Different forms of separability of second-order equations

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1 Introduction: a classical example

By way of introducing the subject of this contribution, consider the motion of a charged particle in a constant electromagnetic field, for which the governing force is the Lorentz force $\mathbf{F} = e\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. As is customarily done in most textbooks, we teach our students that a clever way to simplify the problem of integrating the equations of motion is to choose e.g. the z-axis of an inertial frame of reference along the constant magnetic field $\mathbf{B}$. This way, the differential equations to be solved are of the form

$$\begin{aligned} m\ddot{x} &= e\,E_x + e\,\dot{y}B, \\ m\ddot{y} &= e\,E_y - e\,\dot{x}B, \\ m\ddot{z} &= e\,E_z, \end{aligned}$$

and actually have two characteristic features (apart from being linear) which greatly simplify the problem. Indeed, first of all, the equation for z is decoupled from the rest of the system and secondly, the remaining two equations happen to combine into a single complex equation for $\zeta = x + iy$.

Suppose, however, that no special reference frame would be selected from the outset. The equations of motion then would be of the form

$$\begin{aligned} \ddot{q}_1 &= a_1 + b_3\dot{q}_2 - b_2\dot{q}_3, \\ \ddot{q}_2 &= a_2 + b_1\dot{q}_3 - b_3\dot{q}_1, \\ \ddot{q}_3 &= a_3 + b_2\dot{q}_1 - b_1\dot{q}_2, \end{aligned} \tag{1}$$

for some constants a_i, b_i. The question we want to address is the following: if a system of differential equations is given in a form such as (1), how would one figure out that special coordinates exist which produce decoupling of the equations in the above sense?

There are of course two aspects which a general approach to such a question must be able to tackle, if it wants to be of any practical value. Firstly, concerning the existence problem, the practical issue is whether we can test by means of procedures on the given data whether suitable coordinates exist. Secondly, if they exist, we should be able to figure out how to construct them.

We shall show in what follows that a quite general problem which encompasses the one introduced above, is that of complete decoupling of a general system of second-order ordinary differential equations. This problem was first addressed by Martínez *et al* in [1]. Its solution requires knowing about the differential geometry related to second-order equations. In particular, the tests for the existence of suitable decoupling coordinates turn out to be of an algebraic nature and involve certain intrinsically defined endomorphisms associated to the given system. Diagonalizability of the corresponding matrices is a key issue in the theory. In [1], where the focus was on complete decoupling into individual equations, we were led to assume that all eigenfunctions of the matrices under consideration were real. In recent work, in collaboration with Gerard Thompson [2], we have discovered how to cope also with the occurrence of complex eigenfunctions. The type of maximal decoupling which corresponds to such cases is exactly of the kind that we can see in the example of charged particle motion. Before presenting a brief survey of the theory here, we shall illustrate on this example what the practical implementation of the theory does.

2 The theory at work

The first important object to analyse in searching for maximal decoupling, is a type (1,1) tensor field Φ, called the *Jacobi endomorphism*. Its matrix representation, in the case of system (1), is given by

$$\Phi = \frac{1}{4}\begin{pmatrix} b_2^2+b_3^2 & -b_1b_2 & -b_1b_3 \\ -b_1b_2 & b_1^2+b_3^2 & -b_2b_3 \\ -b_1b_3 & -b_2b_3 & b_1^2+b_2^2 \end{pmatrix}$$

and happens to be symmetric. The test properties to be satisfied are that Φ must be algebraically diagonalizable (which is obviously the case here), and that some "differential concomitants" of Φ (see later) must vanish. Also these are easy to verify for the example at hand. The theory then guarantees simultaneous integrability of all eigendistributions of Φ, which in turn means that Φ is actually diagonalizable via a coordinate transformation.

