through a suitable coordinate transformation. The constructive nature of the theory is reflected in two specific features: firstly, testing whether such a coordinate transformation exists, can be done, in principle, by testing a number of purely algebraic conditions on the given data; secondly, if the system passes all tests, the theory also shows how separation variables can be constructed.

Most of the algebraic tests for separability come from conditions on a matrix Φ^i_j, the components of the so-called *Jacobi endomorphism* Φ of the given SODE; only when this tensor field is too trivial to provide any information, has one to appeal to other algebraic data, which in the case of autonomous systems come from the *tension* field. That Φ should be an important tool for getting qualitative information about the system, is geometrically evident because Φ completely determines the curvature of the non-linear connection associated with the SODE (the tension provides information about homogeneity properties of this connection). Another application, for example, in which the analysis of Φ plays a key role, is the inverse problem of Lagrangian mechanics (see [3, 10]). In fact, also in that context one encounters a "separable case", identified by Douglas [4] for the case of two degrees of freedom, and generalized to arbitrary n in [2]. Although Douglas's notion of a separable case refers to a situation where partial differential equations for determining the existence of a Lagrangian for a give SODE decouple, it was shown in [2] that there is an underlying form of separability of the SODE itself behind this, which does not necessarily, however, preserve the second-order character of the given system and thus is less restrictive.

The first condition on Φ for the existence of separation variables in both situations referred to above is its (pointwise) algebraic diagonalizability. For the sake of simplicity both in [8] and [2], this was understood to mean that none of the eigenvalues of the real matrix Φ would turn out to be complex. It is this limitation which we intend to remove here for the case of complete separability in the sense of [8]. It will involve developing a good understanding of the sometimes rather subtle interplay between the given dynamics and certain integrable almost complex structures which can be constructed on the integral submanifolds of distributions, corresponding to pairs of complex conjugate eigenvalues of Φ.

In the next section, we recall the intrinsic operations on vector fields and forms along the projection $\tau : TM \to M$, which entered the separability analysis in [8], and focus on their role in characterizing invariance properties of certain type (1,1) tensor fields on TM. We discuss almost complex structures and complex differential equations in Section 3. Section 4 is the main section, where we closely follow the previously established separability results and investigate to what extent the theory can be adapted to allow for tensors with complex eigenvalues. As in the case of real eigenvalues, the results are fairly straightforward when all eigenvalues are distinct, whereas much more care is needed when there is degeneracy. Some examples are presented in Section 5.

2 The calculus along the tangent bundle projection associated to a second-order equation field

Consider a system of second-order autonomous ordinary differential equations. It is modelled on a tangent bundle TM by a vector field

$$\Gamma = v^i \frac{\partial}{\partial x^i} + f^i(x,v) \frac{\partial}{\partial v^i} . \tag{1}$$

As such, Γ defines a non-linear connection on $\tau : TM \to M$, i.e. a horizontal distribution, locally spanned by

$$H_i = \frac{\partial}{\partial x^i} - \Gamma^j_i \frac{\partial}{\partial v^j} \qquad \text{with} \quad \Gamma^j_i = -\tfrac{1}{2}\frac{\partial f^j}{\partial v^i} . \tag{2}$$

By means of this connection, every vector field on TM has a unique decomposition into a horizontal and vertical part. It will be convenient for our purposes to think of these as coming from a vector field along τ by a corresponding lifting process. Vector fields along τ are sections of the pull back bundle $\tau^*TM \to TM$ and most operations of interest resulting in vector fields on TM, when one looks at the decomposition into horizontal and vertical components of the result, will be seen to come essentially from operations on $\mathcal{X}(\tau)$ or $\bigwedge(\tau)$ (the $C^\infty(TM)$-module of differential forms along τ). A complete study of derivations of scalar and vector-valued forms along τ was carried out for that reason in [6, 7]. We will frequently refer to results of those papers, but one can quickly get acquainted with the main operations that we need as follows (the reader may wish to consult the introductions of, for example, [8, 3] for a somewhat more complete picture).

Let X^H, Y^V denote respectively the horizontal and vertical lift of elements $X, Y \in \mathcal{X}(\tau)$. We have

$$\begin{aligned} [X^H, Y^V] &= (\mathrm{D}^H_X Y)^V - (\mathrm{D}^V_Y X)^H, && (3)\\ [\Gamma, X^H] &= (\nabla X)^H + (\Phi(X))^V. && (4) \end{aligned}$$

We recognize in the right-hand sides degree zero derivations D^H_X, D^V_Y and ∇, which can be extended to 1-forms along τ by duality, and subsequently to tensor fields along τ of arbitrary type. In fact these are covariant derivative type derivations (D^H_X and D^V_X depend $C^\infty(TM)$-linearly on X); they comprise the essential ingredients of a *linear* connection on $\tau^*TM \to TM$, which is in some sense the linearization of the non-linear connection defined by (2). The other term in (4) reveals a type (1,1) tensor field Φ along τ, called the Jacobi endomorphism, whose components are given by

$$\Phi^i_j = -\frac{\partial f^i}{\partial x^j} - \Gamma^k_j \Gamma^i_k - \Gamma(\Gamma^i_j). \qquad (5)$$

One of the main advantages conferred by this formalism is precisely that it enables Φ^i_j to be thought of as the components of a tensor field. Calculations thus can be performed more efficiently because the redundant information is sifted out.

To be able to do coordinate calculations with the derivations identified above, it suffices to know their action on functions F on TM and on the coordinate vector fields and 1-forms on M, regarded as (basic) vector fields and forms along τ. In fact we have

$$\mathrm{D}^V_X F = X^i \frac{\partial F}{\partial v^i}, \quad \mathrm{D}^V_X \left(\frac{\partial}{\partial x^j}\right) = 0, \quad \mathrm{D}^V_X (dx^i) = 0\,, \qquad (6)$$

$$\mathrm{D}^H_X F = X^i H_i(F), \quad \mathrm{D}^H_X \left(\frac{\partial}{\partial x^j}\right) = \left(X^k \frac{\partial \Gamma^i_j}{\partial v^k}\right) \frac{\partial}{\partial x^i}, \quad \mathrm{D}^H_X (dx^i) = -\left(X^k \frac{\partial \Gamma^i_j}{\partial v^k}\right) dx^j\,, \qquad (7)$$

$$\nabla F = \Gamma(F), \quad \nabla \left(\frac{\partial}{\partial x^j}\right) = \Gamma^i_j \frac{\partial}{\partial x^i}, \quad \nabla(dx^i) = -\Gamma^i_j dx^j\,. \qquad (8)$$

Two other frequently occurring Lie bracket operations are

$$\begin{aligned} [X^V, Y^V] &= (\mathrm{D}^V_X Y - \mathrm{D}^V_Y X)^V && (9)\\ [X^H, Y^H] &= (\mathrm{D}^H_X Y - \mathrm{D}^H_Y X)^H + (R(X,Y))^V\,, && (10) \end{aligned}$$

where R is the curvature of the connection (2) seen here as a vector-valued 2-form along τ. Complementing the relation (4), we further mention:

$$[\Gamma, X^V] = -X^H + (\nabla X)^V. \tag{11}$$

Finally, let us recall that there exists a canonical vertical exterior derivative d^V on $\bigwedge(\tau)$ (and on $V(\tau)$, the module of vector-valued forms along τ) defined independently of any connection. The latter allows one to define a corresponding horizontal exterior derivative as well. By way of example, if $U \in V^1(\tau)$, $d^V U$ and $d^H U$, both elements of $V^2(\tau)$, are given by

$$d^V U(X,Y) = \mathrm{D}^V_X U(Y) - \mathrm{D}^V_Y U(X), \qquad d^H U(X,Y) = \mathrm{D}^H_X U(Y) - \mathrm{D}^H_Y U(X). \tag{12}$$

The importance of the tensor field Φ can now be underscored by observing that it completely determines the curvature and its 'dynamical evolution', according to the properties

$$d^V \Phi = 3\, R, \qquad d^H \Phi = \nabla R. \tag{13}$$

Various ways of lifting elements of $V^1(\tau)$ to $V^1(TM)$ were discussed in [7], and the Lie derivative with respect to Γ of the lifted objects was computed in terms of corresponding actions on the original (1,1) tensor field along τ. Not surprisingly, in all such computations, an important role is played by the Jacobi endomorphism Φ and the dynamical covariant derivative ∇. The particular lifting process of interest here is one which depends on the given Γ, and will be denoted by the same symbol J_Γ irrespective of whether it applies to vectors, forms or endomorphisms. Thus, for $X \in \mathcal{X}(\tau)$, $\alpha \in \bigwedge^1(\tau)$ and $U \in V^1(\tau)$, we have

$$\begin{aligned} J_\Gamma X &= X^H + (\nabla X)^V && (14)\\ J_\Gamma \alpha &= (\nabla \alpha)^H + \alpha^V && (15)\\ J_\Gamma U &= U^H + (\nabla U)^V, && (16) \end{aligned}$$

where the horizontal and vertical lift of a type (1,1) tensor field U along τ are determined by

$$\begin{aligned} U^H(X^H) &= U(X)^H, \qquad & U^H(X^V) &= U(X)^V && (17)\\ U^V(X^H) &= U(X)^V, \qquad & U^V(X^V) &= 0\,. && (18) \end{aligned}$$

