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COSYMPLECTIC REDUCTION OF CONSTRAINED SYSTEMS WITH SYMMETRY

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A general model for time-dependent constrained systems is proposed within the framework of cosymplectic geometry. The model covers, in particular, the case of time-dependent nonholonomic mechanical systems. The theory of reduction of these systems in the presence of symmetry is discussed.

Key words: cosymplectic structures, time-dependent constraints, nonholonomic systems, symmetry, reduction.

1. Introduction

The reduction theory of time-independent nonholonomic systems with symmetry has already been discussed at great length in the literature, at least for the case where the symmetry is defined in terms of a regular (i.e. free and proper) Lie group action: see e.g. [2, 3, 5, 8, 9, 11, 12, 15, 21]. In [6], this theory was reconsidered in the framework of a general model for constrained dynamical systems on a symplectic manifold.

The purpose of the present paper is to extend the symmetry and reduction theory of nonholonomic mechanics to the time-dependent setting, where the constraints as well as the Lagrangian (or Hamiltonian) may exhibit an explicit time-dependence. We will thereby consider the general case of constraints with a possibly nonlinear dependence on the velocities. A natural geometric environment for dealing with time-dependent mechanics is provided by cosymplectic geometry. In analogy with the treatment of the autonomous case in [6], we will propose a general model for constrained systems on a cosymplectic manifold which covers, in particular, the description of time-dependent nonholonomic mechanics. The reduction theory of time-dependent nonholonomic systems with symmetry will then be discussed in terms of this general model. This work can be seen as an extension of the cosymplectic reduction procedure described in [1], to time-dependent systems with constraints.

The scheme of the paper is as follows. In Section 2 we recall some general aspects on cosymplectic structures and their reduction under symmetry. In Section 3 we then propose a model for constrained systems on a cosymplectic manifold, and discuss the existence and uniqueness of a constrained dynamics. Sections 4 and 5 are devoted to the general theory of symmetry and reduction of this model. Finally, in Section 6, a particular case is considered of symplectic reduction under a 1-parameter group of symmetries.

Throughout this paper, all geometric structures and maps are supposed to be of class C^∞. We will make no notational distinction between a vector bundle over a manifold and the module of its smooth sections: if $\pi : F \to N$ denotes a vector bundle over a manifold N and X is a C^∞-section of π, we will simply denote this by $X \in F$. From the context it should always be clear whether we are referring to a point (vector) of the bundle, or to a section. The tangent map of a smooth map f between manifolds will be denoted by Tf. Finally, if U is a linear subspace (resp. vector subbundle) of a linear space (resp. vector bundle) V, its annihilator in the dual space (resp. dual bundle) V^* will be denoted by U^o.

2. Cosymplectic structures

For more details about the concepts and properties mentioned in this section, we refer to the paper by C. Albert [1] (see also [4]).

A *cosymplectic vector space* is given by a triple (V, Ω, η) consisting of a $(2n+1)$-dimensional real linear space V (with $n \geq 1$), an exterior 2-form $\Omega \in \bigwedge^2 V^*$ and a 1-form (co-vector) $\eta \in V^*$ such that $\Omega^n \wedge \eta \neq 0$.

A *cosymplectic manifold* is a triple (M, Ω, η), where

- M is a $(2n+1)$-dimensional real smooth manifold ($n \geq 1$);
- Ω is a closed 2-form and η a closed 1-form on M (i.e. $d\Omega = 0$ and $d\eta = 0$);
- $(T_m M, \Omega_m, \eta_m)$ is a cosymplectic vector space for all $m \in M$.

From the definition it immediately follows that $\Omega^n \wedge \eta \neq 0$ everywhere, i.e. $\Omega^n \wedge \eta$ is a volume form on M. Moreover, it is not difficult to check that a cosymplectic structure induces a linear bundle isomorphism

$$\chi_{\Omega,\eta} : TM \longrightarrow T^*M, \quad v \in T_m M \longmapsto i_v \Omega_m + \eta_m(v)\eta_m \ . \tag{1}$$

On a cosymplectic manifold there exists a distinguished vector field, called the *Reeb vector field*, defined by $R = \chi^{-1}_{\Omega,\eta} \circ \eta$ or, equivalently, by

$$i_R \Omega = 0 \ , \quad i_R \eta = 1 \ .$$

One can prove that in a neighbourhood of each point of a cosymplectic manifold (M, Ω, η), one can define *canonical coordinates* (t, q^i, p_i) (with $i = 1, \ldots, n$) in terms of which Ω, η and R read (cf. [1]):

$$\Omega = dq^i \wedge dp_i \ , \quad \eta = dt \ , \quad R = \frac{\partial}{\partial t} \ .$$

Next, with any smooth function $f \in C^\infty(M)$ one can associate a *Hamiltonian vector field* X_f, defined by

$$i_{X_f}\eta = 0 \ , \quad i_{X_f}\Omega = df - R(f)\eta \ . \tag{2}$$

In terms of canonical coordinates, this vector fields is of the form

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} \ .$$

Note that, in spite of the denomination, X_f does not represent the dynamics of a time-dependent Hamiltonian system with Hamiltonian f. The latter is described instead by the so-called evolution vector field $E_f = X_f + R$ (see e.g. [4]).

