Leafwise holonomy of connections over a bundle map

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Abstract

In this paper we introduce a generalisation of the notion of holonomy for connections over a bundle map on a principal fibre bundle. We prove that, as in the standard theory on principal connections, the holonomy groups are Lie subgroups of the structure group of the principle fibre bundle and we also derive a straightforward generalisation of the Reduction Theorem.

Key words: holonomy groups, generalised connections
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1 Introduction

The standard notion of a connection (see e.g. [8]) has been generalised along many different lines. To mention a few of these generalisations, we may refer to the so-called "pseudo-connections" (see F. Etayo [2] for a review) and, in particular, to the "partial connections" as studied, for instance, by F. Kamber et al. [7]. More recently, a notion of connection on Lie algebroids has been introduced and studied, among others, by R.L. Fernandes [4,5]. The importance of these generalisations can be illustrated, for instance, by the fact that partial connections were used to prove the vanishing of some cohomology classes on manifolds admitting a regular integrable distribution, and the theory of Lie algebroid connections has lead to the construction of a generalised Chern-Weil homomorphism onto the set of Lie algebroid cohomology classes.

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In an attempt to establish a unified framework for the various types of connections mentioned above, we have introduced in a recent paper a notion of generalised connections over a vector bundle map [1]. In some subsequent papers, we have investigated possible applications of these generalised connections: to nonholonomic mechanics [9], to the study of length minimising curves in sub-Riemannian geometry [10] and to the formulation and proof of a geometric version of the maximum principle in control theory [11].

In [1] we managed to associate a notion of "parallelism" and "covariant derivation" with a generalised connection over a bundle map. However, torsion and curvature are in general not well defined unless the bundles under consideration admit some additional geometric structures, such as in the case of a pre-Lie algebroid. In this paper we present a notion of "holonomy" for these generalised connections and we derive a version of the Reduction Theorem [8, p 83]. It should be mentioned that holonomy has already been studied for partial connections in the framework of (contact) sub-Riemannian geometry, see for instance [3], and for generalised connections in the framework of Lie algebroids [5].

The structure of the paper is as follows. In Section 2 we introduce the notion of an anchored bundle and discuss some of its basic properties. The structure of an anchored bundle is encountered in sub-Riemannian geometry, control theory, nonholonomic mechanics and also in the theory of (affine) Lie algebroids [6,13,14]. Therefore, we believe it is worth to study this structure in its own right. In Section 3 we introduce the notion of a lift over an anchored bundle, which can be regarded as a right invariant anchor map on a principal fibre bundle, commuting with a given anchor map on the base space. Furthermore, the notion of "leafwise holonomy" of a lift over an anchored bundle is defined. In Section 5 we prove that the generalised holonomy groups are Lie subgroups of the structure group of the given principal fibre bundle. A generalisation of the Reduction Theorem is then easily obtained. To conclude this paper we briefly discuss some possible applications in sub-Riemannian geometry.

All manifolds considered in this paper are real, finite dimensional smooth manifolds without boundary, and by smooth we will always mean of class C^∞. The set of (real valued) smooth functions on a manifold B will be denoted by $C^\infty(B)$, the set of smooth vector fields by $\mathcal{X}(B)$ and the set of smooth one-forms by $\mathcal{X}^*(B)$. The set of all smooth (local or global) sections of an arbitrary fibre bundle $\tau : E \to B$ will be denoted by $\Gamma(\tau)$.

2   Anchored bundles

In this section we describe the basic structure on which our study of generalised connections is based, namely that of an anchored bundle. Let M denote an arbitrary n-dimensional manifold with tangent bundle $\tau_M : TM \to M$. The conceptual idea of an anchored bundle is that one considers a bundle over M which is related to TM, in such a way that, for further developments, the bundle can be taken as an alternative to the tangent bundle of M. The notion of an anchored bundle is also encountered in the work of P. Popescu [16], who also uses the denomination "relative tangent space".

Definition 1 *An anchored bundle on M is a pair (ν, ρ) where, $\nu : N \to M$ denotes a fibre bundle over M, and $\rho : N \to TM$ is a bundle map, fibred over the identity on M. We call ρ the* anchor map *of the anchored bundle.*
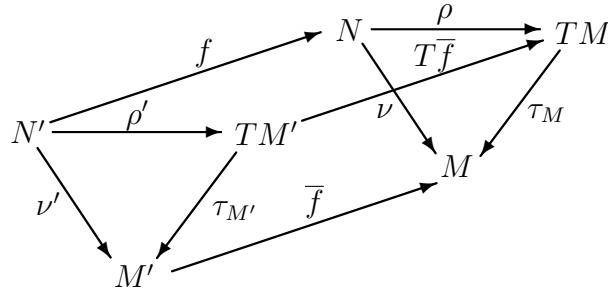
The following diagram is commutative:

$$
\begin{array}{ccc}
N & \xrightarrow{\quad \rho \quad} & TM \\
& \searrow{\scriptstyle \nu} \quad \swarrow{\scriptstyle \tau_M} & \\
& M &
\end{array}
$$

We say that an anchored bundle (ν, ρ) is *linear*, if ν is a vector bundle and ρ is a linear bundle morphism.

Consider two anchored bundles (ν', ρ') and (ν, ρ) with base manifolds respectively M' and M. An anchored bundle morphism $(f, \overline{f})$ from (ν', ρ') to (ν, ρ) consists of a smooth mapping $\overline{f} : M' \to M$ and a bundle morphism $f : N' \to N$ fibred over $\overline{f}$, in such a way that the following equation holds:

$$
T\overline{f} \circ \rho' = \rho \circ f.
$$

We say that f is an anchored bundle isomorphism if f is a bundle isomorphism (see e.g. [17]), and if, in addition, f^{-1} is also an anchored bundle morphism. In this case we can write $\rho' = T(\overline{f})^{-1} \circ \rho \circ f$ and conversely $\rho = T\overline{f} \circ \rho' \circ f^{-1}$. If $\overline{f}$ is an injective immersion, then we say that (ν', ρ') is an anchored subbundle of (ν, ρ). Note that ρ' is completely determined by $\rho' = T(\overline{f})^{-1} \circ \rho \circ f$, which is well defined since $\overline{f}$ is an immersion. Assume that both anchored bundles are linear. Then, we say that f is a *linear homomorphism* if $f : N' \to N$ is a linear bundle map. The following commutative diagram represents an anchored bundle morphism:

$$
\begin{array}{c}
N \xrightarrow{\ \rho\ } TM \\
\end{array}
$$

$N' \xrightarrow{\ \rho'\ } TM'$, f , $T\overline{f}$, ν , τ_M , ν' , $\tau_{M'}$, $\overline{f}$, M , M'

2.1   *The foliation on anchored bundles*

In this section we need some elements of the theory of integrability of distributions, developed by H.J. Sussmann [18] (see also [12]). We first briefly recall the basic definitions and main results on distributions, before applying them to anchored vector bundles. We also use this section to fix some notations regarding composite flows and concatenations of integral curves of vector fields.

Consider a manifold M and assume that F is a differentiable distribution on M, i.e. F is a subset of TM such that, for any point $x \in M$, the fibre $F_x = F \cap T_x M$ is a linear subspace of $T_x M$ and such that F_x is spanned by a finite number of vector fields in F evaluated at x (we say that $X \in \mathcal{X}(M)$ is a vector field in F if $X(y) \in F_y$, for arbitrary $y \in M$).

The *rank* of the distribution F at a point $x \in M$ is the dimension of F_x. Note that, in the above definition, a distribution need not have, in general, constant rank. If F has constant rank, we say that F is a *regular distribution*.

A distribution is said to be completely integrable if, given any $x \in M$, then there exists an immersed connected submanifold $i : L \hookrightarrow M$ containing x and such that $T_y L = F_y$, for each $y \in L$. A submanifold L satisfying the above conditions, is called a leaf if it is maximal, in the sense that, given any other submanifold L', verifying the above conditions, and which contains L then $L' = L$. It can be proven that these leaves are unique and determine a partition on M which is called the foliation induced by the completely integrable distribution. Note that, by definition, the distribution F has constant rank on the points of the leaf L.

Assume that $\mathcal{F}$ is a family of vector fields on M, each defined on an open subset of M. We say that $\mathcal{F}$ is everywhere defined if, given any $x \in M$, there exists an element X of $\mathcal{F}$ containing x in its domain. An everywhere defined family of vector fields $\mathcal{F}$ generates a distribution F in the following way:

$$F_x = \operatorname{span}\{X(x) \mid X \in \mathcal{F}, x \in \operatorname{dom} X\}.$$

It is readily seen that F is a differentiable distribution. H.J. Sussmann has shown that, given of an everywhere defined family of vector fields $\mathcal{F}$, generating a distribution F, one can always construct the smallest completely integrable distribution containing F. In order to discuss this construction, we need the notion of a composite flow.

Assume that we have fixed an ordered ℓ-tuple $\mathcal{X} = (X_\ell, \ldots, X_1)$ of vector fields on M, and let us represent the flow of X_i by $\{\phi^i_t\}$.

