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GENERALISED CONNECTIONS ON AFFINE LIE ALGEBROIDS

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We investigate a geometric model for a certain class of first-order differential equations on an affine bundle, called pseudo-SODEs. We mention a generalised version of the concept of connection. Further, if the affine bundle is related to a Lie algebroid, we give a definition for torsion and curvature for such a generalised connection. Next, we show how a pseudo-SODE generates a generalised connection and we characterise this construction by means of the vanishing of torsion.

Keywords: affine bundle, Lie algebroid, pseudo-SODE, generalised connection.

1. Introduction

Consider an affine bundle $\pi : E \to M$ (modeled on a vector bundle $\overline{\pi} : \overline{E} \to M$) with a given affine bundle map $\rho : E \to TM$. In this context we will refer to ρ as *anchor map* and to (π, ρ) as an *anchored affine bundle*. Let x^a denote local coordinates for $m \in M$. In order to coordinatise E, we choose a zero section e_0 in $Sec(\pi)$ and a local basis $\{\boldsymbol{e}_\alpha\}$ for $Sec(\overline{\pi})$. Then, every point $e \in E_m$ can be rewritten as $e_0(m) + y^\alpha \boldsymbol{e}_\alpha(m)$, for some (affine) fibre coordinates y^α. Locally, $\rho : (x, y) \mapsto (x, \rho^a_\alpha(x) y^\alpha + \rho^a_0(x))$ for functions ρ^a_α and ρ^a_0 on M. Consider a curve ψ in E, locally given by $t \mapsto (x(t), y(t))$. This paper concerns the geometrical modeling of dynamical systems of the form

$$\left\{ \begin{array}{rcl} \dot{x}^a &=& \rho^a_\alpha(x) y^\alpha + \rho^a_0(x), \\ \dot{y}^\alpha &=& f^\alpha(x, y), \end{array} \right. \qquad (1)$$

for a certain set of functions $f^\alpha \in C^\infty(E)$.

First, we will say a few words to explain our interest in this kind of differential equations. In [16], Weinstein was the first to study a certain generalisation of the class of autonomous Lagrangian systems. The setup for this extension is an arbitrary Lie algebroid whose vector bundle replaces the tangent bundle $TM \to M$ as carrying space, while its anchor map serves as a bridge between these two bundles. In addition, an extension of the Lie bracket of the algebroid takes the role of the canonical Lie bracket structure on $TTM \to TM$. Weinstein's ideas were further explored in e.g. [2, 3, 8, 10].

In previous papers [14, 11] we have modified this setup in such a way that also time dependent Lagrangians fit in. This involved the use of an affine bundle $E \to M$ rather than a vector bundle. As a consequence, we had to investigate how one can introduce a structure on E that is similar to a Lie algebroid. We will come back to this *affine Lie algebroid* in section 3.2. However, the 'Lagrangian equations' that we found for a Lagrangian $L(x, y)$ take the form

$$\left\{ \begin{array}{rcl} \dot{x}^a &=& \rho^a_\alpha y^\alpha + \rho^a_0, \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial y^\alpha}\right) &=& \rho^a_\alpha \dfrac{\partial L}{\partial x^a} - (C^\gamma_{\alpha\beta} y^\beta - C^\gamma_{0\alpha}) \dfrac{\partial L}{\partial y^\gamma} \end{array} \right. \tag{2}$$

where the structure functions ρ^a_α, ρ^a_0, $C^\gamma_{\alpha\beta}$, $C^\gamma_{0\alpha} \in C^\infty(M)$ further satisfy suitable compatibility conditions. Obviously, for a regular Lagrangian (meaning that the rank of $\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}$ equals the fibre dimension of π) the system (2) can be transformed to (a special case of) the equations (1).

In the special case that the affine bundle under investigation is the first jet bundle $J^1M \to M$ of a bundle $M \to \mathbb{R}$, ρ is the natural injection $J^1M \to TM$ (i.e. $\rho^a_\alpha = 1$, $\rho^a_0 = 0$) and the bracket structure is the canonical bracket one on $TJ^1M \to J^1M$ (i.e. $C^\gamma_{0\alpha} = 0$ and $C^\gamma_{\alpha\beta} = 0$) we recover in (2) the Lagrangian equations for a time-dependent Lagrangian. In this special situation, one uses adapted coordinates $(t, x^i, \dot{x}^i)$ for (x^a, y^α). With these assumptions on the anchor map, the system (1) takes the form of a set of time-dependent second-order ordinary differential equations (in the following called a SODE in short)

$$\ddot{x}^i = f^i(t, x, \dot{x}). \tag{3}$$

The class of dynamical systems of the form (1) can thus be seen as an extension of the class of SODEs. In the case that the affine bundle under consideration is in fact a vector bundle (with a Lie algebroid structure on it), Weinstein has called the equations (1) "second-order equations on a Lie algebroid". It is clear that these equations are truly second-order differential equations only when the base manifold and the fibres have the same dimension and ρ is injective. Therefore we will adopt the name *'pseudo'*-SODE for a system of the form (1).

