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AUTONOMOUS OPTIMAL CONTROL PROBLEMS

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A geometric version of the maximum principle for autonomous optimal control problems is derived and applied to the length-minimising problem in sub-Riemannian geometry and to Lagrangian mechanics on Lie-algebroids.

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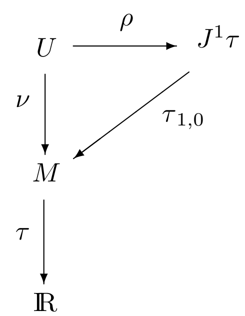
1. Introduction

The development of a differential geometric setting for optimal control theory has been carried out, among other, by H.J. Sussmann in [12], where a coordinate-free formulation of the maximum principle is given.

In [4] we have given a proof of a coordinate-free version of the maximum principle for (time-dependent) optimal control systems with fixed endpoint conditions, relying on an approach due to L.S. Pontryagin et al.in [5]. As a side result of our approach, we were able to give some necessary and sufficient conditions for the existence of so-called *(strictly) abnormal extremals* (for an example of a stricly abnormal extremal, we refer to [7]). In this paper, it is our goal to prove the maximum principle for autonomous optimal control problems and apply it to sub-Riemannian geometry ([9, 10]) and Lagrangian systems on Lie-algebroids ([1, 6, 13]).

The outline of the paper is as follows. In the remainder of this section we first recall the notion of a geometric control structure as described in [4] in a time-dependent setting, as well as the notions of an optimal control problem with fixed and variable endpoints, respectively. We then consider an adapted version of these notions for the autonomous case. In Section 2. we briefly review the approach to the maximum principle presented in [3] and [4] and use it as a starting point to derive a version of the maximum principle for autonomous optimal control problems. Section 3. contains some specific results for linear autonomous optimal control problems and in Section 4. we discuss some applications.

We start by recalling the definition of *a geometric control structure* (see [4] for more details). A geometric control structure is a triple (τ, ν, ρ) where (i) $\tau : M \to \mathbb{R}$ and (*ii*) $\nu : U \to M$ are fibre bundles, with typical fibres denoted by, respectively, Q and C, where Q is called the *configuration space* and C the *control domain.* The control bundle U is related to the first jet bundle of τ by means of a bundle map (*iii*) $\rho : U \to J^1\tau$, with $\tau_{1,0} \circ \rho = \nu$ (where $\tau_{1,0} : J^1\tau \to M$ denotes the standard projection, see for instance [8]). This is represented schematically in the following commuting diagram:



Let u denote a smooth section of $\tau \circ \nu$, i.e. $u : I = [a, b] \to U$ such that $\tau(\nu(u(t))) = t$ (we assume that I is a compact interval in $\mathbb{R}$ and that u admits a smooth extension to an open interval containing I). Then u is a *control* if $\rho \circ u = j^1c$ with $c = \nu \circ u$ the *base section of* u. Given any section c of τ, then c is called a *controlled* section if c is the base section of a control. We say that the control u *takes the point* $c(a)$ *to the point* $c(b)$, and is represented schematically by $c(a) \xrightarrow{u} c(b)$. In order to fix the ideas we first investigate locally the notion of smooth controls. Let (t, x^i) denote bundle adapted coordinates on M, and, similarly, let (t, x^i, u^a) denote bundle adapted coordinates on U. Then, a control u locally satisfies:

$$\rho^i(t, c^i(t), u^a(t)) = \dot{c}^i(t),$$

for all t. It is easily seen that these equations correspond to the definition of a control in [5].

However, it turns out that the class of smooth controls should be further extended to sections admitting (a finite number of) discontinuities in the form of certain 'jumps' in the fibres of ν, such that the corresponding base section is piecewise smooth. For instance, assume that $u_1 : [a, b] \to U$ and $u_2 : [b, c] \to U$ are two smooth controls with respective bases c_1 and c_2, such that $c_1(b) = c_2(b)$. The *composite control* $u_2 \cdot u_1 : [a, c] \to U$ of u_1 and u_2 is defined by:

$$u_2 \cdot u_1(t) = \left\{ \begin{array}{ll} u_1(t) & t \in [a, b], \\ u_2(t) & t \in]b, c]. \end{array} \right.$$

It is readily seen that, although in general $u_2 \cdot u_1$ is, discontinuous at $t = b$, the base section $\nu \circ (u_2 \cdot u_1)$ is continuous. This definition can easily be extended to any finite number of smooth controls, yielding what we shall call in general a *control* (a detailed definition can be found in [4]).