For later use we still introduce the following notation: given a subbundle $U \subset TM$, denote the *cosymplectic orthogonal complement* by,

$$U^\perp = \chi^{-1}_{\Omega,\eta}(U^o) \ . \tag{3}$$

We now turn to the symmetry and reduction theory for cosymplectic structures. Let (M, Ω, η) be a cosymplectic manifold, with Reeb vector field R. Let G be a Lie group, with Lie algebra $\mathfrak{g}$, and assume there exists an action Φ of G on M by automorphisms of the cosymplectic structure, i.e. $\Phi : G \times M \to M$ defines a smooth Lie group action such that for all $g \in G$, with $\Phi_g := \Phi(g, .)$, we have

$$\Phi^*_g\Omega = \Omega \quad \text{and} \quad \Phi^*_g\eta = \eta.$$

We will call such an action Φ a *cosymplectic action.* For any vector $\xi \in \mathfrak{g}$, the corresponding infinitesimal generator of Φ on M will be denoted by ξ_M.

A cosymplectic action Φ is said to admit a *momentum map* if there exists a smooth map $J : M \longrightarrow \mathfrak{g}^*$, with $\mathfrak{g}^*$ the dual space of the Lie algebra $\mathfrak{g}$, such that

$$\xi_M = X_{J_\xi} \ , \forall \xi \in \mathfrak{g} \quad \text{and} \quad R(J) = 0 \ , \tag{4}$$

where $J_\xi := \langle \xi, J\rangle \in C^\infty(M)$ and X_{J_ξ} is the corresponding Hamiltonian vector field. One can show that if Φ admits a momentum map J, one can always define a representation ψ of G on $\mathfrak{g}^*$ such that J is equivariant with respect to Φ and ψ (cf. [1], Proposition 6). If $\psi = \mathrm{Ad}^*$ (i.e. the coadjoint action of G on $\mathfrak{g}^*$), we will say that J is Ad^*-equivariant.

Assume now we have a cosymplectic action Φ on (M, Ω, η) which is free and proper, admitting a momentum map J which is equivariant with respect to Φ and a representation ψ of G on $\mathfrak{g}^*$. Let $\mu \in \mathfrak{g}^*$ be a (weakly) regular value of J and G_μ the isotropy group of ψ in μ, then $J^{-1}(\mu)$ is fibred over the orbit space $J^{-1}(\mu)/G_\mu$, with projection denoted by $\pi_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$. Putting $j_\mu : J^{-1}(\mu) \hookrightarrow M$, the natural injection, we arrive at the following result.

THEOREM 1. (Cosymplectic reduction theorem [1]). *Under the above assumptions there exists a cosymplectic structure (Ω_μ, η_μ) on the orbit space $J^{-1}(\mu)/G_\mu$, which is uniquely determined by*

$$j^*_\mu\Omega = \pi^*_\mu\Omega_\mu \ , \quad j^*_\mu\eta = \pi^*_\mu\eta_\mu \ .$$

Under some conditions, it may also be possible to find a reduction of a cosymplectic manifold (M, Ω, η) to a symplectic manifold. In its simplest form this may come about as follows. Assume there exists a complete vector field T on M satisfying

$$i_T\Omega = dH \ , \quad i_T\eta = 1 \ ,$$

for some $H \in C^\infty(M)$. In particular, T induces a cosymplectic action of $\mathbb{R}$ on M. Define the following 2-form on M:

$$\Omega_0 = \Omega - dH \wedge \eta \ .$$

Note, in passing, that (Ω_0, η) is again a cosymplectic structure on M (see [7], Lemma 5.5). Assume, in addition, that $M/\mathbb{R}$ is a smooth manifold. It is easily verified that both Ω_0 and H are projectable onto $M/\mathbb{R}$, with projections denoted by, respectively, $\widetilde{\Omega_0}$ and $\widetilde{H}$.

THEOREM 2. (Symplectic reduction theorem [1]). *The 2-form $\widetilde{\Omega_0}$ defines a symplectic structure on $M/\mathbb{R}$, and the Reeb vector field R of (M, Ω, η) projects onto the Hamiltonian vector field $X_{\widetilde{H}}$ on the symplectic quotient space $(M/\mathbb{R}, \widetilde{\Omega_0})$.*

3. A general model for time-dependent constrained systems

In the recent literature one can find various approaches to the geometric formulation of time-dependent nonholonomic mechanics (see e.g. [7, 10, 13, 14, 16, 17, 18, 19, 20]). We will give a very brief outline of such a formulation, mainly as a motivation for the general model we are going to present subsequently.

