On the geometry of Randers manifolds

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Abstract. Randers manifolds are studied in the framework of the pullback bundle formalism, with the aid of intrinsic methods only. After checking a sufficient condition for a Randers manifold to be a Finsler manifold, we provide a systematic description of the Riemann-Finsler metric, the canonical spray, the Barthel endomorphism, the Berwald connection, the Cartan tensors and the Cartan vector field in this new setting. Finally, as an application of the new tools and geometric ideas developed here, we present an intrinsic proof of the celebrated theorem about a criterion for a Randers manifold to become a Berwald manifold.

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1 Introduction

As their name suggests, Randers metrics were introduced by G. Randers in 1941, and named after him for the first time by R. S. Ingarden. The original interest for Randers manifolds came from physics: in optics, Randers metrics were found to describe the motion of a relativistic electron, but also in other physical areas (see e.g. [1, 2]) many applications followed. Not only physicists, but also pure geometers started to show interest in the subject, because Randers manifolds supply one of the most basic examples of Finsler manifolds: by adding a 1-form, their fundamental function perturbs the fundamental function of a Riemann manifold. A lot of invariants in Finsler geometry were explicitly calculated for the first time for Randers manifolds. Randers metrics were seen in a more general class of metrics which emerged in the study of what are now called (α, β)-metrics. For a general survey of results and applications of Randers manifolds, we refer to [2] and [12].

Since their first introduction, Randers manifolds have been studied intensively and the theory has already reached a significant level of development. Important textbooks which contain sections about Randers manifolds are [1, 2]. Among the many papers on the subject we cite only a few [8, 12, 14, 15, 16] which have a direct bearing on the subject matter of this paper. What these works have in common, and in fact what is common to the bulk of the literature on Finsler geometry, is that the analysis is almost entirely based on computations in local coordinates. There is no doubt that classical tensor calculus still is a very important tool for discovering and proving intrinsic features in most fields of applied differential geometry. It is our believe, however, that it is of interest also to develop purely coordinate-free methods in such fields. Quite often, the more abstract approach reveals much better the geometric structures which are at work and thus paves the way to learning from these structures in related theories or applications. For example, coordinate-free intrinsic methods are indispensable tools to obtain classification theorems, see e.g. Szabó's results on the description of positive definite Berwald manifolds in [17].

More directly related to the results we want to present, we can cite important work by Grifone [6, 7] and by Crampin [3], whose intrinsic methods for describing the geometry of a tangent bundle have had a great influence on many subsequent developments. For the former, this is true also concerning its contribution to Finsler geometry and recent developments which have been inspired by it are, for example, [18, 19, 20, 21, 22]. Also our present analysis is essentially based on the techniques of tangent bundle geometry. But we add an additional feature to it: it has been observed in the past that for many important geometrical aspects, the vector and tensor fields of interest are vertical vector valued or, equivalently, can be identified with tensor fields along the tangent bundle projection $\pi : TM \to M$. So, working with sections of the pullback bundle π^*TM rather than sections of $TTM \to TM$ can avoid unnecessary duplication of formulae and this is the line of approach we will follow here (see e.g. [4, 9, 10, 13] for earlier work in this direction).

The main novelty of our contribution lies in the purely intrinsic character of the methods and results. The paper is organised as follows. In the second section, we briefly review some elementary concepts of Finsler geometry, translated into the pullback bundle language. After an introduction of the basic tools of a Randers manifold in Section 3, we investigate the condition under which a Randers manifold is a Finsler manifold in Section 4. The fifth section establishes the essential relation between the spray of a Randers manifold and the spray of its underlying Riemannian structure. We further present a formulation of the Barthel endomorphism and the Berwald connection and list the Cartan tensors. The strength of our coordinate-free calculus becomes evident if one looks at the compact global form of the expressions in the propositions 3, 4, 5, 6 and 9 (that are listed in this context for the first time). Indeed, Bao, Chern and Shen admit in [2] (at the end of section 11.4) that they are not able to write down an explicit formulation, because of the complexity of the involved coordinate calculations. An important theorem on Randers manifolds is discussed in Section 6: it gives a necessary and sufficient condition for a Randers manifold to be of Berwald type. Although our coordinate free proof is admittedly longer and maybe even somewhat more complicated than the coordinate version of [2], an interesting point here is hat we use an entirely different argumentation than in the coordinate proofs.

2 Finsler manifolds

In this section, we (very briefly) recall a few basic concepts of Finsler geometry that we shall need later on. This background material only intends to fix some notational conventions and is meant to make the paper more self-contained paper.

Throughout this paper, we work on a n-dimensional ($n \geq 3$) smooth manifold M. The special case of Randers surfaces needs in many ways a different approach (see e.g. [8, 22]) and therefore we will exclude it here, although, of course, many results remain valid for any dimension. By $C^\infty(M)$ and $\mathcal{X}(M)$ we denote respectively the ring of smooth functions and the $C^\infty(M)$-module of vector fields on M. $\tau_M := (TM, \pi, M)$ is the tangent bundle of M; $\overset{\circ}{T}M$ is the set of all non-zero tangent vectors. The pullback of τ_M by the projection π is a bundle over TM with target space $\pi^*TM = TM \times_M TM := \{(v, w) \in TM \times TM \mid \pi(v) = \pi(w)\}$. Sections of this bundle can be represented by smooth maps $\tilde{X} : TM \to TM$ such that $\pi \circ \tilde{X} = \pi$. The set of such sections is denoted by $\mathcal{X}(\pi)$ and elements of it will be called *vector fields along the projection* π. A special class of vector fields along the projection is formed by the sections of the form $\hat{X} := X \circ \pi$, where X is a vector field on M. For obvious reasons, they will be called *basic* vector fields along the projection and they can be used to obtain a *local* basis of $\mathcal{X}(\pi)$: if $(X_1, \ldots, X_n)$ is a local basis of $\mathcal{X}(M)$, then $(\hat{X}_1, \ldots, \hat{X}_n)$ is a local basis for $\mathcal{X}(\pi)$. This observation will simplify a great deal of calculations. Besides the class of basic vector fields along π, an important role will be played by *the canonical vector field along the projection* $\mathbf{T} : v \mapsto v$.

Observe the difference in notations (that we will consistently use from now on) for the concepts $X \in \mathcal{X}(M)$, $\hat{X} = X \circ \pi \in \mathcal{X}(\pi)$ and an arbitrary $\tilde{X} \in \mathcal{X}(\pi)$. We will adopt an analogous notation for *forms along the projection.* A 1-form along the projection is a smooth map $\tilde{\theta} : TM \to T^*M$ that satisfies the condition $\pi^* \circ \tilde{\theta} = \pi$ (π^* is the natural projection $T^*M \to M$). Given a 1-form θ on M, it is easy to obtain a 1-form $\hat{\theta}$ along the projection: $\hat{\theta} := \theta \circ \pi$. Without going into technical details, we can say that, in general, a *tensor field along the projection* will consist of a combination of ($C^\infty(TM)$-linear) tensorial products of 1-forms and vector fields along the projection.

The following short exact sequence, displays the canonical structures of the pullback bundle

$$0 \to \pi^*TM \xrightarrow{i} TTM \xrightarrow{j} \pi^*TM \to 0. \tag{1}$$

All the manifolds in (1) are fibred over TM. There is a corresponding short exact sequence for the module of sections of these bundles and we will use the same symbols for the corresponding maps, i.e., we write

$$0 \to \mathcal{X}(\pi) \xrightarrow{i} \mathcal{X}(TM) \xrightarrow{j} \mathcal{X}(\pi) \to 0. \tag{2}$$

To make a long story short: the map i is a $C^\infty(TM)$-linear extension of the vertical lift to general vector fields along the projection: $i(f\hat{X}) = fX^v$, for $f \in C^\infty(TM)$. The vector field $C := i \circ \mathbf{T}$ on TM is called the *Liouville vector field.* The map $j : z \in T_vTM \mapsto j(z) := (v, T\pi(z))$ can be used to define the (1,1) tensor field $J := i \circ j$ on TM, called the *vertical endomorphism.*

Let a function $E : TM \to \mathbb{R}$ be given.

Definition 1. *The pair (M, E) is said to be a Finsler manifold if the following conditions hold.*

(1) For any vector $v \in \overset{\circ}{T}M, E(v) > 0$; $E(0) = 0$,

(2) E is of class C^1 on TM and smooth on $\overset{\circ}{T}M$,

(3) $CE = 2E$, i.e, E is homogeneous of degree 2,

(4) the so-called fundamental form $\omega := d(dE \circ J)$ is symplectic.

Of course, d stands here for the exterior derivative on TM. The function E is the *energy* of the Finsler manifold; $L := \sqrt{2E}$ is its *fundamental function.* For every Finsler manifold, there exists a *canonical spray* S, defined by the relation $i_S\omega = -dE$. An interesting property is $S(L) = 0$. The canonical spray generates a *horizontal endomorphism* on M, given by

$$h = \frac{1}{2}(1_{\mathcal{X}(TM)} + [J, S]) \tag{3}$$

and called the *Barthel endomorphism* (the bracket [.,.] in (3) is the Frölicher-Nijenhuis bracket of vector forms). For further details concerning horizontal endomorphisms, especially the Barthel endomorphism see e.g. [18] and [19]). h determines canonically a splitting $\mathcal{H} : \pi^*TM \to TTM$ of (1) (and of (2), of course) such that $h = \mathcal{H} \circ j$. $X^h := \mathcal{H} \circ \hat{X}$ is the *horizontal lift* of the vector field $X \in \mathcal{X}(M)$. The Barthel endomorphism is *conservative* which means that $X^h(E) = 0$ for any vector field X on M. We denote by $\mathcal{V}$ the complementary left splitting to $\mathcal{H}$, i.e. the bundle map $\mathcal{V} : TTM \to \pi^*TM$ satisfying $Im\mathcal{H} = Ker\mathcal{V}$ and $\mathcal{V} \circ i = 1_{\pi^*TM}$.

The mapping $g : \mathcal{X}(\pi) \times \mathcal{X}(\pi) \to C^\infty(TM)$ with

$$g(\tilde{X}, \tilde{Y}) := \omega(i\tilde{X}, \mathcal{H}\tilde{Y})$$

is a well-defined, non-degenerate and symmetric (0,2)-tensor along the projection. It will be called the *Riemann-Finsler metric* of (M, E). For basic vector fields the definition implies

$$g(\hat{X}, \hat{Y}) = X^v(Y^vE).$$

The *angular metric* k is defined by the formula $k(\tilde{X}, \tilde{Y}) = Ld(dL \circ J)(\mathcal{H}\tilde{X}, i\tilde{Y})$. For basic vector fields this reduces to

$$k(\hat{X}, \hat{Y}) = -LY^v(X^vL).$$

A number of linear connections were constructed for Finsler manifolds. We will give their characterization on the pullback bundle, see e.g. [4]. The *Berwald connection* $(\overset{\circ}{D}, \mathcal{H})$ of the Finsler manifold (M, E) can be represented by the following two direct formulae:

$$\overset{\circ}{D}_{i\tilde{X}}\tilde{Y} = j[i\tilde{X}, \mathcal{H}\tilde{Y}] \qquad \text{and} \qquad \overset{\circ}{D}_{\mathcal{H}\tilde{X}}\tilde{Y} = \nu[\mathcal{H}\tilde{X}, i\tilde{Y}].$$

In particular, for basic vector fields

$$\overset{\circ}{D}_{X^v}\hat{Y} = 0 \qquad \text{and} \qquad \overset{\circ}{D}_{X^h}\hat{Y} = \nu[X^h, Y^v]. \tag{4}$$

The Berwald connection is a *normal* connection, which means that it satisfies

$$\overset{\circ}{D}_{X^v}\mathbf{T} = \hat{X} \qquad \text{and} \qquad \overset{\circ}{D}_{X^h}\mathbf{T} = 0.$$

Before introducing a second linear connection on the pullback bundle, we will give a definition of the very important *Cartan tensors*, which will make use of the Berwald connection and the Riemann-Finsler metric. Following [4], the *first* Cartan tensor $\mathcal{C}$ of the Finsler manifold (M, E) is defined by

$$g(\mathcal{C}(\hat{X}, \hat{Y}), \hat{Z}) = \big(\overset{\circ}{D}_{X^v} g\big)(\hat{Y}, \hat{Z}) \qquad\qquad (X, Y \in \mathcal{X}(M)).$$

Remark that, because of the local basis property, it is only necessary to define a tensor along the projection by its action on basic vector fields, since its action on arbitrary vector fields along π follows from the $C^\infty(TM)$-linear character of the tensor. In the following, we will use this property repeatedly. The first Cartan tensor is symmetric and $\mathcal{C}(\mathbf{T}, \tilde{X}) = 0$, for all $\tilde{X} \in \mathcal{X}(\pi)$. The tensor $\mathcal{C}_\flat(\hat{X}, \hat{Y}, \hat{Z}) := \big(\overset{\circ}{D}_{X^v} g\big)(\hat{Y}, \hat{Z}) = X^v(Y^v(Z^v E))$ is called the *lowered* Cartan tensor. The *second* Cartan tensor $\mathcal{C}'$ of the Finsler manifold (M, E) has an analogous definition:

$$g(\mathcal{C}'(\hat{X}, \hat{Y}), \hat{Z}) = \big(\overset{\circ}{D}_{X^h} g\big)(\hat{Y}, \hat{Z}), \tag{5}$$

and its relation to the first Cartan tensor is $\mathcal{C}' = -\overset{\circ}{D}_S \mathcal{C}$ (see e.g. [4]). Again, the second Cartan tensor is symmetric and $\mathcal{C}'(\mathbf{T}, \tilde{X}) = 0$.