In the remainder of this section we focus our attention on *optimal control problems.* Assume that a *cost* function L on the control space U is given, i.e. $L \in C^\infty(U)$. With any control $u : [a, b] \to M$ we are now able to define its *cost* $\mathcal{J}(u)$:

$$\mathcal{J}(u) = \int_a^b L(u(t))dt.$$

A control u taking $x = \nu(u(a))$ to $y = \nu(u(b))$ is said to be *optimal* if, for any other control u' taking x to y we have

$$\mathcal{J}(u) \leq \mathcal{J}(u').$$

The problem of finding the optimal controls taking a given point to another given point is called an *optimal control problem with fixed endpoint conditions.*

On the other hand, assume that two immersed submanifolds $i : S_i \to M$ and $j : S_f \to M$ are given. A control u taking a point $x \in i(S_i)$ to a point $y \in j(S_f)$ is said to be (S_i, S_f)*-optimal* if, given any other control u' taking $x' \in i(S_i)$ to $y' \in j(S_f)$, then $\mathcal{J}(u) \leq \mathcal{J}(u')$. The problem of finding the (S_i, S_f)-optimal controls taking a point in S_i to a point in S_f is called an *optimal control problem with variable endpoint conditions* (see [3]).

An *autonomous geometric control structure* consists of a pair $(\tilde{\nu}, \tilde{\rho})$ where $\tilde{\nu} : C \to Q$ is a bundle and $\tilde{\rho} : C \to TQ$ a bundle map fibred over the identity on Q. We can then consider the following geometric control structure (τ, ν, ρ), in the sense defined above, associated with $(\tilde{\nu}, \tilde{\rho})$:

1. $\tau : M = \mathbb{R} \times Q \to \mathbb{R} : (t, q) \mapsto \tau(t, q) = t$,
2. $\nu : U = \mathbb{R} \times C \to M : (t, p) \mapsto \nu(t, p) = (t, \tilde{\nu}(p))$, and
3. $\rho : U \to J^1\tau : (t, p) \mapsto (t, \tilde{\rho}(p))$, since $J^1\tau \cong \mathbb{R} \times TQ$.

Let $\tilde{u} : [a, b] \to C$ denote a curve in the control domain C. Then $\tilde{u}$ is called $\tilde{\rho}$-*admissible* if $\tilde{\rho}(\tilde{u}(t)) = \dot{\tilde{c}}(t)$, where $\tilde{c}(t) = \tilde{\nu}(\tilde{u}(t))$ is called the base of $\tilde{u}$.

We now translate concepts defined in the autonomous geometric control structure $(\tilde{\nu}, \tilde{\rho})$ to known concepts in the associated geometric control structure (τ, ν, ρ). For instance, it is an easy exercise to see that every $\tilde{\rho}$-admissible curve $\tilde{u} : [a, b] \to C$ determines a smooth control $u : [a, b] \to U$ with $u(t) = (t, \tilde{u}(t))$. On the other hand, if we assume that $u : [a, b] \to U$ is a smooth control, then u can be written as $u(t) = (t, \tilde{u}(t))$ since u is a section of $\tau \circ \nu$. The curve $\tilde{u} : [a, b] \to C$ is $\tilde{\rho}$-admissible since $\rho(u(t)) = (t, \tilde{\rho}(\tilde{u}(t))$ and $j^1_t c = (t, \dot{\tilde{c}}(t))$ (where $c(t) = (t, \tilde{c}(t))$ denotes the base of u). This correspondence between $\tilde{\rho}$-admissible curves and smooth controls is easily extended to the more general notion of composite controls. Similar to the control setting, we say that the $\tilde{\rho}$-admissible curve $\tilde{u}$ takes $\tilde{c}(a)$ to $\tilde{c}(b)$.