The image sets of the operator J_Γ are interesting, if only because they contain the Γ-invariant objects such as symmetry vector fields, adjoint symmetries (invariant 1-forms) and recursion operators (invariant type (1,1) tensor fields). In fact, these image sets of objects on TM were identified as such before in [9]: $J_\Gamma(\mathcal{X}(\tau))$ was the subset of $\mathcal{X}(TM)$ denoted by $\mathcal{X}_\Gamma$; likewise $J_\Gamma(\bigwedge^1(\tau))$ was the set $\mathcal{X}^*_\Gamma$ and $J_\Gamma(V^1(\tau))$ is just the set of type (1,1) tensor fields on TM, which preserve $\mathcal{X}_\Gamma$ and $\mathcal{X}^*_\Gamma$. We know therefore that, for example, for $U \in V^1(\tau)$, $J_\Gamma U \in V^1(TM)$ is completely characterized by the properties:

$$J_\Gamma U \circ S = S \circ J_\Gamma U, \qquad S \circ \mathcal{L}_\Gamma(J_\Gamma U) = 0, \tag{19}$$

where S is the canonically defined vertical endomorphism on TM.

A second reason why the image sets under J_Γ are of interest is the fact that they generalize and thus contain the complete lifts of fields on the base manifold M. Adding an extra condition which will ensure that the tensor field along τ under consideration is actually basic, will thus provide a characterization of tensor fields on TM which are complete lifts. In particular, in the case of a type (1,1) tensor field, the extra condition to impose is

$$\mathcal{L}_\Delta(J_\Gamma U) = 0, \tag{20}$$

plus smoothness on the zero section, where Δ is the Liouville (or dilation) vector field, which incidentally is the vertical lift of the canonical vector field $\mathbf{T}$ along τ:

$$\Delta = \mathbf{T}^V, \qquad \mathbf{T} = v^i \frac{\partial}{\partial x^i}. \tag{21}$$

Alternatively, expressing that U is a basic tensor field can be done by requiring that $\mathrm{D}^V_X U = 0, \forall X \in \mathcal{X}(\tau)$.

We will, in particular, be interested here in tensor fields which are invariant under Γ. Recall from [7] that

$$\mathcal{L}_\Gamma(J_\Gamma U) = 0 \quad \Longleftrightarrow \quad \nabla U = 0 \;\text{ and }\; [U, \Phi] = 0. \tag{22}$$

The same conditions of course apply to the particular case of complete lifts. For readers who are perhaps more familiar with computations involving 'honest' tensor fields on the full space TM let us list properties which are very easy to deduce from the analysis in [7] and give equivalent (but less transparent) representations of the two conditions in (22). We have

$$\nabla U = 0 \quad \Leftrightarrow \quad [\mathcal{L}_\Gamma S, J_\Gamma U] = 0, \tag{23}$$

and

$$\left\{\begin{array}{l} [\mathcal{L}_\Gamma S, J_\Gamma U] = 0 \\ [\mathcal{L}^2_\Gamma S, J_\Gamma U] = 0 \end{array}\right. \quad \Longleftrightarrow \quad \left\{\begin{array}{l} \nabla U = 0 \\ [U, \Phi] = 0 \end{array}\right. \tag{24}$$

We end this section with the observation that in the case of a complete lift, i.e. when $J_\Gamma U = U^c$, it is easy to verify (and in fact well known, see e.g. [12]) that the Nijenhuis tensor $\mathcal{N}_{U^c}$ is zero, if and only if $\mathcal{N}_U = 0$ on M.

3 Complex second-order equation fields

We now focus on almost complex structures. If J is an almost complex structure on M, then it is well known that J^c is an almost complex structure on TM (see e.g. [12]). By way of example of the use of the calculus summarised in the previous section, we show how this result easily generalizes when the components of the original J are allowed to depend on points on TM. We say that $J \in V^1(\tau)$ is an *almost complex structure along* τ if $J^2 = -I$.

Proposition 1. *If J is an almost complex structure along τ, then $J_\Gamma J$ is an almost complex structure on TM.*

PROOF. We evaluate $J_\Gamma J$ on horizontal and vertical lifts of vector fields along τ, using the defining relation (16) and the properties (17)(18).

$$\begin{aligned} J_\Gamma J(X^H) &= J^H(X^H) + (\nabla J)^V(X^H) \\ &= J(X)^H + \nabla J(X)^V \end{aligned}$$

and likewise

$$J_\Gamma J(X^V) = J(X)^V.$$

It follows that

$$\begin{aligned} (J_\Gamma J)^2(X^H) &= J^H(J(X)^H + \nabla J(X)^V) + (\nabla J)^V(J(X)^H + \nabla J(X)^V) \\ &= J^2(X)^H + J \circ \nabla J(X)^V + \nabla J \circ J(X)^V \\ &= -X^H, \end{aligned}$$

since $J^2 = -I$ obviously implies $J \circ \nabla J + \nabla J \circ J = 0$. Showing that $(J_\Gamma J)^2(X^V) = -X^V$ is even more direct. $\square$

In fact, the proof would even become a bit shorter still if one would use vector fields of the form $J_\Gamma X$ and X^V to generate a local basis for $\mathcal{X}(\tau)$, using the property that for any $U \in V^1(\tau)$ and $X \in \mathcal{X}(\tau)$: $J_\Gamma U(J_\Gamma X) = J_\Gamma(UX)$.

Computing the Nijenhuis tensor of $J_\Gamma J$ is interesting in its own right, but is much more involved. Since it will turn out that the separability analysis of the next section does not require this level of generality, we will refrain from doing this computation here and turn to the special case where J is basic, i.e. is an almost complex structure on M. It then follows that if J is integrable, so is $J_\Gamma J = J^c$.

Let then J be an integrable almost complex structure on M (so M has dimension $2n$ say), and assume further that J satisfies the conditions (22). We wish to see what this implies for the given SODE Γ. Appealing first to the Newlander-Nirenberg theorem (see e.g. [1]), we know that M then is actually a complex manifold which, expressed in real terms, means that there exist coordinates (x^i, y^i) on M such that J takes the standard form

$$J = \frac{\partial}{\partial y^i} \otimes dx^i - \frac{\partial}{\partial x^i} \otimes dy^i\,. \tag{25}$$

If we write $(x^\alpha)_{1\leq\alpha\leq 2n}$ for the collective coordinates and $\ddot{x}^\alpha = F^\alpha(x^\beta, \dot{x}^\beta)$ for the differential equations corresponding to the given Γ, we know that the connection coefficients $\Gamma^\alpha_\beta = -\frac{1}{2}\partial F^\alpha/\partial\dot{x}^\beta$ determine the dynamical covariant derivative ∇ as in (8). Let us put $(F^\alpha) = (f^i, g^i)$ and

$$\nabla \frac{\partial}{\partial x^i} = {}^1\Gamma^j_i \frac{\partial}{\partial x^j} + {}^2\Gamma^j_i \frac{\partial}{\partial y^j} = -\tfrac{1}{2}\frac{\partial f^j}{\partial \dot{x}^i}\frac{\partial}{\partial x^j} - \tfrac{1}{2}\frac{\partial g^j}{\partial \dot{x}^i}\frac{\partial}{\partial y^j} \tag{26}$$

$$\nabla \frac{\partial}{\partial y^i} = {}^1\Gamma^j_{n+i} \frac{\partial}{\partial x^j} + {}^2\Gamma^j_{n+i} \frac{\partial}{\partial y^j} = -\tfrac{1}{2}\frac{\partial f^j}{\partial \dot{y}^i}\frac{\partial}{\partial x^j} - \tfrac{1}{2}\frac{\partial g^j}{\partial \dot{y}^i}\frac{\partial}{\partial y^j}\,. \tag{27}$$

In agreement with these notational conventions, we thus write the matrix of connection coefficients formally as follows:

$$\left(\Gamma^{\alpha}_{\beta}\right) = \begin{pmatrix} {}^1\Gamma_1 & {}^1\Gamma_2 \\ {}^2\Gamma_1 & {}^2\Gamma_2 \end{pmatrix} . \tag{28}$$

Regarded as endomorphism on vector fields J itself accordingly has the matrix representation