We define the *Cartan connection* $(D, \mathcal{H})$ from the Berwald connection by making use of the Cartan tensors as soldering forms:

$$D_{i\tilde{X}}\tilde{Y} = \overset{\circ}{D}_{i\tilde{X}}\tilde{Y} + \frac{1}{2}\mathcal{C}(\tilde{X}, \tilde{Y}) \qquad \text{and} \qquad D_{\mathcal{H}\tilde{X}}\tilde{Y} = \overset{\circ}{D}_{\mathcal{H}\tilde{X}}\tilde{Y} + \frac{1}{2}\mathcal{C}'(\tilde{X}, \tilde{Y}).$$

Some authors put the factor $\frac{1}{2}$ already in the definition of the Cartan tensors. In comparing our results with the literature, one should keep this in mind. The Cartan connection is *metrical*, i.e. $Dg = 0$, and normal.

At the end of this paper, we will deal with *curvature* of the Berwald and the Cartan connections. The next three expressions define respectively the *vertical*, *mixed* and *horizontal* curvature of the Cartan connection:

$$
\begin{aligned}
\mathcal{Q}(\tilde{X},\tilde{Y},\tilde{Z}) &= D_{i\tilde{X}}D_{i\tilde{Y}}\tilde{Z} - D_{i\tilde{Y}}D_{i\tilde{X}}\tilde{Z} - D_{[i\tilde{X},i\tilde{Y}]}\tilde{Z} \\
\mathcal{P}(\tilde{X},\tilde{Y},\tilde{Z}) &= D_{\mathcal{H}\tilde{X}}D_{i\tilde{Y}}\tilde{Z} - D_{i\tilde{Y}}D_{\mathcal{H}\tilde{X}}\tilde{Z} - D_{[\mathcal{H}\tilde{X},i\tilde{Y}]}\tilde{Z} \\
\mathcal{R}(\tilde{X},\tilde{Y},\tilde{Z}) &= D_{\mathcal{H}\tilde{X}}D_{\mathcal{H}\tilde{Y}}\tilde{Z} - D_{\mathcal{H}\tilde{Y}}D_{\mathcal{H}\tilde{X}}\tilde{Z} - D_{[\mathcal{H}\tilde{X},\mathcal{H}\tilde{Y}]}\tilde{Z}
\end{aligned}
$$

If $\overset{\circ}{\mathcal{Q}}, \overset{\circ}{\mathcal{P}}$ and $\overset{\circ}{\mathcal{R}}$ denote the corresponding curvatures of the Berwald connection, it follows from (4) that $\overset{\circ}{\mathcal{Q}} = 0$.

3 Randers Manifolds

We start this section with an important class of Finsler manifolds. Let α be a Riemannian metric on M (i.e. a positive definite, symmetric $(0,2)$-tensor on M). As mentioned above, $\hat{\alpha} = \alpha \circ \pi$ will be a (positive definite, symmetric) $(0,2)$-tensor along the projection. As in [9], we can define a 1-form $\bar{\alpha}$ along the projection: $\bar{\alpha} : TM \to T^*M; v \mapsto \bar{\alpha}(v)$, by means of

$$\bar{\alpha}(v)(w) := \hat{\alpha}_v(v,w) = \alpha_{\pi(v)}(v,w),$$

for every $w \in TM$. This form can also be interpreted as a map $\bar{\alpha} : \mathcal{X}(\pi) \to C^\infty(TM)$; $\tilde{X} \mapsto \bar{\alpha}(\tilde{X}) := \hat{\alpha}(\mathbf{T},\tilde{X})$. The next step is to consider the (square of the) function $L_\alpha^2 := \bar{\alpha}(\mathbf{T}) = \hat{\alpha}(\mathbf{T},\mathbf{T})$ on TM. Pointwise its definition means that for every vector $v \in TM$

$$L_\alpha^2(v) = \hat{\alpha}_v(\mathbf{T}(v),\mathbf{T}(v)) = \alpha_{\pi(v)}(v,v).$$

As α is a Riemannian metric, it will give rise to a metrical, torsion-free connection ∇ on M, the Levi-Civita connection.

Let $E := \frac{1}{2}L_\alpha^2$. The pair (M, E_α) is called a *Riemannian Finsler manifold* and its canonical spray S_α and Barthel endomorphism h_α are supposed to be well-known. The Riemann-Finsler metric of (M, E_α) is nothing but $\hat{\alpha}$ and its two Cartan tensors vanish, which implies that the Berwald and Cartan connections will be identical; let us denote them commonly by $\overset{\alpha}{D}$. The relation with the Levi-Civita connection ∇ of the Riemannian manifold (M,α) is

$$\overset{\alpha}{D}_{X^{h_\alpha}}\hat{Y} = \widehat{\nabla_X Y} = \nu[X^{h_\alpha}, Y^v] \qquad \text{and} \qquad \overset{\alpha}{D}_{X^v}\hat{Y} = 0,$$

so $\overset{\alpha}{D}$ is just the 'lift' of ∇ into $\pi^*TM \cong VTM$. In particular, $\overset{\alpha}{D}$ of (M, E_α) is metrical, i.e. $\overset{\alpha}{D}\hat{\alpha} = 0$ and normal. Besides the vertical curvature, the mixed curvature of $\overset{\alpha}{D}$ also vanishes: $\overset{\alpha}{\mathcal{Q}} = \overset{\alpha}{\mathcal{P}} = 0$.

The first lemma provides some immediate consequences of the above definitions.

Lemma 1. *For every vector fields X and Y on M, we obtain*

(1) $X^v L_\alpha^2 = 2\bar{\alpha}(\hat{X})$, (2) $X^v L_\alpha = \frac{1}{L_\alpha}\bar{\alpha}(\hat{X})$,
(3) $Y^v(X^v L_\alpha^2) = 2[\alpha(X,Y)]^v = 2\hat{\alpha}(\hat{X},\hat{Y})$, (4) $Y^v(X^v L_\alpha) = \frac{1}{L_\alpha}\hat{\alpha}(\hat{X},\hat{Y}) - \frac{1}{L_\alpha^3}\bar{\alpha}(\hat{X})\bar{\alpha}(\hat{Y})$.

Proof: We use the connection $\overset{\alpha}{D}$ to write $X^v L_\alpha^2 = X^v[\hat{\alpha}(\mathbf{T},\mathbf{T})] = (\overset{\alpha}{D}_{X^v}\hat{\alpha})(\mathbf{T},\mathbf{T}) + 2\hat{\alpha}(\mathbf{T},\overset{\alpha}{D}_{X^v}\mathbf{T})$. Then the first result follows from the fact that $\overset{\alpha}{D}$ is normal and metrical. The other properties can be obtained by analogous reasonings. □

Suppose now that, in addition to the Riemannian metric α on M, we have a non-zero 1-form β on M at our disposal. Again, $\hat{\beta} := \beta \circ \pi$ will denote the corresponding 1-form along the projection. Its contraction with the canonical vector field gives rise to a function $\bar{\beta}$ on TM: $\bar{\beta} := \hat{\beta}(\mathbf{T})$, or explicitly, for every $v \in TM$:

$$\bar{\beta}(v) := \hat{\beta}_v(\mathbf{T}(v)) = \beta_{\pi(v)}(v).$$

Definition 2. *Under the preceding assumptions, consider the function $L := L_\alpha + \bar{\beta}$ and let $E := \frac{1}{2}L^2$. Then (M,E) is said to be the Randers manifold constructed from the Riemannian manifold (M,α) by perturbation with $\bar{\beta}$.*

Note that *not always* $L \equiv \sqrt{2E}$ and that a Randers manifold is not necessarily a Finsler manifold. A sufficient condition for this to be true will be presented in the next section.

First, we would like to prove some basic properties.

Lemma 2. *For every vector field X on M, $X^v\bar{\beta} = [\beta(X)]^v = \hat{\beta}(\hat{X})$.*
Proof: The first equality is nothing but the definition of the vertical lift of a vector field X on M. For the second equality, we have for $v \in TM$: $\hat{\beta}(\hat{X})(v) = (\beta\circ\pi)_v[\hat{X}(v)] = \beta_{\pi(v)}[X(\pi(v))] = [\beta(X)\circ\pi](v) = [\beta(X)]^v(v)$. □

Recall that we assumed our manifold M to have dimension at least three.

Lemma 3. *(1) There is no 1-form μ on M such that $\alpha = \beta \odot \mu$.*

(2) There is no 1-form μ on M such that $L_\alpha^2 = \bar{\mu}\bar{\beta}$, with $\bar{\mu} = \hat{\mu}(\mathbf{T})$.
Proof: (1) Suppose that there exist a μ anyway. The kernels of μ and β have dimension $n-1$, and therefore have a non-empty intersection. Then, for any $X \neq 0$ such that $\beta(X) = \mu(X) = 0$, we would have $\alpha(X,Y) = 0$ for all $Y \in \mathcal{X}(M)$, which is in conflict with the non-degeneracy of α.

(2) If L_α^2 were of the form $\bar{\mu}\bar{\beta}$, we would have

$$\begin{aligned} X^v(Y^v(L_\alpha^2)) &= X^v\left[Y^v(\bar{\mu})\bar{\beta} + \bar{\mu}Y^v(\bar{\beta})\right] \\ &= X^v[Y^v\bar{\mu}]\bar{\beta} + Y^v(\bar{\mu})X^v(\bar{\beta}) + X^v(\bar{\mu})Y^v(\bar{\beta}) + \bar{\mu}X^v[Y^v\bar{\beta}] \end{aligned}$$

or, in view of Lemmas 1 and 2,

$$\hat{\alpha}(\hat{X},\hat{Y}) = \frac{1}{2}\big(\hat{\mu}(\hat{X})\hat{\beta}(\hat{Y}) + \hat{\mu}(\hat{Y})\hat{\beta}(\hat{X})\big),$$

which is impossible because of the first statement. □

We define the musical isomorphism $\sharp_\alpha$ with respect to the metric tensor $\hat{\alpha}$ along the projection in the usual manner: if $\tilde{\theta} \in \mathcal{X}(\pi)$, then $\tilde{\theta}^{\sharp_\alpha}$ is the unique vector field along π such that for any $\tilde{X} \in \mathcal{X}(\pi)$,

$$\tilde{\theta}(\tilde{X}) = \hat{\alpha}(\tilde{\theta}^{\sharp_\alpha}, \tilde{X}). \tag{6}$$

In what follows, a prominent role will be played by the vector field $\hat\beta^{\sharp_\alpha}$ along the projection. It will give rise to a function on TM (which is in fact the vertical lift of a function on M), defined by

$$||\,\hat\beta\,||^2_\alpha := \hat\beta(\hat\beta^{\sharp_\alpha}) = \hat\alpha(\hat\beta^{\sharp_\alpha}, \hat\beta^{\sharp_\alpha}),$$

which is nothing but the (square of the) norm of $\hat\beta^{\sharp_\alpha}$ with respect to the metric $\hat\alpha$. Applying $\bar\alpha$ on $\hat\beta^{\sharp_\alpha}$ gives $\bar\alpha(\hat\beta^{\sharp_\alpha}) = \hat\alpha(\mathbf{T}, \hat\beta^{\sharp_\alpha}) = \hat\beta(\mathbf{T}) = \bar\beta$.

Starting with the Levi-Civita connection ∇ of the Riemannian metric α, we can look at its action on the 1-form β. It gives rise to a (0,2)-tensor $\nabla\beta$ on M, with the associated (0,2)-tensor $\widehat{\nabla\beta}$ along the projection. In fact:

Lemma 4. *Let $\tilde X$ and $\tilde Y$ be vector fields along π. Then*

$$(1)\ \overset{\alpha}{D}_{i\tilde X}\hat\beta = 0 \qquad\qquad (2)\ \big(\overset{\alpha}{D}_{\mathcal{H}_\alpha\tilde X}\hat\beta\big)(\tilde Y) = \widehat{\nabla\beta}(\tilde X, \tilde Y).$$

Proof: (1) For any vector field Y on M, $\overset{\alpha}{D}_{i\tilde X}\hat\beta(\hat Y) = i\tilde X\big([\beta(Y)]^v\big) - \hat\beta(\overset{\alpha}{D}_{i\tilde X}\tilde Y) = 0$. Due to $C^\infty(TM)$-linearity on both sides, this property is also valid for general vector fields $\tilde Y$ along the projection.