Consider a mechanical system described by a regular Lagrangian L, defined on the first jet bundle $J^1\pi$ of a fibred manifold $\pi : E \to \mathbb{R}$ (the 'space-time' manifold of the system). Local bundle coordinates on E, resp. $J^1\pi$, will be denoted by (t, q^A), resp. $(t, q^A, \dot{q}^A)$. On $J^1\pi$, there exists a canonical type-(1,1) tensor field S (the 'vertical endomorphism') which in coordinates reads: $S = (dq^A - \dot{q}^A dt) \otimes (\partial/\partial\dot{q}^A)$. The Poincaré-Cartan 2-form associated with L is given by $\Omega_L = -d(Ldt + S^*(dL))$, and one can show that regularity of L is equivalent to (Ω_L, dt) defining a cosymplectic structure on $J^1\pi$. Now, assume the system is subjected to some, possibly nonlinear, nonholonomic constraints, represented by a submanifold C of $J^1\pi$. To preclude the existence of 'holonomic' constraints, we require that $\pi_{1,0}(C) = E$, where $\pi_{1,0} : J^1\pi \to E$ is the natural projection. When adopting the so-called Chetaev approach to nonholonomic mechanics, it follows that the bundle of 'constraint (or reaction) forces' along C is given by the co-distribution $S^*(TC^o)$ (cf. [13, 16, 20]). It is customary to make the following *admissibility assumption*: $\dim(S^*(TC^o)) = \dim(TC^o)$. The nonholonomic problem then consists in finding a section X of $TJ^1\pi_{|C}$ such that the following relations hold (see e.g. [7, 14]):

$$(i_X\Omega_L)_{|C} \in S^*(TC^o) \ , \quad i_X dt_{|C} = 1 \ , \quad X \in TC \ . \tag{5}$$

This is an intrinsic way of representing the classical constrained Euler-Lagrange equations:

$$\left\{ \begin{array}{l} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}^A}\right) - \dfrac{\partial L}{\partial q^A} = \displaystyle\sum_i \lambda^i \frac{\partial \phi_i}{\partial \dot{q}^A} \ , \\ \phi_i(t, q^A, \dot{q}^A) = 0 \ , \end{array} \right.$$

where the λ^i are the Lagrange multipliers and ϕ_i are some independent constraint functions, locally defining C. Finally, in case L is 'hyperregular' we can pass from the Lagrangian to an equivalent (global) Hamiltonian description of the given nonholonomic system, which exhibits a structure similar to (5). For details, see [14, 19, 20].

In view of the above description of time-dependent nonholonomic mechanics, we now propose a general model for time-dependent constrained systems, similar to the one considered in [6] for the autonomous case. This model is built on the following ingredients:

- a cosymplectic manifold (M, Ω, η);
- a closed embedded submanifold C of M (the *constraint submanifold*);
- a smooth distribution F on M of constant rank, defined along C; i.e. F is a vector subbundle of $TM_{|C}$.

For simplicity, both M and C will always be assumed to be connected. We are then interested in the following problem: find a smooth section X of the restricted tangent bundle $TM_{|C} \to C$, such that

$$\begin{cases} (i_X\Omega)_{|C} \in F^o \ , \\ (i_X\eta)_{|C} = 1 \ , \\ X \in TC \ , \end{cases} \tag{6}$$

with F^o the annihilator of F in $T^*M_{|C}$. In particular, we see that X, if it exists, induces a vector field on C. The Lagrangian (and Hamiltonian) description of nonholonomic mechanics is indeed recovered as a special case, as is readily seen upon comparing (5) and (6).

With R denoting the Reeb vector field associated to (Ω, η), we immediately have that for any solution X of (6), $(i_X\Omega)_{|C}(R_{|C}) = 0$. Since $(i_X\Omega)_{|C} \in F^o$, this justifies the following additional condition we will impose on the given model:

$$R_{|C} \in F \ . \tag{7}$$

Note that this condition is fulfilled in the case of nonholonomic mechanics.

The following proposition now characterises the existence and uniqueness of a solution of the constrained problem (6).

Proposition 1. *A general time-dependent constrained system of the form (6) admits a solution X if and only if $\eta_{|C} \in (F \cap TC^\perp)^o$. In addition, this solution will be unique if and only if $F^\perp \cap TC = 0$.*

Proof: Suppose that $X \in TM_{|C}$ is a solution of (6). For any $Y \in TC^\perp$ we then have, by definition, that

$$0 = \chi_{\Omega,\eta}(Y)(X) = \Omega(Y, X) + i_Y\eta \, .$$

Hence, $(\eta - i_X\Omega)_{|C} \in (TC^\perp)^o$, from which it follows that $\eta_{|C} \in (TC^\perp)^o + F^o = (F \cap TC^\perp)^o$. Conversely, if $\eta_{|C}$ belongs to $(F \cap TC^\perp)^o$, then there exists a $\beta \in F^o$ such that

$\eta_{|C} - \beta \in (TC^\perp)^o$. Consider the section $W = \chi^{-1}_{\Omega,\eta} \circ (\beta + \eta_{|C})$ of $TM_{|C}$. Taking into account (7), contraction of the 1-form $\chi_{\Omega,\eta}(W)$ with R gives $\eta(W) = \chi_{\Omega,\eta}(W)(R_{|C}) = (\beta+\eta)(R_{|C}) = 1$. Consequently, $(i_W\Omega)_{|C} = \beta \in F^o$. In addition, we have that

$$\chi_{\Omega,\eta}(Y)(W) = \Omega(Y,W) + \eta(Y) = (\eta - \beta)(Y) = 0\,, \quad \forall Y \in TC^\perp\,.$$

Since, by definition, $\chi_{\Omega,\eta}(TC^\perp) = TC^o$, this implies that $W \in TC$ and, therefore, W is a solution of (6).