The *composite flow* of $\mathcal{X}$ is the map

$$\Phi : V \subset \mathbb{R}^\ell \times B \to B : ((t_\ell, \ldots, t_1), x) \mapsto \phi^\ell_{t_\ell} \circ \ldots \circ \phi^1_{t_1}(x),$$

defined on some open subset V of $\mathbb{R}^\ell \times B$. For brevity we shall write $\Phi_T(x)$ in stead of $\Phi(T, x)$, where $T = (t_\ell, \ldots, t_1)$. We shall sometimes refer to T as the *composite flow parameter*. For each fixed T, Φ_T determines a diffeomorphism from an open subset of M (which may be empty) to another open subset of M. It can be proven that, if we fix a point $x \in M$, then the map $T' \mapsto \Phi_{T'}(x)$ is smooth and defined on an open neighbourhood of T. For further details on the domain of composite flows, we refer the reader to [12].

Assume that we have fixed two composite flows: Φ of $\mathcal{X} = (X_\ell, \ldots, X_1)$ and Ψ of $\mathcal{Y} = (Y_{\ell'}, \ldots, Y_1)$. The *composition of* Φ and Ψ is the composite flow $\Psi \star \Phi$ of the $\ell' + \ell$-tuple $(Y_{\ell'}, \ldots, Y_1, X_\ell, \ldots, X_1)$. Using these notations, it is easily seen that, for instance, Φ equals $\phi^\ell \star \ldots \star \phi^1$. If T is a composite flow parameter for Φ and T' for Ψ, then $T' \star T = (T', T) \in \mathbb{R}^{\ell' + \ell}$ is a composite flow parameter for $\Psi \star \Phi$.

The composite flow Φ of $\mathcal{X} = (X_\ell, \ldots, X_1)$ is said to be *generated* by an everywhere defined family of vector fields $\mathcal{F}$ if $\mathcal{X}$ is an ordered ℓ-tuple of elements of $\mathcal{F}$. Using all composite flows generated by $\mathcal{F}$, we can define an equivalence relation on the points of M, denoted by $\leftrightarrow$.

Definition 2 *Assume that $x, y \in M$. Then $x \leftrightarrow y$ if there exists a composite flow Φ generated by $\mathcal{F}$ and a composite flow parameter T such that $\Phi_T(x) = y$.*

It is easily seen that the relation $\leftrightarrow$ is transitive (see the above definition of the composition of composite flows) and reflexive (take $T = (0, \ldots, 0)$). If Φ is a composite flow of $\mathcal{X} = (X_\ell, \ldots, X_1)$ and $\Phi_T(x) = y$ for some $T = (t_\ell, \ldots, t_1)$, then the composite flow $\widetilde{\Phi}$ of $\widetilde{\mathcal{X}} = (X_1, \ldots, X_\ell)$ and the composite flow parameter $\widetilde{T} = (-t_1, \ldots, -t_\ell)$ satisfy $\widetilde{\Phi}_{\widetilde{T}}(y) = x$. Since $\widetilde{\Phi}$ is generated by $\mathcal{F}$, this makes the relation symmetric. Assume in the following that the distribution F is generated by the family $\mathcal{F}$.

Theorem 3 *The smallest completely integrable distribution $\widetilde{F}$ containing F is the distribution generated by the everywhere defined family $\widetilde{\mathcal{F}}$ containing all*

elements of the form $\Phi_T^* Y$ where $Y \in \mathcal{F}$ and Φ is a composite flow generated by $\mathcal{F}$.

The leaves of the distribution $\widetilde{F}$ are the equivalence classes of the equivalence relation $\leftrightarrow$.

Consider the distribution $\widetilde{F}$ and let $[X, Y]$ denote the Lie bracket of two vector fields in $\mathcal{F}$. It is easily seen that $[X, Y]$ is a vector field in $\widetilde{F}$. Indeed, let $\{\phi_t\}$ be the flow of X and observe that $\phi_t^* Y$ is in $\widetilde{\mathcal{F}}$. Then, for each $x \in M$, the curve $t \mapsto \phi_t^* Y(x)$ is entirely contained in the linear space $\widetilde{F}_x$, and so is its tangent vector:

$$\left. \frac{d}{dt} \right|_0 (\phi_t^* Y(x)) = [X, Y](x).$$

This reasoning can easily be extended to any finite number of iterated Lie brackets of vector fields in $\mathcal{F}$. In fact, this observation is rather important since it leads to an alternative proof for the Theorem of Chow (see [18]).

Assume that $\mathcal{X} = (X_\ell, \ldots, X_1)$ is an arbitrary finite ordered family of vector fields, with composite flow Φ. Fix a value $(t_\ell, \ldots, t_1)$ of the composite flow parameter T. The *concatenation of integral curves* of $\mathcal{X}$ trough $x \in M$ is the piecewise smooth curve $\gamma : [a, a + |t_1| + \ldots + |t_\ell|] \to M$ defined as follows, where $a_i = a + \sum_{j=1}^i |t_j|$, $\mathrm{sgn}(t_i) = \frac{t_i}{|t_i|}$ for $t_i \neq 0$ and $\mathrm{sgn}(0) = 0$,

$$\gamma(t) = \begin{cases} \phi^1_{\mathrm{sgn}(t_1)(t-a)}(x) & \text{for } t \in [a, a_1] \\ \phi^2_{\mathrm{sgn}(t_2)(t-a_1)}(\phi^1_{t_1}(x)) & \text{for } t \in]a_1, a_2] \\ \ldots & \\ \phi^\ell_{\mathrm{sgn}(t_\ell)(t-a_{\ell-1})}(\ldots (\phi^1_{t_1}(x)) \ldots) & \text{for } t \in]a_{\ell-1}, a_\ell], \end{cases}$$

Note that, if $t \in]a_{i-1}, a_i[$ then $\dot\gamma(t) = \mathrm{sgn}(t_i) X_i(\gamma(t))$ and, hence, the restriction of γ to $]a_{i-1}, a_i[$ is an integral curve of X_i if $t_i > 0$ (or $-X_i$ if $t_i < 0$). Note that $\gamma(a_\ell) = \Phi_T(x)$, i.e. the endpoint of γ coincides with the image of x under the composite flow Φ_T. It is easily seen that in the specific case where $\mathcal{X}$ is generated by $\mathcal{F}$, the concatenation of integral curves of $\mathcal{X}$ through $x \in M$ is entirely contained in the leaf L_x through x.

Let us now proceed towards the construction of an everywhere defined family of vector fields on M, given an anchored bundle (ν, ρ) on M. Consider an arbitrary (local) section σ of ν, i.e. $\sigma : M \to N$ is a smooth map with $(\nu \circ \sigma)(x) = x$. Using the anchor map we can define the following vector field on M: $\rho \circ \sigma$. Let $\mathcal{D}$ denote the set of all vector fields of the form $\rho \circ \sigma$. Clearly, $\mathcal{D}$ is everywhere defined and using the notations as described above, the manifold M is equipped with a distribution D generated by $\mathcal{D}$ (with $D = \mathrm{im}\,\rho$ if (ν, ρ) is linear) and the smallest completely integrable distribution $\widetilde{D}$ containing D. The leaf on M through x, induced by $\widetilde{D}$, is denoted by L_x.

Consider the immersion $i : L_x \hookrightarrow M$, and let $\nu' : N' = L_x \times_M N \to L_x$ denote the pull-back bundle of ν under i, i.e. $(y, s) \in N'$ if $i(y) = \nu(s)$. Since i is an immersion, we can define an anchor map $\rho' : N' \to TL_x$ as follows: $T_y i(\rho'(y, s)) = \rho(s)$, given any $(y, s) \in N'$. The projection $\pi_2 : N' \to N$ of N' onto the second factor, determines an anchored bundle morphism, fibred over the immersion i, i.e. (ν', ρ') is an anchored subbundle of (ν, ρ). We shall call (ν', ρ') *the pull-back anchor bundle under i*.

Before passing to the next section, we first give two examples of an anchored bundle and the distribution induced by it. The first example is taken from [15], where it was used in the context of sub-Riemannian geometry to construct length-minimising strictly abnormal extremals. The other example is taken from [12] and provides a non-trivial completely integrable distribution on $\mathbb{R}^2$.

Example 4 Assume that $M = \mathbb{R}^3$ (we use cylindrical coordinates on $\mathbb{R}^3$), and that $\nu : N = \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^3$ is a trivial bundle over M. Consider the following two vector fields on M: $X_1 = \frac{\partial}{\partial r}$ and $X_2 = \frac{\partial}{\partial \theta} - p(r)\frac{\partial}{\partial z}$, where $p(r)$ is a function on $\mathbb{R}$ with a single non degenerate maximum at $r = 1$, i.e. p satisfies:

$$\frac{d}{dr}p(r)\Big|_{r=1} = 0 \quad \text{and} \quad \frac{d^2}{dr^2}p(r)\Big|_{r=1} < 0.$$

Such a function can always be constructed (take, for instance, $p(r) = \frac{1}{2}r^2 - \frac{1}{4}r^4$). Let ρ denote the map defined by $\rho(x, u^1, u^2) = u^1 X^1(x) + u^2 X^2(x)$, with $x = (r, \theta, z) \in M$. It is easily seen that (ν, ρ) is a linear anchored bundle. The flows of X_1, X_2 are denoted by $\{\phi_t\}, \{\psi_t\}$, respectively. In particular, we have $\phi_t(r, \theta, z) = (t + r, \theta, z)$, $\psi_t(r, \theta, z) = (r, \theta + t, z - p(r)t)$. The foliation induced by $\text{im}\,\rho$ is trivial. Indeed, all iterated Lie brackets of the two vector fields X^1 and X^2 span the total tangent space at each point, implying that $\widetilde{D} = TM$ and M itself is the only leaf.