(2) Next, with vector fields X and Y on M,

$$\begin{aligned}(\overset{\alpha}{D}_{X^{h_\alpha}}\hat\beta)(\hat Y) &= X^{h_\alpha}[\hat\beta(\hat Y)] - \hat\beta[\overset{\alpha}{D}_{X^{h_\alpha}}\hat Y] = X^{h_\alpha}[\beta(Y)]^v - \hat\beta(\widehat{\nabla_X Y})\\ &= [X(\beta(Y))]^v - [\beta(\nabla_X Y)]^v = \widehat{\nabla\beta}(\hat X, \hat Y).\end{aligned}$$

Again, due to $C^\infty(TM)$-linearity on both sides, we can extend this property to general vector fields $\tilde X, \tilde Y$ along the projection. $\square$

Let us make now some notational conventions which will repeatedly be used. In the special case that the first argument of $\widehat{\nabla\beta}$ is always $\mathbf{T}$, we can obtain a 1-form $\overline{\nabla\beta}$ along the projection:

$$\overline{\nabla\beta}(\tilde Y) := \widehat{\nabla\beta}(\mathbf{T}, \tilde Y).$$

The symmetric and skew-symmetric extension of $\widehat{\nabla\beta}$ will be denoted respectively by $Sym\widehat{\nabla\beta}$ and $Alt\widehat{\nabla\beta}$:

$$\begin{aligned}Sym\widehat{\nabla\beta}(\tilde X, \tilde Y) &:= \frac{1}{2}\Big(\widehat{\nabla\beta}(\tilde X, \tilde Y) + \widehat{\nabla\beta}(\tilde Y, \tilde X)\Big) \quad \text{and}\\ Alt\widehat{\nabla\beta}(\tilde X, \tilde Y) &:= \frac{1}{2}\Big(\widehat{\nabla\beta}(\tilde X, \tilde Y) - \widehat{\nabla\beta}(\tilde Y, \tilde X)\Big).\end{aligned}$$

In the special case of $\tilde X = \mathbf{T}$, we will use again e.g. $\overline{Alt\nabla\beta}(\tilde Y) := Alt\widehat{\nabla\beta}(\mathbf{T}, \tilde Y)$. Since $\overline{Alt\nabla\beta}$ is a 1-form along the projection, it is possible to define the vector field $\overline{Alt\nabla\beta}^{\sharp_\alpha}$ along the projection, in the sense of (6).

Since $\big(\overset{\alpha}{D}_{\mathcal{H}_\alpha\tilde X}\beta\big)(\tilde Y) = \widehat{\nabla\beta}(\tilde X, \tilde Y)$, we could also rewrite these newly-made conventions in terms of $\overset{\alpha}{D}$, but we prefer to keep the above notations (with the Levi-Civita symbol ∇ in it), because they reflect more the original form in which the results on Randers manifolds followed from local tensor calculations.

Especially for the main theorem in Section 6, we shall need:

Lemma 5. *For any vector fields X, Y on M we have:*

$$\begin{array}{ll} (1)\, X^v\big(\widehat{\nabla\beta}(\mathbf{T},\hat\beta^{\sharp_\alpha})\big) = \widehat{\nabla\beta}(\hat X,\hat\beta^{\sharp_\alpha}), & (2)\, X^v\big(\widehat{\nabla\beta}(\hat\beta^{\sharp_\alpha},\mathbf{T})\big) = \widehat{\nabla\beta}(\hat\beta^{\sharp_\alpha},\hat X), \\ (3)\, X^v\big(\widehat{\nabla\beta}(\mathbf{T},\mathbf{T})\big) = 2Sym\widehat{\nabla\beta}(\mathbf{T},\hat X), & (4)\, Y^v\big[X^v\big(\widehat{\nabla\beta}(\mathbf{T},\mathbf{T})\big)\big] = 2Sym\widehat{\nabla\beta}(\hat X,\hat Y). \end{array}$$

Proof: Let us calculate for example the derivative $X^v\big(\widehat{\nabla\beta}(\mathbf{T},\hat\beta^{\sharp_\alpha})\big)$, the other properties can be obtained analogously.

$$X^v\big(\widehat{\nabla\beta}(\mathbf{T},\hat\beta^{\sharp_\alpha})\big) = X^v\big((\overset{\alpha}{D}_{S_\alpha}\hat\beta)(\hat\beta^{\sharp_\alpha})\big) = \overset{\alpha}{D}_{X^v}\big(\overset{\alpha}{D}_{S_\alpha}\big(\hat\beta(\hat\beta^{\sharp_\alpha})\big)\big) - \overset{\alpha}{D}_{X^v}\big(\hat\beta(\overset{\alpha}{D}_{S_\alpha}\hat\beta^{\sharp_\alpha})\big).$$

We have mentioned before that for Riemannian Finsler manifolds, the mixed curvature of the connection $(\overset{\alpha}{D},\mathcal{H}_\alpha)$ vanishes. We will use this fact for interchanging the X^v and S_α derivations in the previous relations.

$$\overset{\alpha}{D}_{X^v}\overset{\alpha}{D}_{S_\alpha}||\,\hat\beta\,||^2_\alpha = \overset{\alpha}{D}_{S_\alpha}\overset{\alpha}{D}_{X^v}||\,\hat\beta\,||^2_\alpha - \overset{\alpha}{D}_{[S_\alpha,X^v]}||\,\hat\beta\,||^2_\alpha = -\overset{\alpha}{D}_{-X^{h_\alpha}}||\,\hat\beta\,||^2_\alpha,$$

since for vertical vector fields V, $\overset{\circ}{D}_V||\,\hat\beta\,||^2_\alpha = 0$. For the other term, we get

$$\begin{aligned} \overset{\alpha}{D}_{X^v}\big(\hat\beta(\overset{\alpha}{D}_{S_\alpha}\hat\beta^{\sharp_\alpha})\big) &= \big(\overset{\alpha}{D}_{X^v}\hat\beta\big)\big(\overset{\alpha}{D}_{S_\alpha}\hat\beta^{\sharp_\alpha}\big) + \hat\beta\big(\overset{\alpha}{D}_{X^v}\overset{\alpha}{D}_{S_\alpha}\hat\beta^{\sharp_\alpha}\big) \\ &= \hat\beta\big(\overset{\alpha}{D}_{S_\alpha}\overset{\alpha}{D}_{X^v}\hat\beta^{\sharp_\alpha}\big) - \hat\beta\big(\overset{\alpha}{D}_{[S_\alpha,X^v]}\hat\beta^{\sharp_\alpha}\big) = \hat\beta\big(\overset{\alpha}{D}_{X^{h_\alpha}}\hat\beta^{\sharp_\alpha}\big), \end{aligned}$$

since $\overset{\alpha}{D}_{X^v}\hat\beta = 0$ and for vertical vector fields V, $\overset{\alpha}{D}_V\hat\beta^{\sharp_\alpha} = 0$. Subtracting the two terms then gives

$$X^v\big(\widehat{\nabla\beta}(\mathbf{T},\hat\beta^{\sharp_\alpha})\big) = \overset{\alpha}{D}_{X^{h_\alpha}}||\,\hat\beta\,||^2_\alpha - \hat\beta\big(\overset{\alpha}{D}_{X^{h_\alpha}}\hat\beta^{\sharp_\alpha}\big) = \big(\overset{\alpha}{D}_{X^{h_\alpha}}\hat\beta\big)\big(\hat\beta^{\sharp_\alpha}\big) = \widehat{\nabla\beta}(\hat X,\hat\beta^{\sharp_\alpha}). \qquad \square$$

4 Randers Manifolds of Finsler type

Let us turn back to the definition of a Randers manifold. Obviously, the second item of Definition 1 is satisfied for a Randers manifold (M,E) and also the homogeneity of the energy can easily be checked. Indeed, using the metrical and normal properties of $(\overset{\alpha}{D},\mathcal{H}_\alpha)$ and taking Lemma 4 into account we obtain:

$$\begin{aligned} CL_\alpha &= C\big(\sqrt{\hat\alpha(\mathbf{T},\mathbf{T})}\big) = \frac{1}{2L_\alpha}C\big(\hat\alpha(\mathbf{T},\mathbf{T})\big) = \frac{1}{2L_\alpha}\big(\overset{\alpha}{D}_C\hat\alpha\big)(\mathbf{T},\mathbf{T}) + \frac{1}{L_\alpha}\hat\alpha\big(\overset{\alpha}{D}_C\mathbf{T},\mathbf{T}\big) \\ &= \frac{1}{L_\alpha}\hat\alpha(\mathbf{T},\mathbf{T}) = L_\alpha. \\ C\bar\beta &= C(\hat\beta(\mathbf{T})) = (\overset{\alpha}{D}_C\hat\beta)(\mathbf{T}) + \hat\beta(\overset{\alpha}{D}_C\mathbf{T}) = \hat\beta(\mathbf{T}) = \bar\beta. \end{aligned}$$

So, L is homogeneous of degree 1 and, consequently, $E = \frac{1}{2}L^2$ is homogeneous of degree 2. Of course, for $v = 0$, the energy of a Randers manifold vanishes, but the other items of the definition are more difficult to check. The next proposition gives a sufficient and necessary criterion for the function L to be the fundamental function $\sqrt{2E}$.

Proposition 1. *The fundamental function L of a Randers manifold is strictly positive if and only if $||\,\hat{\beta}\,||_\alpha < 1$.*

PROOF: Take a non-zero $v \in TM$. Suppose that $||\,\hat{\beta}\,||_\alpha < 1$. Using the Cauchy-Schwartz inequality (with respect to $\hat{\alpha}_v$) for v and $\hat{\beta}^{\sharp_\alpha}(v)$ it follows that $-\bar{\beta}(v) < L_\alpha(v)$. Conversely, suppose that L is strictly positive. Of course, also for $w := -\hat{\beta}^{\sharp_\alpha}(v)$, $L(w) > 0$. This will lead to $||\,\hat{\beta}\,||_\alpha < 1$. □

An important consequence of this proposition is the following: under the condition $||\,\hat{\beta}\,||_\alpha < 1$, item 1 of Definition 1 is satisfied. In fact, we have more:

Proposition 2. *Let (M,E) be the Randers manifold arising from the Riemannian manifold (M,α) by perturbation with $\bar{\beta}$. If $||\,\hat{\beta}\,||_\alpha < 1$ then (M,E) is a Finsler manifold. In this case, the Riemann-Finsler metric of a Randers manifolds (M,E) can be represented in the form*

$$g = \frac{L}{L_\alpha}\hat{\alpha} - \frac{\bar{\beta}}{L_\alpha^3}\bar{\alpha}\otimes\bar{\alpha} + \frac{1}{L_\alpha}\bar{\alpha}\odot\hat{\beta} + \hat{\beta}\otimes\hat{\beta}. \tag{7}$$

PROOF: We first establish the expression (7) for the metric. Next we will show that it defines a positive definite (0,2)-tensor along the projection.

The general formula for the Riemann-Finsler metric of a Finsler manifold reduces in the case of two basic vector fields $\hat{X}, \hat{Y}$ to

$$\begin{aligned} g(\hat{X},\hat{Y}) &= X^v(Y^v E) = \frac{1}{2}X^v(Y^v L_\alpha^2) + X^v(Y^v(L_\alpha\bar{\beta})) + \frac{1}{2}X^v(Y^v\bar{\beta}^2) \\ &= \frac{1}{2}X^v(Y^v L_\alpha^2) + (X^v L_\alpha)(Y^v\bar{\beta}) + (X^v\bar{\beta})(Y^v L_\alpha) + \bar{\beta}X^v(Y^v L_\alpha) \\ &\quad +(X^v\bar{\beta})(Y^v\bar{\beta}), \end{aligned}$$

taking into account that $X^v(Y^v\bar{\beta}) = X^v\big(\hat{\beta}(\hat{Y})\big) = X^v\big(\beta(Y)\big)^v = 0$. Using the results of Lemmas 1 and 2 now gives

$$\begin{aligned} g(\hat{X},\hat{Y}) &= \hat{\alpha}(\hat{X},\hat{Y}) + \frac{1}{L_\alpha}\bar{\alpha}(\hat{X})\hat{\beta}(\hat{Y}) + \frac{1}{L_\alpha}\hat{\beta}(\hat{X})\bar{\alpha}(\hat{Y}) - \frac{\bar{\beta}}{L_\alpha^3}\bar{\alpha}(\hat{X})\bar{\alpha}(\hat{Y}) + \frac{\bar{\beta}}{L_\alpha}\hat{\alpha}(\hat{X},\hat{Y}) \\ &\quad +\hat{\beta}(\hat{X})\hat{\beta}(\hat{Y}) \\ &= \Big(\frac{L}{L_\alpha}\hat{\alpha} - \frac{\bar{\beta}}{L_\alpha^3}\bar{\alpha}\otimes\bar{\alpha} + \frac{1}{L_\alpha}\bar{\alpha}\odot\hat{\beta} + \hat{\beta}\otimes\hat{\beta}\Big)(\hat{X},\hat{Y}). \end{aligned}$$

from which (7) follows. This expression can equivalently be rewritten as

$$g(\hat{X},\hat{Y}) = \Big(\frac{L}{L_\alpha^3}(L_\alpha^2\hat{\alpha} - \bar{\alpha}\otimes\bar{\alpha}) + \frac{1}{L_\alpha^2}\bar{\alpha}\otimes\bar{\alpha} + \frac{1}{L_\alpha}\bar{\alpha}\odot\hat{\beta} + \hat{\beta}\otimes\hat{\beta}\Big)(\hat{X},\hat{Y}).$$