Consider a function $\tilde{L}$ on C. Then we can define a cost function on U by considering $L = p_C^*\tilde{L}$, where $p_C : U \to C$ denotes the natural projection. We define a functional on the set of $\tilde{\rho}$-admissible curves by:

$$\mathcal{J}(\tilde{u}) = \int_a^b \tilde{L}(\tilde{u}(t))dt,$$

where we have used the same notation as in the previous section since $\mathcal{J}(\tilde{u}) = \mathcal{J}(u)$, with u the control associated with $\tilde{u}$.

A $\tilde{\rho}$-admissible curve $\tilde{u} : [a, b] \to C$, taking p to q, is called *optimal* if, given any $\tilde{\rho}$-admissible curve $\tilde{u}' : [a, b] \to C$ taking p to q, then $\mathcal{J}(\tilde{u}) \leq \mathcal{J}(\tilde{u}')$. Note that $\tilde{u}$ is optimal with respect to $\tilde{L}$ iff the associated control $u(t) = (t, \tilde{u}(t))$ is optimal with respect to L.

We say that $\tilde{u}$ is *strong optimal* if, given any other control $\tilde{u}' : [a', b'] \to C$ taking p to q, then $\mathcal{J}(\tilde{u}) \leq \mathcal{J}(\tilde{u}')$. Consider $S_p = \mathbb{R}$ and $i_p : S_p \to \mathbb{R} \times Q : s \mapsto (s, p)$. Similarly we put $S_q = \mathbb{R}$ with $j_q : S_q \to \mathbb{R} \times Q : s \mapsto (s, q)$. Using the notations from above, we can write $S_i = S_p$ and $S_f = S_q$. It is now easily seen that a $\tilde{\rho}$-admissible curve $\tilde{u}$, taking p to q is strong optimal iff the associated control u is (S_p, S_q)-optimal with respect to $p_C^* \tilde{L}$. The problem of finding the optimal $\tilde{\rho}$-admissible curves taking p to q, is called the *autonomous optimal control problem.*

2. The Maximum principle

2.1. Non-autonomous optimal control problems

We now proceed towards the formulation of the maximum principle for non-autonomous optimal control problems, proven in [4], providing necessary conditions for optimal controls. We first define the notion of a multiplier of a control u. For that purpose, we construct a 1-parameter family of closed two-forms on $U \times_M V^*\tau$ (where $V\tau = \ker T\tau$ is the *vertical* subbundle of TM and $V^*\tau$ its dual). Let $\tilde{\omega}$ be the closed two-form on the fibred product $U \times_M T^*M$, obtained by pulling back the canonical symplectic form on T^*M by the projection $U \times_M T^*M \to T^*M$. Next, for any real number λ we can define a section σ_λ of the fibration $U \times_M T^*M \to U \times_M V^*\tau$ in the following way. Take $u_m \in U_m, \eta_m \in V_m^*\tau$ and put $\sigma_\lambda(u_m, \eta_m) = (u_m, \alpha_m)$, where $\alpha_m \in T_m^*M$ is uniquely determined by the conditions (i) $\langle \alpha_m, \mathbf{T}(\rho(u_m)) \rangle + \lambda L(u_m) = 0$ and (ii) α_m projects onto η_m. As usual, $\mathbf{T} : J^1\tau \to TM$ represents the total time derivative defined by $\mathbf{T}(j_t^1 c) = T_t c(\partial_t)$, for $j_t^1 c \in J^1\tau$ arbitrary and with ∂_t the standard vector field on $\mathbb{R}$. The mapping σ_λ is smooth, as can be easily seen from the following coordinate expression: putting $u_m = (t, x^i, u^a)$ and $\eta_m = p_i \, dx^i_{|m}$, a straightforward computation gives

$$\sigma_\lambda(t, x^i, u^a, p_i) = \left(t, x^i, u^a, -\rho^i(t, x^i, u^a) p_i - \lambda L(t, x^i, u^a), p_i\right).$$

We can now use σ_λ to pull-back the closed two-form $\tilde{\omega}$ to a closed two-form on $U \times_M V^*\tau$, which will be denoted by $\omega_\lambda = \sigma_\lambda^* \tilde{\omega}$. Herewith, we can introduce the following definition of a multiplier.