Example 5 Let $M = \mathbb{R}^2$ and let $N = M \times \mathbb{R}^2$, with $\rho(x, u^1, u^2) = u^1 X(x) + u^2 Y(x)$, where

$$X = \frac{\partial}{\partial x} \quad \text{and} \quad Y = y\frac{\partial}{\partial y}.$$

The distribution F on M defined by $F = \text{im}\,\rho$ satisfies $F = \widetilde{F}$, since $[X, Y] = 0$, i.e. F is completely integrable. The two 2-dimensional submanifolds $\{y < 0\}$, $\{y > 0\}$ and the 1-dimensional submanifold $\{y = 0\}$ are the leaves of the foliation on M. We use this example to show that Lemma 7 in the following section is non-trivial. We shall construct a curve, which is tangent to F, i.e. has tangent vector everywhere contained in F, but, the curve itself is not entirely contained in a single leaf. Indeed, consider $\tilde{c} : \mathbb{R} \to M : t \mapsto (t, t^3)$. It is readily seen that $\dot{\tilde{c}}(t) = X(\tilde{c}(t)) + 3t^{-1}Y(\tilde{c}(t)) \in F_{\tilde{c}(t)}$ for $t \neq 0$ and $\dot{\tilde{c}}(0) = X(0, 0) \in F_x$. However $\tilde{c}$ passes through the three leaves of F.

We introduce here the notion of a *ρ-admissible curve*. By a smooth curve in a manifold M we will always mean a C^∞ map $c : I \to M$, where $I \subseteq \mathbb{R}$ may be either an open or a closed (compact) interval. In the latter case, the denominations "path" or "arc" are also frequently used in the literature but, for simplicity, we will make no distinction in terminology between both cases. For a curve defined on a closed interval, say $[a, b]$, it is tacitly assumed that it admits a smooth extension to an open interval containing $[a, b]$. Fix an anchored bundle (ν, ρ) on M.

Definition 6 *Let $c : [a, b] \to N$ denote a smooth curve in N, and let $\widetilde{c} = \nu \circ c$ denote the projected curve in M, called the base curve of c. Then c is called a smooth ρ-admissible curve if $\rho \circ c = \dot{\widetilde{c}}$.*

Local coordinates on M will be denoted by (q^i) and corresponding bundle adapted coordinates on N by (q^i, u^a), with $i = 1, \dots, n$ and $a = 1, \dots, k$, where k is the dimension of the typical fibre of N. If we write the bundle map ρ locally as

$$\rho(q^i, u^a) = \gamma^i(q^j, u^a) \frac{\partial}{\partial q^i} \tag{2.1}$$

Then a smooth ρ-admissible curve $c(t) = (q^i(t), u^a(t))$ locally satisfies

$$\gamma^i(q^j(t), u^a(t)) = \dot{q}^i(t).$$

In order to introduce a suitable concept of "leafwise holonomy" in the framework of principal ρ-lifts, it turns out that the class of ρ-admissible curves in N should be further extended to curves admitting (a finite number of) discontinuities in the form of certain 'jumps' in the fibres of N, such that the corresponding base curve is piecewise smooth. In order to define these "piecewise" ρ-admissible curves we first consider the composition of smooth ρ-admissible curves.

The *composition* of a finite number of, say ℓ, smooth ρ-admissible curves $c_i : [a_{i-1}, a_i] \to N$ for $i = 1, \dots, \ell$, satisfying the conditions $\widetilde{c}_i(a_i) = \widetilde{c}_{i+1}(a_i)$ for $i = 1, \dots, \ell - 1$, is the map $c_\ell \cdot \ldots \cdot c_1 : [a_0, a_\ell] \to N$ defined by

$$(c_\ell \cdot \ldots \cdot c_1)(t) = \begin{cases} c_1(t) & t \in [a_0, a_1], \\ \ldots \\ c_\ell(t) & t \in {]a_{\ell-1}, a_\ell]}. \end{cases} \tag{2.2}$$

Note that the base curve of $c_\ell \cdot \ldots \cdot c_1$ is a piecewise smooth curve. However, in general $c_\ell \cdot \ldots \cdot c_1$ is discontinuous at $t = a_i$, $i = 1, \dots, \ell - 1$. The composition $c = c_\ell \cdot \ldots \cdot c_1$ is called *a piecewise ρ-admissible curve*, or simply a *ρ-admissible*

curve. We now proceed towards the following important result, saying that the base curve of a ρ-admissible curve is always entirely contained in a leaf of the foliation on M, induced by the everywhere defined family of vector fields $\mathcal{D}$ on M (see the previous section).

Lemma 7 *The base curve $\widetilde{c}$ of a ρ-admissible curve $c : [a, b] \to N$ is entirely contained in the leaf L_x, with $x = \widetilde{c}(a)$.*

Proof. It is sufficient to prove this result for c smooth. For any point $y \in M$, consider a coordinate neighbourhood U of y with coordinates $(q^1, \ldots, q^n)$, adapted to the foliation induced by $\mathcal{D}$, such that: (1) if $q^{p+1}(z), \ldots, q^n(z) = 0$ then $z \in L_y$ and (2) the coordinate functions $q^1, \ldots, q^p$ determine local coordinates on the leaf L_y (this is always possible since L_y is an immersed submanifold). Upon restricting U to a smaller subset, if necessary, we may always assume, in addition, that the fibre bundle ν is trivial over U, and we denote the adapted bundle coordinates by (q^i, u^a), for $i = 1, \ldots, n$ and $a = 1, \ldots, k$. In the following we only consider such coordinate charts. Recall the definition of the pull-back anchored bundle (ν', ρ') under $i : L_y \hookrightarrow M$. Note that $(q^1, \ldots, q^p, u^1, \ldots, u^k)$ is a bundle adapted coordinate chart on N'.

Fix a coordinate chart (in the sense specified above) containing the point $x = \widetilde{c}(a)$ and assume that c is written in these coordinates as $(\widetilde{c}^i(t), u^a(t))$. Let $\widetilde{d}$ denote the solution in L_y of the following differential equation, in the anchored bundle (ν', ρ'):

$$\dot{\widetilde{d}}(t) = \rho'(\widetilde{d}^1(t), \ldots, \widetilde{d}^p(t), u^1(t), \ldots, u^k(t)),$$

with initial condition $\widetilde{d}(a) = x$. From standard arguments we know that $\widetilde{d}$ is defined on some interval, say $[a, t + \epsilon[$, with $\epsilon > 0$.

Consider the curve $\widetilde{d}' = i \circ \widetilde{d} : [a, t + \epsilon[\to M$ in M, through y at time t. Then we have, by uniqueness of solutions to differential equations, that $\widetilde{d}' = \widetilde{c}|_{[a,t+\epsilon[}$, since the curves $\widetilde{d}'$ and $\widetilde{c}$ both solve $\dot{\widetilde{c}} = \rho(\widetilde{c}^j, u^a)$. Indeed, for $\widetilde{c}$ this is trivially satisfied and for $\widetilde{d}'$ we have

$$\dot{\widetilde{d}}'(t) = Ti(\rho'(\widetilde{d}(t), c(t))) = \rho(\widetilde{d}^i, 0, u^a(t')).$$

Therefore we conclude that $\widetilde{c}|_{[a,t+\epsilon[}$ is contained in the leaf L_x, since by making use of the coordinate system, we have $\widetilde{c}^i(t) = 0$ for $t \in [a, t + \epsilon[$ and $i = p + 1, \ldots, n$. Taking the limit from the left at $t = a + \epsilon$, we obtain that $\widetilde{c}^i(a + \epsilon) = 0$ for $i = p + 1, \ldots, n$, or $\widetilde{c}(a + \epsilon) \in L_x$. We can repeat the above reasoning for the curve $c|_{[a+\epsilon,b]}$, i.e. we start from the point $\widetilde{c}(a + \epsilon)$ in stead of the point x. We thus obtain that $\widetilde{c}(t) \in L_x$ for all $t \in [a, a + \epsilon + \epsilon']$ for some $\epsilon' > 0$. Continuing this way, we eventually obtain that the entire curve $\widetilde{c}$ is contained in L_x, concluding the proof. $\square$

It can be seen that the curve $\tilde{c}$ constructed in Example 5 does not contradict the previous lemma although $\tilde{c}$ is a curve tangent to the distribution $\operatorname{im}\rho$. Indeed, $\tilde{c}$ can not be written as the base curve of a ρ-admissible curve, since, at $t = 0$ a singularity is encountered.