To show that (under the condition $||\,\hat{\beta}\,||_\alpha < 1$) this is the Riemann-Finsler metric of a Finsler manifold, it suffices to verify that $g(\hat{X},\hat{X}) > 0$ is satisfied for every basic non-zero vector field $\hat{X}$ along the projection. If we look at this pointwise, it means that for every (non-zero) vector $v \in TM$, we should have

$$[g(\hat{X},\hat{X})](v) = g_v(\hat{X}(v),\hat{X}(v)) > 0$$

In the case of Randers manifolds we find that

$$\begin{aligned} g_v(\hat{X}(v), \hat{X}(v)) &= \frac{L(v)}{L_\alpha^3(v)}\Big(L_\alpha^2(v)\hat{\alpha}_v\big(\hat{X}(v), \hat{X}(v)\big) - \big[\bar{\alpha}(\hat{X}(v))\big]^2\Big) \\ &\quad + \frac{1}{L_\alpha^2(v)}\big[\bar{\alpha}_v(\hat{X}(v))\big]^2 + \frac{2}{L_\alpha(v)}\bar{\alpha}_v\big(\hat{X}(v)\big)\hat{\beta}_v\big(\hat{X}(v)\big) + \big[\hat{\beta}_v(\hat{X}(v))\big]^2. \end{aligned}$$

The last line is clearly the positive number $\Big(\frac{1}{L_\alpha(v)}\bar{\alpha}_v\big(\hat{X}(v)\big) + \hat{\beta}_v\big(\hat{X}(v)\big)\Big)^2$. For the first line we will use the Cauchy-Schwartz inequality (for $\hat{\alpha}$) again. Since by definition $L_\alpha^2 = \hat{\alpha}(\mathbf{T}, \mathbf{T})$ and $\bar{\alpha}(\hat{X}) = \hat{\alpha}(\mathbf{T}, \hat{X})$, we obtain

$$\big[\hat{\alpha}_v(\mathbf{T}(v), \mathbf{T}(v))\big]\big[\hat{\alpha}_v(\hat{X}(v), \hat{X}(v))\big] - \big[\hat{\alpha}_v(\mathbf{T}(v), \hat{X}(v))\big]^2 > 0.$$

By the observation of Proposition 1, L is a positive function if $||\,\hat{\beta}\,||_\alpha < 1$, so the proof is complete. ☐

From now on, we will only work with Randers manifolds which are Finsler manifolds. The condition $||\,\hat{\beta}\,||_\alpha < 1$ can for example be found in [2]. Sometimes (e.g. in [15]) we find the condition $B < 1$, with $B(p) = \sup_{\substack{v\in T_pM \\ v\neq 0}} \frac{\bar{\beta}(v)}{L_\alpha(v)}$ for $p \in M$. We shall show that $||\,\hat{\beta}\,||_\alpha = B^v$. First we check that $||\,\hat{\beta}\,||_\alpha$ is an upper bound. Like in the proof of Proposition 1, we use the Cauchy-Schwartz inequality and $\bar{\beta}(v) = \hat{\alpha}_v(\hat{\beta}^{\sharp_\alpha}(v), v) \leq |\hat{\alpha}_v(\hat{\beta}^{\sharp_\alpha}(v), v)|$ to obtain that

$$\bar{\beta}(v) \leq \sqrt{\hat{\alpha}_v(\hat{\beta}^{\sharp_\alpha}(v), \hat{\beta}^{\sharp_\alpha}(v))}\sqrt{\hat{\alpha}_v(v,v)} = ||\,\hat{\beta}\,||_\alpha(v)L_\alpha(v).$$

So $||\,\hat{\beta}\,||_\alpha$ is indeed an upper bound. Let us now consider the vector $u := +\hat{\beta}^{\sharp_\alpha}(v)$. It has the property $\pi(u) = \pi(v)$, so

$$\bar{\beta}(u) = \hat{\beta}_u(u) = \hat{\beta}_{\hat{\beta}^{\sharp_\alpha}(v)}(\hat{\beta}^{\sharp_\alpha}(v)) = \hat{\beta}_v(\hat{\beta}^{\sharp_\alpha}(v)) = \hat{\beta}(\hat{\beta}^{\sharp_\alpha})(v) = ||\,\hat{\beta}\,||_\alpha^2(v).$$

On the other hand

$$L_\alpha(u) = \sqrt{\hat{\alpha}_u(u,u)} = \sqrt{\alpha_v(\hat{\beta}^{\sharp_\alpha}(v), \hat{\beta}^{\sharp_\alpha}(v))} = \sqrt{\hat{\alpha}(\hat{\beta}^{\sharp_\alpha}, \hat{\beta}^{\sharp_\alpha})(v)} = ||\,\hat{\beta}\,||_\alpha(v).$$

This means that for every non-zero $v \in TM$, there exists a non-zero u such that $\frac{\bar{\beta}(u)}{L_\alpha(u)} = ||\,\hat{\beta}\,||_\alpha(v)$, and therefore any other upper bound will be less than $||\,\hat{\beta}\,||_\alpha$. So, $||\,\hat{\beta}\,||_\alpha$ is the smallest upper bound, i.e. the supremum.

For later use, we list two special evaluations of g. The proof is immediate by (7).

Corollary 1. *For any vector field $\tilde{X}$ along the projection:*

(1) $g(\mathbf{T}, \tilde{X}) = \frac{L}{L_\alpha}\bar{\alpha}(\tilde{X}) + L\bar{\beta}(\tilde{X})$,

(2) $g(\hat{\beta}^{\sharp_\alpha}, \tilde{X}) = \frac{L+\bar{\beta}+L_\alpha||\,\hat{\beta}\,||_\alpha^2}{L_\alpha}\bar{\beta}(\tilde{X}) - \frac{\bar{\beta}^2 - L_\alpha^2||\,\hat{\beta}\,||_\alpha^2}{L_\alpha^3}\bar{\alpha}(\tilde{X})$. ☐

In the case of Randers metrics, the angular metric takes the form

$$k = \frac{L}{L_\alpha}\hat{\alpha} - \frac{L}{L_\alpha}\bar{\alpha} \otimes \bar{\alpha}, \tag{8}$$

(use Lemma 1), and its relation to the Riemann-Finsler metric is

$$k = g - (\frac{1}{L_\alpha}\bar{\alpha} + \hat{\beta}) \otimes (\frac{1}{L_\alpha}\bar{\alpha} + \hat{\beta}).$$

5 Horizontal structure and Cartan tensors

In this section, we list expressions for the canonical spray, the Barthel endomorphism, the Berwald connection, the Cartan tensors and the Cartan vector field in the setting of Randers manifolds (notations as above). Related coordinate expressions can be found e.g. in [2], but their compact global forms as in Propositions 3, 4, 5, 6 and 9 appear here for the first time. With the conventions of Section 3, the meaning of the composing parts of the spray become clear.

Proposition 3. *The canonical sprays of a Randers manifold (M, E) and its associated Riemannian Finsler manifold (M, E_α) are related as follows:*

$$S = S_\alpha + \frac{1}{L}\Big(2L_\alpha \overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha}) - \overline{\nabla}\beta(\mathbf{T})\Big)C - 2L_\alpha i\big(\overline{Alt\nabla\beta}\big)^{\sharp_\alpha}. \tag{9}$$

Proof: S_α is a solution of $i_{S_\alpha} d(dE_\alpha \circ J) = -dE_\alpha$. The defining relation for the canonical spray (with $E = E_\alpha + L_\alpha\bar{\beta} + \frac{1}{2}\bar{\beta}^2$) leads to

$$i_S d(dE \circ J) = i_{S_\alpha} d(dE_\alpha \circ J) - \bar{\beta} dL_\alpha - L_\alpha d\bar{\beta} - \bar{\beta} d\bar{\beta}. \tag{10}$$

In this expression, we will replace $i_{S_\alpha} d(dE_\alpha \circ J)$ again by $i_{S_\alpha} d(dE \circ J) - i_{S_\alpha} d\big(d(L_\alpha\bar{\beta}) \circ J\big) - i_{S_\alpha} d\big(d(\frac{1}{2}\bar{\beta}^2) \circ J\big)$, but first we take some side-steps. For S_α (and more generally for any semispray), we have

$$i_{S_\alpha}(dL_\alpha \circ J) = dL_\alpha\big(J(S_\alpha)\big) = dL_\alpha(C) = CL_\alpha = L_\alpha \tag{11}$$

and likewise

$$i_{S_\alpha}(d\bar{\beta} \circ J) = \bar{\beta}. \tag{12}$$

On the other hand, for the canonical spray S_α of (M, E_α)

$$i_{S_\alpha} dL_\alpha = S_\alpha(L_\alpha) = 0, \tag{13}$$

since the Barthel endomorphism of a Finsler manifold is conservative. We use properties (11, 12, 13) to obtain the next expression for $i_{S_\alpha} d(d(L_\alpha\bar{\beta}) \circ J)$:

$$\begin{aligned} i_{S_\alpha} d\big(d(L_\alpha\bar{\beta}) \circ J\big) &= i_{S_\alpha} dL_\alpha \wedge (d\bar{\beta} \circ J) - dL_\alpha \wedge i_{S_\alpha}(d\bar{\beta} \circ J) + L_\alpha i_{S_\alpha} d(d\bar{\beta} \circ J) \\ &\quad + i_{S_\alpha} d\bar{\beta} \wedge (dL_\alpha \circ J) - d\bar{\beta} \wedge i_{S_\alpha}(dL_\alpha \circ J) + \bar{\beta} i_{S_\alpha} d(dL_\alpha \circ J) \\ &= -\bar{\beta} dL_\alpha + L_\alpha i_{S_\alpha} d(d\bar{\beta} \circ J) + S_\alpha(\bar{\beta})(dL_\alpha \circ J) - L_\alpha d\bar{\beta} \end{aligned}$$

and likewise for $i_{S_\alpha} d\big(d(\frac{1}{2}\bar\beta)\circ J\big)$

$$i_{S_\alpha} d\big(d(\tfrac{1}{2}\bar\beta^2)\circ J\big) = S_\alpha(\bar\beta)(d\bar\beta\circ J) - \bar\beta d\bar\beta + \bar\beta i_{S_\alpha} d(d\bar\beta\circ J).$$

Substituting these results in (10) gives:

$$i_{S-S_\alpha} d(dE\circ J) = -L i_{S_\alpha} d(d\bar\beta\circ J) - S_\alpha(\bar\beta)(dL\circ J) + \bar\beta i_{S_\alpha} d(dL_\alpha\circ J). \tag{14}$$

The left-hand side of this expression is $i_{S-S_\alpha}\omega$. We will now try to re-write also the right-hand side in terms of ω. The last term in (14) is easy. Replacing E_α by $\frac{1}{2}L_\alpha^2$ in the defining relation for the spray S_α gives

$$i_{S_\alpha} d\big(L_\alpha dL_\alpha\circ J\big) = -L_\alpha dL_\alpha,$$

or

$$i_{S_\alpha} dL_\alpha\wedge(dL_\alpha\circ J) - dL_\alpha\wedge i_{S_\alpha}(dL_\alpha\circ J) + L_\alpha i_{S_\alpha} d(dL_\alpha\circ J) = -L_\alpha dL_\alpha.$$

From (11,13) it is obvious then that $i_{S_\alpha} d(dL_\alpha\circ J) = 0$.

The second term of the right-hand side of equation (14) can easily be related to ω:

$$dL\circ J = \frac{1}{L} dE\circ J = \frac{1}{L} i_C\omega$$

and we will also re-write its coefficient $S_\alpha(\bar\beta)$. Taking $\tilde X$ and $\tilde Y$ both equal to $\mathbf{T}$ in the second item of Lemma 4, we find that $S_\alpha\bar\beta = \big(\overset{\alpha}{D}_{S_\alpha}\hat\beta\big)(\mathbf{T}) + \hat\beta\big(\overset{\alpha}{D}_{S_\alpha}\mathbf{T}\big) = \widehat{\nabla\beta}(\mathbf{T},\mathbf{T}) = \overline{\nabla\beta}(\mathbf{T})$ (since $\overset{\alpha}{D}$ is normal).