Here, Φ has the double eigenvalue $\lambda_1 = \lambda_2 = \frac{1}{4}b^2$ (with $b^2 = \sum b_i^2$), and $\lambda_3 = 0$. An eigenvector for λ_3 is the vector $\mathbf{b}$; all vectors orthogonal to $\mathbf{b}$ are eigenvectors for $\lambda_1 = \lambda_2$. Let us choose, for example, a matrix of eigenvectors which are orthonormal as follows (with $l = \sqrt{b_1^2+b_2^2}$)

$$U = \frac{1}{bl}\begin{pmatrix} b_2b & b_1b_3 & b_1l \\ -b_1b & b_2b_3 & b_2l \\ 0 & -l^2 & b_3l \end{pmatrix} \quad \in SO(3).$$

The coordinate transformation $q = U\,Q$ diagonalizes Φ and in fact transforms the given system into

$$\ddot{Q}_1 \;=\; b\,\dot{Q}_2 + \frac{1}{l}(a_1b_2 - a_2b_1),$$

$$
\begin{aligned}
\ddot{Q}_2 &= -b\,\dot{Q}_1 + \frac{1}{bl}\left[(a_1 b_1 + a_2 b_2)b_3 - l^2 a_3\right], \qquad (2)\\
\ddot{Q}_3 &= \frac{1}{b}\sum a_i b_i,
\end{aligned}
$$

which is seen to have the desired features of decoupling!

The choice of this transformation matrix U, however, is slightly misleading! It is a coincidence here that Φ has properties which suggest to select eigenvectors in a special way and there is no reason to expect from our theory that passing to the Q-variables will achieve the maximal features of decoupling. Assume, for example, that we do not insist on normalizing the eigenvectors and make the following alternative choice of a transformation matrix:

$$
\tilde{U} = \begin{pmatrix} b_2 & b_1 b_3 & b_1 \\ -b_1 & b_2 b_3 & b_2 \\ 0 & -(b_1^2 + b_2^2) & b_3 \end{pmatrix}.
$$

Then, the transformation $q = \tilde{U}\,\tilde{Q}$ gives rise to the system

$$
\begin{aligned}
\ddot{\tilde{Q}}_1 &= b^2\,\dot{\tilde{Q}}_2 + \frac{1}{l^2}(a_1 b_2 - a_2 b_1), nonumber \qquad (3)\\
\ddot{\tilde{Q}}_2 &= -\dot{\tilde{Q}}_1 + \frac{1}{b^2 l^2}\left[(a_1 b_1 + a_2 b_2)b_3 - l^2 a_3\right], \qquad (4)\\
\ddot{\tilde{Q}}_3 &= \frac{1}{b^2}\sum a_i b_i,
\end{aligned}
$$

and the first two equations this time do not combine into a single complex equation! What then should be done next?

The answer can be found in the analysis of another tensor with intrinsic meaning: the so-called *tension field* $\mathbf{t}$. In the original q-coordinates, $\mathbf{t}$ has the matrix representation

$$
\mathbf{t} = \frac{1}{2}\begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix},
$$

whereas in the new variables $\tilde{Q}$, we have

$$
\tilde{\Phi} = \frac{1}{4}\begin{pmatrix} b^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \tilde{\mathbf{t}} = \frac{1}{2}\begin{pmatrix} 0 & -b^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

The problem is that the transition $q \to \tilde{Q}$ has not, in some sense, simultaneously diagonalized Φ and $\mathbf{t}$. Our theory in fact guarantees now that there exists another transformation which will bring $\mathbf{t}$ in real Jordan normal form. To construct it, we have to integrate the distributions spanned by the real and imaginary parts of the complex eigenvectors of $\mathbf{t}$. If we do so, we will be led to consider the further transformation $\tilde{Q}_1 = b\,\hat{Q}_2$, $\tilde{Q}_2 = \hat{Q}_1$. It has the effect of transforming equations (4) into the form

$$
\ddot{\hat{Q}}_1 = -b\,\dot{\hat{Q}}_2 + \frac{1}{b^2 l^2}\left[(a_1 b_1 + a_2 b_2)b_3 - l^2 a_3\right],
$$

$$\rightsquigarrow \quad \ddot{\hat{Q}}_2 = b\,\dot{\hat{Q}}_1 + \frac{1}{bl^2}(a_1b_2 - a_2b_1), \tag{5}$$
$$\ddot{\hat{Q}}_3 = \frac{1}{b^2}\sum a_i b_i,$$

which has all the desired characteristics even though (5) is not the same as our first lucky draw (2).