Consider two anchored bundles (ν', ρ') and (ν, ρ), and a anchored bundle morphism f between them, i.e. $f : N' \to N$ fibred over $\overline{f} : M' \to M$. Let c' denote a ρ'-admissible curve. Consider the curve $c = f \circ c'$ in N, and let $\tilde{c}$, resp. $\tilde{c}'$, denote the base curve of c, resp. c'. Then, we have that c is ρ-admissible, since

$$\rho \circ c = \rho \circ f \circ c' = T\overline{f} \circ \rho' \circ c' = T\overline{f} \circ \dot{\tilde{c}}' = \dot{\tilde{c}}.$$

Let c denote a ρ-admissible curve. If $x = \tilde{c}(a)$ and $y = \tilde{c}(b)$, then we say that c *takes x to y*, and we write $x \xrightarrow{c} y$ (shortly $x \to y$ if we do not want to mention the ρ-admissible curve explicitly). The relation $\to$ on M is transitive, and is preserved by an anchored bundle morphism, i.e. if $x' \to y'$ then $\overline{f}(x') \to \overline{f}(y')$ for $x', y' \in M'$. The set of points y such that $x \to y$ for some fixed x is denoted by R_x and is called *the set of reachable points from x*. Until now, we have proven that the base curve of a ρ-admissible curves is contained in a leaf L_x of the foliation on M, i.e. $R_x \subset L_x$. It is interesting to wonder if every point in L_x can be reached from x following a ρ-admissible curve. In general this is not the case. However, if we consider the composition of ρ- and $(-\rho)$-admissible curves, then every point in L_x can be reached.

Definition 8 *Given an anchored bundle (ν, ρ). The* inverse anchored bundle *is defined as $(\nu, -\rho)$, where $-\rho : N \to TM : s \mapsto -\rho(s)$.*

An anchored bundle (ν, ρ) is related to its inverse in the following way. Assume that c is a ρ-admissible curve taking x to y, i.e. $x \xrightarrow{c} y$. Then the curve $c^* : [a, b] \to N : t \mapsto c((b - t) + a)$ is $(-\rho)$-admissible and takes y to x. We shall call this curve *the $(-\rho)$-admissible curve associated with c*, or simply the *reverse of c*. Note that, using these notations, $(c^*)^* = c$. If we write, the relation on M induced by the inverse anchored bundle as $\leftarrow$, we have the following equivalence:

$$x \xrightarrow{c} y \text{ iff } y \xleftarrow{c^*} x.$$

Note that the family of vector fields on M defined by the inverse anchored bundle equals $-\mathcal{D} = \{-\rho \circ \sigma \mid \sigma \in \Gamma(\nu)\}$, and therefore produce the same distribution D and the same foliation as $\mathcal{D}$. We now consider the composition of ρ- and $(-\rho)$-admissible curves. Thus, assume that we have ℓ curves $c_i : [a_{i-1}, a_i] \to N$ for $i = 1, \ldots, \ell$ such that $c_{i-1}(a_{i-1}) = c_i(a_{i-1})$ and such that c_i is either ρ-admissible or $(-\rho)$-admissible. The composition of the curves c_i (defined as in Equation 2.2) $c = c_\ell \cdot \ldots \cdot c_1$ is called a $\pm\rho$-*admissible curve*.

The projection $\widetilde{c}$ of c onto M is a piecewise smooth curve which is called *the base curve of the $\pm\rho$-admissible curve*. If $\widetilde{c}(a_0) = x$ and $\widetilde{c}(a_\ell) = y$ we say that the $\pm\rho$-admissible curve takes x to y. Note that, in this case, the $\pm\rho$-admissible curve c^*, defined by $c^* = (c_1)^* \cdot \ldots \cdot (c_\ell)^*$, takes y to x.

We thus obtain an alternative characterisation of the leaves of the foliation generated by the anchored bundle (ν, ρ).

Theorem 9 *We have that $x \leftrightarrow y$, or $y \in L_x$, iff there exists a $\pm\rho$-admissible curve taking x to y.*

Proof. The 'if'-part of the proof follows straightforwardly from Lemma 7. The 'only if'-part is proven by the following reasoning. Assume that $y \in L_x$ and consider a composite flow Φ of $\mathcal{X} = (X_\ell, \ldots, X_1)$, with $X_i = \rho \circ \sigma_i$ and $\sigma_i \in \Gamma(\nu)$ (Φ is generated by $\mathcal{D}$) such that $\Phi_T(x) = y$. Consider the following curves,
$$c_i : [a_{i-1}, a_i] \to N : t \mapsto \sigma_i \circ \gamma|_{[a_{i-1}, a_i]},$$
where γ is the concatenation associated with $\mathcal{X}$ and T through x (where we have used the notations from the preceding section). It is easily seen that c_i is ρ-admissible if $\operatorname{sgn}(t_i) > 0$, and $(-\rho)$-admissible if $\operatorname{sgn}(t_i) < 0$. If we put $c = c_\ell \cdot \ldots \cdot c_1$, then c takes x to y and is $\pm\rho$-admissible. $\square$

The proof of the following theorem now easily follow from Theorem 9. Note that any anchored bundle morphism f between (ν', ρ') and (ν, ρ), which is fibred over $\overline{f} : M' \to M$, is also a morphism of the corresponding inverted anchored bundles, i.e. $f : (\nu', -\rho') \to (\nu, -\rho)$. This implies that, if $x' \leftarrow y'$ then $\overline{f}(x') \leftarrow \overline{f}(y')$, for $x', y' \in M'$.

Theorem 10 *Let f denote a morphism between (ν', ρ') and (ν, ρ), fibred over $\overline{f} : M' \to M$. Then $\overline{f}(L_{x'}) \subset L_{\overline{f}(x')}$. If (ν', ρ') is the pull-back bundle along $i : L_x \hookrightarrow M$ and $f = \pi_2$, then $i(\overline{L}_x) = L_{i(x)}$.*

It is interesting to consider the special case of *linear* anchored bundles.

Theorem 11 *Let (ν, ρ) denote a linear anchored bundle on M and take any $x, y \in M$. Then $y \in L_x$ or $x \leftrightarrow y$ iff there exists a ρ-admissible curve that takes x to y, i.e. we have $R_x = L_x$.*

This theorem follows from the fact that, given a linear anchored bundle, then $x \to y$ iff $y \to x$. Indeed, assume that $c : [a, b] \to N$ is a ρ-admissible curve taking x to y. Then the curve $c^{-1} : [a, b] \to N : t \mapsto -c((b - t) + a)$ is also ρ-admissible and takes y to x. Note that $c^{-1} = -c^*$. The curve c^{-1} is called *the inverse of c*. In particular, the base curve of a $\pm\rho$-admissible curve is the base curve of a ρ-admissible curve on a linear anchored bundle, which proves

the above theorem. Let $c : [a, b] \to N$ denote a smooth ρ-admissible curve. We now prove that any "reparameterisation" of $\widetilde{c}$ is again a base curve of a ρ-admissible curve. Assume that $\phi : [a, b] \to [c, d]$ is a diffeomorphism satisfying $\phi(a) = c$ and $\phi(b) = d$. Consider the following curve $c' : [c, d] \to N$ defined by

$$c'(s) = \frac{d\phi^{-1}}{ds}(s)c(\phi^{-1}(s)).$$

From elementary calculations, it is easily seen that c' is ρ-admissible, and that the base curve equals $\widetilde{c}(\phi^{-1}(s))$, i.e. the reparameterisation of c. Note that the above definitions are only valid if (ν, ρ) is a linear anchored bundle.

2.3 $\pm\rho$-admissible loops

Consider a point $x \in M$ and let $C(x, N)$ denote the set of all $\pm\rho$-admissible curves taking x to itself. Elements of $C(x, N)$ are called, with some abuse of terminology, $\pm\rho$-admissible loops with base point x. Indeed, in general a $\pm\rho$-admissible loop need not be continuous, nor closed.

Let $\pi_1(x, M)$ denote the first homotopy group of M with reference point x and consider the map $C(x, N) \to \pi_1(x, M)$, associating to the base curve of a $\pm\rho$-admissible loop c, its homotopy class in $\pi_1(x, M)$, i.e. if $\widetilde{c}$ is the base curve of $c = c_\ell \cdot \ldots \cdot c_1 \in C(x, N)$, then $\widetilde{c}$ is mapped onto $[\widetilde{c}]$. It is easily seen that the image of $C(x, N)$ determines a subgroup of $\pi_1(x, M)$, which is denoted by $\pi_1^N(x, M)$. Indeed, assume that $c = c_\ell \cdot \ldots \cdot c_1$ and $d = d_{\ell'} \cdot \ldots \cdot d_1$ are elements of $C(x, N)$, with homotopy classes $[\widetilde{c}]$ and $[\widetilde{d}]$ in $\pi_1(x, M)$. Then, the product $[\widetilde{c}] \cdot [\widetilde{d}]$ in $\pi_1(x, M)$ is the homotopy class of the base curve of

$$c_\ell \cdot \ldots \cdot c_1 \cdot d_{\ell'} \cdot \ldots \cdot d_1.$$

On the other hand, if $c = c_\ell \cdot \ldots \cdot c_1$ is a $\pm\rho$-admissible loop with base point x, then the curve $c^* = (c_1)^* \cdot \ldots \cdot (c_\ell)^*$ is also contained in $C(x, N)$, and the homotopy class of the base curve of c^* is precisely the inverse $[\widetilde{c}]^{-1}$ of $[\widetilde{c}]$. Therefore, the $\pm\rho$-admissible loops generate a subgroup of $\pi_1(x, M)$ which is denoted by $\pi_1^N(x, M)$. Note that, if (ν, ρ) is linear, then $\pi_1^N(x, M)$ is generated by the set of ρ-admissible loops with base point x, i.e. ρ-admissible curves taking x to itself.