To prove uniqueness, consider two solutions X_1, X_2 of (6). Then, for all $Z \in F$, we have that $\chi_{\Omega,\eta}(X_1 - X_2)(Z) = 0$, that is, $X_1 - X_2 \in F^\perp \cap TC$. Hence, if $F^\perp \cap TC = 0$, the solution to (6) is unique. For the converse, we will prove that in case $F^\perp \cap TC \neq 0$, a solution of (6), if it exists, will not be unique. Indeed, let X be a solution of (6) and take any $0 \neq \hat{X} \in F^\perp \cap TC$. Then, $\eta(\hat{X}) = \chi_{\Omega,\eta}(\hat{X})(R) = 0$ and, therefore, $(i_{\hat{X}}\Omega)_{|C} = \chi_{\Omega,\eta}(\hat{X}) \in F^o$. It follows that $X + \hat{X}(\neq X)$ is also a solution of (6). $\boxed{QED}$

Remark 1: If for a given constrained system we have that $\dim F^o = \dim TC^o$, which is sometimes called the 'admissibility condition' in nonholonomic mechanics (see above), it readily follows that $F^\perp \cap TC = 0$ implies $TM_{|C} = F^\perp \oplus TC$. From the latter we deduce that $F \cap TC^\perp = 0$ or, equivalently, $(F \cap TC^\perp)^o = T^*M$. Consequently, if the admissibility condition holds, Proposition 1 tells us that $F^\perp \cap TC = 0$ is the necessary and sufficient condition for the existence of a unique solution of (6).

Remark 2: If a nonholonomic system (5) admits a unique solution X, the latter satisfies the second-order equation condition, i.e. $S(X)(x) = 0$ at each point $x \in C$ (cf.[7, 14, 20]). From this property one can then easily deduce that $X \in F$ (since $F^o = S^*(TC^o)$ is spanned by certain linear combinations of the contact forms $dq^A - \dot{q}^A dt$).

4. Symmetry and reduction

In the sequel we always consider a constrained system (M,Ω,η,C,F), as introduced in the previous section, satisfying the additional condition (7). Let there be given a cosymplectic action $\Phi : G \times M \longrightarrow M$ of a Lie group G on the cosymplectic manifold (M,Ω,η) (cf. Section 2), such that the constraint submanifold C and the vector subbundle F of $TM_{|C}$ are G-invariant, i.e. for all $g \in G$ we have:

- $\Phi_g^*\Omega = \Omega$, $\Phi_g^*\eta = \eta$;
- $\Phi_g(C) \subseteq C$;
- $T\Phi_g(F_x) = F_{\Phi_g(x)}$ for all $x \in C$.

If the constrained problem (6) admits a solution X, it is routine to verify that Φ_g^*X will also be a solution for each $g \in G$. This still means that at each point $x \in C$, $(\Phi_g^*X)(x) - X(x) \in F_x^\perp \cap T_xC$. In particular, in case (6) has a unique solution, the latter will automatically be G-invariant. For the further analysis we confine ourselves to the case where G is connected and Φ is free and proper.

We will first present a general reduction scheme which is the analogue of the one established by Bates and Śniatycki [2] for autonomous nonholonomic Hamiltonian systems with symmetry. Although some of the subsequent results also hold under weaker

assumptions, for the sake of clarity and conciseness we will always assume that:
(i) the given problem (6) admits a unique solution X, (such that, in particular, the conditions of Proposition 1 hold);
(ii) X is a section of F (i.e. $X \in F$).
From the remarks at the end of the previous section it follows that, at least from the point of view of nonholonomic mechanics, both assumptions are not very restrictive. From the above we then also learn that X is G-invariant.

Since Φ is assumed to be a free and proper action, the orbit space $\bar{M} = M/G$ is a differentiable manifold and $\rho : M \to \bar{M}$ is a principal fibre bundle with structure group G, whereby ρ denotes the natural projection. We will denote by $\mathcal{V}$ the subbundle of TM whose fibres are the tangent spaces to the G-orbits, i.e. $\mathcal{V}_x = T_x(Gx)$ or, equivalently, $\mathcal{V} = \ker T\rho$. The assumed invariance of C under the given action implies, in particular, that $\mathcal{V}_x \subset T_xC$ for all $x \in C$, i.e. $\mathcal{V}_{|C} \subset TC$.

We now define a generalised distribution U on M along C by putting for each $x \in C$:

$$U_x = \{v \in F_x \cap T_xC \mid \Omega(v, \tilde{\xi}) = 0\ ,\quad \text{for all } \tilde{\xi} \in \mathcal{V}_x \cap F_x\} \tag{8}$$

In the sequel, we will always tacitly assume that U has constant rank, determining a genuine vector bundle over C. It is readily seen that U is G-invariant and, hence, projects onto a subbundle $\bar{U}$ of $T\bar{M}_{|\bar{C}}$.