Definition 1. Given a control $u : [a, b] \to U$, a pair (η, λ) consisting of a piecewise smooth section η of $V^*\tau$ along $c = \nu \circ u$ and a real number λ, is called a multiplier of u if the following conditions are satisfied:

1. $i_{(\dot{u}(t), \dot{\eta}(t))} \omega_\lambda = 0$ on every smooth part of the curve $(u(t), \eta(t))$,
2. given any $t_0 \in [a, b]$, and if we put $\sigma_\lambda(u(t_0), \eta(t_0)) = (u(t_0), \alpha_0)$ then the function $u' \mapsto \langle \alpha_0, \mathbf{T}(\rho(u')) \rangle + \lambda L(u')$, defined on $\nu^{-1}(c(t_0))$, attains a global maximum for $u' = u(t_0)$,

3. $(\eta(t), \lambda) \neq (0,0)$ for all $t \in [a,b]$.

We then have the following result (cf. [4]).

THEOREM 1. *Assume that $x \xrightarrow{u} y$ and that u is optimal. Then there exists a multiplier (η, λ) with $\lambda \leq 0$.*

In [3] we have considered optimal control problems with variable endpoints. Assume that S_i and S_f denote two immersed submanifolds of M, where either S_i or S_f reduce to a point (note that if we assume u, with $x \xrightarrow{u} y$, to be (S_i, S_f)-optimal, then u is also (S'_i, S'_f)-optimal where $S'_i = S_i$ and $S'_f = \{y\}$ or $S'_f = S_f$ and $S'_i = \{x\}$). Let us denote the annihilator of a linear subspace W in a vector space V by W^0.

THEOREM 2. *Let $x \xrightarrow{u} y$ with $x \in S_i$ and $y \in S_f$. If u is (S_i, S_f)-optimal then there exists a multiplier (η, λ) such that*

1. $\lambda \leq 0$,
2. $\sigma_\lambda(u(a), \eta(a)) \in (Ti(T_x S_i))^0$, *if* $S_f = \{y\}$ *or* $\sigma_\lambda(u(b), \eta(b)) \in (Tj(T_y S_f))^0$, *if* $S_i = \{x\}$.

2.2. Autonomous optimal control problems

Consider an autonomous optimal control problem as defined in Section 1. and how it was related to a non-autonomous optimal control problem. Let ω_Q denote the canonical symplectic form on T^*Q, and consider the closed two-form $\tilde{\omega}_Q$ on $C \times_Q T^*Q$, which is the pull-back of ω_Q under the projection $p_{T^*Q} : C \times_Q T^*Q \to T^*Q$. We can now prove the following version of the maximum principle for autonomous optimal control problems.

THEOREM 3. *If a $\tilde{\rho}$-admissible curve $\tilde{u} : [a,b] \to C$ is optimal with respect to $\tilde{L}$ then there exists a piecewise smooth one-form $\tilde{\eta}(t)$ along $\tilde{c}(t) = \tilde{\nu}(\tilde{u}(t))$ and a real number $\lambda \leq 0$ such that:*

1. $i_{(\dot{\tilde{u}}(t), \dot{\tilde{\eta}}(t))}\tilde{\omega}_Q = -h_\lambda(\tilde{u}(t), \tilde{\eta}(t))$ *on every smooth part of the curve* $(\tilde{u}, \tilde{\eta})(t)$, *with* $h_\lambda \in C^\infty(C \times_Q T^*Q)$ *defined by* $h_\lambda(\tilde{u}'_p, \zeta_p) = \langle \zeta_p, \tilde{\rho}(\tilde{u}'_p)\rangle + \lambda L(\tilde{u}'_p)$ *for arbitrary* $(\tilde{u}'_p, \zeta_p) \in C \times_Q T^*Q$,
2. *given any* $t \in I$, *the function* $\tilde{u}' \mapsto h_\lambda(\tilde{u}', \tilde{\eta}(t))$ *on* $C_{c(t)} = \tilde{\nu}^{-1}(c(t))$ *attains a global maximum for* $\tilde{u}' = \tilde{u}(t)$,
3. $h_\lambda(\tilde{u}(t), \tilde{\eta}(t)) = const.$ *for all* t *and,*
4. $(\tilde{\eta}(t), \lambda) \neq 0$ *for all* $t \in I$.