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{29}$$

and let us similarly represent the matrix of components of Φ, as determined in (5) by

$$\left(\Phi^{\alpha}_{\beta}\right) = \begin{pmatrix} {}^1\Phi_1 & {}^1\Phi_2 \\ {}^2\Phi_1 & {}^2\Phi_2 \end{pmatrix} . \tag{30}$$

Then, it is easy to verify that

$$\nabla J = 0 \quad \Longleftrightarrow \quad \begin{cases} {}^1\Gamma_1 = {}^2\Gamma_2 \\ {}^1\Gamma_2 = -{}^2\Gamma_1 \end{cases} . \tag{31}$$

In other words, in this matrix representation $\nabla J = 0$ just means that the matrix of connection coefficients commutes with J. Of course, for the remaining condition imposed by (22) on J, we likewise have

$$[\Phi, J] = 0 \quad \Longleftrightarrow \quad \begin{cases} {}^1\Phi_1 = {}^2\Phi_2 \\ {}^1\Phi_2 = -{}^2\Phi_1 \end{cases} . \tag{32}$$

Observe now finally that the conditions (31) are equivalent to saying that

$$\frac{\partial f^j}{\partial \dot{x}^i} = \frac{\partial g^j}{\partial \dot{y}^i}, \qquad \frac{\partial f^j}{\partial \dot{y}^i} = -\frac{\partial g^j}{\partial \dot{x}^i}, \tag{33}$$

and taking these equalities into account in the definition (5) of the components of Φ, the conditions (32) are equivalent to

$$\frac{\partial f^j}{\partial x^i} = \frac{\partial g^j}{\partial y^i}, \qquad \frac{\partial f^j}{\partial y^i} = -\frac{\partial g^j}{\partial x^i}. \tag{34}$$

Recognizing the Cauchy-Riemann conditions in (33) and (34), we are led to introduce the following concept.

Definition. *A second-order differential equation field Γ on the tangent bundle of an even dimensional manifold M is said to be complex if M admits an integrable almost complex structure J, such that $\nabla J = 0$ and $[\Phi, J] = 0$ (or equivalently $\mathcal{L}_\Gamma J^c = 0$).*

The results of the preceding calculations can then be summarized as follows.

Proposition 2. *For any complex Γ on the tangent bundle of an even dimensional manifold M, there exist complex coordinates z^i on M such that the differential equations corresponding to Γ represent the real and imaginary parts of a system of complex differential equations $\ddot{z}^i = \mathcal{F}^i(z, \dot{z})$.* □

It is of some interest to look at the particular case where the complex SODE Γ is a quadratic spray with zero curvature, more specifically for the purpose of understanding in detail how the construction of 'flat coordinates' for the real equations associated to Γ corresponds to the construction of flat holomorphic coordinates for the associated complex equations $\ddot{z}^i = \mathcal{F}^i(z, \dot{z})$.

A complex quadratic spray, expressed in coordinates which bring J into the form (25), is a system of second-order differential equations of the form

$$\begin{aligned} \ddot{x}^i &= f^i = \tfrac{1}{2}F^i_{jk}(\dot{x}^j\dot{x}^k - \dot{y}^j\dot{y}^k) + \tfrac{1}{2}G^i_{jk}(\dot{x}^j\dot{y}^k + \dot{x}^k\dot{y}^j) && (35)\\ \ddot{y}^i &= g^i = \tfrac{1}{2}G^i_{jk}(\dot{y}^j\dot{y}^k - \dot{x}^j\dot{x}^k) + \tfrac{1}{2}F^i_{jk}(\dot{y}^j\dot{x}^k + \dot{y}^k\dot{x}^j), && (36)\end{aligned}$$

with $F^i_{jk} = F^i_{kj}, \quad G^i_{jk} = G^i_{kj}$ and

$$\frac{\partial F^i_{jk}}{\partial x^l} = -\frac{\partial G^i_{jk}}{\partial y^l}, \qquad \frac{\partial F^i_{jk}}{\partial y^l} = \frac{\partial G^i_{jk}}{\partial x^l}. \tag{37}$$

The specific structure (35-36) of the quadratic spray, plus the above properties satisfied by its connection components are of course simply dictated by the fact that we want the Cauchy-Riemann conditions (33) and (34) to be satisfied. The corresponding complex differential equations are of the form

$$\ddot{z}^i = \mathcal{F}^i(z, \dot{z}) = \tfrac{1}{2}\mathcal{F}^i_{jk}\dot{z}^j\dot{z}^k\,, \tag{38}$$

where

$$\mathcal{F}^i_{jk} = F^i_{jk} - i\,G^i_{jk}\,. \tag{39}$$

For this complex representation of the original spray, curvature zero means that

$$\frac{\partial \mathcal{F}^i_{jl}}{\partial z^k} + \tfrac{1}{2}\mathcal{F}^i_{jm}\mathcal{F}^m_{kl} - (j \longleftrightarrow k) = 0\,. \tag{40}$$

This gives rise to two conditions (coming from the real and imaginary part) which, taking the properties of the F^i_{jl} and G^i_{jl} into account, can be verified to coincide identically with the zero curvature conditions for the quadratic spray (35),(36) we started from – there are in fact 12 conditions, in principle, if one looks at all the different curvature components for the spray (35),(36), but again, from the properties of the connection components, these eventually reduce to just two independent conditions. By the complex version of the classical theorem about linear connections with zero torsion and curvature (see e.g. [5]), we know that there exists a holomorphic change of variables $z = z(z')$ to a representation with zero connection components. In real terms, the change of variables $(x, y) \leftrightarrow (x', y')$ therefore has a Jacobian which commutes with J. For later use, we formulate this simple result as a lemma.

Lemma 1. *Given a complex quadratic spray on the tangent bundle of an even dimensional real manifold M which has zero curvature, there exists a coordinate transformation which preserves the almost complex structure J and will transform the given* SODE *into trivial equations with zero right-hand side.* □

4 Separability of second-order equations when the Jacobi endomorphism has complex eigenvalues

The purpose of this section is to generalize the results of [8] about separable SODE's to the case where the Jacobi endomorphism Φ of the given system is allowed to have complex eigenvalues. One should keep in mind of course that the word eigenvalue here and in what follows will generally refer to a (locally defined) function on TM.

So, Γ at the start is an arbitrary SODE on TM and M can have arbitrary dimension. As in [8], a large part of the analysis can be carried out in the context of diagonalizability and separability of type (1,1) tensor fields U in general; the full separability of Γ then emerges when we specialize to the case $U = \Phi$ (possibly supplemented by information coming from another tensor field, called the *tension*). We will closely follow the path set out in [8] and actually go through some of the features in reasonable detail again, because we need to assure that eigendistributions corresponding to complex eigenvalues can be dealt with in a manner that is consistent with the real parts.

Firstly, we consider type (1,1) tensor fields along $\tau : TM \to M$ which are diagonalizable in an algebraic sense. To be precise: $U \in \mathcal{X}(\tau)$ is said to *diagonalizable* if for all $v \in TM$, the linear map $U(v) : T_{\tau(v)}M \to T_{\tau(v)}M$ is diagonalizable and this can be done smoothly in a neighbourhood of v, leading to eigendistributions which have constant dimension. Eigenvalues μ_A of U, however, could have multiplicity greater than one and are allowed this time to be complex-valued. Naturally, since the original matrix $U^i_j(x,v)$ is real valued, complex eigenvalues will come in pairs of complex conjugate ones. The eigendistribution corresponding to μ_A will be denoted by $\mathcal{D}_A$ and its dimension by d_A. Such distributions, when μ_A is real, are spanned by vector fields $\{X_{A\alpha}\}_{\alpha=1,\ldots,d_A}$ along τ. To make a notational distinction, let us agree to write $(\lambda_A, \overline{\lambda}_A)$ for pairs of complex conjugate eigenvalues and denote a local basis for their eigendistributions by $\{Z_{A\alpha}\}$ and $\{\overline{Z}_{A\alpha}\}$ respectively. Thus $Z_{A\alpha}$ for example, as a complex vector field along τ, is a smooth section of the complexified bundle $\tau^*TM \otimes \mathbb{C} \to TM$. To fix notations further, we will typically write

$$\lambda_\alpha = \rho_\alpha + i\sigma_\alpha, \qquad Z_\alpha = V_\alpha + iW_\alpha \tag{41}$$

for the real and imaginary parts of complex eigenvalues or eigenvectors (omitting the extra subscript A referring to the distribution in question when there is no danger of confusion). Knowing that $U(Z_\alpha) = \lambda_\alpha Z_\alpha$, it then follows that

$$U(V_\alpha) = \rho_\alpha V_\alpha - \sigma_\alpha W_\alpha, \qquad U(W_\alpha) = \sigma_\alpha V_\alpha + \rho_\alpha W_\alpha\,. \tag{42}$$