We now elaborate on how the above defined structures on anchored bundles behave under homomorphisms. From Section 2.2, we already now that $\pm\rho$-admissible curves are preserved under anchored bundle morphisms. Similarly, $\pm\rho$-admissible loops are preserved, taking us to a group morphism between the corresponding subgroups of the first fundamental group of the base manifolds. More precisely, assume that f denotes a homomorphism between two anchored bundles (ν', ρ') and (ν, ρ), fibred over $\overline{f}$. Then, if $[\overline{f}]$ denotes the corresponding

group morphism from $\pi_1(x', M')$ to $\pi_1(\overline{f}(x'), M)$, we have that $[\overline{f}]$ can be restricted to a morphism from $\pi_1^{N'}(x', M')$ to $\pi_1^N(\overline{f}(x'), M)$.

Consider the pull-back setting under $i : L_x \hookrightarrow M$, and let $\pi_2 : N' = i^*N \to N$ denote the associated anchored bundle morphism. From the above, we now that $[i]$ maps the subgroup $\pi_1^{N'}(y, L_x)$ of $\pi_1(L_x)$ to the subgroup $\pi_1^N(y, M)$ $\pi_1(y, M)$ (note that L_x is connected, allowing us to omit the reference point in the first homotopy group of L_x). We now prove that $[i] : \pi_1^{N'}(y, L_x) \to \pi_1^N(y, M)$ is onto. Consider an arbitrary element of $\pi_1^N(y, M)$ associated with some $c \in C(y, N)$. From Lemma 7 we know that $\widetilde{c}$ is contained in the leaf $L_y = L_x$, which in turn implies that there exists a $\pm\rho$-admissible loop $c' \in C(y, N')$ such that $\pi_2 \circ \widetilde{c}' = c$. In particular, we have that $[i]([\widetilde{c}']) = [\widetilde{c}]$, and, hence, $[i]$ is onto, when restricted to $\pi_1^{N'}(y, L_x)$.

Moreover, from Theorem 11, we know that any two points in L_x can be connected by the base curve of a $\pm\rho$-admissible curve. This implies, using standard arguments, that we can omit the reference point in $\pi_1^{N'}(y, L_x)$, and from now on, we use the notation $\pi_1^N(L_x)$ for $\pi_1^{N'}(y, L_x)$. Similarly, we write $\pi_1^N(L_x, M)$ for $\pi_1^N(x, M)$.

3   Principal ρ-lifts

Let us first briefly recall the notion of a connection over a bundle map in the context of principal fibre bundles and describe some elementary properties. For further details we refer to [1]. Let (ν, ρ) denote an anchored bundle on M and let $\pi : P \to M$ denote a principal fibre bundle with structure group G. Consider the pull-back bundle $\widetilde{\pi}_1 : \pi^*N \to P$ and let $\widetilde{\pi}_2 : \pi^*N \to N$ denote the projection onto the second factor.

Definition 12 *A principal lift over the bundle map ρ, simply a principal ρ-lift, is a bundle map $h : \pi^*N \to TP$, fibred over the identity on P, such that in addition the following conditions are satisfied for any $(u, s) \in \pi^*N$:*

(1) $TR_g(h(u, s)) = h(ug, s)$, and
(2) $T\pi \circ h = \rho \circ \widetilde{\pi}_2$.

If (ν, ρ) is a linear anchor bundle, the bundle $\widetilde{\pi}_1 : \pi^*N \to P$ can be given linear structure. In this case, we say that a principal ρ-lift h is a principal ρ-connection if $h : \pi^*N \to TP$ is, in addition, a linear bundle morphism from $\widetilde{\pi}_1$ to τ_P

It is easily seen from the definition of a principal ρ-lift h that $(\widetilde{\pi}_1, h)$ determines an anchored bundle and that the projection $\widetilde{\pi}_2 : \pi^*N \to N$, which a bundle

morphism fibred over $\pi : P \to M$, determines an anchored bundle morphism between $(\tilde{\pi}_1, h)$ and (ν, ρ). Moreover, if h is a ρ-connection, we have that $(\tilde{\pi}_1, h)$ is a linear anchored bundle and that $\tilde{\pi}_2$ is a linear anchored bundle morphism. The situation is illustrated by the following diagram:
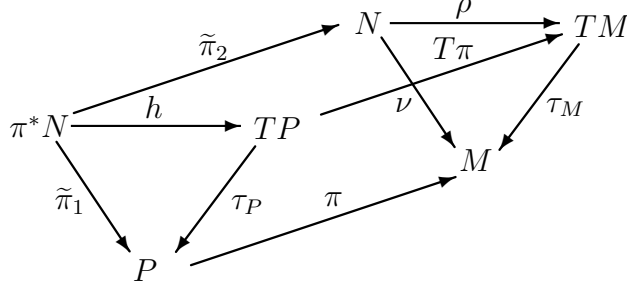
We will now apply the tools from the previous section to the study of principal ρ-lifts. We first fix some notations and make some preliminary comments. The everywhere defined family of vector fields on P generated by $(\tilde{\pi}_1, h)$ is denoted by $\mathcal{Q}$, and correspondingly, the distribution on P generated by $\mathcal{Q}$ is denoted by Q. We refrain from calling Q a horizontal distribution since for arbitrary $u \in P$ it may be that Q_u has non-zero intersection with the distribution of vertical tangent vectors $V_u\pi = \ker T\pi$. Moreover, in general $\mathcal{Q}_u + V_u\pi \neq T_uE$, i.e. $\mathcal{Q}_u$ and $V_u\pi$ do not necessarily span the full tangent space T_uP. The smallest integrable distribution containing Q is denoted by denoted by $\widetilde{Q}$. The leaf of $\widetilde{Q}$ through an arbitrary point $u \in P$ is written as $H(u)$. The principal ρ-lift h can be used to lift several kinds of objects from the anchored bundle (ν, ρ) on M to the anchored bundle $(\tilde{\pi}_1, h)$ on P. For instance, given any (local) section s of ν, we can define a mapping $s^h : P \to TP$ by

$$s^h(u) = h(u, s(\pi(u))). \tag{3.1}$$

It is seen that, by construction, s^h is smooth and verifies $\tau_P(s^h(u)) = u$, i.e. s^h is a (local) vector field on P, called the *lift of the section s with respect to h*, or simply *the lift of s* if no confusion can arise. Let us denote by $\mathcal{D}^h$ the everywhere defined family of vector fields on P defined by the lift of (local) sections of ν.

Next, we recall some definitions and results on principal fibre bundles and principal connections from [8] since they will be used extensively in the following sections. Let $\pi : P \to M$ denote a principle fibre bundle with structure group G. The Lie algebra of G is denoted by $\mathfrak{g}$.

Consider the smooth map $\sigma_u : G \to P$ for each $u \in P$ defined by $\sigma_u(g) = ug$. Then we have $T_e\sigma_u : \mathfrak{g} \to V_u\pi$ and $TR_g \circ T_e\sigma_u = T_e\sigma_{ug} \circ Ad_{g^{-1}}$, where $Ad_h : \mathfrak{g} \to \mathfrak{g}$ denotes the adjoint action of G on its Lie algebra. Given any $A \in \mathfrak{g}$, let $\sigma(A)$ denote the vertical vector field on P defined by $\sigma(A)(u) = T_e\sigma_u(A)$. It is easily seen that $(R_g)_*\sigma(A) = \sigma(Ad_{g^{-1}}A)$.

A standard principal connection on P is defined by *a connection form ω on P*, i.e. ω is a $\mathfrak{g}$-valued one form on P satisfying the following two conditions: (1) for any $A \in \mathfrak{g}$, $\omega(\sigma(A)) = A$, and (2) for any $g \in G$, $R_g^*\omega = Ad_{g^{-1}} \cdot \omega$. It is well known that ω is equivalently defined by a *horizontal lift $h^\omega : P \times_M TM \to TP$*, where h^ω and ω are related in the following way: $h^\omega(u, X) = \widetilde{X} - T_e\sigma_u(\omega(\widetilde{X}))$ for any $\widetilde{X} \in T_uP$ satisfying $T\pi(\widetilde{X}) = X$. From (2) it follows that h^ω is right invariant, i.e. $TR_g(h^\omega(u, X)) = h^\omega(ug, X)$ for $X \in T_{\pi(u)}M$ and $g \in G$ arbitrary. For the sake of completeness, we mention that, equivalently, a principal connection can be defined by the right invariant distribution spanned by the image of h^ω, determining a direct decomposition of TP, i.e. if $\operatorname{im} h^\omega = H\pi$, then $TP = H\pi \oplus V\pi$.