Let us now denote by Ω_U the restriction of Ω to (sections of) U. Clearly, Ω_U is also G-invariant and since, moreover, $i_{\tilde{\xi}}\Omega_U = 0$ for all $\tilde{\xi} \in \mathcal{V} \cap U$, the 2-form Ω_U reduces to a 2-form $\Omega_{\bar{U}}$ on $\bar{U}$ (i.e. $\Omega_{\bar{U}}$ only acts on vectors belonging to $\bar{U}$). In order to complete the reduction procedure, we have to make the additional assumption that $\eta_{|\mathcal{V}\cap F} = 0$. Together with the G-invariance of η, this implies that the restriction of η to U can also be pushed forward under ρ to a 1-form $\eta_{\bar{U}}$ on $\bar{U}$. Note that neither $\Omega_{\bar{U}}$ nor $\eta_{\bar{U}}$ are genuine differential forms on $\bar{C}$: they are exterior forms on the vector bundle $\bar{U}$ over $\bar{C}$, with smooth dependence on the base point.

Under the previous assumptions we can now prove the following reduction result.

THEOREM 3. *The G-invariant solution X of (6) projects onto $\bar{C}$, and its projection $\bar{X}$ is a section of $\bar{U}$ satisfying the equations*

$$\begin{cases} i_{\bar{X}}\Omega_{\bar{U}} = 0\ , \\ i_{\bar{X}}\eta_{\bar{U}} = 1\ . \end{cases} \tag{9}$$

Proof: From (6) we have that

$$(i_X\Omega)_{|C} = \beta\ , \quad (i_X\eta)_{|C} = 1\ ,$$

for some $\beta \in F^o$. It follows that for any section $\tilde{\xi}$ of $\mathcal{V} \cap F$, we have $\beta(\tilde{\xi}) = 0$ and, therefore, we can indeed conclude that $X \in U$. Hence, we have $i_X\Omega_U = 0$, and the remainder of the proof now readily follows from the symmetry assumptions and from previous considerations. $\boxed{QED}$

REMARK 3: In case $\dim F^o = \dim TC^o$ one can show, taking into account the properties described above (cf. Remark 1), that the reduced structure $(\Omega_{\bar{U}}, \eta_{\bar{U}})$ is nondegenerate in the following sense: if $\bar{v} \in \bar{U}_{\bar{x}}$ (for some $\bar{x} \in \bar{C}$) satisfies $\Omega_{\bar{U}}(\bar{v}, \bar{w}) + \eta_{\bar{U}}(\bar{v})\eta_{\bar{U}}(\bar{w}) = 0$ for all $\bar{w} \in \bar{U}_{\bar{x}}$, then $\bar{v} = 0$. Together with (9) this implies that for each $\bar{x} \in \bar{C}$, the triple $(\bar{U}_{\bar{x}}, \Omega_{\bar{U}}(\bar{x}), \eta_{\bar{U}}(\bar{x}))$ is a cosymplectic vector space, with Reeb vector $\bar{X}(\bar{x})$. In particular, in this case $\bar{X}$ is uniquely determined by (9).

5. A classification of constrained systems with symmetry

We again consider a constrained system (6) with symmetry, verifying the assumptions of the previous section. For each infinitesimal generator ξ_M of the given cosymplectic action on M, its restriction to C is precisely the infinitesimal generator ξ_C of the induced action on C. If ξ_C is a section of $\mathcal{V} \cap F$, we will call it a *horizontal symmetry* of the given constrained system [2, 3, 6]. The following classification of constrained systems with symmetry, which is similar to the one introduced by Bloch *et al* [3] for autonomous nonholonomic systems with linear or affine nonholonomic constraints, is based on the relative positioning of the subspaces $\mathcal{V}_x$ and F_x.

1. *The purely kinematic (or 'principal') case*: $\mathcal{V}_x \cap F_x = \{0\}$ and $T_xC = \mathcal{V}_x + (F_x \cap T_xC)$, for all $x \in C$.
2. *The case of horizontal symmetries*: $\mathcal{V}_x \cap F_x = \mathcal{V}_x$, for all $x \in C$ (which is equivalent to $\mathcal{V}_x \subseteq F_x$, for all $x \in C$).
3. *The general case*: $\{0\} \subset \mathcal{V}_x \cap F_x \subset \mathcal{V}_x$ (with strict inclusions) for all $x \in M$.

5.1. The purely kinematic case

The assumptions of the purely kinematic case imply that $T_xC = \mathcal{V}_x \oplus (F_x \cap T_xC)$. In other words, observing that in this case $U = F \cap TC$, we have $TC = \mathcal{V}_{|C} \oplus U$. Since the bundle U is G-invariant, this decomposition defines a principal connection Υ on the principal G-bundle $\rho_{|C} : C \to \bar{C}$, with horizontal subspace U_x at $x \in C$. In what follows we let X again denote a fixed G-invariant solution of (6) which, moreover, belongs to F. This still means that X is horizontal, i.e. $X \in U$.