Definition. *A distribution $\mathcal{D}$ along τ, real or complex, is said to be basic when it can be spanned by vector fields on M. A real basic distribution along τ is said to be involutive, if the corresponding distribution on M is involutive.*

Lemma 2. *A distribution $\mathcal{D}$ along τ is basic if and only if $\mathrm{D}^V_X(\mathcal{D}) \subset \mathcal{D}$, for all $X \in \mathcal{X}(\tau)$.*

PROOF. We merely check that the arguments used in Proposition 3.3 of [8] remain valid when $\mathcal{D}$ is complex. Clearly, if $\mathcal{D}$ is basic, it is D^V-invariant. Conversely, let $\{\tilde{Z}_\alpha\}_{\alpha=1,\ldots,d}$

be a local basis for $\mathcal{D}$, and $\{X_i\}_{i=1,\ldots,n}$ a local basis for $\mathcal{X}(M)$. Then, we have $\tilde{Z}_\alpha = \Lambda^i_\alpha X_i$, where the rank of the matrix of complex valued functions Λ^i_α is d. Assuming, without loss of generality, that the upper left-hand $(d\times d)$-submatrix is regular, we can multiply by its inverse to obtain a new local basis $\{Z_\alpha\}_{\alpha=1,\ldots,d}$ for $\mathcal{D}$, of the form $Z_\alpha = X_\alpha + \sum_{i=d+1}^n \sigma^i_\alpha X_i$. It follows that for arbitrary X, $\mathrm{D}^V_X Z_\alpha = \sum_{i=d+1}^n (\mathrm{D}^V_X \sigma^i_\alpha) X_i$. For this to belong to $\mathcal{D}$, we must have $\mathrm{D}^V_X \sigma^i_\alpha = 0$, meaning that the Z_α (and of course also the $\overline{Z}_\alpha$ spanning $\overline{\mathcal{D}}$) will be basic. □

A fairly obvious result, the proof of which remains unaltered in the presence of complex eigenvalues, is the following.

Lemma 3. *If D is any self-dual derivation on $V(\tau)$ and $U \in V^1(\tau)$ is diagonalizable, then the eigendistributions of U are D-invariant if and only if $[DU, U] = 0$. In such a case, DU is simultaneously diagonalizable and the eigenvalues of DU are the D-derivatives of the eigenvalues of U.* □

Whenever $(\mathcal{D}_A, \overline{\mathcal{D}}_A)$ is a pair of complex conjugate eigendistributions of U, spanned by $(Z_{A\alpha}, \overline{Z}_{A\alpha})$, we will be interested in integrability aspects of the $2d_A$-dimensional real distribution $\mathcal{D}^r_A = \mathrm{sp}\,\{V_{A\alpha}, W_{A\alpha}\}$. That means, of course, that we have the real Jordan normal form of U in mind, rather than its diagonal form, and most of the analysis is about ensuring that such a real Jordan form can be achieved via a coordinate transformation. Under the conditions of the preceding lemma, i.e. when $\mathcal{D}_A$ is D-invariant, then so is $\mathcal{D}^r_A$. In fact, we can be a bit more precise about this. If we put

$$DZ_{A\alpha} = \sum_{\beta=1}^{d_A} c^{A\beta}_{A\alpha}\, Z_{A\beta}, \qquad \text{with} \quad c^{A\beta}_{A\alpha} = a^{A\beta}_{A\alpha} + i\, b^{A\beta}_{A\alpha}\,, \tag{43}$$

then we find:

$$DV_{A\alpha} = \sum_{\beta=1}^{d_A} \left(a^{A\beta}_{A\alpha} V_{A\beta} - b^{A\beta}_{A\alpha} W_{A\beta}\right), \quad DW_{A\alpha} = \sum_{\beta=1}^{d_A} \left(b^{A\beta}_{A\alpha} V_{A\beta} + a^{A\beta}_{A\alpha} W_{A\beta}\right). \tag{44}$$

Having diagonalized the tensor field U in some neighbourhood, we have a set of complementary distributions at our disposal, say n_1 real eigendistributions $\mathcal{D}_A$ and n_2 even dimensional distributions of the type $\mathcal{D}^r_A$, such that for each v in the neighbourhood under consideration, we have

$$T_{\tau(v)}M = \oplus_{A=1}^{n_1} \mathcal{D}_A(v) \oplus_{A=1}^{n_2} \mathcal{D}^r_A(v)\,. \tag{45}$$

We define a type (1,1) tensor field J along τ in terms of the constructed local bases of vector fields for all these distributions, via the relations

$$\begin{cases} J(V_{A\alpha}) = W_{A\alpha}, \quad J(W_{A\alpha}) = -V_{A\alpha}, \quad \forall\, V_{A\alpha}, W_{A\alpha} \in \mathcal{D}^r_A\,, \\ J(X_{A\alpha}) = 0, \quad \forall\, X_{A\alpha} \in \mathcal{D}_A\,. \end{cases} \tag{46}$$

Clearly, J is a degenerate almost complex structure, in the sense that

$$J^2\Big|_{\mathcal{D}^r_A} = -I\,. \tag{47}$$

It follows from Eqns. (42) that

$$[U, J] = 0\,. \tag{48}$$

Lemma 4. *If U is diagonalizable and $[DU, U] = 0$ for some given self-dual derivation D, then $DJ = 0$.*

PROOF. From (44) and (46), it follows that

$$J(DV_{A\alpha}) = DW_{A\alpha}, \qquad J(DW_{A\alpha}) = -DV_{A\alpha}\,. \tag{49}$$

As a result we get

$$DJ(V_{A\alpha}) = D(JV_{A\alpha}) - J(DV_{A\alpha}) = 0\,,$$

and likewise $DJ(W_{A\alpha}) = 0$. Obviously, for the real eigendistributions we also have $DJ(X_{A\alpha}) = 0$. The result follows. □

We now turn to the degree zero derivations described in Eqns. (6-8). As in [8], we introduce the type (1,2) tensor field C^V_U along τ, defined by

$$C^V_U(X, Y) = [\mathrm{D}^V_X U, U](Y)\,, \tag{50}$$

and recall also that the commutator of the derivations ∇ and D^V_X generates D^H_X, according to the property:

$$[\nabla, \mathrm{D}^V_X] = \mathrm{D}^V_{\nabla X} - \mathrm{D}^H_X\,. \tag{51}$$

We thus come to the first main theorem.

Theorem 1. *Let $U \in V^1(\tau)$ be diagonalizable and satisfy the extra conditions $C^V_U = 0$ and $[\nabla U, U] = 0$. Then, there exists a coordinate transformation on M which will bring U into its real Jordan normal form in a neighbourhood of an arbitrary point of TM.*

PROOF. From Lemma 3, $C^V_U = 0$ implies that the eigendistributions are D^V-invariant, which according to Lemma 2 means that they can be spanned by basic vector fields. Furthermore, $[\nabla U, U] = 0$ implies that they are also ∇-invariant and hence, from the property (51), also D^H-invariant. Rather then continuing the reasoning with eigendistributions, we replace the complex eigendistributions with the real distributions denoted by $\mathcal{D}^r_A$ before and look at the direct sum decomposition (45). Naturally, we may assume now that a local basis $\{V_{A\alpha}, W_{A\alpha}\}$ for $\mathcal{D}^r_A$ has been selected, consisting of basic vector fields. The D^H-invariance, in particular, means that for arbitrary X and for $Y \in \mathcal{D}_A$ (or $\mathcal{D}^r_A$), $\mathrm{D}^H_X Y$ belongs to the same distribution, implying further that for two vector fields X, Y belonging to two different distributions, the combination $\mathrm{D}^H_X Y - \mathrm{D}^H_Y X$ will belong to the sum of these two distributions. This will in particular be true for the basic vector fields spanning all distributions, but when X and Y are vector fields on M, $\mathrm{D}^H_X Y - \mathrm{D}^H_Y X$ is just their Lie bracket. All associated basic distributions are therefore involutive and will be simultaneously integrable by the Frobenius theorem (such a situation was analysed in detail in Lemma 3.4 in [8]). We may already conclude that there exist coordinates on M such that each of the distributions in (45) will be spanned by a corresponding number of coordinate vector fields. The coordinate vector fields spanning real eigendistributions will be eigenvectors themselves and so U will already be diagonal in those parts. The situation is different for the $\mathcal{D}^r_A$, as $\{V_{A\alpha}, W_{A\alpha}\}$ may so far be combinations of all coordinate vector fields spanning the distribution.

Turning our attention to the tensor field J defined by the relations (46), we prove the following intermediate result.