If $\tilde{u}$ is strong optimal then condition (3) is to be replaced by $h_\lambda(\tilde{u}(t), \tilde{\eta}(t)) = 0$.

Proof. Since we already know from Section 1. that if $\tilde{u}$ is optimal, then the associated control u is also optimal in the non-autonomous setting. This correspondence admits us to apply Theorem 1. The remainder of this proof consists of translating the necessary conditions from the time-dependent setting to the autonomous setting.

First, note that $V^*\tau \cong \mathbb{R} \times T^*Q$ and that the section σ_λ can be written as $\sigma_\lambda(u', \zeta) = -h_\lambda(p_C(u'), \zeta)dt + \zeta$ where $(u', \zeta) \in U \times_M V^*\tau$. Recalling the definition of a multiplier then we know that there exists a piecewise smooth one-form $\eta(t) = (t, \tilde{\eta}(t))$ and a $\lambda \leq 0$ such that, of u denotes the control associated with $\tilde{u}$,

1. $i_{(\dot{u}(t), \dot{\eta}(t))}\omega_\lambda = 0$ on every smooth part of the curve $(u, \eta)(t)$,
2. given any $t \in I$, then the function $u' \mapsto \langle \sigma_\lambda(u(t), \eta(t)), \mathbf{T}(\rho(u')) \rangle + \lambda L(u')$ defined on $\nu^{-1}(c(t))$ attains a global maximum for $u' = u(t)$,
3. $(\eta(t), \lambda) \neq 0$ for all $t \in I$.

The closed two form ω_λ equals (with a slight abuse of notations) $\tilde{\omega}_Q - dh_\lambda \wedge dt$. The function

$$u' \mapsto \langle \sigma_\lambda(u(t), \eta(t)), \mathbf{T}(\rho(u')) \rangle + \lambda L(u'),$$

equals $-h_\lambda(\tilde{u}(t), \tilde{\eta}(t)) + h_\lambda(p_C(u'), \tilde{\eta}(t))$. We conclude that $h_\lambda(p_C(u'), \tilde{\eta}(t))$ also attains a global maximum for $u' = u(t)$ or for $\tilde{u}' = p_C(u') = \tilde{u}(t)$.

Given any tangent vector $X = \tilde{X} + X^t \frac{\partial}{\partial t} \in T(U \times_M V^*\tau)$, with $\tilde{X} \in T(C \times_Q T^*Q)$, then $i_X \omega_\lambda = i_{\tilde{X}} \tilde{\omega}_Q - dh_\lambda(\tilde{X})dt + X^t dh_\lambda$. If we assume that $X = (\dot{u}(t), \dot{\eta}(t))$, then $X^t = 1$ and $\tilde{X} = (\dot{\tilde{u}}(t), \dot{\tilde{\eta}}(t))$, and equation (1) is equivalently rewritten as: $i_{(\dot{\tilde{u}}(t), \dot{\tilde{\eta}}(t))}\tilde{\omega}_Q = -dh_\lambda$.

Since $(\tilde{u}, \tilde{\eta})$ solves the implicit Hamiltonian system with Hamiltonian h_λ, the function h_λ is constant on every smooth part of the curve $(\tilde{u}(t), \tilde{\eta}(t))$. Thus, it remains to prove that $h_\lambda(\tilde{u}(t), \tilde{\eta}(t))$ is continuous. Consider therefore a discontinuous point (at $t = t_0$) of $\tilde{u}(t)$, and assume that we have fixed an adapted coordinate chart containing it. Then,

$$h_\lambda(\tilde{c}^i(t), \tilde{u}^a(t), \tilde{\eta}_i(t)) \geq h_\lambda(\tilde{c}^i(t), w^a, \tilde{\eta}_i(t)),$$

for all w^a. If we consider this inequality and take successively the limit from the left and from the right for $t \to t_0$, we obtain the continuity of $h_\lambda(\tilde{c}^i(t), \tilde{u}^a(t), \tilde{\eta}_i(t))$ as a function of t. It now remains to prove, in the case of strong optimality, that h_λ is zero on $(\tilde{u}(t), \tilde{\eta}(t))$. We make use of Theorem 2. From the fact that