Denote by $\mathbf{h} : TC \to U$ and $\mathbf{v} : TC \to \mathcal{V}_{|C}$ the horizontal and vertical projectors associated with the decomposition $TC = \mathcal{V}_{|C} \oplus U$. The curvature of Υ is the type-(1,2) tensor field on C given by

$$\Lambda = \frac{1}{2}[\mathbf{h}, \mathbf{h}]\ ,$$

where [,] denotes the Nijenhuis bracket of type (1,1) tensor fields. Taking into account that in the present case $\bar{U} = T\bar{C}$, and applying the method developed in Section 4., we obtain that the forms $\Omega_{\bar{U}}$ and $\eta_{\bar{U}}$ now become genuine differential forms on $\bar{C}$, denoted by $\bar{\Omega}$ and $\bar{\eta}$, respectively, and such that the projection $\bar{X}$ of X verifies

$$\left\{ \begin{array}{l} i_{\bar{X}}\bar{\Omega} = 0\ . \\ i_{\bar{X}}\bar{\eta} = 1\ . \end{array} \right. \tag{10}$$

It should be emphasised that the reduced forms $\bar{\Omega}$ and $\bar{\eta}$ in general need not be closed, as demonstrated by the following lemma.

LEMMA 1. *For arbitrary vector fields Y_1, Y_2, Y_3 on $\bar{C}$, we have*

1. $d\bar{\Omega}(Y_1,Y_2,Y_3) = \Omega(Y_1^h,\Lambda(Y_2^h,Y_3^h)) + \Omega(Y_2^h,\Lambda(Y_3^h,Y_1^h)) + \Omega(Y_3^h,\Lambda(Y_1^h,Y_2^h))$,
2. $d\bar{\eta}(Y_1,Y_2) = -\eta(\Lambda(Y_1^h,Y_2^h))$,

where Y_i^h denotes the horizontal lift of Y_i with respect to the connection Υ.

(The proof of the lemma follows by straightforward computation, using the properties of a principal connection.)

Assume for the remainder of this section that $\dim F^o = \dim TC^o$ holds. Then, it follows from Remark 3 that the pair $(\bar{\Omega},\bar{\eta})$ defines what we may call an 'almost cosymplectic' structure on $\bar{C}$. Under the stronger assumption for the given G-action that $\eta_{|\mathcal{V}} = 0$, and assuming the 2-form Ω on M is exact, one can construct another form of the reduced equations, equivalent to (10), but now in terms of a pair of closed forms defining a genuine cosymplectic structure on $\bar{C}$. Again, this is similar to the situation encountered in the autonomous (symplectic) case (see e.g. [6, 9]).

Assume $\Omega = d\theta$ for some 1-form θ on M. Denote by θ' the 1-form on C defined by $\theta' = j_C^*\theta$, where $j_C : C \hookrightarrow M$ is the canonical inclusion. By means of the given solution X of (6) we can construct a 1-form α_X on C as follows:

$$\alpha_X = i_X(d\mathbf{h}^*\theta' - \mathbf{h}^*d\theta')\ . \tag{11}$$

We then have the following interesting result.

THEOREM 4. *If the given action Φ leaves the 1-form θ invariant, then the 1-forms $\mathbf{h}^*\theta'$ and α_X are projectable, with projections denoted by $\bar{\theta}'_h$ and $\overline{\alpha_X}$, respectively. Moreover, the reduced 1-form $\bar{\eta}$ is closed and the projection $\bar{X}$ of X satisfies the system*

$$\left\{\begin{array}{l} i_{\bar{X}}d\bar{\theta}'_h = \overline{\alpha_X}\ , \\ i_{\bar{X}}\bar{\eta} = 1\ . \end{array}\right. \tag{12}$$

Proof: The projectability of the 1-forms $\mathbf{h}^*\theta'$ and α_X is easily established by showing that each infinitesimal generator ξ_C is a characteristic vector field of both forms. In view of the assumption, $\eta_{|\mathcal{V}} = 0$, Lemma 1 immediately implies that $\bar{\eta}$ is closed. The derivation of the reduced equations of motion (12) proceeds as follows.

Recall that X satisfies an equation of the form $i_X d\theta = \beta$, for some $\beta \in F^o$. Putting $\beta' = j_C^*\beta$ and taking into account that X is tangent to C, we can take the pull-back of that equation to C, i.e. $i_X d\theta' = \beta'$. Since X is horizontal, i.e. $\mathbf{h}X = X$, it follows that $\mathbf{h}^*(i_X d\theta') = i_X\mathbf{h}^*d\theta'$. Furthermore, it is readily seen that $\mathbf{h}^*\beta' = 0$. The horizontal projection of the equation of motion on C therefore becomes $i_X\mathbf{h}^*d\theta' = 0$. In view of the definition of the 1-form α_X, we then obtain $i_X d\mathbf{h}^*\theta' = \alpha_X$. All objects in this equation are projectable onto $\bar{C}$, and since we already have that $i_{\bar{X}}\bar{\eta} = 1$, the reduced dynamics indeed satisfies (12). $\boxed{QED}$

In terms of mechanics, Theorem 4 describes a situation where a time-dependent nonholonomic system with symmetry admits a reduction to an unconstrained time-dependent system (12), but with an additional "nonconservative force" given by $\overline{\alpha_X}$.

