Connections in sub-Riemannian geometry

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Abstract. We introduce the notion of a connection over a bundle map and apply it to a sub-Riemannian geometry. It is shown that the concepts of normal and abnormal extremals of a sub-Riemannian structure, can be characterized as parallel transported sections with respect to these generalized connections. Using this formalism we are able to give necessary and sufficient conditions for the existence of a specific class of abnormal extremals.

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1 Sub-Riemannian structures on a manifold

A *sub-Riemannian structure* (M, Q, h) is a triple where M is a pathwise connected manifold, Q is a regular distribution on M and h is a Riemannian bundle metric on Q (considered as a linear bundle over M). The fact that one can assign a notion of length to any curve tangent vector in Q (see below), is an important property associated with a sub-Riemannian structure. Indeed, once the length of a curve is defined, one can look for those curves which minimize length. This problem has been solved locally using the Maximum principle. For instance, in the paper by R. Strichartz, see [10] (and its erratum in [11]), one can find necessary conditions for length minimizing curves. In this section we shall give all preliminary definitions and properties in order to arrive at a formulation of these conditions.

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The module of smooth (i.e. class C^∞) sections of any bundle $\pi : E \to M$ over a manifold M is denoted by $\Gamma(E)$. With any Riemannian bundle metric h on E we can associate the bundle isomorphism $\flat_h : E \to E^*$ with inverse $\sharp_h : E^* \to E$.

Definition 1 *Let (M, Q, h) be a sub-Riemannian structure. The linear bundle mapping $g : T^*M \to TM$, fibred over the identity, is defined by $g = i^* \circ \sharp_h \circ i$, where $i : Q \hookrightarrow TM$ is the natural inclusion.*

Denote the module of 1-forms on M by $\mathcal{X}^*(M) = \Gamma(T^*M)$. The following properties are easily proven: $\ker g = Q^0$ (the annihilator of Q), $\operatorname{Im} g = Q$ and for any $\alpha, \beta \in \mathcal{X}^*(M)$,

$$\langle \beta, g(\alpha) \rangle = \langle \alpha, g(\beta) \rangle = h(g(\alpha), g(\beta)).$$

As a consequence of the last identity we introduce the symmetric tensor $\overline{g} \in \Gamma(TM \otimes TM)$, defined by $\overline{g}(\alpha, \beta) := \langle g(\alpha), \beta \rangle$. Then $\overline{g}(\alpha, \beta) = h(g(\alpha), g(\beta))$.

A curve $c : I = [a, b] \to M$ is always assumed to be the restriction of a smooth (i.e. class C^∞) mapping defined on an open interval containing I. We also assume that c is an injective immersion (in particular $\dot{c}(t) \neq 0$, for any $t \in I$).

Definition 2 *A curve c is tangent to Q if $\dot{c}(t) \in Q_{c(t)}$, for any $t \in I$. A g-admissible curve α is a curve in T^*M such that $g(\alpha(t)) = \dot{c}(t)$, where $c(t) = \pi_M(\alpha(t))$.*

Since $h(\dot{c}(t), \dot{c}(t))$ exists for any curve c tangent to Q, the following notion of length of c is well defined.

Definition 3 *For any curve $c : [a, b] \to M$ tangent to Q, the length of c is given by:*

$$L(c) = \int_a^b h(\dot{c}(t), \dot{c}(t)).$$

It will be interesting to consider a Riemannian metric G on M, such that the restriction of G to Q equals h. In this case, we say that G *restricts to h on Q*. The length of a curve tangent to Q then equals the length measured using the Riemannian metric G. In [10] it is proven that, given any sub-Riemannian structure (M, Q, h), there always exists a Riemannian metric

G that restricts to h on Q. With G we can associate projection mappings π and $\pi^{\perp}$ of TM onto Q and $Q^{\perp}$, respectively, where $Q^{\perp}$ is the orthogonal complement of Q (with respect to the metric G). The projections of T^*M onto Q^0 and $(Q^{\perp})^0 \equiv \flat_G(Q)$ are denoted by $\tau^{\perp}$ and τ, respectively. Using some elementary manipulations the following identities can be proven:

$$\begin{aligned}
&\tau^{\perp} = \flat_G \circ \pi^{\perp} \circ \sharp_G, \\
&\tau = \flat_G \circ \pi \circ \sharp_G, \\
&g \circ \flat_G(X) = X \text{ for all } X \in Q, \\
&g = \pi \circ \sharp_G.
\end{aligned}$$