For the first term in the right-hand side of (14), we need some calculation. $d(d\bar\beta\circ J)$ vanishes on vertical vector fields, and for $X, Y\in\mathcal{X}(M)$ we can write

$$\begin{aligned}
d(d\bar\beta\circ J)(X^h,Y^h) &= X^h(Y^v\bar\beta) - Y^h(X^v\bar\beta) - J[X^h,Y^h]\\
&= \big(X(\beta(Y))\big)^v - \big(Y(\beta(X))\big)^v - \big(\beta([X,Y])\big)^v\\
&= \big(X(\beta(Y)) - Y(\beta(X)) - \beta(\nabla_X Y) + \beta(\nabla_Y X)\big)^v\\
&= \big(\nabla\beta(X,Y) - \nabla\beta(Y,X)\big)^v.
\end{aligned}$$

In going to the third line we have invoked that the torsion of ∇ vanishes. So, we have found that

$$d(d\bar\beta\circ J)(X^h,Y^h) = 2Alt\widehat{\nabla\beta}(\hat X,\hat Y).$$

In view of the $C^\infty(TM)$-bilinearity of $d(d\bar\beta\circ J)$, we can in particular write that

$$d(d\bar\beta\circ J)(S_\alpha,\mathcal{H}\tilde Y) = 2Alt\widehat{\nabla\beta}(\mathbf{T},\tilde Y).$$

It is our intention to relate this first part of the right-hand side of equation (14) to ω. We would like to find the (unique vertical) vectorfield $V = i\tilde G\in\mathcal{X}(TM)$ (or the unique vector field $\tilde G$ along the projection), such that for any $\hat Y\in\mathcal{X}(M)$

$$-Ld(d\bar\beta\circ J)(S_\alpha,\mathcal{H}\tilde Y) = \omega(V,\mathcal{H}\tilde Y) = g(\tilde G,\tilde Y),$$

or such that

$$-2LAlt\widehat{\nabla\beta}(\mathbf{T},\tilde{Y}) = g(\tilde{G},\tilde{Y}), \tag{15}$$

because the equation (14) then reduces to

$$i_{S-S_\alpha}\omega = i_{i\tilde{G}}\omega - i_{\frac{1}{L}\overline{\nabla\beta}(\mathbf{T})C}\omega$$

or

$$S = S_\alpha + i\tilde{G} - \frac{1}{L}\overline{\nabla\beta}(\mathbf{T})C.$$

The computation of this $\tilde{G}$ is very technical. Therefore we will limit ourselves here to giving the expression for $\tilde{G}$ and checking that it indeed satisfies (15). We state that

$$\tilde{G} = \Big(\frac{2L_\alpha}{L}\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha})\mathbf{T} - 2L_\alpha\overline{Alt\nabla\beta}^{\sharp_\alpha}\Big).$$

Using (among others) that for a 1-form $\tilde{\theta}$ along the projection

$$\hat{\beta}(\tilde{\theta}^{\sharp_\alpha}) = \hat{\alpha}(\hat{\beta}^{\sharp_\alpha},\tilde{\theta}^{\sharp_\alpha}) = \tilde{\theta}(\hat{\beta}^{\sharp_\alpha}),$$

it is easy to obtain the composing parts of $g(\tilde{G},\tilde{Y})$:

$\frac{L}{L_\alpha}\hat{\alpha}(\tilde{G},\tilde{Y}) = 2\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha})\bar{\alpha}(\tilde{Y}) - 2L\overline{Alt\nabla\beta}(\tilde{Y})$, $\quad -\frac{\bar{\beta}}{L_\alpha^3}\bar{\alpha}(\tilde{G},\tilde{Y}) = -\frac{2\bar{\beta}}{L}\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha})\bar{\alpha}(\tilde{Y})$,
$\frac{1}{L_\alpha}\bar{\alpha}(\tilde{G})\hat{\beta}(\tilde{Y}) = \frac{2L_\alpha^3}{L}\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha})\hat{\beta}(\tilde{Y})$, $\quad \frac{1}{L_\alpha}\bar{\alpha}(\tilde{Y})\hat{\beta}(\tilde{G}) = -\frac{2L_\alpha}{L}\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha})\bar{\alpha}(\tilde{Y})$,
$\hat{\beta}(\tilde{G})\hat{\beta}(\tilde{Y}) = \frac{2L_\alpha\bar{\beta}-L_\alpha L}{L}\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha})\hat{\beta}(\tilde{Y})$.

Taken together, all terms cancel out except $-2L\overline{Alt\nabla\beta}(\tilde{Y})$, which demonstrates that $\tilde{G}$ is the unique vector field along the projection that we are looking for. So finally, the spray takes the form (9). ☐

For later use, we deduce now

Corollary 2. *(1)* $S(L_\alpha) = -S(\bar{\beta}) = \frac{L_\alpha}{L}\Big(2L_\alpha\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha}) - \overline{\nabla\beta}(\mathbf{T})\Big)$,

(2) $S(||\,\hat{\beta}\,||_\alpha^2) = 2\overline{\nabla\beta}(\hat{\beta}^{\sharp_\alpha})$.

Proof: (1) The first equality simply expresses that $S(L) = 0$. Using (9), we obtain $S(L_\alpha) = S_\alpha(L_\alpha) + \frac{L_\alpha}{L}\Big(2L_\alpha\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha}) - \overline{\nabla\beta}(\mathbf{T})\Big) - 2L_\alpha i\big(\overline{Alt\nabla\beta}^{\sharp_\alpha}\big)(L_\alpha)$. Of course, also $S_\alpha L_\alpha = 0$ for the Riemannian Finsler manifold and, due to Lemma 2,

$$\begin{aligned} i\big(\overline{Alt\nabla\beta}^{\sharp_\alpha}\big)(L_\alpha) &= \frac{1}{L_\alpha}\bar{\alpha}(\overline{Alt\nabla\beta}^{\sharp_\alpha}) = \frac{1}{L_\alpha}\hat{\alpha}(\overline{Alt\nabla\beta}^{\sharp_\alpha},\mathbf{T}) = \frac{1}{L_\alpha}\overline{Alt\nabla\beta}(\mathbf{T}) \\ &= \frac{1}{L_\alpha}Alt\widehat{\nabla\beta}(\mathbf{T},\mathbf{T}) = 0, \end{aligned}$$

which implies the required result.

(2) First of all, note that, since $||\,\hat{\beta}\,||_{\alpha}^{2}$ is in fact the vertical lift of $\beta(\beta^{\sharp_\alpha}) \in C^\infty(M)$, $C(||\,\hat{\beta}\,||_{\alpha}^{2}) = 0$ and $i\big(\overline{Alt\nabla\beta}^{\sharp_\alpha}\big)(||\,\hat{\beta}\,||_{\alpha}^{2}) = 0$, so we only need to calculate $S_\alpha(||\,\hat{\beta}\,||_{\alpha}^{2})$.

$$
\begin{aligned}
S_\alpha(||\,\hat{\beta}\,||_{\alpha}^{2}) &= S_\alpha\big(\hat{\alpha}(\hat{\beta}^{\sharp_\alpha},\hat{\beta}^{\sharp_\alpha})\big) = \big(\overset{\alpha}{D}_{S_\alpha}\hat{\alpha}\big)(\hat{\beta}^{\sharp_\alpha},\hat{\beta}^{\sharp_\alpha}) + 2\hat{\alpha}\big(\overset{\alpha}{D}_{S_\alpha}\hat{\beta}^{\sharp_\alpha},\hat{\beta}^{\sharp_\alpha}\big)\\
&= 2\hat{\beta}\big(\overset{\alpha}{D}_{S_\alpha}\hat{\beta}^{\sharp_\alpha}\big) = 2\overset{\alpha}{D}_{S_\alpha}\big(\hat{\beta}(\hat{\beta}^{\sharp_\alpha})\big) - 2\big(\overset{\alpha}{D}_{S_\alpha}\hat{\beta}\big)(\hat{\beta}^{\sharp_\alpha})\\
&= 2S_\alpha(||\,\hat{\beta}\,||_{\alpha}^{2}) - 2\big(\overset{\alpha}{D}_{S_\alpha}\hat{\beta}\big)(\hat{\beta}^{\sharp_\alpha}),
\end{aligned}
$$

or $S_\alpha(||\,\hat{\beta}\,||_{\alpha}^{2}) = 2\big(\overset{\alpha}{D}_{S_\alpha}\hat{\beta}\big)(\hat{\beta}^{\sharp_\alpha}) = 2\widehat{\nabla}\beta(\mathbf{T},\hat{\beta}^{\sharp_\alpha}) = 2\overline{\nabla\beta}(\hat{\beta}^{\sharp_\alpha})$. □

Lemma 5 will now be used in the construction of the Barthel endomorphism of Randers manifolds.

Proposition 4. *The relation between the horizontal lift mapping coming from the Barthel endomorphism of the Randers manifold and that of its underlying Riemannian Finsler manifold is given by*

$$
\begin{aligned}
X^h &= X^{h_\alpha} + \frac{\mu}{2L}X^v - \frac{\mu}{2L^2}P(\hat{X})C + \frac{1}{L}Q(\hat{X})C\\
&\quad -\frac{1}{L_\alpha}\bar{\alpha}(\hat{X})i\big(\overline{Alt\nabla\beta}^{\sharp_\alpha}\big) - L_\alpha\big[X^v, i\big(\overline{Alt\nabla\beta}^{\sharp_\alpha}\big)\big],
\end{aligned}
\tag{16}
$$

where X is any vector field on M, and

$$
\mu := 2L_\alpha\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha}) - \overline{\nabla\beta}(\mathbf{T}), \qquad P(\hat{X}) := \frac{1}{L_\alpha}\bar{\alpha}(\hat{X}) + \hat{\beta}(\hat{X}),
$$

$$
Q(\hat{X}) := \frac{1}{L_\alpha}\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha})\bar{\alpha}(\hat{X}) + L_\alpha Alt\widehat{\nabla}\beta(\hat{X},\hat{\beta}^{\sharp_\alpha}) - Sym\widehat{\nabla}\beta(\hat{X},\mathbf{T}).
$$

PROOF: We only sketch the basic steps of the reasoning. Applying the definition (3) (for both h and h_α) on the complete lift X^c of a vector field $X \in \mathcal{X}(M)$ gives

$$
X^h = \frac{1}{2}(X^c + [X^v, S]) \qquad \text{and} \qquad X^{h_\alpha} = \frac{1}{2}(X^c + [X^v, S_\alpha]).
$$

Subtracting these two relations we get

$$
X^h = X^{h_\alpha} - \frac{1}{2}[X^v, S_\alpha - S].
$$

The difference $S_\alpha - S$ is given in expression (9). For the computation of the bracket $-\frac{1}{2}\big[X^v, -\frac{1}{L}\big(2L_\alpha\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha}) - \overline{\nabla\beta}(\mathbf{T})\big)C\big]$, we use Lemmas 1, 2 and 5 to find the first line of (16). The second line of (16) is nothing but $-\frac{1}{2}\big[X^v, 2L_\alpha i\big(\overline{Alt\nabla\beta}^{\sharp_\alpha}\big)\big]$. □

Proposition 5. *Let $X, Y \in \mathcal{X}(M)$. The Berwald connection $(\overset{\circ}{D}, \mathcal{H})$ for a Randers man-*

ifold is given by

$$\begin{aligned}
\overset{\circ}{D}_{X^v}\hat{Y} &= 0,\\
\overset{\circ}{D}_{X^h}\hat{Y} &= \overset{\alpha}{D}_{X^{h_\alpha}}\hat{Y} + \Big(\frac{\mu}{2L^2}P(\hat{X}) - \frac{1}{L}Q(\hat{X})\Big)\hat{Y} + \Big(\frac{\mu}{2L^2}P(\hat{Y}) - \frac{1}{L}Q(\hat{Y})\Big)\hat{X}\\
&\quad + \Big(\frac{1}{L^2}P(\hat{X})Q(\hat{Y}) + \frac{1}{L^2}Q(\hat{X})P(\hat{Y}) - \frac{\mu}{L^3}P(\hat{X})P(\hat{Y})\Big)\mathbf{T}\\
&\quad - \frac{1}{LL_\alpha}\Big(\bar{\alpha}(\hat{X})Alt\widehat{\nabla\beta}(\hat{Y},\hat{\beta}^{\sharp_\alpha}) + \bar{\alpha}(\hat{Y})Alt\widehat{\nabla\beta}(\hat{X},\hat{\beta}^{\sharp_\alpha})\Big)\mathbf{T}\\
&\quad + \frac{1}{L}Sym\widehat{\nabla\beta}(\hat{X},\hat{Y})\mathbf{T} - \frac{1}{L^2}h(\hat{X},\hat{Y})\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha})\mathbf{T} + \frac{1}{L}h(\hat{X},\hat{Y})\overline{Alt\nabla\beta}^{\sharp_\alpha}\\
&\quad + \frac{1}{L_\alpha}\bar{\alpha}(\hat{X})\nu_\alpha\big[Y^v, i\big(\overline{Alt\nabla\beta}^{\sharp_\alpha}\big)\big] + \frac{1}{L_\alpha}\bar{\alpha}(\hat{Y})\nu_\alpha\big[X^v, i\big(\overline{Alt\nabla\beta}^{\sharp_\alpha}\big)\big]\\
&\quad + L_\alpha\nu_\alpha\big[X^v, \big[Y^v, i\big(\overline{Alt\nabla\beta}^{\sharp_\alpha}\big)\big]\big]. \qquad (17)
\end{aligned}$$

Proof: Using (16) in the definition (4), we only need to calculate $\nu[X^h, Y^v]$. Using Lemmas 1, 2 and 5 again, this is a routine task. □

Proposition 6. *The first lowered Cartan tensor $\mathcal{C}_\flat$ and the first Cartan tensor $\mathcal{C}$ of a Randers manifold (M,L) can be represented as follows:*

$$\mathcal{C}_\flat = \frac{1}{L_\alpha}\Big(\hat{\beta}\odot\hat{\alpha} - \frac{\bar{\beta}}{L_\alpha^2}\bar{\alpha}\odot\hat{\alpha} - \frac{1}{L_\alpha^2}\hat{\beta}\odot\bar{\alpha}\otimes\bar{\alpha} + \frac{\bar{\beta}}{L_\alpha^4}\bar{\alpha}\odot\bar{\alpha}\otimes\bar{\alpha}\Big), \qquad (18)$$

$$\begin{aligned}
\mathcal{C} &= \frac{3\bar{\beta} + L_\alpha\|\,\hat{\beta}\,\|_\alpha^2}{L^2L_\alpha^3}\bar{\alpha}\otimes\bar{\alpha}\otimes\mathbf{T} - \frac{\bar{\beta} + L_\alpha\|\,\hat{\beta}\,\|_\alpha^2}{L^2L_\alpha}\hat{\alpha}\otimes\mathbf{T} + \frac{1}{L}\hat{\alpha}\otimes\hat{\beta}^{\sharp_\alpha} - \frac{1}{LL_\alpha^2}\bar{\alpha}\otimes\bar{\alpha}\otimes\hat{\beta}^{\sharp_\alpha}\\
&\quad + \frac{\bar{\beta} - L_\alpha}{L^2L_\alpha^2}\bar{\alpha}\odot\hat{\beta}\otimes\mathbf{T} - \frac{\bar{\beta}}{LL_\alpha^2}\bar{\alpha}\odot id + \frac{1}{L}\hat{\beta}\odot id - \frac{2}{L^2}\hat{\beta}\otimes\hat{\beta}\otimes\mathbf{T}. \qquad (19)
\end{aligned}$$