$$\sigma_\lambda(u(a), \eta(a)) = -h_\lambda(\tilde{u}(a), \tilde{\eta}(a))dt + \tilde{\eta}_i(a)dx^i \in (TS_p)^0 = V^*\tau,$$

we obtain $h_\lambda(\tilde{u}(a), \tilde{\eta}(a)) = 0$. ■

Before proceeding to the following section, we introduce some additional definitions. Assume that a $\tilde{\rho}$-admissible curve $\tilde{u}$ is given. A couple $(\tilde{\eta}, \lambda)$, where $\tilde{\eta}$ denotes a one form along the base of $\tilde{u}$ and a real number λ, is called a *local multiplier* if the conditions (1), (3) and (4) from Theorem 3 are satisfied. If, in addition, condition (2) is satisfied then $(\tilde{\eta}, \lambda)$ is called a *global multiplier.* Note that the implicit Hamiltonian system in condition (1), implies that h_λ attains a local extremum which justifies the above definitions. It is well known from literature, that the $\tilde{\rho}$-admissible curve $\tilde{u}$ is called a *global (local) extremal* if it admits a global (local) multiplier $(\tilde{\eta}, \lambda)$ with $\lambda \leq 0$. Furthermore if $\lambda = 0$, then $\tilde{u}$ is called an *abnormal* extremal, and if $\lambda < 0$, then $\tilde{u}$ is called a *normal* extremal.

Using these definitions, Theorem 3 says that any optimal $\tilde{\rho}$-admissible curve is a global extremal. Note that, given any global multiplier $(\tilde{\eta}, \lambda)$, then for any $\alpha > 0$, the pair $(\alpha\tilde{\eta}, \alpha\lambda)$ is also a multiplier. Therefore, we shall henceforth always assume that the multiplier $(\tilde{\eta}, \lambda)$ is 'normalised', in the sense that λ equals 0, 1 or -1.

3. Linear autonomous control problems

In the following we shall concentrate on *linear autonomous geometric optimal control structures*, i.e. autonomous geometric optimal control structures satisfying the additional conditions that $\tilde{\nu} : C \to Q$ is a linear bundle and $\tilde{\rho}$ is a linear bundle map.

We first consider the maximality condition derived in Theorem 3. Fix any $\zeta^0 \in T^*Q$ and let $x_0 = \pi_Q(\zeta^0)$. The function $\tilde{u} \mapsto h_\lambda(\tilde{u}, \zeta^0)$ for any $\tilde{u} \in C_{x_0}$, attains a local extremum at $\tilde{u} = \tilde{u}_0$ iff

$$\left.\frac{\partial}{\partial u^a}\right|_{\tilde{u}_0^a} \left(\tilde{\rho}_a^i u^a \zeta_a^0 + \lambda L(x_0^i, u^a, \zeta_a^0)\right) = 0,$$

or equivalently $\tilde{\rho}^*(\zeta) = -\lambda \mathbb{F}L(u)$, where $\mathbb{F}L : C \to C^*$ denotes the fibre derivative of L and $\tilde{\rho}^* : T^*Q \to C^*$ is the dual of the linear bundle map $\tilde{\rho}$.

If $\tilde{u}$ is an abnormal local extremal (i.e. there exists a local multiplier $(\tilde{\eta}, \lambda)$ with $\lambda = 0$), then $\tilde{\rho}^*(\tilde{\eta}(t)) = 0$ for any t or, equivalently, $\tilde{\eta}(t) \in (\tilde{\rho}(C_{q(t)}))^0$ (the annihilator space of the image of $C_{q(t)}$ under $\tilde{\rho}$). Moreover, in this specific case, the function $\tilde{u}' \mapsto h_0(\tilde{u}', \tilde{\eta}(t))$ equals 0 for all $\tilde{u}' \in C_{q(t)}$, which implies that $\tilde{u}$ is a global abnormal extremal. We conclude that for linear autonomous control problems, the abnormal local extremals are global abnormal extremals.