Lemma 5. *If a diagonalizable $U \in V^1(\tau)$ satisfies $C^V_U = 0$ and $[\nabla U, U] = 0$, then the corresponding tensor field J is integrable.*

PROOF. We know from Lemma 4 that under the given hypotheses on U, J will be a basic tensor field, which further is ∇-invariant and from (51) also D^H-invariant. Considering the Nijenhuis tensor $\mathcal{N}_J$, we have for two of the vector fields of type V_α, possibly belonging to different distributions $\mathcal{D}^r_A$ and $\mathcal{D}^r_B$,

$$\mathcal{N}_J(V_{A\alpha}, V_{B\beta}) = [W_{A\alpha}, W_{B\beta}] + J^2\left([V_{A\alpha}, V_{B\beta}]\right) - J\left([W_{A\alpha}, V_{B\beta}] + [V_{A\alpha}, W_{B\beta}]\right).$$

We can write the brackets of all such basic vector fields in terms of D^H-derivatives and make use of the properties (49) of J with respect to the appropriate D^H-derivatives. By way of example, one can write

$$J\left([W_{A\alpha}, V_{B\beta}]\right) = J\left(\mathrm{D}^H_{W_{A\alpha}} V_{B\beta} - \mathrm{D}^H_{V_{B\beta}} W_{A\alpha}\right) = \mathrm{D}^H_{W_{A\alpha}} W_{B\beta} + \mathrm{D}^H_{V_{B\beta}} V_{A\alpha}\,.$$

This way, it is easy to see that all terms will cancel out. Obviously, the computation will be completely similar in the case of two W_α-type arguments, or one V_α and one W_β. The result is trivial when both arguments are of X_α-type, because J is zero on real eigenspaces. Finally,

$$\begin{aligned}
\mathcal{N}_J(X_{A\alpha}, V_{B\beta}) &= J^2\left([X_{A\alpha}, V_{B\beta}]\right) - J\left([X_{A\alpha}, W_{B\beta}]\right) \\
&= J^2\left(\mathrm{D}^H_{X_{A\alpha}} V_{B\beta} - \mathrm{D}^H_{V_{B\beta}} X_{A\alpha}\right) - J\left(\mathrm{D}^H_{X_{A\alpha}} W_{B\beta} - \mathrm{D}^H_{W_{B\beta}} X_{A\alpha}\right) \\
&= J^2\left(\mathrm{D}^H_{X_{A\alpha}} V_{B\beta}\right) + \mathrm{D}^H_{X_{A\alpha}} V_{B\beta} = 0\,,
\end{aligned}$$

and likewise for $\mathcal{N}_J(X_{A\alpha}, W_{B\beta})$. □

Returning now to the proof of the theorem we have, in particular, that for each of the $\mathcal{D}^r_A$ distributions, the restriction of J to that distribution will be an integrable almost complex structure. It follows that there exists a further coordinate transformation among the coordinates whose tangent fields span the distribution, such that $J|_{\mathcal{D}^r_A}$ is of the form (25), and this can be done for all the $\mathcal{D}^r_A$ distributions simultaneously. In other words, in such coordinates a basis for the distribution $\mathcal{D}^r_A$ is given by

$$V_{A\alpha} = \frac{\partial}{\partial x^{A\alpha}}, \qquad W_{A\alpha} = \frac{\partial}{\partial y^{A\alpha}}\,. \tag{52}$$

One final remark is in order now. Writing an almost complex structure in the standard form (25), makes further calculations such as those following Eqn. (25) much easier, but with the choice (52), the tensor field U does not quite acquire what one usually calls its real Jordan normal form. Passing from one representation to the other, however, is merely a matter of renumbering the variables. Explicitly, it corresponds to writing the basis (52) for $\mathcal{D}^r_A$ in the order

$$V_{A1},\, W_{A1},\, \ldots\,,\, V_{Ad},\, W_{Ad}\,. \tag{53}$$

This concludes the proof of Theorem 1. $\square$

It is interesting to observe that whenever a SODE Γ admits a tensor field U satisfying the conditions of Theorem 1, we already know a number of separability properties of the equations with respect to the velocity variables. Indeed, since all the distributions involved in the decomposition (45) are simultaneously integrable and are ∇-invariant, it follows from the action of ∇ on coordinate vector fields (see (8)) that the connection components, in the right coordinates, will satisfy

$$\Gamma^{A\alpha}_{B\beta} = 0, \qquad \text{for} \quad A \neq B. \tag{54}$$

This means that the 'force functions' $f^{A\alpha}$ in the general representation (1) of Γ will not depend on the velocities $v^{B\beta}$. Moreover, inside each of the 'complex parts' of U, where J is non-trivial and has the standard form, we know from $\nabla J = 0$ that we are in a situation as in (31). This means that the corresponding part of the differential equations will be of the form

$$\begin{aligned} \ddot{x}^i &= f^i(\ldots, \dot{x}, \dot{y}), \\ \ddot{y}^i &= g^i(\ldots, \dot{x}, \dot{y}), \end{aligned}$$

where the right-hand sides have the properties (33). There is no information, however, about the dependence on position variables. In fact, the line dots could refer to all coordinates, including those coming from the other eigendistributions. More info will be available if U is the important type (1,1) tensor field which canonically comes with the SODE, namely the Jacobi endomorphism Φ. Let us first look at the situation where there are no further complications coming from the degeneracy (or multiplicity) of eigenvalues.

Theorem 2. *Suppose that Φ is (algebraically) diagonalizable with distinct (but possibly complex) eigenvalues. Assume further that Φ satisfies the conditions $C^{\mathrm{V}}_{\Phi} = 0$ and $[\nabla\Phi, \Phi] = 0$. Then, there exist coordinates with respect to which the second-order equations decouple into scalar equations (one for each real eigenvalue) and pairs of equations, not coupled with the rest, which are the real and imaginary parts of a single complex equation (one for each pair of complex conjugate eigenvalues).*

PROOF. Using the results obtained before, applied to $U = \Phi$, we already know that (54) holds in appropriate coordinates, where $\mathcal{D}_A$ is one-dimensional for real eigenvalues and $\mathcal{D}^r_A$ is two-dimensional for complex conjugate ones. We further know that the forces satisfy Cauchy-Riemann conditions with respect to the velocity variables inside each $\mathcal{D}^r_A$. Moreover, Φ is in real Jordan normal form. If, referring to the coordinate representation (5) of Φ^i_j, the upper index i refers to a line with a real eigenvalue, we thus have $\Phi^i_j = 0$, $\forall j \neq i$, which, knowing that also $\Gamma^i_j = 0\ \forall j \neq i$ in that case, implies that $\partial f^i/\partial x^j = 0$, i.e. f^i depends on (x^i, v^i) only.

For each of the invariant subspaces associated to complex eigenvalues, we further exploit the fact that according to (48) $[\Phi, J] = 0$ which, as discussed in the previous section with Eqns. (32-34), eventually implies Cauchy-Riemann conditions with respect to the position variables as well, while independence of all others still follows from the normal form of Φ as above. $\square$

Observe that it follows from the first of the relations in both (12) and (13) that for any two eigenvectors X_k of Φ, corresponding to (distinct) eigenvalues λ_k,

$$\begin{aligned} 3R(X_i, X_j) &= \mathrm{D}^{\scriptscriptstyle V}_{X_i}\Phi(X_j) - \mathrm{D}^{\scriptscriptstyle V}_{X_j}\Phi(X_i) \\ &= \mathrm{D}^{\scriptscriptstyle V}_{X_i}(\lambda_j X_j) - \mathrm{D}^{\scriptscriptstyle V}_{X_j}(\lambda_i X_i) - \Phi(\mathrm{D}^{\scriptscriptstyle V}_{X_i}X_j - \mathrm{D}^{\scriptscriptstyle V}_{X_j}X_i) \\ &= (\mathrm{D}^{\scriptscriptstyle V}_{X_i}\lambda_j)X_j - (\mathrm{D}^{\scriptscriptstyle V}_{X_j}\lambda_i)X_i, \end{aligned}$$

where the simplification in the last line comes from the fact that e.g. $\mathrm{D}^{\scriptscriptstyle V}_{X_i}X_j$ is proportional to X_j as a result of the assumption $C^{\scriptscriptstyle V}_{\Phi} = 0$. In the case of real eigenvalues, still under the assumptions of Theorem 2, we know that the λ_i, in the coordinates which diagonalize Φ, will depend on the variables (x^i, v^i) only. It follows that $\mathrm{D}^{\scriptscriptstyle V}_{X_i}\lambda_j = 0$ for $i \neq j$. For complex eigenvalues it may not be so obvious at first sight that the curvature will still be zero inside each block of (real) dimension 2 determined by sp $\{V_i, W_i\}$. However, the above computation remains formally valid in a complex representation, with $\lambda_j = \bar{\lambda}_i$ say. The conclusion coming from Theorem 2 is that in suitable complex coordinates we will have $X_i = \partial/\partial z^i$ and λ_i will not depend on $\bar{z}^i$, so that R is still zero. The 'real version' of this argument (which is easy to verify explicitly) is that the curvature of a single complex SODE

$$\begin{aligned} \ddot{x} &= f(x, y, \dot{x}, \dot{y}) \\ \ddot{y} &= g(x, y, \dot{x}, \dot{y}) \end{aligned}$$

is zero as a result of the Cauchy-Riemann properties (33-34) satisfied by f and g. We therefore draw the following important conclusion.