Remarks: $\mathbf{t}$ has to be looked at only when Φ has multiple eigenvalues. In this example, Φ has real eigenvalues and the complex nature of the final decoupling in fact comes from the complex eigenvalues of $\mathbf{t}$.

3 Sketch of the theory

A. Generalities

An autonomous system of second-order differential equations (SODE for short)

$$\ddot{q}^i = f^i(q, \dot{q}) \qquad i = 1, \ldots, n, \tag{6}$$

is governed by a vector field

$$\Gamma = v^i \frac{\partial}{\partial q^i} + f^i(q, v)\frac{\partial}{\partial v^i} \tag{7}$$

on the tangent bundle TM of a manifold M. It defines a horizontal distribution on TM, locally spanned by the vector fields

$$H_i = \frac{\partial}{\partial q^i} - \Gamma^j_i \frac{\partial}{\partial v^j}, \quad \text{where} \quad \Gamma^j_i = -\frac{1}{2}\frac{\partial f^j}{\partial v^i}. \tag{8}$$

This "non-linear connection" on $\tau : TM \to M$ in turn defines a linear connection on the pull-back bundle $\tau^* TM \to TM$. [The linear connection under consideration actually is one of "Berwald type" and Berwald-type connections are well known in the context of Finsler geometry.] Our SODE-Berwald-type connection essentially has three constituents: a horizontal and vertical covariant derivative and what we call the *dynamical covariant derivative* ∇. These are degree zero derivations of (vector-valued) forms along the projection $\tau : TM \to M$.

To fix some notations and basic formulas, we write $\mathcal{X}(\tau)$ for the set of vector fields along τ. Locally, an element $X \in \mathcal{X}(\tau)$ is of the form

$$X = X^i(q, v)\frac{\partial}{\partial q^i}. \tag{9}$$

Such an X can be horizontally and vertically lifted to vector fields X^H and X^V on TM, defined by

$$X^H = X^i\, H_i, \qquad X^V = X^i\, V_i = X^i\, \frac{\partial}{\partial v^i}. \tag{10}$$

A concise way to see the covariant derivative operators D^V_X, D^H_X and ∇ come into the picture is to look at the identities (see for example [1])

$$
\begin{aligned}
[X^H, Y^V] &= (\mathrm{D}^H_X Y)^V - (\mathrm{D}^V_Y X)^H & (11)\\
[\Gamma, X^H] &= (\nabla X)^H + \Phi(X)^V, & (12)
\end{aligned}
$$

which should be understood as follows. A vector field such as $[X^H, Y^V]$ on TM has a unique decomposition into a vertical and horizontal part, each of which is a corresponding lift of a certain vector field along τ. By looking at the way the latter are related to the original $X, Y \in \mathcal{X}(\tau)$, one discovers the action of the horizontal covariant derivative operators on the $C^\infty(TM)$-module $\mathcal{X}(\tau)$. In the same way, the second relation identifies the dynamical covariant derivative in its horizontal part, whereas its vertical part turns out to depend linearly on X and this way defines the *Jacobi endomorphism* Φ. In coordinates, with $F \in C^\infty(TM)$, we have

$$
\mathrm{D}^V_X(F) = X^V(F), \quad \mathrm{D}^H_X = X^H(F), \quad \nabla(F) = \Gamma(F), \tag{13}
$$

$$
\mathrm{D}^V_X \frac{\partial}{\partial q^i} = 0, \quad \mathrm{D}^H_X \frac{\partial}{\partial q^i} = X^V(\Gamma^j_i)\frac{\partial}{\partial q^j}, \quad \nabla \frac{\partial}{\partial q^i} = \Gamma^j_i \frac{\partial}{\partial q^j}, \tag{14}
$$

whereas the action on 1-forms along τ then follows by duality. The Jacobi endomorphism Φ, a type (1,1) tensor field along τ, has components

$$
\Phi^i_j = -\frac{\partial f^i}{\partial q^j} - \Gamma^i_k \Gamma^k_j - \Gamma(\Gamma^i_j). \tag{15}
$$