On the other hand, if $\tilde{u}$ is a normal local extremal i.e. $\lambda = -1$, then $\tilde{\rho}^*(\zeta) = \mathbb{F}L(u)$. We say that L is a *regular cost* if the fibre derivative of L is invertible. We say that a curve $\tilde{\eta}(t)$ in T^*Q *generates* a curve $\tilde{u}(t)$ in C, if $\tilde{u}(t) = (\mathbb{F}L)^{-1}(\tilde{\rho}^*(\tilde{\eta}(t)))$. In this case, if $\tilde{\eta}$ is piecewise smooth, then the curve $\tilde{u}$ generated by $\tilde{\eta}$ is also continuous. Therefore, a normal local extremal is a piecewise smooth curve in C. Moreover, from the following proposition it follows that it is a smooth curve.

THEOREM 4. *Assume that L denotes a regular cost. Every normal local extremal $\tilde{u}$ is generated by an integral curve $\tilde{\eta}(t)$ of the hamiltonian vector field X_G on T^*Q associated with the function $G(\zeta) = h_\lambda(\mathbb{F}L^{-1}(\tilde{\rho}^*(\zeta), \zeta)$, for $\zeta \in T^*Q$ and with $\lambda = -1$. The converse also holds, i.e. every integral curve of X_G generates a normal local extremal.*

Proof. We assume that $\tilde{u}$ is a local extremal and fix a local multiplier $(\tilde{\eta}, \lambda)$ with $\lambda = -1$. Consider the function $\mathcal{L} : T^*Q \to C \times_Q T^*Q$, defined by:

$$\mathcal{L}(\zeta) = (\mathbb{F}L^{-1}(\tilde{\rho}^*(\zeta)), \zeta).$$

Note that $\mathcal{L}$ is a section of the bundle $p_{T^*Q} : C \times_Q T^*Q \to T^*Q$, with p_{T^*Q} the projection onto the second factor. Then it is easily seen that $\mathcal{L}^* h_\lambda = G$ and that $\mathcal{L}^* \tilde{\omega}_Q = \omega_Q$. Recall the implicit Hamiltonian system: a multiplier has to satisfy: $i_{(\dot{\tilde{u}}(t), \dot{\tilde{\eta}}(t))} \tilde{\omega}_Q =$

$-dh_\lambda(\tilde{u}(t),\tilde{\eta}(t))$, and consider the tangent vectors $X = (\dot{\tilde{u}}(t),\dot{\tilde{\eta}}(t)) = T\mathcal{L}(\dot{\tilde{\eta}}(t))$ and $Y = T\mathcal{L}(w) \in T(C \times_Q T^*Q)$ for $w \in T(T^*Q)$ arbitrary, then $\tilde{\omega}_Q(X,Y) = \omega_Q(\dot{\tilde{\eta}}(t),w)$. By substituting $\mathcal{L}^*h_\lambda = G$ we obtain that $i_{\dot{\tilde{\eta}}(t)}\omega_Q = -dG(\tilde{\eta}(t))$ for every smooth part of $\tilde{\eta}$. By uniqueness of solutions to differential equations, it follows that $\tilde{\eta}$ is smooth (and therefore we have that $\tilde{u}$ is smooth).

On the other hand, assume that $i_{\dot{\tilde{\eta}}(t)}\omega_Q = -dG(\tilde{\eta}(t))$ and consider the smooth curve $\tilde{u}(t) = (\mathbb{F}L)^{-1}(\tilde{\rho}^*(\tilde{\eta}(t)))$ in C. Then, by reversing the above arguments, we obtain that $\tilde{\omega}_Q(X,Y) = -dh_\lambda(Y)$ with $X = (\dot{\tilde{u}}(t),\dot{\tilde{\eta}}(t)) = T\mathcal{L}(\dot{\tilde{\eta}}(t))$ and $Y = T\mathcal{L}(w) \in T(C\times_Q T^*Q)$ for $w \in T(T^*Q)$ arbitrary and $\lambda = -1$. It remains to check that this is also valid for arbitrary $Y \in T(C\times_Q T^*Q)$. Since $\mathcal{L}$ is a section of p_{T^*Q}, any element Y in $T(C\times_Q T^*Q)$ can be written as $Y = T\mathcal{L}(w) + Z$ where $w = Tp_{T^*Q}(Y) \in T(T^*Q)$ and $Z \in \ker Tp_{T^*Q}$. Since $\tilde{\omega}_Q = p^*_{T^*Q}\omega_Q$ it is easily seen that $i_Z\tilde{\omega}_Q = 0$. From this we conclude that $i_X\tilde{\omega}_Q = -dh_\lambda(\tilde{u}(t),\tilde{\eta}(t))$ for arbitrary t. ■