5.2. The case of horizontal symmetries

The symmetry assumption now is that $\mathcal{V}_x \cap F_x = \mathcal{V}_x$, for all $x \in C$ or, equivalently, $\mathcal{V}_{|C} \subset F$. In particular, every infinitesimal generator of the given group action is a horizontal symmetry, in the sense defined at the beginning of this section.

For the further analysis of this case we assume that the given cosymplectic action Φ on M admits an Ad^*-equivariant momentum map $J : M \to \mathfrak{g}^*$. Let $\mu \in \mathfrak{g}^*$ be a (weakly) regular value of J, such that we can apply Theorem 1, i.e. $(M_\mu := J^{-1}(\mu)/G_\mu, \Omega_\mu, \eta_\mu)$ is a cosymplectic manifold, with Ω_μ and η_μ uniquely determined by $\pi_\mu^*\Omega_\mu = j_\mu^*\Omega$, $\pi_\mu^*\eta_\mu = j_\mu^*\eta$.

It follows from the definition of the momentum map that $\xi_M = \chi_{\Omega,\eta}^{-1}(dJ_\xi)$. Taking into account that in the case under consideration $\mathcal{V}_{|C} \subset F$, we find that, along the constraint submanifold C, the solution X of (6), satisfies

$$X(J_\xi) = 0 \ ,$$

i.e. the components of the momentum mapping are conserved quantities for the constrained dynamics. This is a version of Noether's theorem for time-dependent constrained systems. (For the analogous result in the case of time-independent nonholonomic mechanics, see e.g. [3, 6, 21]).

Imposing a condition of clean intersection of C and $J^{-1}(\mu)$, we have that $C' = C \cap J^{-1}(\mu)$ is a submanifold of $J^{-1}(\mu)$ which is G_μ-invariant. Passing to the quotient we then obtain a submanifold $C_\mu = C'/G_\mu$ of M_μ. Next, we can define a distribution F' on M along C' by putting

$$F'_{x'} = T_{x'}(J^{-1}(\mu)) \cap F_{x'} \ , \ \forall x' \in C' \ .$$

We now make the further simplifying assumption that F' has constant rank. It is then obvious that F' is a G_μ-invariant subbundle of $TP_{|C'}$ and, hence, it projects onto a subbundle F_μ of TM_μ along C_μ.

THEOREM 5. *Suppose that X is a G-invariant solution of (6). Then, X induces a section X_μ of $(TM_\mu)_{|C_\mu}$, such that*

$$\left\{ \begin{array}{l} (i_{X_\mu}\Omega_\mu)_{|C_\mu} \in F_\mu^o \ , \\ (i_{X_\mu}\eta_\mu)_{|C_\mu} = 1 \ , \\ X_\mu \in TC_\mu \ . \end{array} \right. \qquad (13)$$

Proof: First of all, notice that $X' = X_{|C'}$ is everywhere tangent to C', since both $J^{-1}(\mu)$ and C are invariant submanifolds of X. Pulling back (6) to $J^{-1}(\mu)$, we find that X' satisfies

$$\left\{ \begin{array}{l} (i_{X'}j_\mu^*\Omega)_{|C'} = \beta \ , \\ (i_{X'}j_\mu^*\eta)_{|C'} = 1 \ , \end{array} \right.$$

for some section β of F'^o. Since X is G-invariant, and taking into account the other symmetry assumptions, it follows that both X' and β are G_μ-equivariant sections of

TC' and F'^o, respectively. Moreover, from the fact that we are dealing with horizontal symmetries we may deduce that for all $\xi \in \mathfrak{g}_\mu$ (= the Lie algebra of G_μ), $(\xi_M)_{|C'}$ is a section of F'. Therefore, β projects onto a section of F^o_μ. Using a standard argument, it now readily follows that X' projects onto a vector field on C_μ for which (13) holds.

$\boxed{QED}$

In the case of horizontal symmetries we have thus proved that, under the appropriate assumptions, the given constrained problem defined by (M, Ω, η, C, F) reduces to a constrained problem of the same type, defined by $(M_\mu, \Omega_\mu, \eta_\mu, C_\mu, F_\mu)$, i.e. the reduction process preserves the category of constrained cosymplectic systems.

5.3. The general case

We now consider the case where, along C, $\{0\} \neq \mathcal{V}_x \cap F_x \neq \mathcal{V}_x$. To study this case, we assume again that the 2-form Ω is exact: $\Omega = d\theta$. It is known that the given cosymplectic action then admits a Ad^*-equivariant momentum map $J : M \to \mathfrak{g}^*$, defined by $< J(x), \xi >= -\theta_x(\xi_M(x))$. It is no longer true, however, that J is a conserved quantity for the constrained dynamics. In fact, we will see that, as in the time-independent case [3, 6], only the symmetries 'compatible' with the constraints provide relevant information about the dynamics.