It is of interest to observe that Φ completely determines the curvature R of the non-linear connection. Indeed, we have:

$$
3\, R(X, Y) = \mathrm{D}^V_X \Phi(Y) - \mathrm{D}^V_Y \Phi(X) \Big(\;=\; d^V \Phi(X, Y) = \Big). \tag{16}
$$

The tension $\mathbf{t}$ is the type (1,1) tensor field along τ, with components

$$
\mathbf{t}^i_j = \Gamma^i_j - v^k \frac{\partial \Gamma^i_j}{\partial v^k}. \tag{17}
$$

One way of defining it intrinsically is as

$$
\mathbf{t} = -d^H \mathbf{T} \quad (\text{or } -d^V \nabla \mathbf{T}), \tag{18}
$$

where $\mathbf{T} = v^i \partial/\partial q^i$ is the canonical vector field along τ and the horizontal exterior derivative d^H can be defined in a way which is similar to the d^V in (16). The way the tensor fields Φ and $\mathbf{t}$ are introduced here, as type (1,1) tensor fields along τ, is taken from the study of derivations of forms along τ in [3, 4]. We refer to these papers for a more thorough discussion of their properties. Note, however, that coordinate expressions of the same tensor fields were obtained, through the method of equivalence, as early as in the work of Kosambi [5], Cartan [6] and Chern [7], to which other speakers in the same session are referring.

For a general type (1,1) tensor Ψ along τ, we define the 'concomitant' C^V_Ψ (a type (1,2) tensor field along τ) by

$$C^V_\Psi(X,Y) = [\mathrm{D}^V_X\Psi, \Psi](Y). \tag{19}$$

We are now ready to describe the main results about separability of SODEs.

B. Complete decoupling of a second-order system

For a good picture of the meaning of all assumptions involved in the main theorem about decoupling, we will present the path to the final result in three distinct stages.

($\mathbf{H_1}$) Assume that Φ is diagonalizable and satisfies the conditions $C^V_\Phi = 0$ and $[\nabla\Phi, \Phi] = 0$.

Theorem 1 Under the hypothesis (H_1), *if all eigenfunctions of* Φ *are distinct*, there exist coordinates with respect to which the equations decouple into scalar equations (one for each real eigenfunction) and pairs of equations which are the real and imaginary parts of a complex equation (one for each pair of complex conjugate eigenfunctions).

Note that the construction of appropriate separation coordinates boils down to integrating distributions which are Frobenius integrable. It is further worthwhile to observe that it follows from the assumptions in Theorem 1 that the curvature is zero.

When Φ has degenerate eigenvalues (real or complex), the situation becomes more complicated and we should start by imposing zero curvature as additional hypothesis:

($\mathbf{H_2}$) Assume that $R = 0$.

The immediate result from this extra assumption is that degenerate eigenvalues of Φ necessarily must be constant. Without any further assumptions at this stage, one can show that at least a form of partial decoupling will apply:

- for each real eigenvalue with multiplicity k, a k-dimensional subsystem will decouple from the rest in suitable variables;
- for each complex eigenvalue with multiplicity k, there will be a corresponding complex subsystem of dimension $2k$, which is not coupled with the rest.

If we then insist on having further decoupling inside each of these subsystems, it is quite clear that Φ will not have any further insight to offer, so that we have to turn to another source of information which is the tension!

($\mathbf{H_3}$) Assume $\mathbf{t}$ is diagonalizable and $C^V_{\mathbf{t}} = 0$.

Theorem 2 Under the assumptions (H_1), (H_2) and (H_3), there exist coordinates which will maximally decouple the given SODE into a number of individual equations and a number of pairs of complex equations.

The conditions for maximal decoupling are necessary and sufficient. Also, pairs of complex equations will occur as soon as either Φ or $\mathbf{t}$ have complex eigenvalues.