From these results it is easily seen that any curve c tangent to Q admits a g-admissible curve with base c (for instance, take $\alpha(t) = \flat_G(\dot{c}(t))$). From now on, we assume that Q is bracket generating, i.e. the iterated Lie-brackets of section of Q pointwise generate the full tangent space to M. According to a theorem of Chow [2], this condition guarantees that any two points in M can be connected using a concatenation of curves tangent to Q. The next theorem is taken from [10] and gives necessary conditions for "absolutely continuous curves" tangent to Q connecting two given points to be length minimizing. For simplicity we only consider length minimizing curves that are smooth. All results in this paper can be extended to a more general class of curves (namely piecewise smooth curves): this will be discussed in detail in a forthcoming paper.

Definition 4 *Let $c(t)$ be a curve tangent to Q, contained in a coordinate neighborhood U. We say that c is a normal extremal if there exists a section ψ of T^*M along c such that*

$$\left\{\begin{array}{l} \dot{\psi}_i(t) = -\frac{1}{2}\dfrac{\partial g^{jk}}{\partial x^i}(c(t))\psi_j(t)\psi_k(t), \\ g(\psi(t)) = \dot{c}(t), \end{array}\right. \tag{1}$$

c is said to be an abnormal extremal if there exists a section ψ along c such that

$$\left\{\begin{array}{l} \dot{\psi}_i(t) = -\dfrac{\partial g^{jk}}{\partial x^i}(c(t))\psi_j(t)\alpha_k(t), \\ g(\psi(t)) = 0, \end{array}\right. \tag{2}$$

where $\alpha(t)$ is any g-admissible curve with base c.

Theorem 1 *Let $c : [a,b] \to M$ be a curve tangent to Q contained in a coordinated neighborhood. If c minimizes length, then c is either a normal or abnormal extremal.*

Note that if c is normal, then ψ is a g-admissible curve. On the other hand if c is abnormal, then ψ lies in Q^0. At first sight, the definition of an abnormal extremal depends on the choice of α. However, this is not the case as will become clear later on.

2 Connections over a bundle map

We first develop the more general setting in which connections over a bundle are defined (a more detailed description will be given in a forthcoming paper [1]). The concept of connections over a bundle map is inspired on the work by R.L. Fernandes [3, 4] and, as we have recently found out, is closely related to work done by M. Popescu and P. Popescu (see [9] and references therein). Denote the tangent bundle over M by $\tau_M : TM \to M$. Consider two linear bundles $\nu : N \to M$ and $\pi : E \to M$ and a linear bundle mapping $\rho : \nu \to \tau_M$ fibred over the identity:

$$\begin{tikzcd} N \arrow[rr, "\rho"] \arrow[dr, "\nu"'] & & TM \arrow[dl, "\tau_M"] \\ & M & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[d, "\pi"] \\ M \end{tikzcd}$$

We use the notation $\mathcal{F}(M)$ to denote the ring of smooth functions on a manifold M.

Definition 5 *A connection over the bundle map ρ on the bundle $\pi : E \to M$ (shortly,* a ρ-connection on E*), is defined as a mapping $\nabla : \Gamma(N) \times \Gamma(E) \to \Gamma(E), (s, \sigma) \mapsto \nabla_s \sigma$ such that the following properties hold:*

1. *∇ is $\mathbb{R}$-linear in both arguments;*
2. *∇ is $\mathcal{F}(M)$-linear in s;*
3. *for any $f \in \mathcal{F}(M)$ and for all $s \in \Gamma(N)$ and $\sigma \in \Gamma(E)$ one has: $\nabla_s(f\sigma) = f\nabla_s\sigma + (\rho \circ s)(f)\sigma$.*

Let k and ℓ denote the fibre dimensions of N and E, respectively, and let $\{s^\alpha : \alpha = 1, \ldots, k\}$, resp. $\{\sigma^A : A = 1, \ldots, \ell\}$, be a local basis for the $\mathcal{F}(M)$-module of sections of $\nu : N \to M$, resp. $\pi : E \to M$, defined on a common open neighborhood $U \subset M$. Then we have $\nabla_{s^\alpha}\sigma^A = \Gamma^{\alpha A}_B \sigma^B$, for some functions $\Gamma^{\alpha A}_B \in \mathcal{F}(U)$, called the connection coefficients of the given ρ-connection. In order to associate a notion of parallel transport with linear ρ-connections, we first need to introduce a special class of curves in N. A curve $\tilde{c} : I = [a,b] \to N$ is called *ρ-admissible* if $\dot{c}(t) = (\rho \circ \tilde{c})(t)$, for any $t \in I$, where c is assumed to be the *base curve* of $\tilde{c}$, i.e. $c = \nu \circ \tilde{c}$. Note that, in principle, a base curve may reduce to a point.