In the next section we recall a geometric model for a pseudo-SODE and in the first part of the third section we investigate (generalised) connections on anchored affine bundles and the notion of an affine Lie algebroid. Such concepts were fully explored in [14, 11, 12]. We limit ourselves here to the main features which are needed to set the stage for some new elements which will be developed in the present contribution. First, for the kind of generalised connections under consideration, we look at the way torsion and curvature can be introduced. More specifically, we show that such concepts become apparent only within the new formalism introduced in [12], that of regarding the connection as a splitting of a short exact sequence related to a prolonged bundle. Next, we show that any given pseudo-SODE on an affine Lie algebroid gives rise in a natural way to an associated generalised connection and prove that it can be characterised by a zero torsion property.

2. Pseudo-Sodes

A curve ψ in E which satisfies the condition $\dot{x}^a = \rho^a_\alpha y^\alpha + \rho^a_0$ can be characterised by the condition $T(\pi \circ \psi) = \rho \circ \psi$. Such a curve is called *ρ-admissible.* Obviously, a pseudo-Sode is a vector field Γ on E whose integral curves are all admissible: $\Gamma = \big(\rho^a_\alpha(x)y^\alpha + \rho^a_0(x)\big)\frac{\partial}{\partial x^a} + f^\alpha(x,y)\frac{\partial}{\partial y^\alpha}$ for some functions f^α. There exists a direct formulation of this property:

Definition 1. $\Gamma \in \mathcal{X}(E)$ is a pseudo-Sode if $T\pi \circ \Gamma = \rho$.

It is possible to develop a second characterisation. The property that for every $e \in E$, $T\pi(\Gamma(e)) = \rho(e)$, indicates that the couple $(e, \Gamma(e))$ is actually a point of the manifold

$$T^\rho E = \{(a, X_e) \in E \times TE \mid \rho(a) = T\pi(X_e)\}.$$

This manifold is the total manifold of the pullback bundle $\pi^2 : \rho^* TE \to E : (a, X_e) \mapsto a$. We will use the notation ρ^1 for the projection on the second argument of a couple, i.e. $\rho^1(a, X_e) = X_e$. We prefer to use the notation $T^\rho E$ for $\rho^* TE$ because, as in [7], we would like to look at it as being fibred over E via the projection $\pi^1 = \tau_E \circ \rho^1$ (τ_E is the tangent projection and thus $\pi^1(a, X_e) = e$). We can then consider a pseudo-Sode as a section $e \mapsto (e, \Gamma(e))$ of π^1, i.e. a section for which the projection π^2 always coincides with the bundle projection π^1.