For each $x \in C$, consider the vector subspace of $\mathfrak{g}$,

$$\mathfrak{g}_x = \{\xi \in \mathfrak{g} \mid \xi_C(x) \in F_x\}\,.$$

The disjoint union $\mathfrak{g}_C = \bigsqcup_{x\in C} \mathfrak{g}_x$ of all these subspaces determines a generalised vector bundle over C. Henceforth, we assume that $\mathfrak{g}_C$ is of constant rank, i.e. it determines a genuine vector bundle over C.

We now consider only the momentum components provided by the symmetry directions along the vector bundle F. That is, we define

$$\begin{array}{rccccc} J^c: & C & \longrightarrow & \mathfrak{g}^*_C & & \\ & x & \longmapsto & J^c(x): & \mathfrak{g}_x \;\to & \mathbb{R} \\ & & & & \xi \;\mapsto & < J(x), \xi > \,. \end{array}$$

If $\bar\xi : C \to \mathfrak{g}_C$ is a global section of the bundle $\mathfrak{g}_C \to C$, we can define a function $J^c_{\bar\xi} : C \to \mathbb{R}$ by $J^c_{\bar\xi} =< J^c, \bar\xi >$. Moreover, associated to $\bar\xi$, there is a 'fundamental vector field' which has a double dependence on the base point, namely: $\Xi(x) = (\bar\xi(x))_C(x) \in F_x$, $x \in C$. With all these ingredients we are now able to derive the following nonautonomous version of the so-called momentum equation (cf. [3, 6]).

PROPOSITION 2. *If $\bar\xi : C \longrightarrow \mathfrak{g}_C$ is a global section of $\mathfrak{g}_C \longrightarrow C$, then*

$$X(J^c_{\bar\xi}) = -(\mathcal{L}_\Xi\theta)(X)\,. \tag{14}$$

Proof: We have that

$$X(J^c_{\bar\xi}) = dJ^c_{\bar\xi}(X) = -i_X d i_\Xi j^*\theta = i_X i_\Xi j^*\Omega - i_X \mathcal{L}_\Xi j^*\theta\,.$$

Since $j^*\Omega(\Xi, X) = -(i_X\Omega)(\Xi)$ and $\Xi \in F$ this further becomes

$$X(J^c_\xi) = -i_X\mathcal{L}_\Xi j^*\theta = i_{[\Xi,X]}j^*\theta - \mathcal{L}_\Xi i_X j^*\theta = -(\mathcal{L}_\Xi\theta)(X)\,.$$

$\boxed{QED}$

For nonholonomic Lagrangian systems, we have that $\theta = Ldt + S^*(dL)$ and X satisfies the second-order equation condition. As a consequence, $(\mathcal{L}_\Xi\theta)(X)$ can be obtained without having to compute explicitly the dynamics X.

In the last section, we will briefly consider the possibility of reducing a cosymplectic constrained system with a 1-parameter symmetry group, to a constrained Hamiltonian system on a symplectic manifold.

6. Symplectic reduction

Starting again, as before, with a constrained system (M, Ω, η, C, F) for which (6) admits a unique solution X belonging to F, we now assume that there exists a free and proper cosymplectic action Φ of $\mathbb{R}$ on M such that for all $t \in \mathbb{R}$

$$\Phi_t(C) \subset C\,, \quad T\Phi_t(F_x) = F_{\Phi_t(x)}\,, \quad \forall x \in C\,.$$

We then know that Φ defines a symmetry of X. In addition, we assume that the infinitesimal generator T of Φ satisfies

$$i_T\eta = 1\,, \quad i_T\Omega = dH\,,$$

for some $H \in C^\infty(M)$. We can now distinguish two cases, depending on whether or not $T_{|C}$ is a horizontal symmetry.

(i) If $T_{|C} \in F$, we can invoke Theorem 2. In particular, recall that the cosymplectic manifold (M, Ω, η) reduces to a symplectic manifold $(\widetilde{M} := M/\mathbb{R}, \widetilde{\Omega_0})$, where $\widetilde{\Omega_0}$ is the projection of the closed 2-form $\Omega_0 = \Omega - dH \wedge dt$. It is now rather straightforward to check that, in view of the given symmetry assumptions, the constraint manifold C projects onto a submanifold $\widetilde{C}$ of $\widetilde{M}$. Moreover, due to the fact that T is a horizontal symmetry, it easily follows that the subbundle F of $TM_{|C}$ projects onto a subbundle $\widetilde{F}$ of $T\widetilde{M}_{|\widetilde{C}}$. Finally, it is routine to verify that the projection $\widetilde{X}$ of X verifies

$$\left\{\begin{array}{rcl} (i_{\widetilde{X}}\widetilde{\Omega_0} - d\widetilde{H})_{|\widetilde{C}} & \in & \widetilde{F}^o \\ \widetilde{X}_{\widetilde{C}} & \in & T\widetilde{C}\,, \end{array}\right.$$

where $\widetilde{H}$ denotes the projection of H on $\widetilde{M}$. The reduced system therefore takes the form of a constrained Hamiltonian system on a symplectic manifold, as described in [6].

(ii) If $T_{|C}$ is not a section of F, we have to apply the general reduction procedure for the purely kinematic case, as developed in Subsection 5.1.

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