As in standard connection theory, with any linear ρ-connection ∇ on a vector bundle $\pi : E \to M$, and any ρ-admissible curve $\tilde{c} : [a,b] \to N$, one can associate an operator $\nabla_{\tilde{c}}$, acting on sections of π defined along the base curve $c = \nu \circ \tilde{c}$. More precisely, let σ be such a section, i.e. $\sigma : [a,b] \to E$ with $\pi \circ \sigma = c$ and let $f \in \mathcal{F}([a,b])$, then the operator $\nabla_{\tilde{c}}$ is uniquely determined from ∇ if it satisfies

1. $\nabla_{\tilde{c}}$ is $\mathbb{R}$ linear;
2. $\nabla_{\tilde{c}} f\sigma = \dot{f}\sigma + f\nabla_{\tilde{c}}\sigma$;
3. $\nabla_{\tilde{c}}\sigma(t) = \nabla_{\tilde{c}(t)}\overline{\sigma}$, for $\overline{\sigma} \in \Gamma(E)$ such that $\overline{\sigma}(c(t)) = \sigma(t)$ for all $t \in [a,b]$.

Definition 6 *A section σ of π, defined along the base curve c of a ρ-admissible curve $\tilde{c}$, will be called parallel along $\tilde{c}$ if and only if $\nabla_{\tilde{c}}\sigma(t) = 0$ for all t.*

Using the notation from above and putting $\sigma(t) = r_A(t)\sigma^A(c(t))$ in such a coordinate chart, we have

$$\nabla_{\tilde{c}}\sigma(t) = \left(\dot{r}_A(t) + \Gamma^{\alpha B}_A(\tilde{c}(t)) r_B(t)\tilde{c}_\alpha(t)\right)\sigma^A(c(t)) = 0,$$

for all $t \in I$. This is a set of linear differential equations in the components of σ and therefore, given any ρ-admissible curve and an initial element in $E_{\tilde{c}(a)}$, a unique parallel transported section of E along $\tilde{c}$ can be found, defined on the whole of $[a,b]$.

We will now apply the theory of ρ-connections to a sub-Riemannian structure (M,Q,h). Using the notations from above we now take $N = T^*M$, $\rho = g$ and $E = T^*M$ (note that the notion of a g-admissible curve, introduced in the previous section, coincides with the notion of a ρ-admissible curve for $\rho = g$). Our main goal is to characterize the concepts of normal and abnormal extremals making use of g-connections.

Definition 7 *Given a sub-Riemannian structure (M, Q, h), we define the following bracket of 1-forms on M:*

$$\{\alpha, \beta\} = \mathcal{L}_{g(\alpha)}\beta + \mathcal{L}_{g(\beta)}\alpha - d(\overline{g}(\alpha, \beta)), \textit{ for } \alpha, \beta \in \mathcal{X}^*(M).$$

It is easily seen that this bracket satisfies the following properties:

1. $\{\alpha, \beta\} = \{\beta, \alpha\}$,
2. the bracket is $\mathbb{R}$ linear in both arguments,
3. $\{f\alpha, \beta\} = g(\beta)(f)\alpha + f\{\alpha, \beta\}$, with $f \in \mathcal{F}(M)$,
4. $\{\alpha, \eta\} = \mathcal{L}_{g(\alpha)}\eta$, with $\eta \in \Gamma(Q^0)$ and equals zero if α is also contained in $\Gamma(Q^0)$.