The bundle $\pi^1 : T^\rho E \to E$ is affine again. Now, in order to avoid the technical difficulties related for example to working with (vector valued) forms on affine bundles, it is convenient to regard π^1 as imbedded in a larger vector bundle which is constructed in the following way. Let us have a look first at the affine bundle π again. Consider the set $E^\dagger_m := \mathrm{Aff}(E_m, \mathbb{R})$ of all affine functions from E_m to $\mathbb{R}$. The union of all such spaces forms a manifold $E^\dagger = \bigcup_{m \in M} E^\dagger_m$ and leads to the vector bundle $\pi^\dagger : E^\dagger \to M$, called the extended dual. The dual of $\pi^\dagger$ is again a vector bundle, denoted by $\tilde{\pi} : \tilde{E} := (E^\dagger)^* \to M$. This new bundle contains both E and $\overline{E}$ since there exist canonical injections $\iota : E \to \tilde{E}$ and $\boldsymbol{\iota} : \overline{E} \to \tilde{E}$. The map ι is defined by $\iota(e)(\phi) = \phi(e)$ for $e \in E_m$ and $\phi \in \mathrm{Aff}(E_m, \mathbb{R})$. It is an affine map and $\boldsymbol{\iota}$ is its associated linear map. It can be shown that every element of $\tilde{E}_m$ is either of the form $\lambda\iota(e)$ (for a unique $\lambda \in \mathbb{R}$ and $e \in E_m$) or $\boldsymbol{\iota}(\boldsymbol{e})$ (for a unique $\boldsymbol{e} \in \overline{E}_m$). As a consequence it follows that the set $\{e_0 = \iota(e_0), e_\alpha = \boldsymbol{\iota}(\boldsymbol{e}_\alpha)\}$ forms a local basis for $Sec(\tilde{\pi})$. Let us use $\boldsymbol{\rho} : \overline{E} \to TM$ for the linear map that is associated to ρ. It is convenient to extend ρ to a map $\tilde{\rho} : \tilde{E} \to TM$, defined by means of $\tilde{\rho}(\lambda\iota(e)) := \lambda\rho(e)$ and $\tilde{\rho}(\boldsymbol{\iota}(\boldsymbol{e})) := \boldsymbol{\rho}(\boldsymbol{e})$. In coordinates, $\tilde{\rho} : (x; y^0, y^\alpha) \mapsto (x; \rho^a_\alpha y^\alpha + \rho^a_0 y^0)$.

We can now come to the construction of the required vector bundle. Consider now the following (vector) bundle over E with total space

$$T^{\tilde{\rho}} E := \tilde{\rho}^* TE = \{(\tilde{e}, X_e) \in \tilde{E} \times TE \mid \tilde{\rho}(\tilde{e}) = T\pi(X_e)\}$$

and projection $\tilde{\pi}^1 := \tau_E \circ \tilde{\rho}^1 : T^{\tilde{\rho}} E \to E : (\tilde{e}, X_e) \mapsto e$ (again, we use the notations $\tilde{\pi}^2(\tilde{e}, X_e) = \tilde{e}$ and $\tilde{\rho}^1(\tilde{e}, X_e) = X_e$). In our special example, $E = J^1M$, $\tilde{E} = TM$, $\tilde{\rho} = id$ and $T^{id} J^1 M \cong TJ^1M$. In general, π^1 is an affine subbundle of $\tilde{\pi}^1$ (see [11]), but it contains also a canonical vector subbundle. An element of $T^{\tilde{\rho}} E$ is called *vertical* if it is in the kernel of the projection $\tilde{\pi}^2$ and we will denote the whole set with $\mathcal{V}^{\tilde{\rho}}E$. Remark that for any $(0, X_e) \in \mathcal{V}^{\tilde{\rho}}E$, X_e is also vertical as a tangent vector to T_eE, since $T\pi(X_e) = \tilde{\rho}(0) = 0$.

A lot of properties become more visible, once we have selected an appropriate basis for the sections of $T^{\tilde{\rho}}E$. Evidently, the local sections $\mathcal{V}_\alpha$, given by $\mathcal{V}_\alpha(e) = \Big(0, \frac{\partial}{\partial y^\alpha}\Big|_e\Big)$, span the vertical sections. We can extend this set to a full basis for $Sec(\tilde{\pi}^1)$ by making use of the sections $\mathcal{X}_A$, given by $\mathcal{X}_A(e) = \big(e_A(x), \rho^a_A(x)\frac{\partial}{\partial x^a}\big|_e\big)$, which project to the original basis $\{e_A\}$ of $Sec(\tilde{\pi})$ (Here, and in what follows x denote coordinates for $\pi(e) \in M$ and indices such as A stand for either 0 or α). An arbitrary section of $\tilde{\pi}^1$ can thus be represented by

$$\mathcal{Z} = \zeta^A \mathcal{X}_A + Z^\alpha \mathcal{V}_\alpha$$

(with $\zeta^A, Z^\alpha \in C^\infty(E)$). The dual basis in $Sec(\tilde{\pi}^{1*})$ is denoted by $\{\mathcal{X}^A, \mathcal{V}^\alpha\}$. It can be shown that $\mathcal{X}^0$ is a globally defined 1-form. We have shown in [11] that it is possible to define a (vertical) lift procedure, mapping sections of $\pi^*\tilde{E}$ onto vertical ones. As usual, this results in the definition of the vertical endomorphism S, which is a (1,1)-tensor field on $T^{\tilde{\rho}}E$ whose coordinates expression reads

$$S = (\mathcal{X}^\alpha - y^\alpha \mathcal{X}^0) \otimes \mathcal{V}_\alpha. \tag{4}$$

After this brief introduction of the essential machinery on $T^{\tilde{\rho}}E$, we can give two equivalent formulations for a pseudo-Sode as a section of $\tilde{\pi}^1$. Hereto, we will use the same symbol Γ as its image under $\tilde{\rho}^1$ in Definition 1.