Before discussing some features of the proof of these results, it is worth mentioning the following intrinsic characterization of complex SODEs: *a second-order vector field* Γ *on the tangent bundle of an even dimensional manifold* M *is complex, if* M *admits an integrable almost complex structure* J*, such that*

$$\nabla J = 0, \quad \textit{and} \quad [\Phi, J] = 0. \tag{20}$$

In the case of pairs of complex eigenvalues $(\lambda, \overline{\lambda})$ of Φ, if we denote a basis of corresponding eigenvectors by $\{Z_A, \overline{Z}_A\}$, with $Z_A = V_A + iW_A$, let us agree that the term 'eigendistribution' in the enumeration which follows refers to sp $\{V_A, W_A\}$.

- The assumption $C^V_\Phi = 0$ entails that all eigendistributions of Φ are D^V-invariant, which means that they are spanned by vector fields on M.
- The ∇-invariance of the eigendistributions which follows from the commutation of Φ and $\nabla\Phi$ then further guarantees that the eigendistributions are simultaneously integrable.
- Putting $J(V_A) = W_A, \quad J(W_A) = -V_A$, we define an almost complex structure for each block coming from complex eigenfunctions and prove that J is integrable, i.e. that its Nijenhuis torsion N_J is zero. As a result, the Newlander-Nirenberg theorem applies and gives rise to holomorphic coordinates for each almost complex submanifold.
- A further consequence of the condition $[\nabla\Phi, \Phi] = 0$ is that the connection coefficients Γ^i_j, relating to different eigendistributions are zero. This leads to decoupling w.r.t. velocity coordinates between the distinct blocks and Cauchy-Riemann conditions w.r.t. velocity variables inside each block coming from complex eigenfunctions.
- When the eigenvalues are non-degenerate, the transformation of Φ to real Jordan form gives rise to decoupling w.r.t. position variables as well, plus the remaining Cauchy-Riemann conditions for the complex parts.
- Whenever an eigenvalue is degenerate, $R = 0$ guarantees that it is constant, so we are down to studying systems for which either $\Phi = \mu\, I$ or $\Phi = \alpha\, I - \beta\, J$ (with μ or α, β constant). At this point, the tension comes into play.
- The assumptions on $\mathbf{t}$ now lead to a similar integrability analysis for its 'eigendistributions'. This involves a number of extra technicalities when also $\mathbf{t}$ has degenerate eigenvalues, upon which we will not further dwell here.

4 Some further examples

Let us first return to the example of charged particle motion and put the theory to a final test now by ignoring all physical insight and routinely apply what the above described procedures dictate.

Starting from the original system (1), having obtained the eigenvalues of Φ, we now make the simplest possible choice of eigenvectors, leading to the transformation matrix

$$U = \begin{pmatrix} 0 & -b_3 & b_1 \\ b_3 & 0 & b_2 \\ -b_2 & b_1 & b_3 \end{pmatrix}.$$

The coordinate transformation $q = UQ$ produces the new system:

$$\begin{aligned} \ddot{Q}_1 &= -\frac{b_1 b_2}{b_3}\,\dot{Q}_1 + \frac{b_1^2+b_3^2}{b_3}\,\dot{Q}_2 + \cdots \\ \ddot{Q}_2 &= \frac{b_1 b_2}{b_3}\,\dot{Q}_2 - \frac{b_2^2+b_3^2}{b_3}\,\dot{Q}_1 + \cdots \\ \ddot{Q}_3 &= \frac{1}{b^2}\sum a_i b_i, \end{aligned} \tag{21}$$

where the dots refer to constant terms which are quite irrelevant for our discussion. We observe that everything the theory predicts holds true: the equation for the non-degenerate eigenvalue is decoupled from the rest, but there is no reason why anything further could be said about the first two equations. We observe that they are NOT the real and imaginary parts of a complex equation.