Corollary. *Under the assumptions of Theorem 2, the curvature R of the connection determined by the* SODE Γ *is zero.* □

Allowing for degeneracy in the eigenvalues, a somewhat stronger condition will be needed to ensure further separability in each block. For an intermediate result then, we go back to the general discussion on type (1,1) tensor fields U and identify, as in [8], conditions which will ensure that U really projects onto all distributions $\mathcal{D}_A$ and $\mathcal{D}^r_A$. Such a U is said to be *separable*.

Theorem 3. *Let $U \in V^1(\tau)$ be diagonalizable and satisfy the conditions: (i) $C^{\scriptscriptstyle V}_U = 0$, (ii) $[\nabla U, U] = 0$, (iii) $d^{\scriptscriptstyle V}U = 0$, (iv) $d^{\scriptscriptstyle H}U = 0$. Then U is separable and the eigenvalues of multiplicity greater than one moreover are constant.*

Proof. Let $X \in \mathcal{D}_A$ and $Y \in \mathcal{D}_B$ be any two eigenvectors of U (real or complex). Computing $d^{\scriptscriptstyle V}U(X,Y)$ as in (12), we obtain, using the results of Lemma 3 for the case that D is $\mathrm{D}^{\scriptscriptstyle V}_X$ or $\mathrm{D}^{\scriptscriptstyle V}_Y$:

$$d^{\scriptscriptstyle V}U(X,Y) = (\mathrm{D}^{\scriptscriptstyle V}_X\mu_B)\,Y - (\mathrm{D}^{\scriptscriptstyle V}_Y\mu_A)\,X\,.$$

Since X and Y are linearly independent, the vanishing of this expression leads to the following conclusions. Firstly, we have $\mathrm{D}^{\scriptscriptstyle V}_X\mu_B = 0$ for all $X \in \mathcal{D}_A$ with $A \neq B$. Secondly, when $A = B$, i.e. when the dimension of $\mathcal{D}_A$ is at least 2 (in the sense of the complexified tangent space when μ_A is complex), then in fact $\mathrm{D}^{\scriptscriptstyle V}_X\mu_A = 0$ for all $X \in \mathcal{X}(\tau)$. Condition (iv) gives rise to exactly the same conclusions with $\mathrm{D}^{\scriptscriptstyle H}$-derivatives. It follows therefore that the only, possibly non-constant eigenvalues are the non-degenerate ones and that in the coordinates which bring U in its normal form, these eigenvalues will depend on the coordinates and velocities associated to the corresponding distribution only (i.e. U is separable). $\square$

IMPORTANT REMARKS. We have again formally left these considerations, which are of an algebraic nature only, in a (potentially) complex set-up. So, strictly speaking, for a complex eigenvector Z_α as in (41), $\mathrm{D}^{V}_{Z_\alpha}$ should be understood as $\mathrm{D}^{V}_{V_\alpha} + i\,\mathrm{D}^{V}_{W_\alpha}$. Complex eigenvalues λ_A likewise should be thought of as being represented as in (41). For a real eigenvalue μ_A acted upon by a complex $\mathrm{D}^{V}_{Z_{B\alpha}}$, we will then get both $\mathrm{D}^{V}_{V_{B\alpha}}\mu_A = 0$ and $\mathrm{D}^{V}_{W_{B\alpha}}\mu_A = 0$ (and likewise for D^{H}-derivatives). In the case of a complex eigenvalue $\lambda_A = \rho_A + i\sigma_A$ it is easy to verify that $\mathrm{D}^{V}_{Z_{B\beta}}\lambda_A = 0$ implies that all of the functions $\mathrm{D}^{V}_{V_{B\beta}}\rho_A$, $\mathrm{D}^{V}_{W_{B\beta}}\rho_A$, $\mathrm{D}^{V}_{V_{B\beta}}\sigma_A$ and $\mathrm{D}^{V}_{W_{B\beta}}\sigma_A$ will be zero. Hence, the same conclusions hold for the basic vector fields spanning the distributions $\mathcal{D}^{r}_{A}$ and for the $(\rho_\alpha, \sigma_\alpha)$ which determine λ_α as in (41). We enter into such detail here because the result will in fact be used in this form later on.

Observe also that this result is the analogue of Theorem 4.5 in [8] and that the weaker version in Theorem 4.4 of that paper, which characterizes separability of U only (without regard to the constancy of multiple eigenvalues), would equally apply in the present context. That weaker version will, however, not be needed in what follows.

In the case where Φ is the tensor field U under consideration, the two extra conditions (iii) and (iv), in view of (13), reduce to the vanishing of the curvature. We can safely impose such a restriction as part of a set of sufficient conditions for arriving at maximal decoupling, because we know by the corollary of Theorem 2 that it is also a necessary condition. We thus arrive at the following intermediate result.

Theorem 4. *If Φ is diagonalizable and is such that $C^{V}_{\Phi} = 0$, $[\nabla\Phi, \Phi] = 0$ and $R = 0$, then the differential equations split into a number of decoupled blocks, one for each real or pair of complex conjugate eigenvalues, which are constant when there is degeneracy; the blocks corresponding to complex eigenvalues give rise to complex second-order systems in the sense defined in Section 3.*

PROOF. Theorem 3 applies and ensures that Φ is separable in appropriate coordinates. The normal form structure of Φ in those coordinates further says that $\Phi^{A\alpha}_{B\beta} = 0$ for $A \neq B$, while we already saw in (54) that the connection coefficients have the same structure. It then follows again from the local structure (5) of Φ that also $\partial f^{A\alpha}/\partial x^{B\beta} = 0$. This provides a blockwise decoupling of the given SODE. Now the arguments about J used in the proof of Theorem 2 also remain valid, i.e. we have $\nabla J = 0$ and $[\Phi, J] = 0$. It follows from the general considerations of Section 3 that for each separate block coming from complex conjugate eigenvalues, Cauchy-Riemann conditions of type (33-34) will be satisfied, meaning that the projected even dimensional SODE for each such block will be complex. $\square$

If we want the maximal decoupling of equations, i.e. further decoupling inside each block coming from multiple eigenvalues, it remains to investigate separately the cases where Φ has one of the following two structures: $\Phi = \mu\, I$ or $\Phi = \rho\, I - \sigma\, J$, with μ, ρ and σ real constants. Evidently, for such systems, no information about decoupling can come from

Φ itself. Fortunately, there is another tensor field available which is the *tension* field $\mathbf{t}$ along τ, whose components are given by

$$\mathbf{t}^i_j = \Gamma^i_j - v^k \frac{\partial \Gamma^i_j}{\partial v^k}. \tag{55}$$

It can be defined intrinsically from the canonical element $\mathbf{T}$ of $\mathcal{X}(\tau)$, as

$$\mathbf{t} = -d^H \mathbf{T}. \tag{56}$$

The tension satisfies the identity $d^V \mathbf{t} = 0$. In both of the cases to be investigated, Φ is basic and consequently $R = 0$. This is all we needed in Lemma 5.4 of [8] to conclude that $\nabla \mathbf{t} = 0$. It then follows that also $d^H \mathbf{t} = 0$. Therefore, if we assume that $\mathbf{t}$ is diagonalizable (of course allowing for complex eigenvalues again) and that $C^V_{\mathbf{t}} = 0$, the tension satisfies all assumptions of Theorem 3; we then conclude that $\mathbf{t}$ is separable and that its multiple eigenvalues are constant. The coordinate transformation involved in separating $\mathbf{t}$ will have no effect on a Φ which is a multiple of the identity, so let's deal with this situation first. As before, it follows from the general conclusions of Theorem 1 that the connection components have the property (54), A and B referring of course to different eigendistributions of $\mathbf{t}$ now. The form of Φ then further implies that the 'force functions' $f^{A\alpha}$ depend on the variables $(x^{A\alpha}, v^{A\alpha})$ only. If $\mathbf{t}$ has no multiple eigenvalues we are finished and reach a conclusion similar to that of Theorem 2. In the opposite case, we are reduced to analysing each subsystem corresponding to an invariant subspace with multiple eigenvalues of $\mathbf{t}$ separately. The only new situation to look at here (since the case that also $\mathbf{t}$ is a multiple of the identity was already analysed in [8]) is the case of multiple complex eigenvalues of $\mathbf{t}$. In other words, we still have to investigate separately systems for which $\Phi = \mu I$ and $\mathbf{t}$ is of the form $\mathbf{t} = \alpha I - \beta J$, with α and β constant (choosing of course to write $\mathbf{t}$ in a basis such as (52), rather than labelling the vectors in the order (53)). We postpone this analysis for a moment.