The trace of the first Cartan tensor can be expressed in the form

$$tr\mathcal{C} = (n+1)\frac{1}{L}\Big(\hat{\beta} - \frac{\bar{\beta}}{L_\alpha^2}\bar{\alpha}\Big). \qquad (20)$$

Proof: (1) Again, we will work with vector fields X, Y and Z on M. The computation of the lowered Cartan tensor then reduces to

$$\begin{aligned}
\mathcal{C}_\flat(\hat{X},\hat{Y},\hat{Z}) &= X^v[Y^v(Z^vE)] = X^v[g(\hat{Y},\hat{Z})]\\
&= X^v\Big[\frac{L}{L_\alpha}\hat{\alpha}(\hat{Y},\hat{Z}) - \frac{\bar{\beta}}{L_\alpha^3}\bar{\alpha}(\hat{Y})\bar{\alpha}(\hat{Z}) + \frac{1}{L_\alpha}\big(\bar{\alpha}(\hat{Y})\hat{\beta}(\hat{Z}) + \hat{\beta}(\hat{Y})\bar{\alpha}(\hat{Z})\big)\\
&\quad + \hat{\beta}(\hat{Y})\hat{\beta}(\hat{Z})\Big]\\
&= -\frac{L}{L_\alpha^2}X^v(L_\alpha)\hat{\alpha}(\hat{Y},\hat{Z}) + \frac{1}{L_\alpha}X^v(L)\alpha(\hat{Y},\hat{Z}) - \frac{1}{L_\alpha^3}X^v(\bar{\beta})\bar{\alpha}(\hat{Y})\bar{\alpha}(\hat{Z})\\
&\quad + \frac{3\bar{\beta}}{L_\alpha^4}X^v(L_\alpha)\bar{\alpha}(\hat{Y})\bar{\alpha}(\hat{Z}) - \frac{\bar{\beta}}{L_\alpha^3}X^v[\bar{\alpha}(\hat{Y})]\bar{\alpha}(\hat{Z}) - \frac{\bar{\beta}}{L_\alpha^3}X^v[\bar{\alpha}(\hat{Z})]\bar{\alpha}(\hat{Y})\\
&\quad + \frac{1}{L_\alpha^2}X^v(L_\alpha)\bar{\alpha}(\hat{Y})\hat{\beta}(\hat{Z}) - \frac{1}{L_\alpha}X^v[\bar{\alpha}(\hat{Y})]\hat{\beta}(\hat{Z}) + \frac{1}{L_\alpha^2}X^v(L_\alpha)\bar{\alpha}(\hat{Z})\hat{\beta}(\hat{Y})\\
&\quad - \frac{1}{L_\alpha}X^v[\bar{\alpha}(\hat{Z})]\hat{\beta}(\hat{Y}),
\end{aligned}$$

since we have already seen that $X^v(Y^v\bar{\beta}) = 0$ and likewise $X^v[\hat{\alpha}(\hat{Y}, \hat{Z})] = 0$. Again, the results of Lemmas 1 and 2 are useful for rearranging the above terms into:

$$\begin{aligned}
\mathcal{C}_\flat(\hat{X}, \hat{Y}, \hat{Z}) &= -\frac{\bar{\beta}}{L_\alpha}\Big(X^v(L_\alpha)Y^v[Z^v(L_\alpha)] + Y^v(L_\alpha)X^v[Z^v(L_\alpha)] + Z^v(L_\alpha)X^v[Y^v(L_\alpha)]\Big) \\
&\quad + X^v(\bar{\beta})Y^v[Z^v L_\alpha] + Y^v(\bar{\beta})X^v[X^v L_\alpha] + Z^v(\bar{\beta})X^v[Y^v(L_\alpha)]. \\
&= \Big(\frac{1}{L_\alpha}\hat{\beta}\odot\hat{\alpha} - \frac{1}{L_\alpha^3}\hat{\beta}\odot(\bar{\alpha}\otimes\bar{\alpha}) - \frac{\bar{\beta}}{L_\alpha^3}\bar{\alpha}\odot\hat{\alpha} + \frac{\bar{\beta}}{L_\alpha^5}\bar{\alpha}\odot(\bar{\alpha}\otimes\bar{\alpha})\Big)\Big(\tilde{X}, \tilde{Y}, \tilde{Z}\Big).
\end{aligned}$$

This is the desired result.

(2) The easiest way to prove (19) is to check that it satisfies the relation

$$g(\mathcal{C}(\tilde{X}, \tilde{Y}), \tilde{Z}) = \mathcal{C}_\flat(\tilde{X}, \tilde{Y}, \tilde{Z}).$$

which uniquely determines $\mathcal{C}$. This is also a simple but lengthy calculation.

(3) We repeatedly use the properties of the trace operator (adapted from [23] to forms along the projection) to find

$$\begin{aligned}
tr\mathcal{C}(\tilde{X}) &= \frac{3\bar{\beta} + L_\alpha\|\,\hat{\beta}\,\|_\alpha^2}{L^2 L_\alpha^3}\bar{\alpha}(\mathbf{T})\bar{\alpha}(\tilde{X}) - \frac{\bar{\beta} + L_\alpha\|\,\hat{\beta}\,\|_\alpha^2}{L^2 L_\alpha}\hat{\alpha}(\mathbf{T}, \tilde{X}) + \frac{1}{L}\hat{\alpha}(\hat{\beta}^{\sharp_\alpha}, \tilde{X}) \\
&\quad - \frac{1}{LL_\alpha^2}\bar{\alpha}(\hat{\beta}^{\sharp_\alpha})\bar{\alpha}(\tilde{X}) - \frac{\bar{\beta} - L_\alpha}{L^2 L_\alpha^2}(\bar{\alpha}(\mathbf{T})\hat{\beta}(\tilde{X}) + \bar{\alpha}(\tilde{X})\hat{\beta}(\mathbf{T})) \\
&\quad - (n+1)\frac{\bar{\beta}}{LL_\alpha^2}\bar{\alpha}(\tilde{X}) + (n+1)\frac{1}{L}\hat{\beta} - \frac{2}{L^2}\hat{\beta}(\mathbf{T})\hat{\beta}(\tilde{X}).
\end{aligned}$$

Regrouping the terms in $\bar{\alpha}(\tilde{X})$ and $\hat{\beta}(\tilde{X})$ now produces the desired result. □

The lowered Cartan tensor has an easy decomposition, using the angular metric (8).

Corollary 3. *The lowered Cartan tensor is related to the angular metric and the trace of the first Cartan tensor by* $\mathcal{C}_\flat = \frac{1}{n+1}(k \odot tr\mathcal{C})$. □

Finsler manifolds having this property are called *$\mathcal{C}$-reducible.* In dimension $n > 2$, the only class of $\mathcal{C}$-reducible Finsler manifolds, next to the Randers manifolds, are Kropina manifolds, with fundamental function $L(v) = \frac{L_\alpha^2(v)}{\beta(v)}$ [12].

Let us take a step backwards and consider an arbitrary Finsler manifold (M, E). The following concept is well-defined.

Definition 3. *The Cartan vector field $\mathcal{C}^*$ of a Finsler manifold is the unique vector field along the projection such that*

$$g(\mathcal{C}^*, \tilde{X}) = (tr\mathcal{C})(\tilde{X})$$

for any vector field $\tilde{X} \in \mathcal{X}(\pi)$.

Lemma 6. *The Cartan vector field is orthogonal (with respect to the metric g) to the canonical vector field.*

PROOF: The defining relation implies that $g(\mathcal{C}^*, \mathbf{T}) = tr\mathcal{C}(\mathbf{T}) = i_{\mathbf{T}}(tr\mathcal{C}) = tr(i_{\mathbf{T}}\mathcal{C})$, which is zero since $\mathcal{C}(\mathbf{T}, .) = 0$ for every Finsler manifold. □

The next Proposition is valid only for Randers manifolds.

Proposition 7. *Consider a Randers manifold (M, E). A vector field $\tilde{X}^\perp$ along the projection is orthogonal to both the canonical vector field and the Cartan vector field along the projection if and only if it is in the kernel of both $\bar{\alpha}$ and $\hat{\beta}$: $\bar{\alpha}(\tilde{X}^\perp) = 0$ and $\hat{\beta}(\tilde{X}^\perp) = 0$.*
PROOF: Using Corollary 1, we have

$$\begin{aligned} g(\mathbf{T}, \tilde{X}^\perp) &= \frac{L}{L_\alpha}\bar{\alpha}(\tilde{X}^\perp) + L\hat{\beta}(\tilde{X}^\perp), \\ g(\mathcal{C}^*, \tilde{X}^\perp) &= tr\mathcal{C}(\tilde{X}^\perp) = (n+1)\frac{1}{L}\big(\hat{\beta}(\tilde{X}^\perp) - \frac{\bar{\beta}}{L_\alpha^2}\bar{\alpha}(\tilde{X}^\perp)\big), \end{aligned} \tag{21}$$

from which the statement now easily follows. □

Proposition 8. *The canonical vector field, the Cartan vector field and the vector field $\hat{\beta}^{\sharp_\alpha}$ are coplanar in any Randers manifold.*
PROOF: It is enough to show that every vector field $\tilde{X}^\perp$ that is orthogonal to both $\mathbf{T}$ and $\mathcal{C}^*$, is also orthogonal to $\hat{\beta}^{\sharp_\alpha}$, i.e., $g(\hat{\beta}^{\sharp_\alpha}, \tilde{X}^\perp) = 0$. Again, Corollary 1 gives the answer:

$$g(\hat{\beta}^{\sharp_\alpha}, \tilde{X}^\perp) = \frac{L + \bar{\beta} + L_\alpha || \hat{\beta} ||_\alpha^2}{L_\alpha}\hat{\beta}(\tilde{X}^\perp) - \frac{\bar{\beta}^2 - L_\alpha^2 || \hat{\beta} ||_\alpha^2}{L_\alpha^3}\bar{\alpha}(\tilde{X}^\perp) = 0.$$

□

Proposition 9. *The Cartan vector field of a Randers manifold (M, E) can be decomposed as follows:*

$$\mathcal{C}^* = -(n+1)\frac{1}{L^3}(\bar{\beta} + L_\alpha || \hat{\beta} ||_\alpha^2)\mathbf{T} + (n+1)\frac{L_\alpha}{L^2}\hat{\beta}^{\sharp_\alpha}.$$

PROOF: Since the three vector fields $\mathbf{T}, \hat{\beta}^{\sharp_\alpha}$ and $\mathcal{C}^*$ along the projection lie in the same plane, we can find functions F and H in $C^\infty(TM)$, such that $\mathcal{C}^* = F\mathbf{T} + H\hat{\beta}^{\sharp_\alpha}$. Then for any vector field $\tilde{X}$ along the projection $g(\mathcal{C}^*, \tilde{X}) = Fg(\mathbf{T}, \tilde{X}) + Hg(\hat{\beta}^{\sharp_\alpha}, \tilde{X})$. In Corollary 1 and in (21), we have already computed the composing parts of this sum:

$$\begin{aligned} (n+1)\frac{1}{L}\big(\hat{\beta}(\tilde{X}) - \frac{\bar{\beta}}{L_\alpha^2}\bar{\alpha}(\tilde{X})\big) &= F\frac{L}{L_\alpha}\bar{\alpha}(\tilde{X}) + FL\hat{\beta}(\tilde{X}) + H\frac{L + \bar{\beta} + L_\alpha || \hat{\beta} ||_\alpha^2}{L_\alpha}\hat{\beta}(\tilde{X}) \\ &\quad - H\frac{\bar{\beta}^2 - L_\alpha^2 || \hat{\beta} ||_\alpha^2}{L_\alpha^3}\bar{\alpha}(\tilde{X}). \end{aligned} \tag{22}$$

Identifying the coefficients of (22) on both sides, the explicit expression for F and H easily follows. □

Proposition 10. *Consider the first Cartan tensor of a Randers manifold (M, E). Then*

$$\mathcal{C}(\mathcal{C}^*, \mathcal{C}^*) = (n+1)\frac{3}{L^3 L_\alpha}(L_\alpha^2 || \hat{\beta} ||_\alpha^2 - \bar{\beta}^2)\mathcal{C}^* := \frac{\lambda^2}{L^2}\mathcal{C}^*.$$