Definition 8 *A g-connection ∇ is said to be normal if $\nabla_\alpha\beta + \nabla_\beta\alpha = \{a, \beta\}$ for all $\alpha, \beta \in \mathcal{X}^*(M)$.*

It is easily verified that this notion of normal g-connection is well defined. On a local coordinate chart, the connection coefficients for a normal g-connection satisfy:

$$\Gamma_k^{ij} + \Gamma_k^{ji} = \frac{\partial \overline{g}^{ij}}{\partial x^k}, \text{ for all } i, j, k = 1, \ldots, n,$$

or equivalently, for all $\alpha \in T_x^*M$:

$$\Gamma_k^{ij}(x)\alpha_i\alpha_j = \tfrac{1}{2}\frac{\partial \overline{g}^{ij}}{\partial x^k}(x)\alpha_i\alpha_j. \tag{3}$$

In order to state the following theorem we need to introduce the notion of an autoparallel curve of a g-connection. A g-admissible curve $\alpha : I \to T^*M$ is said to be an *autoparallel curve* with respect to a g-connection ∇ if $\nabla_\alpha\alpha(t) = 0$ for all $t \in I$.

Theorem 2 *A normal extremal is the base curve of an autoparallel curve of a normal g-connection.*

The proof immediately follows from the definitions, see equations (3) and (1).

Definition 9 *We say that a g-connection ∇ is bundle adapted (shortly B-adapted) if*

$$\nabla_\alpha \eta = \{\alpha, \eta\} = \mathcal{L}_{g(\alpha)}\eta = i_{g(\alpha)}d\eta$$

for all $\alpha \in \mathcal{X}^*(M)$ and $\eta \in \Gamma(Q^0)$.

Let α be a g-admissible curve with base curve c (contained in a coordinate neighborhood) and η a section of Q^0 along c. If ∇ is a B-adapted g-connection, then the connection coefficients have to satisfy:

$$\Gamma_k^{ij}(x)\beta_i\zeta_j = \frac{\partial \overline{g}^{ij}}{\partial x^k}(x)\beta_i\zeta_j, \text{ for any } \zeta_j dx^j \in Q_x^0,\ \beta_j dx^j \in T_x^*M$$

Therefore $\nabla_\alpha\eta(t)$ is completely determined by the fact that ∇ is B-adapted, as can be seen from the following expression:

$$\nabla_\alpha\eta(t) = \left(\dot{\eta}_i(t) + \frac{\partial \overline{g}^{jk}}{\partial x^i}(c(t))\alpha_j(t)\eta_k(t)\right)dx^i(c(t)).$$

Note that, if $\beta_i dx^i \in Q_x^0$, then $\frac{\partial \overline{g}^{ij}}{\partial x^k}(x)\beta_i\eta_j = 0$. This implies that $\nabla_\alpha\eta(t)$ only depends on $g(\alpha)$. We introduce a new notation ∇^B for this operator acting on sections of Q^0 along curves tangent to Q: $\nabla^B_{\dot{c}}\eta(t) = \nabla_\alpha\eta(t)$, where ∇ is a B-adapted g-connection. Since $\nabla_\alpha\eta(t)$ only depends on $g(\alpha) = \dot{c}$, the notation is justified. By comparing the coordinate expressions for $\nabla^B_{\dot{c}}\eta(t)$ with equation (2), the following theorem can be easily proven.

Theorem 3 *Let $c : I \to M$ be a curve tangent to Q. Then c is an abnormal extremal if and only if there exists a parallel transported section η of Q^0 along c with respect to a B-adapted g-connection, i.e. $\nabla^B_{\dot{c}}\eta(t) = 0$ for all $t \in I$.*

3 Characterizing abnormal extremals

We first mention that, given any curve $c : I \to M$ tangent to Q, contained in a coordinate neighborhood, a vector field $X \in \Gamma(Q)$ can be found such that $\dot{c}(t) = X(c(t))$. This is a special case of a more general result proven by S. Helgason in [5, p. 26]. In the following we always assume that c is contained in a coordinate neighborhood and that it is an integral curve of a vector field tangent to Q (usually integral curves are defined on open intervals, we assume here that c is the restriction to I of an integral curve defined on an open interval containing I).