Before starting our study of principal ρ-lifts, we state the following lemma, taking from [8, p 69].

Lemma 13 *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Let Y_t, for $a \leq t \leq b$, define a continuous curve in $\mathfrak{g}$. Then there exists in G a unique curve g_t of class C^1 such that $g(a) = e$ and $\dot{g}_t g_t^{-1} = Y_t$ for $a \leq t \leq b$.*

Let us now return to the general treatment of principal ρ-lifts. Let (ν, ρ) denote an anchored bundle on M and let P denote a principal fibre bundle on M with structure group G.

Fix a standard principal connection ω on P. In the following we will use the connection form ω in order to obtain an alternative description for a principal ρ-lift h. This alternative description will allow us to easily derive some properties of lifts of ρ-admissible curves with respect to h (see below) using the theory of standard connections. Thus, let h be a given principal ρ-lift and consider the map $\chi : \pi^*N \to \mathfrak{g}$ defined by $\chi(u, s) = \omega(h(u, s))$ for any $(u, s) \in \pi^*N$. Note that the following relation holds $\chi(ug, s) = Ad_{g^{-1}} \cdot \chi(u, s)$ and that $h(u, s) = T_e\sigma_u(\chi(u, s)) + h^\omega(u, \rho(s))$. We shall sometimes refer to χ as *the coefficient of h with respect to ω*. The pair (ω, χ) determines uniquely the principal ρ-lift h, in the following way. Given any connection form ω on P and a map $\chi : \pi^*N \to \mathfrak{g}$, such that χ transforms under the right action in the following way: $\chi(ug, s) = Ad_{g^{-1}} \cdot \chi(u, s)$, then the map $h : \pi^*N \to TP$, defined by $h(u, s) = h^\omega(u, s) + T_e\sigma_u(\chi(u, s))$, determines a principal ρ-lift. Note that the coefficient of h with respect to ω is precisely χ.

Theorem 14 *Given any principal ρ-lift h, then the following properties hold:*

(1) the family $\mathcal{D}^h$ generates the distribution Q, and, hence the integrable distribution $\widetilde{Q}$;

(2) any h-admissible curve is mapped by $\widetilde{\pi}_2$ onto a ρ-admissible curve;

(3) given any ρ-admissible curve c taking x to y and a point $u \in P_x$, then there exists a unique h-admissible curve projecting onto c by $\widetilde{\pi}_2$ and whose base curve in P passes through u.

Proof. Properties 1 and 2 are trivial. In order to prove 3, we fix a principal connection ω and consider the coefficient χ of h with respect to ω.

First we prove that, given any ρ-admissible curve $c : [a,b] \to N$ with base curve $\widetilde{c}$, then there always exists a h-admissible curve whose base curve passes through $u \in P_{\widetilde{c}(a)}$ at $t = a$. First, consider the horizontal lift of $\widetilde{c}$ with respect to the principal connection ω, i.e. $\widetilde{d}^\omega(t)$ is the unique curve satisfying $\dot{d}^\omega(t) = h^\omega(d^\omega(t), \dot{\widetilde{c}}(t))$ and $d^\omega(a) = u$. Let $g(t)$ denote the curve in G satisfying the equation $TR_{g(t)^{-1}}\dot{g}(t) = \chi(d^\omega(t), c(t))$ and $g(a) = e$, with e the unit element of G. From 13 the curve $g(t)$ always exists and is unique. We now prove that $(d(t), c(t)) \in \pi^* N$, with $d(t) = d^\omega(t)g(t)$, is a h-admissible curve. Indeed, we find that:

$$
\begin{aligned}
\dot{d}(t) &= TR_{g(t)}(\dot{d}^\omega(t)) + T_e\sigma_{d^\omega(t)}(\dot{g}(t)), \\
&= TR_{g(t)}h^\omega(d^\omega(t), \dot{\widetilde{c}}(t)) + T_e\sigma_{d(t)}(TL_{g^{-1}(t)} \cdot \dot{g}(t)), \\
&= h^\omega(d(t), \dot{\widetilde{c}}(t)) + T_e\sigma_{d(t)}(Ad_{g^{-1}(t)} \cdot \chi(d^\omega(t), c(t))).
\end{aligned}
$$

From the definition of χ, the tangent vector $\dot{d}(t)$ equals the desired vector $h(d(t), c(t))$. Clearly $(d(t), c(t))$ projects onto $c(t)$ and its base curve $d(t)$ passes through u at $t = a$. It easily follows that $d(t)$ is uniquely determined by these conditions, since it satisfies a first order differential equation, i.e. $\dot{d}(t) = h(d(t), c(t))$, with given initial condition $d(a) = u$. $\quad\square$

h-Displacement and holonomy

Using the notations from the above theorem, we have that $d(t)$ is uniquely determined from the ρ-admissible curve c and a point u in the fibre $P_{\widetilde{c}(a)}$. The curve $d(t)$ is called the *lift of the ρ-admissible curve c through u with respect to h* and we write from now on $c_u^h(t)$ to denote $d(t)$. Similar to standard connection theory, we call the map $c^h : \pi^{-1}(\widetilde{c}(a)) \to \pi^{-1}(\widetilde{c}(b)) : u \mapsto c_u^h(b)$, *the h-displacement along c.* It is easily seen that c^h commutes with R_g for $g \in G$ arbitrary, i.e. $c^h(ug) = c^h(u)g$. Therefore, c^h determines a morphism on the fibres of P. The lift of a composition of ρ-admissible curves, in the sense of Section 2, equals the composition of the corresponding h-admissible curves. Following the constructions described in the previous section, we can also consider the inverse anchored bundles of (ν, ρ) and $(\widetilde{\pi}_1, h)$. We have that $(c^*)^{-h} = (c^h)^{-1}$, i.e. c^h is invertible, that any $\pm h$-admissible curve projects onto a $\pm\rho$-admissible curve and that any ρ-admissible curve is the projection of a $\pm h$-admissible curve. Hence, using Theorem 9, we obtain $\pi(H(u)) = L_{\pi(u)}$. This result is of great importance for the development of a notion of leafwise holonomy for principal ρ-lifts.

Definition 15 *The set of all $g \in G$ such that $ug \in H(u)$, which is called the*

holonomy group with reference point u, *is denoted by* $\Phi(u)$.

The fact that $\Phi(u)$ is a subgroup follows from the following lemma. First, note that, given any $g \in \Phi(u)$, there exists a $\pm h$-admissible curve taking u to $v = ug$, since $v \in H(u)$. This $\pm h$-admissible curve projects onto a $\pm\rho$-admissible loop with base point $\pi(u) = \pi(v) = x$. This implies that v can be reached from u by composing h-admissible curves and $(-h)$-admissible curves. Since a h-admissible curve is a lift of a ρ-admissible curve and since a $(-h)$-admissible curve is a lift of a $(-\rho)$-admissible curve, we obtain that g is determined by composing a finite number of h-displacements along ρ-admissible curves and $(-h)$-displacements along $(-\rho)$-admissible curves. In particular, using the notations from Section 2, we can define a map from the loop space $C(x, N)$ to $\Phi(u)$, which is onto. These observations are used in the proof of the following lemma.

Lemma 16 $\Phi(u)$ *is a subgroup of* G.

Proof. Given any two elements $g, g' \in \Phi(u)$ and let $ug = ((c^\ell)^{\pm h} \circ \ldots \circ (c^1)^{\pm h})(u)$, and $ug' = ((c^{\ell+\ell'})^{\pm h} \circ \ldots \circ (c^{\ell+1})^{\pm h})(u)$, for some $\pm\rho$-admissible curves $c^i, i = 1, \ldots, \ell+\ell'$, and where $(c^i)^{\pm h}$ stands for $(c^i)^h$ if c^i is ρ-admissible, and $(c^i)^{-h}$ if c^i is $(-\rho)$-admissible.