PROOF: Since $\mathcal{C}(\mathbf{T}, .) = 0$, from the decomposition formula we obtain that $\mathcal{C}(\mathcal{C}^*, \mathcal{C}^*) = (n+1)^2 \frac{L_\alpha^2}{L^4} \mathcal{C}(\hat{\beta}^{\sharp_\alpha}, \hat{\beta}^{\sharp_\alpha})$. Now, using (19), Proposition 6 yields

$$\begin{aligned}
\mathcal{C}(\hat{\beta}^{\sharp_\alpha}, \hat{\beta}^{\sharp_\alpha}) &= \frac{3\bar{\beta}^3 + L_\alpha \bar{\beta}^2 || \hat{\beta} ||_\alpha^2}{L^2 L_\alpha^3}\mathbf{T} - \frac{\bar{\beta} || \hat{\beta} ||_\alpha^2 + L_\alpha || \hat{\beta} ||_\alpha^4}{L^2 L_\alpha}\mathbf{T} + \frac{1}{L} || \hat{\beta} ||_\alpha^2 \hat{\beta}^{\sharp_\alpha} - \frac{\bar{\beta}^2}{L L_\alpha}\hat{\beta}^{\sharp_\alpha} \\
&\quad + 2\frac{\bar{\beta}^2 || \hat{\beta} ||_\alpha^2 - L_\alpha \bar{\beta} || \hat{\beta} ||_\alpha^2}{L^2 L_\alpha^2}\mathbf{T} - 2\frac{\bar{\beta}^2}{L L_\alpha^2}\hat{\beta}^{\sharp_\alpha} + 2\frac{|| \hat{\beta} ||_\alpha^2}{L}\hat{\beta}^{\sharp_\alpha} - 2\frac{|| \hat{\beta} ||_\alpha^4}{L^2}\mathbf{T} \\
&= \frac{3}{2 L L_\alpha}(L_\alpha^2 || \hat{\beta} ||_\alpha^2 - \bar{\beta}^2)\Big(\hat{\beta}^{\sharp_\alpha} - \frac{1}{L L_\alpha}(\bar{\beta} + L_\alpha || \hat{\beta} ||_\alpha^2)\mathbf{T}\Big) \\
&= \frac{3L}{(n+1)L_\alpha^3}(L_\alpha^2 || \hat{\beta} ||_\alpha^2 - \bar{\beta}^2)\mathcal{C}^*,
\end{aligned}$$

where, in the last step, we have also used the previous proposition. Putting everything together the result follows. $\square$

6 Randers Manifolds of Berwald and Landsberg type

One of the most important features that distinguishes Riemann geometry from Finsler geometry is the existence of a torsion-free and metrical linear connection (i.e. the Levi-Civita connection) on the base manifold M. Finsler manifolds lack this property and all four famous linear connections (Berwald, Hashiguchi, Chern-Rund and Cartan) can be regarded as several attempts to partially recover this property, either by relaxing the torsion condition, or by softening the metrical condition. Therefore, it is an interesting question to investigate under what condition a Finsler manifold gives rise to a linear connection on M. At a local-coordinates level, the (horizontal) connection coefficients of for example the Berwald connection would form a linear connection on M if they would not explicitly depend on the coordinates of the (tangent) fibre. Equivalently: if we would derive these connection coefficients with respect to the fibre coordinates, we should find zero. Of course, this is nothing but looking for vanishing (mixed) curvature. The following characterizations of some special classes of Finsler manifolds should be interpreted in this regard.

(A) First we recall a basic result which provides five equivalent properties which can be used to define a so-called *Berwald* manifold. For a Finsler mainifold (M, E) the following conditions are equivalent:

(B1) The mixed curvature of the Berwald connection $(\overset{\circ}{D}, \mathcal{H})$ *is zero:* $\overset{\circ}{\mathcal{P}} = 0$.

(B2) There exists a linear connection $\tilde{\nabla}$ *on* M*, such that* $\widehat{\tilde{\nabla}_X Y} = \mathcal{V}[X^h, Y^v]$ *for each* $X, Y \in \mathcal{X}(M)$.

(B3) For all vector fields X, Y *on* M, $[X^h, Y^v] = Z^v$ *for some* Z *on* M.

(B4) The canonical spray S *is* C^∞ *on the whole tangent manifold* TM.

(B5) The Cartan connection $(D, \mathcal{H})$ and the first Cartan tensor $\mathcal{C}$ of the Finsler manifold have the property that $D_{\mathcal{H}\tilde{X}}\mathcal{C} = 0$ for any vector field $\tilde{X}$ along the projection.

Some proofs in a pullback bundle set-up can be found in [5], for the TTM set-up see [19]. Other references are [12, 16, 21].

(B) Let (M, E) be a Finsler manifold, endowed with the Cartan connection $(D, \mathcal{H})$. The following conditions are also equivalent:

(L1) The mixed curvature of the Cartan connection $(D, \mathcal{H})$ is zero: $\mathcal{P} = 0$.

(L2) The second Cartan tensor $\mathcal{C}'$ vanishes.

(L3) The Berwald connection $(\overset{\circ}{D}, \mathcal{H})$ is h-metrical: $\overset{\circ}{D}_{\mathcal{H}\tilde{X}}g = 0$, for all $\tilde{X} \in \mathcal{X}(\pi)$.

(L4) The Berwald connection $(\overset{\circ}{D}, \mathcal{H})$ and the Cartan connection $(D, \mathcal{H})$ coincide for horizontal vector fields: $\overset{\circ}{D}_{\mathcal{H}\tilde{X}}\tilde{Y} = D_{\mathcal{H}\tilde{X}}\tilde{Y}$, $\forall \tilde{X}, \tilde{Y} \in \mathcal{X}(\pi)$.

(L5) The mixed curvature of the Berwald connection is $\overset{\circ}{\mathcal{P}}(\tilde{X}, \tilde{Y}, \tilde{Z}) = -\frac{1}{2}D_{\mathcal{H}\tilde{X}}\mathcal{C}(\tilde{Y}, \tilde{Z})$.

If one (and thus all) of *(L1)-(L5)* is satisfied then (M, E) is called a *Landsberg manifold.* For an intrinsic proof of the equivalence *(L1)-(L5)* we also refer to [5] or [19], while the standard reference for the classical treatment is [11].

(C) From *(B1),(B5)* and *(L5)* it is obvious that *every Berwald manifold is a Landsberg manifold.* From [12] we cite that *any $\mathcal{C}$-reducible Landsberg manifold is a Berwald manifold.*

Coming back to Randers manifolds (which are clearly $\mathcal{C}$-reducible), it should be clear that the notions of Landsberg and Berwald manifolds are equivalent. It then only remains to find a necessary and sufficient condition which characterizes Berwald and Landsberg manifolds in terms of the initial data of the Randers manifold. After the next proposition, we state a theorem that provides an answer to this question.

Proposition 11. *Suppose that a Finsler manifold (M, E) is Berwald and consider the Cartan connection $(D, \mathcal{H})$ of (M, E). Then for any vector field $\tilde{X}$ along the projection,*

$$D_{\mathcal{H}\tilde{X}}\mathcal{C}^* = 0.$$

PROOF: Since the Cartan connection $(D, \mathcal{H})$ of a Finsler manifold is metrical, we find that for every two vector fields $\tilde{X}$ and $\tilde{Y}$ along the projection

$$\begin{aligned} 0 &= (D_{\mathcal{H}\tilde{X}}g)(\tilde{Y}, \mathcal{C}^*) = \mathcal{H}\tilde{X}[g(\tilde{Y}, \mathcal{C}^*)] - g(D_{\mathcal{H}\tilde{X}}\tilde{Y}, \mathcal{C}^*) - g(\tilde{Y}, D_{\mathcal{H}\tilde{X}}\mathcal{C}^*) \\ &= \mathcal{H}\tilde{X}[tr\mathcal{C}(\tilde{Y})] - tr\mathcal{C}(D_{\mathcal{H}\tilde{X}}\tilde{Y}) - g(\tilde{Y}, D_{\mathcal{H}\tilde{X}}\mathcal{C}^*) \\ &= (D_{\mathcal{H}\tilde{X}}tr\mathcal{C})(\tilde{Y}) - g(\tilde{Y}, D_{\mathcal{H}\tilde{X}}\mathcal{C}^*) \\ &= (tr D_{\mathcal{H}\tilde{X}}\mathcal{C})(\tilde{Y}) - g(\tilde{Y}, D_{\mathcal{H}\tilde{X}}\mathcal{C}^*), \end{aligned} \tag{23}$$

where we have used the properties of the trace again. Since the manifold is supposed to be Berwald, $D_{\mathcal{H}\tilde{X}}\mathcal{C} = 0$ and the first term of the last line vanishes. What remains is that for all $\tilde{X}$, $\tilde{Y}$ along the projection: $g(\tilde{Y}, D_{\mathcal{H}\tilde{X}}\mathcal{C}^*) = 0$, which implies that $D_{\mathcal{H}\tilde{X}}\mathcal{C}^* = 0$ for all $\tilde{X}$. □

Now we turn our attention to one of the most famous theorems of the subject. Some history about it can for example be found in [2], section 11.5, where it is referred as *a useful and elegant theorem.* Note that the proof of [2] relies strongly on local coordinate calculations and cannot be readily translated into intrinsic terms (The same remark is true for the other proof mentioned in [2], p 302). Our proof is based on different geometric ideas and is completely intrinsic.

Theorem. *A Randers manifold is a Berwald manifold if and only if $\nabla\beta = 0$, where ∇ is the Levi-Civita connection of the Riemanian manifold (M, α).*

Proof: One implication is obvious: if $\nabla\beta = 0$, then the spray S of the Randers manifold is nothing but the spray S_α, the Barthel endomorphism is h_α and both the Berwald connection and the Cartan connection are reduced to the connection $(\overset{\alpha}{D}, \mathcal{H}_\alpha)$ since the Cartan tensor vanishes. It is obvious that all the characterizations of Berwald (and Landsberg) manifolds are satisfied.

It is harder to prove the converse. The reasoning will be divided into several steps. We will rely on the fact that every vector field along the projection can be decomposed into a component of $Span(\mathbf{T}, \hat{\beta}^{\sharp_\alpha})$ and an orthogonal component $\tilde{X}^\perp$. Having *(B5)* in mind, we will calculate respectively the consequences of the fact that $\big(D_{\mathcal{H}\mathbf{T}}\mathcal{C}\big)(\mathcal{C}^*, \mathcal{C}^*) = 0$, $\big(D_S\mathcal{C}\big)(\tilde{X}^\perp, \mathcal{C}^*) = 0$ and $\big(D_{\mathcal{H}\tilde{X}^\perp}\mathcal{C}\big)(\tilde{Y}^\perp, \mathcal{C}^*) = 0$ in steps 1, 2 and 3. After each step, it will be possible to simplify the expression (17) of the Berwald connection.

Step 1. First we compute $\big(D_{\mathcal{H}\mathbf{T}}\mathcal{C}\big)(\mathcal{C}^*, \mathcal{C}^*) = 0$. Propositions 10 and 11 then imply

$$S(\frac{\lambda^2}{L^2}) = 0 \qquad \text{or} \qquad S(\lambda^2) = 0,$$

since $S(L) = 0$. Computing $S(\lambda^2)$ with the aid of Corollary 2 gives

$$\begin{aligned} S(\lambda^2) \;=\; & \frac{3(n+1)}{L^2 L_\alpha}\Big[\big(2||\,\hat{\beta}\,||_\alpha^2 \overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha}) + 2\overline{\nabla\beta}(\hat{\beta}^{\sharp_\alpha})\big)L_\alpha^3 \\ & +\big(4\bar{\beta}\widehat{\nabla\beta}(\mathbf{T}, \hat{\beta}^{\sharp_\alpha}) - 2\bar{\beta}\widehat{\nabla\beta}(\hat{\beta}^{\sharp_\alpha}, \mathbf{T}) - ||\,\hat{\beta}\,||_\alpha^2\overline{\nabla\beta}(\mathbf{T})\big)L_\alpha^2 \\ & +\big(2\bar{\beta}^2\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha}) - 2\bar{\beta}\overline{\nabla\beta}(\mathbf{T})\big)L_\alpha - \bar{\beta}^2\overline{\nabla\beta}(\mathbf{T})\Big]. \end{aligned} \tag{24}$$

Expressing that $S(\lambda^2)(v) = 0$ and $S(\lambda^2)(-v) = 0$ for arbitrary v, we arrive at the separate equations

$$\big(||\,\hat{\beta}\,||_\alpha^2\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha}) + \overline{\nabla\beta}(\hat{\beta}^{\sharp_\alpha})\big)L_\alpha^2 + \bar{\beta}^2\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha}) - \bar{\beta}\overline{\nabla\beta}(\mathbf{T}) = 0, \tag{25}$$

$$\big(4\bar{\beta}\widehat{\nabla\beta}(\mathbf{T}, \hat{\beta}^{\sharp_\alpha}) - 2\bar{\beta}\widehat{\nabla\beta}(\hat{\beta}^{\sharp_\alpha}, \mathbf{T}) - ||\,\hat{\beta}\,||_\alpha^2\overline{\nabla\beta}(\mathbf{T})\big)L_\alpha^2 - \bar{\beta}^2\overline{\nabla\beta}(\mathbf{T}) = 0. \tag{26}$$