In the Q-variables, Φ becomes diagonal as it should, but

$$\mathbf{t} = \frac{1}{2b_3}\begin{pmatrix} b_1 b_2 & -(b_1^2+b_3^2) & 0 \\ b_2^2+b_3^2 & -b_1 b_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of $\mathbf{t}$ are $\pm\frac{1}{2}ib$. The real and imaginary parts of a set of corresponding basic eigenvectors dictate the further transformation $Q = \tilde{U}\,\tilde{Q}$, with

$$\tilde{U} = \begin{pmatrix} b_1^2+b_3^2 & 0 & 0 \\ b_1 b_2 & b_3 b & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The corresponding system in $\tilde{Q}$-variables now becomes:

$$\begin{aligned} \ddot{\tilde{Q}}_1 &= b\,\dot{\tilde{Q}}_2 + \frac{a_2(b_1^2+b_3^2) - b_2(a_1 b_1 + a_3 b_3)}{b_3 b^2 (b_1^2+b_3^2)}, \\ \ddot{\tilde{Q}}_2 &= -b\,\dot{\tilde{Q}}_1 + \frac{a_3 b_1 - a_1 b_3}{b_3 b (b_1^2+b_3^2)}, \\ \ddot{\tilde{Q}}_3 &= \frac{1}{b^2}\sum a_i b_i, \end{aligned} \tag{22}$$

which has all the features predicted by the theory indeed!

Let us conclude by giving also a couple of examples of non-linear systems, taken from [2]. Consider the system

$$\ddot{q}_1 = a_1\, q_1 {q_2}^2 + b_1\, {q_1}^3$$
$$\ddot{q}_2 = a_2\, {q_1}^2 q_2 + b_2\, {q_2}^3$$

Excluding the trivial case $a_1 = a_2 = 0$ (for which the system would be decoupled from the outset), we have $R = 0$ and $C^V_\Phi = 0$, while $[\nabla\Phi, \Phi] = 0$ requires that

$$a_1 = 3\, b_2, \qquad a_2 = 3\, b_1.$$

If $b_1 b_2 > 0$, Φ has real and distinct eigenvalues. The transformation

$$Q_1 = -b_1\, q_1 + \sqrt{b_1 b_2}\, q_2$$
$$Q_2 = \sqrt{b_1 b_2}\, q_1 + b_2\, q_2$$

reduces the system to

$$\ddot{Q}_1 = {Q_1}^3/b_1\,, \qquad \ddot{Q}_2 = {Q_2}^3/b_2\,.$$

If $b_1 b_2 < 0$, Φ has complex eigenvalues and the transformation

$$Q_1 = -b_1\, q_1$$
$$Q_2 = \sqrt{-b_1 b_2}\, q_2$$

will result in the complex equation

$$\ddot{z} = z^3/b_1, \qquad \text{with} \quad z = Q_1 + iQ_2.$$

Finally, consider the system

$$\begin{aligned}
\ddot{q}_1 &= \dot{q}_1 + \dot{q}_2 - q_1 - \tfrac{1}{2}q_2,\\
\ddot{q}_2 &= \dot{q}_2 - 4\dot{q}_1 - q_2 + 2q_1,\\
\ddot{q}_3 &= \tfrac{1}{2}(\dot{q}_1 + \dot{q}_3)^2 + \dot{q}_3 - \dot{q}_2 - q_1 - 2q_3 + \tfrac{1}{2}q_2.
\end{aligned}$$

Certain numerical factors in these equations are chosen already in such a way that all conditions on Φ are satisfied. The eigenvalues of Φ are: $7/4$ (with multiplicity 2) and $7/4 - q_1 - q_3$. Replacing q_3 by $q_1 + q_3$ diagonalizes Φ and transforms the third equation into

$$\ddot{q}_3 = \tfrac{1}{2}\dot{q}_3^2 + \dot{q}_3 - 2q_3\,.$$

Again, as the theory predicts, there is a partial decoupling so far.

For the subsystem of the first two equations, we now have $\Phi = (7/4)I$. The tension has eigenvalues $\frac{1}{2} \pm i$, and its transformation to real Jordan normal form is achieved simply by multiplying q_1 by -2.

It is easy to verify — but making use of computer algebra procedures is highly recommended for such calculations — that the resulting complex equation now reads:

$$\ddot{z} = -(1+i)z + (1+2i)\dot{z}\,.$$

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