Definition 2. $\Gamma \in Sec(\tilde{\pi}^1)$ is a pseudo-Sode if one of the following equivalent conditions is satisfied: (i) $\tilde{\pi}^2 \circ \Gamma = \iota \circ \tilde{\pi}^1 \circ \Gamma$, (ii) $S(\Gamma) = 0$ and $\mathcal{X}^0(\Gamma) = 1$.

Of course, the first is a direct consequence of what we first found for π^1, while the second is the direct analogue of the way one usually singles out the Sodes in $\mathcal{X}(J^1M)$. In coordinates, a pseudo-Sode can be represented by

$$\Gamma = \mathcal{X}_0 + y^\alpha \mathcal{X}_\alpha + f^\alpha \mathcal{V}_\alpha. \tag{5}$$

3. Generalised connections on π associated with a pseudo-Sode

It is well known (see e.g. [4]) that, given a Sode Γ (3), we can construct a nonlinear connection on $J^1M \to M$. A nonlinear connection is usually characterised as a (right) splitting of the short exact sequence

$$0 \to VJ^1M \to TJ^1M \to \pi^*TM \to 0$$

(VJ^1M being the vertical subbundle in TJ^1M). For the nonlinear connection that is generated by Γ, it is most convenient to present an explicit formula for its associated horizontal projector:

$$P_H = \frac{1}{2}\Big(I - \mathcal{L}_\Gamma S + dt \otimes \Gamma\Big). \tag{6}$$

Remark that in expression (6) some of the most important canonical structures on $TJ^1M \to J^1M$, such as the vertical endomorphism $S = (dx^i - \dot{x}^i dt) \otimes \frac{\partial}{\partial \dot{x}^i}$ and the global 1-form dt, are involved. In addition, the presence of the term $\mathcal{L}_\Gamma S$ demonstrates that the above construction highly relies on the canonical Lie algebroid structure on $TJ^1M \to J^1M$. We will show next how the above construction easily carries over to the more general case of a pseudo-SODE.

3.1. Generalised connections on π

With the vector bundle $\tilde{\pi}^1 : T^{\tilde{\rho}}E \to E$, we have provided an appropriate analogue for the bundle $TJ^1M \to J^1M$. Let's have a look now if we can define a suitable generalisation of a nonlinear connection on it. The vertical subbundle in $T^{\tilde{\rho}}E$ indeed leads to a short exact sequence

$$0 \to \mathcal{V}^{\tilde{\rho}}E \to T^{\tilde{\rho}}E \xrightarrow{j} \pi^*\tilde{E} \to 0. \tag{7}$$

Here, the second arrow denotes the injection of $\mathcal{V}^{\tilde{\rho}}E$ as a subbundle in $T^{\tilde{\rho}}E$. The fibre linear map j is given by $j(\tilde{e}, X_e) = (e, \tilde{e})$. It is surjective and its kernel is $\mathcal{V}^{\tilde{\rho}}E$. A splitting of this sequence is a map ${}^H : \pi^*\tilde{E} \to T^{\tilde{\rho}}E$ such that $j \circ {}^H = id_{\pi^*\tilde{E}}$.

Let $p_{\tilde{E}}$ denote the projection $\pi^*\tilde{E} \to \tilde{E}$. In [12], we have shown that

PROPOSITION 1. *The existence of a linear bundle map* $h : \pi^*\tilde{E} \to TE$ *such that* $T\pi \circ h = \tilde{\rho} \circ p_{\tilde{E}}$ *is equivalent to the existence of a right splitting* ^{H} *of the short exact sequence (7).*