In the last equation, all terms contain a factor $\bar{\beta}$, except one. From Lemma 3, we know that it is impossible that L_α^2 contains a factor $\bar{\beta}$, so there should exist a 1-form ν on M

such that $\overline{\nabla\beta}(\mathbf{T}) = \bar{\nu}\bar{\beta}$. But then equation (26) can be reduced to

$$\bar{\mu}L_\alpha^2 - \bar{\nu}\bar{\beta}^2 = 0,$$

with $\hat{\mu}(\tilde{X}) = 4\widehat{\nabla\beta}(\tilde{X}, \hat{\beta}^{\sharp_\alpha}) - 2\widehat{\nabla\beta}(\hat{\beta}^{\sharp_\alpha}, \tilde{X}) - ||\,\hat{\beta}\,||_\alpha^2\hat{\nu}(\tilde{X})$. This way, however the problem has merely been shifted, because now $\bar{\mu}L_\alpha^2$ should contain $\bar{\beta}$, which is only possible if $\mu = f\beta$ ($f \in C^\infty(M)$). But then

$$f^v L_\alpha^2 - \bar{\nu}\bar{\beta} = 0,$$

which is again impossible, in view of Lemma 3. We conclude that both $\mu = \nu = 0$, or

$$\overline{\nabla\beta}(\mathbf{T}) = \boxed{\widehat{\nabla\beta}(\mathbf{T}, \mathbf{T}) = 0}, \tag{27}$$

and

$$2\widehat{\nabla\beta}(\mathbf{T}, \hat{\beta}^{\sharp_\alpha}) - \widehat{\nabla\beta}(\hat{\beta}^{\sharp_\alpha}, \mathbf{T}) = 0. \tag{28}$$

Lemma 5 shows that if we take the $\hat{\beta}^{\sharp_\alpha}$-vertical derivative of the identity $\overline{\nabla\beta}(\mathbf{T}) = 0$, we find $Sym\widehat{\nabla\beta}(\mathbf{T}, \hat{\beta}^{\sharp_\alpha}) = 0$. Together with (28), we obtain

$$\boxed{\widehat{\nabla\beta}(\mathbf{T}, \hat{\beta}^{\sharp_\alpha}) = \widehat{\nabla\beta}(\hat{\beta}^{\sharp_\alpha}, \mathbf{T}) = 0}. \tag{29}$$

Taking two times the $\hat{\beta}^{\sharp_\alpha}$-vertical derivative of (27) leads to

$$\boxed{\widehat{\nabla\beta}(\hat{\beta}^{\sharp_\alpha}, \hat{\beta}^{\sharp_\alpha}) = 0}. \tag{30}$$

Analogously, by taking the X-vertical derivative of $\widehat{\nabla\beta}(\mathbf{T}, \hat{\beta}^{\sharp_\alpha}) = 0$, we can find with the aid of Lemma 5 that for $X \in \mathcal{X}(M)$, $\widehat{\nabla\beta}(\hat{X}, \hat{\beta}^{\sharp_\alpha}) = 0$ and, by $C^\infty(TM)$-linearity,

$$\widehat{\nabla\beta}(\tilde{X}, \hat{\beta}^{\sharp_\alpha}) = 0, \tag{31}$$

for every $\tilde{X} \in \mathcal{X}(\pi)$. The same reasoning can be applied to $\widehat{\nabla\beta}(\hat{\beta}^{\sharp_\alpha}, \mathbf{T})$ and $\widehat{\nabla\beta}(\mathbf{T}, \mathbf{T})$:

$$\widehat{\nabla\beta}(\hat{\beta}^{\sharp_\alpha}, \tilde{X}) = 0, \qquad (Sym\widehat{\nabla\beta})(\mathbf{T}, \tilde{X}) = 0 \qquad \text{and} \qquad (Sym\widehat{\nabla\beta})(\tilde{X}, \tilde{Y}) = 0. \tag{32}$$

In the special case of $\tilde{X}^\perp$, we find that $\boxed{\widehat{\nabla\beta}(\tilde{X}^\perp, \hat{\beta}^{\sharp_\alpha}) = \widehat{\nabla\beta}(\hat{\beta}^{\sharp_\alpha}, \tilde{X}^\perp) = 0}$.

Consequences of Step 1. The relations (27, 29, 30, 31, 32) have immediate effect on the expressions (17) we had for the Berwald connection of the Randers manifolds. Let us look for example at $\overset{\circ}{D}_S\hat{X}$: for $X \in \mathcal{X}(M)$ one easily checks that: $\overset{\circ}{D}_S\hat{X} - \overset{\alpha}{D}_{S_\alpha}\hat{X} = -\frac{1}{L_\alpha}\bar{\alpha}(\hat{X})\overline{Alt\nabla\beta}^{\sharp_\alpha}$. This relation is tensorial on both sides and will thus also hold if we replace $\hat{X}$ by a general vector field $\tilde{X}$ along the projection. Applying $\hat{\beta}$ on both sides then gives

$$\begin{aligned}\hat{\beta}(\overset{\circ}{D}_S\tilde{X} - \overset{\alpha}{D}_{S_\alpha}\tilde{X}) &= -\frac{1}{L_\alpha}\bar{\alpha}(\tilde{X})\hat{\beta}(\overline{Alt\nabla\beta}^{\sharp_\alpha}) = -\frac{1}{L_\alpha}\bar{\alpha}(\tilde{X})\hat{\alpha}(\overline{Alt\nabla\beta}^{\sharp_\alpha}, \hat{\beta}^{\sharp_\alpha}) \\ &= -\frac{1}{L_\alpha}\bar{\alpha}(\tilde{X})\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_\alpha}) = 0. \end{aligned} \tag{33}$$

In particular for $\tilde{X}^\perp$ (where $\hat{\beta}(\tilde{X}^\perp)=0$), we can conclude that $\big(\overset{\circ}{D}_S\hat{\beta}\big)(\tilde{X}^\perp)=\big(\overset{\alpha}{D}_S\hat{\beta}\big)(\tilde{X}^\perp) = \widehat{\nabla\beta}(\mathbf{T},\tilde{X}^\perp)$. Remember that one of the characterizing properties of a Berwald (and thus Landsberg in the Randers case) manifold was that $D_{\mathcal{H}\tilde{X}}\tilde{Y}=\overset{\circ}{D}_{\mathcal{H}\tilde{X}}\tilde{Y}$, thus also $\big(D_S\hat{\beta}\big)(\tilde{X}^\perp)=\big(\overset{\circ}{D}_S\hat{\beta}\big)(\tilde{X}^\perp)$.

Step 2. Obviously, the next step would be to prove that $\big(D_S\hat{\beta}\big)(\tilde{X}^\perp)=0$. For this purpose, we start by computing

$$0=\big(D_S\mathcal{C}\big)(\tilde{X}^\perp,\mathcal{C}^*)=D_S\big(\mathcal{C}(\tilde{X}^\perp,\mathcal{C}^*)\big)-\mathcal{C}(D_S\tilde{X}^\perp,\mathcal{C}^*)-\mathcal{C}(\tilde{X}^\perp,D_S\mathcal{C}^*),$$

where $\mathcal{C}(\tilde{X}^\perp,\mathcal{C}^*)=\frac{2}{3}\lambda^2\tilde{X}^\perp$ and

$$\begin{aligned}\mathcal{C}\big(D_S\tilde{X}^\perp,\mathcal{C}^*\big) \;=\; & \frac{(n+1)}{L^4L_\alpha}\Big[\big(-\bar{\beta}L\hat{\alpha}(D_S\tilde{X}^\perp)+2L_\alpha^2L\hat{\beta}(D_S\tilde{X}^\perp)\big)\hat{\beta}^{\sharp_\alpha}-\frac{2}{3}\lambda^2D_S\tilde{X}^\perp\\ &+(3\bar{\beta}^2-||\,\hat{\beta}\,||_\alpha^2L_\alpha^2+2||\,\hat{\beta}\,||_\alpha^2L_\alpha\bar{\beta})\hat{\alpha}(D_S\tilde{X}^\perp)\mathbf{T}\\ &+(\bar{\beta}^2-3||\,\hat{\beta}\,||_\alpha^2L_\alpha^2)\hat{\beta}(D_S\tilde{X}^\perp)\mathbf{T}\Big].\end{aligned}$$

Since $S(\lambda^2)=0$ and since all the terms in the direction of $D_S\tilde{X}^\perp$ vanish, we are left with a linear combination of terms in $\hat{\beta}^{\sharp_\alpha}$ and $\mathbf{T}$. But, both vector fields are linearly independent, which means that their coefficients should vanish:

$$\begin{gathered}(3\bar{\beta}^2-||\,\hat{\beta}\,||_\alpha^2L_\alpha^2+2||\,\hat{\beta}\,||_\alpha^2L_\alpha\bar{\beta})\bar{\alpha}(D_S\tilde{X}^\perp)+(\bar{\beta}^2-3||\,\hat{\beta}\,||_\alpha^2L_\alpha^2)\hat{\beta}(D_S\tilde{X}^\perp)=0\\ -\bar{\beta}L\bar{\alpha}(D_S\tilde{X}^\perp)+2L_\alpha^2L\hat{\beta}(D_S\tilde{X}^\perp)=0\end{gathered}$$

The determinant of this system of equations for the unknown $\bar{\alpha}(D_S\tilde{X}^\perp)$ and $\hat{\beta}(D_S\tilde{X}^\perp)$ is not zero (due to Lemma 3), which means that the only solution can be $\bar{\alpha}(D_S\tilde{X}^\perp)=\hat{\beta}(D_S\tilde{X}^\perp)=0$. The last part leads to the desired result

$$\big(D_S\hat{\beta}\big)(\tilde{X}^\perp)=\boxed{\widehat{\nabla\beta}(\mathbf{T},\tilde{X}^\perp)=0}. \tag{34}$$

Consequences of Step 2. Together with (27) and (29), (34) shows that for all $\tilde{X}\in\mathcal{X}(\pi)$, $\widehat{\nabla\beta}(\mathbf{T},\tilde{X})=0$ and automatically, because of $Sym\widehat{\nabla\beta}(\mathbf{T},\tilde{X})=0$, also $\widehat{\nabla\beta}(\tilde{X},\mathbf{T})=0$ and thus $\overline{Alt\nabla\beta}^{\sharp_\alpha}=0$. This last result is very important, because now $S=S_\alpha$ and $h=h_\alpha$, so $D_{X^h}\hat{Y}=\overset{\circ}{D}_{X^h}\hat{Y}=\overset{\alpha}{D}_{X^{h_\alpha}}\hat{Y}=\widehat{\nabla_XY}$.

Step 3. To finish the proof, we only need to show now that for example $\hat{\beta}(D_{\mathcal{H}\tilde{X}^\perp}\tilde{Y}^\perp)=0$. As mentioned before, we will look now at

$$0=\big(D_{\mathcal{H}\tilde{X}^\perp}\mathcal{C}\big)(\tilde{Y}^\perp,\mathcal{C}^*)=D_{\mathcal{H}\tilde{X}^\perp}\big(\mathcal{C}(\tilde{Y}^\perp,\mathcal{C}^*)\big)-\mathcal{C}\big(D_{\mathcal{H}\tilde{X}^\perp}\tilde{Y}^\perp,\mathcal{C}^*\big). \tag{35}$$

Here

$$\begin{aligned}\mathcal{C}\big(D_{\mathcal{H}\tilde{X}^\perp}\tilde{Y}^\perp,\mathcal{C}^*\big) \;=\; & \frac{(n+1)}{L^4L_\alpha}\Big[\big(-\bar{\beta}L\bar{\alpha}(D_{\mathcal{H}\tilde{X}^\perp}\tilde{Y}^\perp)+2L_\alpha^2L\hat{\beta}(D_{\mathcal{H}\tilde{X}^\perp}\tilde{Y}^\perp)\big)\hat{\beta}^{\sharp_\alpha}-\frac{2}{3}\lambda^2D_{\mathcal{H}\tilde{X}^\perp}\tilde{Y}^\perp\\ &+(3\bar{\beta}^2-||\,\hat{\beta}\,||_\alpha^2L_\alpha^2+2||\,\hat{\beta}\,||_\alpha^2L_\alpha\bar{\beta})\bar{\alpha}(D_{\mathcal{H}\tilde{X}^\perp}\tilde{Y}^\perp)\mathbf{T}\\ &+(\bar{\beta}^2-3||\,\hat{\beta}\,||_\alpha^2L_\alpha^2)\hat{\beta}(D_{\mathcal{H}\tilde{X}^\perp}\tilde{Y}^\perp)\mathbf{T}\Big].\end{aligned}$$

Again, the terms in $D_{\mathcal{H}\tilde{X}^\perp}\tilde{Y}^\perp$ will vanish in (35), and we are left with a linear combination of terms in $\tilde{Y}^\perp$, $\mathbf{T}$ and $\hat{\beta}^{\sharp_\alpha}$. These vector fields along π are linearly independent, so their coefficients should vanish. In the same way as before this will lead to

$$\bar{\alpha}(D_{\mathcal{H}\tilde{X}^\perp}\tilde{Y}^\perp) = 0 \qquad \text{and} \qquad \hat{\beta}(D_{\mathcal{H}\tilde{X}^\perp}\tilde{Y}^\perp) = 0.$$

Especially, the last equation is of importance: it means in fact that $0 = \big(D_{\mathcal{H}\tilde{X}^\perp}\beta\big)(\tilde{Y}^\perp)$, or $\boxed{\widehat{\nabla\beta}(\tilde{X}^\perp, \tilde{Y}^\perp) = 0}$. □

The proof of this theorem can also be extended in such a way that it shows that (cf. [12, 16]):

Corollary 4. *For a Randers manifold, the following properties are equivalent*

(1) The manifold is a Landsberg manifold.

(2) The manifold is a Berwald manifold.

(3) $D_{X^h}\hat{Y} = \widehat{\nabla_X Y}$*, for any vector fields* X, Y *on the base manifold.*

(4) $\nabla\beta = 0$.

(5) $D_{\mathcal{H}\tilde{X}}\hat{\beta} = 0$.

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