When Φ is of the form $\rho I - \sigma J$, more care is needed in selecting coordinates which will separate $\mathbf{t}$. We must not forget that we started from the assumption that $\nabla\Phi$ commutes with Φ, which implies because of Lemma 4 that $\nabla J = 0$ (and thus also that $\nabla \Phi = 0$ here). As a result, the connection components have the property (31) (and the system which we start from is actually a complex one). It follows from the explicit form (55) of the tension components that $[\mathbf{t}, J] = 0$ (and therefore also, because of the special form of Φ, that $[\mathbf{t}, \Phi] = 0$). The fact that $\mathbf{t}$ commutes with J has several consequences. Firstly, as soon as we assume that $\mathbf{t}$ is diagonalizable, we can be sure that the diagonalization can be achieved by a similarity transformation preserving J. Let

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

be the matrix of such a transformation. In all generality, for a complex system of the form

$$\ddot{x}^i = f^i, \qquad \ddot{y}^i = g^i, \tag{57}$$

with the right-hand sides satisfying (33) and (34), if we pass to the complex representation

$$\ddot{z}^i = \mathcal{F}^i = f^i + i\, g^i\,, \tag{58}$$

one can easily verify the following properties. With

$$\hat{\Gamma}^i_j = -\tfrac{1}{2}\frac{\partial \mathcal{F}^i}{\partial \dot{z}^j}, \quad \text{and} \quad \hat{\mathbf{t}}^i_j = \hat{\Gamma}^i_j - \dot{z}^k \frac{\partial \hat{\Gamma}^i_j}{\partial \dot{z}^k}\,, \tag{59}$$

we have

$$\left(\hat{\Gamma}^i_j\right) = {}^1\Gamma_1 + i\,{}^2\Gamma_1, \qquad \hat{\mathbf{t}} = {}^1\mathbf{t}_1 + i\,{}^2\mathbf{t}_1\,. \tag{60}$$

Returning to the present situation, one further verifies that $A + iB$ will diagonalize $\hat{\mathbf{t}}$. Of course $C^V_{\mathbf{t}} = 0$ will imply $C^V_{\hat{\mathbf{t}}} = 0$ as well. Likewise, with

$$\hat{\nabla}\frac{\partial}{\partial z^k} = \hat{\Gamma}^l_k \frac{\partial}{\partial z^l}\,, \tag{61}$$

we will have $\hat{\nabla}\hat{\mathbf{t}} = 0$ and so on. Therefore, we can formally apply the results of Theorem 3 to the system $\ddot{z}^i = \mathcal{F}^i$ and its corresponding tension field $\hat{\mathbf{t}}$, forgetting as it were that the z^i are complex variables. In this way, we are guaranteed the existence of a change of variables $z = z(z')$ which will separate $\hat{\mathbf{t}}$ and this in turn implies in the real representation that there is a transformation $(x, y) \leftrightarrow (x', y')$ which separates $\mathbf{t}$ and preserves J.

Another consequence of $[\mathbf{t}, J] = 0$ is that if $\mathbf{t}$ has real eigenvalues, their degeneracy will be even. In the variables (x', y') which separate $\mathbf{t}$, we know as before that $\Gamma^{A\alpha}_{B\beta} = 0$, A and B referring to different eigendistributions of $\mathbf{t}$. Even in the case of real eigenvalues, their even multiplicity will imply compatibility with the given block structure of Φ in the sense that it again will follow that $\Phi^{A\alpha}_{B\beta} = 0$. As a result, we have block decoupling of the complex system that we started from and all eigenvalues of $\mathbf{t}$ with multiplicity greater than one (thus in particular the real ones) are constant. Whether there will be further (full) decoupling in each block coming from multiple eigenvalues of $\mathbf{t}$ remains to be analysed.

The remaining situations to be looked at now can be dealt with simultaneously thanks to Lemma 1 of the preceding section. The list of remaining cases reads as follows:

1. Assume we have a SODE such that $\Phi = \mu\, I$ and $\mathbf{t} = \alpha\, I - \beta\, J$, with $\mu, \alpha, \beta \in \mathbb{R}$. We may further assume without loss of generality that the almost complex structure J is integrable. Indeed, in the complete picture, this will follow by Lemma 5 from the original assumptions on $\mathbf{t}$, namely diagonalizability plus $C^V_{\mathbf{t}} = 0$, knowing further that $\nabla\mathbf{t} = 0$. The form of Φ further ensures that $R = 0$. Trivially, we also have $[\Phi, J] = 0$, while $\nabla\mathbf{t} = 0$ implies $\nabla J = 0$, so that the SODE certainly is complex.

2. Assume we have $\Phi = \rho\, I - \sigma\, J$ and also $\mathbf{t} = \alpha\, I - \beta\, J$, with $\rho, \sigma, \alpha, \beta \in \mathbb{R}$ and allowing possibly for β to be zero. Here, we can further suppose that $\nabla J = 0$ and $\mathrm{D}^V_X J = 0$, since this will follow in the complete picture from the original assumption that $[\nabla\Phi, \Phi] = 0$ and $C^V_{\Phi} = 0$. Again, we are then looking at a complex SODE from the start, with the further properties that $R = 0$ and that the tension is basic.

As argued in [8], if the tension is basic and assumed to be smooth on the zero section, the connection will be affine. Explicitly, we will have

$$\Gamma^i_j(x,v) = \mathbf{t}^i_j + \Gamma^i_{jk}(x)v^k, \tag{62}$$

where the functions $\Gamma^i_{jk}(x)$ define a symmetric linear connection. For both cases, since $R=0$, the Riemann curvature tensor of this linear connection will also vanish. But we are looking here at a linear connection which can be associated to a complex quadratic spray. In view of Lemma 1, therefore, we can find 'flat coordinates', with respect to which we will have $\Gamma^i_j = \mathbf{t}^i_j$, and in such a way that J is preserved, and consequently also Φ and $\mathbf{t}$. With these data, the forces now can be uniquely determined in the new variables. Labelling these variables as (x^i, y^i) and the forces as (f^i, g^i) as in Section 3, the equations $\Gamma^i_j = \mathbf{t}^i_j$ can be integrated explicitly to obtain the velocity dependence of the forces, whereas the arbitrary functions of position variables, obtained in this process, subsequently get fixed as a result of the condition $\Phi = \mu\, I$ or $\Phi = \rho\, I - \sigma\, J$ depending on the case at hand. One easily obtains the following results:

$$f^i = -2\alpha\,\dot{x}^i - 2\beta\,\dot{y}^i + (\beta^2 - \alpha^2 - \rho)x^i + (\sigma - 2\alpha\beta)y^i, \tag{63}$$

$$g^i = 2\beta\,\dot{x}^i - 2\alpha\,\dot{y}^i - (\sigma - 2\alpha\beta)x^i + (\beta^2 - \alpha^2 - \rho)y^i. \tag{64}$$

It is manifestly obvious that the equations then are decoupled into pairs of complex equations in each (x^i, y^i) plane. The above expressions directly cover the second case enumerated before, including the case that $\mathbf{t}$ is diagonal (by putting $\beta = 0$). They also cover the first case, however, for which it suffices to put $\sigma = 0$ and $\rho = \mu$. Note that if we put $\beta = \sigma = 0$, the result is fully consistent with the real situation treated in [8].

Let us summarize the results of the preceding discussion as follows.

Theorem 5. *Let the* SODE Γ *be such that either (i)* $\Phi = \mu I$, $\mu \in \mathbb{R}$, *or (ii)* $\Phi = \rho\, I - \sigma\, J$, $\rho, \sigma \in \mathbb{R}$, *where* J *is an almost complex structure satisfying* $\nabla J = 0$ *and* $\mathrm{D}^{\mathrm{V}}_X J = 0$, $\forall X \in \mathcal{X}(\tau)$. *Assume that* $\mathbf{t}$ *is diagonalizable and satisfies* $C^{\mathrm{V}}_{\mathbf{t}} = 0$. *Then the equations completely decouple. For case (i) one obtains a number of individual equations (as many as the sum of the dimensions of the real eigenspaces of* $\mathbf{t}$*) and a number of pairs of complex equations (as many pairs as half of the total dimension of the complex eigenspaces of* $\mathbf{t}$*). For case (ii) where the dimension of the base manifold is necessarily even, the complete decoupling is into pairs of complex equations.* □

Putting the partial results of Theorems 4 and 5 all together now, we obtain a set of sufficient conditions for an arbitrary SODE with a diagonalizable Φ (real or complex eigenvalues) to be maximally separable. However, for any given SODE which is given in such a maximally separated way, i.e. is the union of a number of individual equations and a number of pairs of complex equations, it is easy to verify that all those conditions will hold true, so that they are also necessary. So, we reach the following main conclusion.