The map h is called a $\tilde{\rho}$-connection on π and is, in this most general form, introduced in [1], where the authors were inspired by earlier work in e.g. [6, 5, 15]. The characterisation of a $\tilde{\rho}$-connection in terms of a horizontal lift ^{H} can be represented locally by

$$(x^a, y^\alpha, u^A)^H = \left((x^a, u^A), u^A\left(\rho^a_A \frac{\partial}{\partial x^a} - \Gamma^\alpha_A \frac{\partial}{\partial y^\alpha}\right)\right)$$

for some connection coefficients $\Gamma^\alpha_A \in C^\infty(E)$. It is convenient to introduce a local basis for the horizontal sections of $\tilde{\pi}^1$ (i.e. those whose image belong to the direct complement of $\mathcal{V}^{\tilde{\rho}}E$), which is given by $\mathcal{H}_A = \mathcal{X}_A - \Gamma^\alpha_A(x,y)\mathcal{V}_\alpha$. A better representation of a section $\mathcal{Z}$, adapted to the given connection, then becomes:

$$\mathcal{Z} = \zeta^A \mathcal{H}_A + (Z^\alpha + \zeta^A \Gamma^\alpha_A)\mathcal{V}_\alpha.$$

The horizontal projector $P_H : Sec(\tilde{\pi}^1) \to Sec(\tilde{\pi}^1)$ is then given by

$$P_H(\mathcal{Z}) = \zeta^A \mathcal{H}_A.$$

3.2. Affine Lie algebroids

So far, we have only talked about anchored affine bundles. In order to introduce torsion and curvature for generalised connections, we will require, next to the anchor map, also the presence of a Lie bracket structure. We first recall the definition of an *affine Lie algebroid* from [14] and [11].

Definition 3. A Lie algebroid structure on the affine bundle π consists of:

1. a real Lie algebra structure on $Sec(\overline{\pi})$;
2. an action by derivations of the affine bundle $Sec(\pi)$ on the real Lie algebra $Sec(\overline{\pi})$ which is compatible with the bracket on $Sec(\overline{\pi})$;
3. an affine anchor map $\rho : E \to TM$ which is such that $Sec(\pi)$ also acts by derivations on $Sec(\overline{\pi})$, regarded as $C^\infty(M)$-module.

Of course, it is necessary to be more precise about the correct interpretation of the terminology used in this compact formulation. The first item is clear and we use the standard bracket notation $[\,,\,]$ for the (real) Lie algebra on $Sec(\overline{\pi})$. If D_ζ is used for the action of $\zeta \in Sec(\pi)$ on $Sec(\overline{\pi})$, then the derivation property referred to in the second item means that we have the properties: $D_\zeta(\lambda\boldsymbol{\sigma}) = \lambda D_\zeta\boldsymbol{\sigma}$ (for all $\boldsymbol{\sigma} \in Sec(\overline{\pi})$ and $\lambda \in \mathbb{R}$); $D_\zeta(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) = D_\zeta\boldsymbol{\sigma}_1 + D_\zeta\boldsymbol{\sigma}_2$ and $D_\zeta[\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2] = [D_\zeta\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2] + [\boldsymbol{\sigma}_1, D_\zeta\boldsymbol{\sigma}_2]$. The compatibility requirement mentioned in the same item further means that $D_{\zeta+\boldsymbol{\sigma}}\boldsymbol{\eta} = D_\zeta\boldsymbol{\eta} + [\boldsymbol{\sigma}, \boldsymbol{\eta}]$. Finally, the compatibility requirement with respect to the anchor map, referred to in the third item, means that for $f \in C^\infty(M)$, $D_\zeta(f\boldsymbol{\sigma}) = \rho(\zeta)(f)\boldsymbol{\sigma} + f\, D_\zeta\boldsymbol{\sigma}$.

As shown in [11], having a Lie algebroid structure on the affine bundle π is equivalent to having a Lie algebroid on the vector bundle $\tilde{\pi} : \tilde{E} \to M$ (with respect to the extended anchor map $\tilde{\rho}$), which is such that the bracket of sections of $\tilde{\pi}$ belonging to the image of ι, belongs to the image of $\boldsymbol{\iota}$. It is then appropriate to use bracket notation also for the action of D_ζ, i.e. to put $[\zeta, \boldsymbol{\sigma}] = D_\zeta\boldsymbol{\sigma}$.

Having chosen a local frame $(e_0; \{\boldsymbol{e}_\alpha\})$ for $Sec(\pi)$, with corresponding local basis $\{e_A\} = \{e_0, e_\alpha\}$ for $Sec(\tilde{\pi})$, there exist structure functions on M, determined by

$$[e_0, e_\alpha] = C^\beta_{0\alpha}(x)e_\beta \qquad \text{and} \qquad [e_\alpha, e_\beta] = C^\gamma_{\alpha\beta}(x)e_\gamma.$$

An important further property, proved in [11], is the following.