Then $g'g^{-1} \in \Phi(u)$ since

$$ug^{-1}g' = ((c^{\ell+\ell'})^{\pm h} \circ \ldots \circ (c^{\ell+1})^{\pm h} \circ ((c^*)^1)^{\pm h} \circ \ldots \circ ((c^*)^\ell)^{\pm h}(u),$$

and, hence, $ug'g^{-1}$ belongs to $H(u)$. $\square$

In the above proof, we used the fact that any $\pm\rho$-admissible loop $c = c^\ell \cdot \ldots \cdot c^1$ with base point $x \in M$, we can associate a map on the fibre $\pi^{-1}(x)$ which commutes with the right action (i.e. such a map is called an automorphism on $\pi^{-1}(x)$). Indeed, for $u \in \pi^{-1}(x)$ and $g \in G$ arbitrary, we have

$$(c^\ell)^{\pm h} \circ \ldots \circ (c^1)^{\pm h}(ug) = (c^\ell)^{\pm h} \circ \ldots \circ (c^1)^{\pm h}(u)g.$$

Using similar arguments as in the above proof, the set of all such automorphisms on the fibre $\pi^{-1}(x)$ forms a group, which is called the *holonomy group with reference point* x and denoted by $\Phi(x)$. We thus have the following commutative diagram:

$$
\begin{array}{ccc}
& C(x, N) & \\
\swarrow & & \searrow \\
\Phi(x) & \longrightarrow & \Phi(u)
\end{array}
$$

Remark 17 In the specific case where h is a principal ρ-connection, the situation becomes more simple. In order to define the concept of holonomy groups it is sufficient to consider only ρ-admissible loops. Indeed, if c is $(-\rho)$-admissible, then $-c$ is ρ-admissible, and $c^{-h} = (-c)^h$. Moreover, we can consider reparameterisations of ρ-admissible curves and the notion of h-displacement does not depend on the parametrisation of c, in the following sense. Assume that $\phi : [a, b] \to [c, d]$ is a diffeomorphism with $\phi(a) = c$ and $\phi(b) = d$, then the curve $c' : [c, d] \to N$, defined by

$$c'(s) = \frac{d\phi^{-1}}{ds}(s)c(\phi^{-1}(s)),$$

is ρ-admissible and, as can be seen from elementary calculations, it follows that h-displacement along c or c' is the same. Recall the definition of the inverse $c^{-1} = -c^*$ of a ρ-admissible curve c. The following identity holds $(c^{-1})^h = (c^h)^{-1}$.

The following properties are well known from the standard theory of holonomy.

Proposition 18 (*i*) *Given any* $v \in H(u)$*, then* $\Phi(u) = \Phi(v)$*.* (*ii*) *Given any* $g \in G$*, then* $\Phi(ug) = I_{g^{-1}}(\Phi(u))$*, where, I denotes the inner automorphism on G (i.e. for $h \in G$, $I_h : G \to G : h' \mapsto hh'h^{-1}$).*

Proof. By definition of $H(u)$, we have that $H(ug) = R_g(H(u))$. Indeed, $H(u)$ is the leaf of a foliation of a distribution generated by right invariant vector fields. Thus, if $h \in \Phi(u)$, then $h^{-1} \in \Phi(u)$ and $uh^{-1} \in H(u)$, or $H(uh^{-1}) = H(u) = H(v)$. Acting on the right by h, we obtain $H(u) = H(vh)$. And since $H(u) = H(v)$, we have $h \in \Phi(v)$, proving (*i*). Since $H(ug) = H(u)g$, we have that, for any $h \in \Phi(u)$, then $H(uhg) = H(ug)$. Thus $g^{-1}hg \in \Phi(ug)$, proving (*ii*). $\square$

4 Mappings between generalised connections

We first fix some notations. Let (ν', ρ') and (ν, ρ) denote anchored bundles with base manifolds, respectively, M' and M and consider an anchored bundle morphism $f : N' \to N$ between (ν', ρ') and (ν, ρ), which is fibred over $\overline{f} : M' \to M$. Assume that $\pi' : P' \to M'$ and $\pi : P \to M$ are principal fibre bundles with structure groups, respectively G' and G. Furthermore, we assume that a principal fibre bundle morphism $F : P' \to P$ between P' and P is given, such that F is also fibred over the map $\overline{f} : M' \to M$ between the base spaces. The group morphism between G' and G, corresponding to F, is denoted by $\overline{F} : G' \to G$, i.e. for all $u' \in P'$ $g' \in G'$, we have $F(u'g') = F(u')\overline{F}(g')$.

The principal fibre bundle morphism F is called a *morphism between the principal ρ'-lift h' and the principal ρ-lift h* if the map (F, f), defined by $(F, f) : (\pi')^* N' \to \pi^* N : (u', s') \mapsto (F(u'), f(s'))$, is an anchored bundle morphism between $(\widetilde{\pi}'_1, h')$ and $(\widetilde{\pi}_1, h)$. More precisely we have that:

$$TF(h'(u', s')) = h(F(u'), f(s')).$$

Theorem 19 *Assume that f is an isomorphism, and that F is a principal fibre bundle morphism from P' to P, fibred over $\overline{f}$. Let h' be a principal ρ'-lift on P'. There exists a unique principal ρ-lift h such that F is a morphism between h' and h. The holonomy group $\Phi(u')$ of h' is mapped by $\overline{F}$ onto $\Phi(F(u'))$.*

Proof. Let u denote an arbitrary point of P, with $\pi(u) = x$. Then fix an element u' in $P'_{\overline{f}^{-1}(x)}$ and an element g in G such that $F(u') = ug$. Define $h(u, s) \in T_u P$, for any $s \in N_{\overline{f}^{-1}(x)}$, by

$$h(u, s) = TR_{g^{-1}}\left(T_{u'}F(h'(u', f^{-1}(s)))\right).$$

This tangent vector in $T_u P$ is well defined, in the sense that it does not depend on the choice of u', since for any other element $v' = u'g'$, then v' satisfies $F(v') = F(u')\overline{F}(g') = uh$ with $h = g\overline{F}(g')$, implying that

$$
\begin{aligned}
h(u, s) &= TR_{h^{-1}}\left(T_{v'}F(h'(v', f^{-1}(s)))\right)\\
&= TR_{h^{-1}}\left(T_{v'}F(TR_{g'}h'(u', f^{-1}(s)))\right)\\
&= TR_{g^{-1}}TR_{\overline{F}(g'^{-1})}TR_{\overline{F}(g')}\left(T_{u'}F(u', f^{-1}(s))\right)\\
&= TR_{g^{-1}}\left(T_{u'}F(h'(u', f^{-1}(s)))\right).
\end{aligned}
$$

In this way, we have constructed a mapping $h : \pi^* P \to TP$, which is clearly right invariant and, by definition, it follows that F is an anchored bundle morphism between $(\widetilde{\pi}'_1, h')$ and $(\widetilde{\pi}_1, h)$. From the fact that f^{-1} maps any $\pm\rho$-admissible curve onto a $\pm\rho'$-admissible curve, we have that $H(u')$ is mapped by F onto $H(F(u'))$, concluding the proof. $\quad\square$

In the specific case that P' is a reduced subbundle of P, i.e. F is an injective immersion and $\overline{F}$ is an monomorphism, then we say that h is *reducible to a principal ρ-lift on P'*. This is important for our treatment of holonomy, where we prove a generalisation of the Reduction Theorem, which says that $H(u)$ is a reduced subbundle with structure group the holonomy group $\Phi(u)$ and that h is reducible to $H(u)$.

For the following theorem we take for (ν', ρ') the pull-back anchored bundle of (ν, ρ) under $i : L_x \hookrightarrow M$, with L_x the leaf through some $x \in M$. Let $P' = i^* P$

and $F : P' \to P$ the projection onto the second factor. Note that the structure group of P' is precisely G.

Theorem 20 *There exists a unique ρ'-lift h' on P' such that F is a morphism between h' and h. Moreover, $F(H(u')) = H(F(u'))$ and, therefore $\Phi(u') = \Phi(F(u'))$.*

Proof. Since F is an injective immersion, we know from Section 2 that a unique anchor map h' on P' can be defined such that F is an anchored bundle morphism between $(\widetilde{\pi}_1', h')$ and $(\widetilde{\pi}_1, h)$. It is trivial to check that h' satisfies the "right invariance" condition making it into a principal ρ-lift.

The fact that the induced foliations coincide follows from the fact that $\pm\rho$-admissible curves are in one-to-one correspondence with the $\pm\rho'$-admissible curves. $\square$

In the following section we prove that the holonomy groups $\Phi(u)$ of a principal ρ-lift is a Lie subgroup of G. In view of the above theorem, we will assume that, without loss of generality, we are working with the ρ'-lift h' on the bundle i^*P, with $i : L_x \hookrightarrow M$. Indeed, the holonomy groups of h and h' are the same.

5 Leafwise Holonomy of a principal ρ-lift

In view of the above comment, we have that $M = L_x$ is a connected manifold and that $\widetilde{D} = TM$. The main consequence of these assumptions is that the distribution $\widetilde{Q}$ generated by a principal ρ-lift h is regular, i.e. has constant rank. We have to prove that $\dim \widetilde{Q}_u = \dim \widetilde{Q}_v$, given two arbitrary points u, v in P. Let $x = \pi(u)$ and $y = \pi(v)$. Then, since $M = L_x$, there exists a composite flow Φ associated with $(\rho \circ \sigma^\ell, \ldots, \rho \circ \sigma^1)$ of vector fields in $\mathcal{D}$ and a composite flow parameter T such that $\Phi_T(x) = y$ (cf. Theorem 3). Consider the vector fields $(\sigma^i)^h$ in $\mathcal{Q}$. The flows of $(\sigma^i)^h$ and $\rho \circ \sigma^i$ are π-related by definition, and therefore, if Φ^h is the composite flow of $((\sigma^\ell)^h, \ldots, (\sigma^1)^h)$, we have $\pi(\Phi_T^h(u)) = y$, or there exists a $g \in G$ such that $\Phi_T^h(u)g = v$. By definition of $\widetilde{Q}$ we have $T\Phi_T^h(\widetilde{Q}_u) = \widetilde{Q}_{\Phi_T^h(u)}$. On the other hand since $\mathcal{D}^h$ consists of right invariant vector fields and since $\mathcal{D}^h$ generates $\widetilde{Q}$, we have $TR_h(\widetilde{Q}_w) = \widetilde{Q}_{wh}$ for any $w \in P$ and $h \in G$. Thus, we obtain $TR_g \circ T\Phi_T^h$ is an isomorphism from $\widetilde{Q}_u$ to $\widetilde{Q}_v$.