Lemma 1 *Let $c(t):[a,b]\rightarrow M$ be an integral curve of a vector field $X\in\Gamma(Q)$, with flow $\{\phi_s\}$. Assume that $\eta(t)$ is a section of Q^0 along $c(t)$. Then the following two equations are equivalent:*

$$\nabla^B_{\dot c}\eta(t)=0 \iff \eta(t)=T^*\phi_{-(t-a)}(\eta(a)).$$

Proof. We first proof that $\nabla^B_{\dot c}\eta(t)=\frac{d}{ds}\big|_0\left(T^*\phi_s(\eta(t+s))\right)$. In coordinates, taking any $\alpha\in\mathcal{X}^*(M)$ such that $g(\alpha)=X$ (for instance $\alpha=\flat_G(X)$), we find:

$$\nabla^B_{\dot c}\eta(t)=\left(\dot\eta_i(t)+\eta_j(t)\frac{\partial X^j}{\partial x^i}(c(t))\right)dx^i(c(t)).$$

On the other hand we have that

$$\begin{aligned}\frac{d}{ds}\bigg|_0\left(T^*\phi_s(\eta(t+s))\right) &= \frac{d}{ds}\bigg|_0\left(\eta_j(t+s)\frac{\partial\phi_s^j}{\partial x^i}(c(t))dx^i(c(t))\right)\\ &= \left(\dot\eta_i(t)+\eta_j(t)\frac{\partial X^j}{\partial x^i}(c(t))\right)dx^i(c(t)).\end{aligned}$$

Assume that $\nabla^B_{\dot c}\eta(t)=0$ then

$$\frac{d}{ds}\bigg|_0\left(T^*\phi_s(\eta(t+s))\right)=0,\ \forall t\in I.$$

For fixed $t\in I$, we apply the linear isomorphism $T\phi_{-(t-a)}$ to this relation and we obtain:

$$\frac{d}{dt}\bigg|_t\left(T^*\phi_{t-a}(\eta(t))\right)=0.$$

This holds for any $t\in I$, implying that $\eta(t)=T^*\phi_{-(t-a)}(\eta(a))$. The converse is simply proven by reversing the above arguments. $\square$

Definition 10 *Let Q denote a distribution on M. Let $\{\phi_s\}$ denote the flow of a vector field $X\in\Gamma(Q)$ with integral curve $c(t)=\phi_{t-a}(c(a)):[a,b]\rightarrow M$. We define a subspace c_t^*Q of $T_{c(t)}M$ as follows:*

$$c_t^*Q:=\mathit{Span}\left\{T\phi_{-(s-t)}(Y_{c(s)})\mid \forall Y_{c(s)}\in Q_{c(s)},\ s\in[a,b]\right\}.$$

Theorem 4 *Let $c(t):[a,b]\rightarrow M$ be an integral curve a vector field X with flow $\{\phi_s\}$. Then $c(t)$ is an abnormal extremal if and only if $c_t^*Q\neq T_{c(t)}M$.*

Proof. Assume $c(t)$ is abnormal, i.e. there exists a section of Q^0 along $c(t)$, say $\eta(t)$. From the preceding lemma we know that $\eta(t) = T^*\phi_{-(t-a)}(\eta(a))$. Since $\eta(t) \in Q^0$ we have

$$\langle \eta(t), Y_{c(t)} \rangle = \langle \eta(a), T\phi_{-(t-a)}(Y_{c(t)}) \rangle = 0,$$

for all $Y_{c(t)} \in Q_{c(t)}$ and $\forall t \in [a,b]$. Following the definition of c_a^*Q we conclude that $\eta(a) \in (c_a^*Q)^0$. This proves one part of the theorem.

Assume that $c_a^*Q \neq T_{c(a)}M$, i.e. there exists a non-trivial $\eta_a \in (c_a^*Q)^0$. The curve $\eta(t)$ defined by $\eta(t) = T^*\phi_{-(t-a)}(\eta_a)$ lies entirely in Q^0 (using the same equation as above). The preceding lemma says that $\nabla^B_{\dot{c}}\eta(t) = 0$ for all $t \in [a,b]$. $\square$

Note that $T\phi_{t-a}$ determines an isomorphism between c_a^*Q and c_t^*Q, implying that the rank of c_t^*Q is constant for every t. Moreover, the Theorem 4 implies that the subspace c_t^*Q is in fact independent of the flow $\{\phi_s\}$ used to define it. Indeed, every element in $(c_t^*Q)^0$ (t fixed) is in a one-to-one correspondence with a parallel transported section η along c with respect to a B-adapted connection, i.e. we have that

$$(c_t^*Q)^0 = \{\eta(t) \mid \nabla^B_{\dot{c}}\eta(s) = 0 \ \forall s \in I\}.$$

This justifies the notation we used.

Remark. In a recent paper by P. Piccione and D.V. Tausk [7] a similar characterization for abnormal extremals was obtained but following a completely different approach.

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