Theorem 6. *Let* Γ *be a* SODE *with Jacobi endomorphism* Φ *and tension field* $\mathbf{t}$, *then a set of necessary and sufficient conditions for the existence of a coordinate transformation which will maximally decouple the equations into a number of individual equations and/or a number of pairs of complex equations, is given by:* Φ *is diagonalizable,* $C^{\mathrm{V}}_{\Phi} = 0$, $[\nabla\Phi, \Phi] = 0$, $R = 0$, $\mathbf{t}$ *is diagonalizable and* $C^{\mathrm{V}}_{\mathbf{t}} = 0$. □

5 Illustrative examples

The tests for maximal decoupling of a given system of second-order differential equations, as developed in the previous section, are all algebraic. We will illustrate here how they can be used in practice. One of the main points to observe is the following: diagonalizability of Φ (or the tension) may be the hardest test to implement; therefore, although it is the first assumption in building up the theory, it is better in practice to leave it as the last condition to test. Typically one will start from a second-order system which contains a number of free parameters (or even functions which are as yet to be determined). One will then first impose the conditions $R = 0$, $C^{\mathrm{v}}_{\Phi} = 0$ and $[\nabla\Phi, \Phi] = 0$. For functions $f^i(x, v)$ which depend polynomially on the velocities, such conditions will often give rise to a large number of restrictions, coming from the coefficients of independent monomials and the diagonalizability test can be postponed until all restrictions coming from the other requirements have been implemented. If Φ turns out to be diagonalizable with distinct eigenvalues, we can rely on Theorem 2; if some of the eigenvalues have multiplicity greater than one, we have to do some more work related to the conditions on the tension $\mathbf{t}$ in Theorem 6. Needless to say, all such calculations, as simple as they may be in principle, are almost impossible to carry out by hand, so that one will seek assistance from computer algebra packages. We have made extensive use of Reduce in doing the computations for the examples below.

Quite a few examples of testing separability have already been given in previous work (see [8, 11]). We limit ourselves here to situations where the new features of the present work occur, i.e. cases with complex eigenvalues. For convenience, variables are labeled by lower indices in the examples.

Example 1. Consider the system

$$\begin{aligned} \ddot{x}_1 &= -x_1 + x_2 + b_1\dot{x}_1 \\ \ddot{x}_2 &= -x_2 - 4x_1 + b_2\dot{x}_2 \end{aligned}$$

where the b_i are constants. We have $R = 0$ and $C^{\mathrm{v}}_{\Phi} = 0$, whereas $[\nabla\Phi, \Phi] = 0 \Leftrightarrow b_1 = b_2$. Φ then has eigenvalues $1 - \frac{1}{4}b_1^2 \pm 2i$ and one easily finds from the eigenvectors that bringing the system into complex form is a simple matter of rescaling the first variable with a factor 2 here. The complex representation of the resulting system (with $b = b_1 = b_2$) then reads

$$\ddot{z} = -(1 + 2i)z + b\dot{z}\,.$$

Example 2. For a system of the form

$$\begin{aligned} \ddot{x}_1 &= -a_1x_1 + b_1x_1x_2 + \dot{x}_2 \\ \ddot{x}_2 &= -a_2x_2 + b_2x_2^3 - 4\dot{x}_1 \end{aligned}$$

it is again $[\nabla\Phi, \Phi] = 0$ which imposes restrictions, namely $b_1 = b_2 = 0$ and $a_1 = a_2 = a$ say. We then have $\Phi = (1 + a)I$ so that the tension has to be invoked to investigate potentially further decoupling. It turns out that $\mathbf{t}$ has eigenvalues $\pm i$ and that one possible choice of an eigenvector for example is $\mathrm{col}\,(-\frac{1}{2}, i)$. This in turn means that multiplication of x_1

by -2, while leaving x_2 unchanged, is a transformation which will bring $\mathbf{t}$ into its Jordan normal form and will accordingly produce a complex system. In complex representation one obtains the equation

$$\ddot{z} = az + 2i\dot{z}\,.$$

EXAMPLE 3. Starting from the system

$$\begin{aligned}\ddot{x}_1 &= a_1 x_1 x_2^2 + b_1 x_1^3 \\ \ddot{x}_2 &= a_2 x_1^2 x_2 + b_2 x_2^3\end{aligned}$$

and excluding the trivially decoupled case $a_1 = a_2 = 0$, we have $R = 0$ and $C^V_\Phi = 0$ again, while $[\nabla\Phi, \Phi] = 0$ requires $a_1 = 3b_2,\ a_2 = 3b_1$. Here, Φ is not constant and is found to be diagonalizable provided that $b_1 b_2 \neq 0$. If $b_1 b_2 > 0$, the eigenvalues are real and distinct so we should be able to achieve complete decoupling. Indeed, the transformation

$$\begin{aligned}\tilde{x}_1 &= -b_1\, x_1 + \sqrt{b_1 b_2}\, x_2 \\ \tilde{x}_2 &= \sqrt{b_1 b_2}\, x_1 + b_2\, x_2\end{aligned}$$

is found to result in the decoupled equations $\ddot{\tilde{x}}_i = \tilde{x}_i^3 / b_i$. If $b_1 b_2 < 0$, Φ has complex eigenvalues and the transformation

$$\begin{aligned}\tilde{x}_1 &= -b_1\, x_1 \\ \tilde{x}_2 &= \sqrt{-b_1 b_2}\, x_2\end{aligned}$$

will give rise to new equations which satisfy the Cauchy-Riemann conditions (33-34). In complex form, the resulting equation is

$$\ddot{z} = z^3 / b_1\,.$$

EXAMPLE 4. Leaving out the sort of preliminary analysis in which some parameters get fixed, another example which satisfies all requirements is given by

$$\begin{aligned}\ddot{x}_1 &= -18x_1^2 + 78x_1x_2 - 78x_2^2 \\ \ddot{x}_2 &= -15x_1^2 + 60x_1x_2 - 57x_2^2\,.\end{aligned}$$

Φ has eigenvalues $-12x_1 + 18x_2 \pm i(6x_1 - 12x_2)$ and one of the eigenvectors, for example, is given by $\mathrm{col}\,(13, 8-i)$. In such a case of constant eigenvectors, by the way, one immediately has the Jacobian of a linear coordinate transformation which will bring Φ into its Jordan form. With the above choice, the transformation in question is of the form $x = U\tilde{x}$, with

$$U = \begin{pmatrix} 13 & 0 \\ 8 & -1 \end{pmatrix}.$$

The resulting complex equation is given by

$$\ddot{z} = 3(2 - 3i)z^2\,.$$

Note in passing that eigenvectors are of course determined to within a factor only. Multiplying the original choice with $(2-3i)/13$, for example, the new choice $\mathrm{col}\,(2-3i, 1-2i)$ would give rise to the coordinate transformation

$$\begin{aligned} x_1 &= 2\tilde{x}_1 - 3\tilde{x}_2\,, \\ x_2 &= \tilde{x}_1 - 2\tilde{x}_2\,, \end{aligned}$$

having the effect of rescaling the complex equation to the form

$$\ddot{z} = 3z^2\,.$$

Example 5. For an example with both real and complex eigenvalues, consider the system

$$\begin{aligned} \ddot{x}_1 &= \dot{x}_1 + \dot{x}_2 - x_1 - \tfrac{1}{2}x_2, \\ \ddot{x}_2 &= \dot{x}_2 - 4\dot{x}_1 - x_2 + 2x_1, \\ \ddot{x}_3 &= \tfrac{1}{2}(\dot{x}_1 + \dot{x}_3)^2 + \dot{x}_3 - \dot{x}_2 - x_1 - 2x_3 + \tfrac{1}{2}x_2. \end{aligned}$$

All conditions on Φ are satisfied. Its eigenvalues are $7/4$ (with multiplicity 2) and $7/4 - x_1 - x_3$. A coordinate transformation which will diagonalize Φ consists in replacing x_3 by $x_1 + x_3$. Its effect is to replace the third equation by

$$\ddot{x}_3 = \tfrac{1}{2}\dot{x}_3^2 + \dot{x}_3 - 2x_3\,.$$

As predicted by the theory, there is partial splitting in the system so far. Continuing with the block of the first two equations for which $\Phi = (7/4)I$, we find that the tension has eigenvalues $\frac{1}{2} \pm i$ and its transformation to real Jordan normal form is achieved for example by multiplying x_1 by -2. The resulting complex equation is

$$\ddot{z} = -(1+i)z + (1+2i)\dot{z}\,.$$

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