4. Applications

1. Consider a sub-Riemannian structure (Q,D,h), where Q is a manifold, D a regular, i.e. constant rank and smooth, distribution on Q and h is a Riemannian bundle metric on D. Let $i : D \to TQ$ denote the natural injection of D into TQ (note that $\tilde{\nu} : D \to Q$ can be considered as a linear bundle on Q). We would like to solve the length-minimising problem in the sub-Riemannian structure (Q,D,h), i.e. we have to solve the autonomous optimal control problem with control structure $(Q,\tilde{\nu},i)$ and cost $E \in C^\infty(D)$, where $E(v) = \frac{1}{2}h(v,v)$. It is easily seen that E is a regular cost, i.e. $\mathbb{F}E = \flat_h$, with $\flat_h$ defined by $h(v,w) = \langle \flat_h(v),w\rangle$ for arbitrary $v,w \in D$. Let $\sharp_h$ denote the inverse of $\flat_h$. The function G, introduced in the above theorem, takes the form: $G(\zeta) = \langle\zeta, i(\sharp_h(i^*(\zeta)))\rangle - \frac{1}{2}h(\sharp_h(i^*(\zeta)),\sharp_h(i^*(\zeta)))$. If we consider the tensor $g \in TQ\otimes TQ$ defined by $g(\zeta,\xi) = h(\sharp_h(i^*(\zeta)),\sharp_h(i^*(\xi)))$ with $\xi,\zeta \in T^*Q$, then $G(\zeta) = \frac{1}{2}g(\zeta,\zeta)$. In [2] we have further investigated the equations of a local extremal using connections over a bundle map and we gave necessary and sufficient conditions for abnormal extremals.

In the next example we consider the case where C is a Lie-algebroid with anchor map $\tilde{\rho}$. We refer to [1, 6, 13] where the importance of this specific case for a generalisation of Lagrangian mechanics is thoroughly investigated. It should be noted that we only derive the Lagrangian equations, using the theory developed above.

2. Assume that C is a Lie-algebroid with anchor map $\tilde{\rho}$ and assume that L is a regular cost. It is a well known fact that the Lie-algebroid structure determines a Poisson structure on C^*, where the bracket on $C^\infty(C^*)$ is denoted by $\{\cdot,\cdot\}_{C^*}$, whereas the Poisson bracket on $C^\infty(T^*Q)$ is denoted by $\{\cdot,\cdot\}$. It is also well known that both Poisson structures are $\tilde{\rho}^*$ connected, i.e. if $\chi = \tilde{\rho}^*$, then, given arbitrary $f,g \in C^\infty(C^*)$ the following equality holds: $\{\chi^*f,\chi^*g\} = \chi^*\{f,g\}_{C^*}$. This implies that any Hamiltonian vector field X_f on C^* is ρ^*-connected to the Hamiltonian vector field X_{χ^*f} on T^*Q. Consider the function G on T^*Q, introduced in the previous section. Define $g \in C^\infty(C^*)$ by $g(\alpha) = \langle\alpha, \mathbb{F}L^{-1}(\alpha)\rangle - L(\mathbb{F}L^{-1}(\alpha))$. Then, it is easily seen that $\chi^*g = G$. This guarantees that, given any integral curve $\eta(t)$ of X_G, then $\tilde{\rho}^*(\eta(t))$ is an integral curve

of X_g and, conversely, any integral curve $\alpha(t)$ of X_g, through a point in the image of $\tilde{\rho}^*$ is the projection under $\tilde{\rho}^*$ of an integral curve of X_G. From this we conclude that, in the case where C^* is a Lie-algebroid, the integral curves of X_g through a point in the image of ρ^* are projections of normal local extremals.

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