Take an arbitrary point $u \in P$ and consider the linear subspace $\mathfrak{g}(u)$ of $\mathfrak{g}$ defined by $T_e\sigma_u(\mathfrak{g}(u)) = V_u\pi \cap \widetilde{Q}_u$.

Proposition 21 *Take $u \in P$ and let $g \in G$ arbitrary. Then, (i) $\mathfrak{g}(u) = \mathfrak{g}(v)$ for any $v \in H(u)$, (ii) $Ad_{g^{-1}}(\mathfrak{g}(u)) = \mathfrak{g}(ug)$ and (iii) $\mathfrak{g}(u)$ is a Lie subalgebra of $\mathfrak{g}$.*

Proof. (i) follows from the fact that $V\pi$ and $\widetilde{Q}$ are invariant under the image of the tangent map to a composite flow associated with vector fields in $\mathcal{D}^h$. (ii) follows from $TR_g \circ T_e\sigma_u = T_e\sigma_{ug} \circ Ad_{g^{-1}}$, $TR_g(V_u\pi) = V_{ug}\pi$ and $TR_g(\widetilde{Q}_u) = \widetilde{Q}_{ug}$. (iii) follows from $[\sigma(A), \sigma(B)] = \sigma([A, B])$, for $A, B \in \mathfrak{g}$ and the fact that $\widetilde{Q}$ is involutive (since it is integrable, by definition). $\square$

These properties allow us to consider the connected Lie group $\Phi^0(u)$ generated by the Lie algebra $\mathfrak{g}(u)$, which is called the *restricted holonomy group*. From the preceding proposition, we have that $\Phi^0(u) = \Phi^0(v)$ for $v \in H(u)$ and $\Phi^0(ug) = I_{g^{-1}}(\Phi^0(u))$.

We prove that $\Phi^0(u)$ is a normal subgroup of $\Phi(u)$ and that $\Phi(u)/\Phi^0(u)$ is countable, implying that $\Phi(u)$ is a Lie-subgroup of G whose identity component is precisely $\Phi^0(u)$, see [8, p 73]. We first prove that $\Phi^0(u)$ is normal subgroup of $\Phi(u)$.

Let $h \in \Phi^0(u)$. By construction of the Lie subgroup $\Phi^0(u)$ (i.e. it is the leaf through e of the left invariant distribution generated by $\mathfrak{g}(u)$), h is obtained from e by a composite flow associated with left invariant vector fields generated by $\mathfrak{g}(u)$. Note that, if $g(t)$ denotes the integral curve through e of the left invariant vector field corresponding to $A \in \mathfrak{g}(u)$, then $ug(t) \in H(u)$, since $\sigma(A)$ determines a vector field tangent to $H(u)$, and hence $g(t) \in \Phi(u)$. We therefore have $\Phi^0(u) < \Phi(u)$. Since $\Phi^0(ug) = I_{g^{-1}}(\Phi^0(u))$ and $\Phi^0(u) = \Phi^0(ug)$ for any $g \in \Phi(u)$ (i.e. $\mathfrak{g}(u) = \mathfrak{g}(ug)$), we may conclude that $\Phi^0(u)$ is a normal subgroup of $\Phi(u)$.

Following a similar reasoning as in [8, p 73], we now prove that $\Phi(u)/\Phi^0(u)$ is countable by constructing a group morphism from $\pi_1^N(L_x)$ to $\Phi(u)/\Phi^0(u)$ which is onto. Since $\pi_1^N(L_x) < \pi_1(M)$ and $\pi_1(M)$ is at most countable, the obtain that the quotient is also countable.

Proof. Let us first make the following basic observation. In order to prove that the map between $C(x, N)$ and $\Phi(u)$ reduces to a well defined morphism from $\pi_1^N(L_x) \to \Phi(u)/\Phi^0(u)$, we must prove that the images of two $\pm\rho$-admissible loops, whose base curves are homotopic, equal up to an element in $\Phi^0(u)$. This is achieved by using some results from standard connection theory. Once we have obtained this morphism $\pi_1^N(L_x) \to \Phi(u)/\Phi^0(u)$ it is easily seen to be onto, which concludes the proof.

Consider a connection ω on P, such that $\operatorname{im} h^\omega$ is a subspace of $\widetilde{Q}$. This is always possible since $\widetilde{Q}$ is regular and $T\pi(\widetilde{Q}) = TM$. Consider the coefficient χ of h with respect to ω (see Section 3). Note that $T_e\sigma_u(\chi(u,s)) = h(u,s) - h^\omega(u, \rho(s))$ is contained in $\widetilde{Q}$ for any $(u,s) \in \pi^*N$. This implies that $\chi(u,s) \in \mathfrak{g}(u)$, for all $s \in N_{\pi(u)}$. On the other hand, the holonomy group with reference point through x of the standard connection ω is a subgroup of $\Phi(u)$ and the restricted holonomy group of ω is a subgroup of $\Phi^0(u)$, since the smallest integrable distribution spanned by $\operatorname{im} h^\omega$ must be contained in $\widetilde{Q}$ (see [8]).

In Section 3 we have proven that the h-lift $c_u^h(t)$ of a ρ-admissible curve through $u \in \pi^{-1}(x)$ equals $c_u^h(t) = d^\omega(t)g(t)$, where $g(t)$ is a curve in G with $g(a) = e$ and $R_{g(t)^{-1}}\dot{g}(t) = \chi(d^\omega(t), c(t))$, and where $\dot{d}^\omega(t) = h^\omega(d^\omega(t), \tilde{\dot{c}}(t))$, with $d^\omega(a) = u$. In particular we have $g(t) \in \Phi^0(u)$ (since the image of χ is contained in $\mathfrak{g}(u)$). This is also valid for the inverted anchored bundles. Thus we can conclude that any element belonging to $\Phi(u)$ can be written as a product of elements belonging to the holonomy group of ω at u and of elements in $\Phi^0(u)$. Moreover, if the base of a $\pm\rho$-admissible curve is homotopic to zero, then the corresponding product of elements is entirely contained in $\Phi^0(u)$, since the restricted holonomy group of ω is a subgroup of $\Phi^0(u)$. This completes the proof. $\square$

Corollary 22 *The holonomy group $\Phi(u)$ is a Lie subgroup of the structure group G with Lie algebra $\mathfrak{g}(u)$.*

We are now able to state a generalisation of the reduction theorem for principal h-lifts.

Theorem 23 *$H(u)$ is a reduced subbundle of P with structure group $\Phi(u)$ and h reduces to a principal ρ-lift on $H(u)$.*

Proof. It is sufficient to prove that, given a point $y \in L_x$, there exists a neighbourhood $U \ni y$ and a section σ of P defined on U such that $\sigma(U) \subset H(u)$. The existence of such a cross-section follows by using a result from [8, p 84] with respect to a connection ω with horizontal distribution contained in $\widetilde{Q}$.

Since $H(u)$ is the leaf of the foliation induced by $\mathcal{Q}$, we can consider the pull-back anchor map of h. Using the fact that $H(u)$ is a principal fibre bundle over $L_{\pi(u)}$ and using Theorem 19, it is easily seen that h is reducible to the pull-back of h. $\square$

Assume that $\dim M \geq 2$. Then, since $H(u)$ is connected, there exists a standard principal connection $\overline{\omega}$ on $H(u)$ whose holonomy group is the structure

group $\Phi(u)$ (see [8, p 90]). Using Theorem 19 from Section 4, then ω can be extended to a connection on P.

Corollary 24 *If* $\dim M \geq 2$, *then there exists a connection* ω *on* P *such that the holonomy groups of* ω *equal the holonomy groups of the lift* h.

6 Possible field of applications

The equations of motion a free particle subjected to linear nonholonomic constraints can be described as the "geodesics" of a unique connection along the natural injection of the constraint distribution into the tangent bundle of the configuration manifold, see [9]. This unique generalised connection admits a notion of holonomy and, consequently, one can wonder wether the holonomy groups may play a role in the study of nonholonomic motions.

Another field of application could be found in sub-Riemannian geometry, see [10]. However, until now, we haven't been able to construct a unique generalised connection in sub-Riemannian geometry. These possible applications of the above developed theory on holonomy groups of generalised connections is left for future work.

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