Proposition 2. *An affine Lie algebroid structure on π prolongs to a vector Lie algebroid structure on the bundle $\tilde{\pi}^1 : T^{\tilde{\rho}}E \to E$, with respect to the anchor map $\tilde{\rho}^1$.*

Locally, the Lie algebroid structure on the prolonged bundle is determined by the bracket relations:

$$[\mathcal{X}_A, \mathcal{X}_B] = C^\gamma_{AB}\mathcal{X}_\gamma, \quad [\mathcal{X}_A, \mathcal{V}_\alpha] = 0, \quad \text{and} \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0.$$

Suppose now again that we have a $\tilde{\rho}$-connection on π. Then, it will often be more suitable to use as local basis for $Sec(\tilde{\pi}^1)$ the set $\{\mathcal{H}_A, \mathcal{V}_\alpha\}$, as discussed in the previous section. For later use, therefore, it is of interest to list the following brackets of horizontal and vertical sections:

$$[\mathcal{H}_A, \mathcal{V}_\alpha] = \frac{\partial \Gamma^\delta_A}{\partial y^\alpha}\mathcal{V}_\delta,\ [\mathcal{H}_A, \mathcal{H}_B] = C^\delta_{AB}\mathcal{H}_\delta + (C^\delta_{AB}\Gamma^\gamma_\delta + \tilde{\rho}^1(\mathcal{H}_B)(\Gamma^\gamma_A) - \tilde{\rho}^1(\mathcal{H}_A)(\Gamma^\gamma_B))\mathcal{V}_\gamma. \quad (8)$$

3.3. Torsion and curvature

Once we have a Lie algebroid structure on a vector bundle, it becomes possible to define a bracket operation also on 'vector-valued forms' on sections of that bundle, in exactly the same way as it is done in the standard Frölicher and Nijenhuis theory (see e.g. [13]). Coming back now to our present situation, we have already come across two type (1,1) tensor fields on the Lie algebroid $T^{\tilde{\rho}}E$, namely the vertical endomorphism S and the horizontal projector P_H. Using the Lie algebroid bracket on $\tilde{\pi}^1$, we thus can define torsion and curvature of the given (non-linear) $\tilde{\rho}$-connection on π in the way this is usually done for connections on a tangent bundle.

Definition 4. The torsion T and curvature R of a $\tilde{\rho}$-connection on π are skew-symmetric, $C^\infty(E)$-bilinear maps: $Sec(\tilde{\pi}^1) \times Sec(\tilde{\pi}^1) \to Sec(\tilde{\pi}^1)$, determined by $T = [P_H, S]$ and $R := \frac{1}{2}[P_H, P_H]$.

We can use (8) to calculate T and R in coordinates.

$$
\begin{aligned}
T &= \frac{1}{2}\left(\frac{\partial \Gamma^\gamma_\alpha}{\partial y^\beta} - \frac{\partial \Gamma^\gamma_\beta}{\partial y^\alpha} - C^\gamma_{\alpha\beta}\right)\mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma \\
&\quad + \left(\frac{\partial \Gamma^\gamma_0}{\partial y^\alpha} - \Gamma^\gamma_\alpha + y^\beta \frac{\partial \Gamma^\gamma_\alpha}{\partial y^\beta} - C^\gamma_{0\alpha}\right)\mathcal{X}^0 \wedge \mathcal{X}^\alpha \otimes \mathcal{V}_\gamma, \\
R &= \frac{1}{2}\Big(\tilde{\rho}^1(\mathcal{H}_\beta)(\Gamma^\gamma_\alpha) - \tilde{\rho}^1(\mathcal{H}_\alpha)(\Gamma^\gamma_\beta) + C^\mu_{\alpha\beta}\Gamma^\gamma_\mu\Big)\mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma \\
&\quad + \Big(\tilde{\rho}^1(\mathcal{H}_\alpha)(\Gamma^\gamma_0) - \tilde{\rho}^1(\mathcal{H}_0)(\Gamma^\gamma_\alpha) + C^\mu_{0\alpha}\Gamma^\gamma_\mu\Big)\mathcal{X}^0 \wedge \mathcal{X}^\alpha \otimes \mathcal{V}_\gamma.
\end{aligned}
$$

In [1] the notions of torsion and curvature of a generalised connection were only introduced under rather special circumstances, namely that the connection is linear (which requires, in the current set-up, that π is in fact a vector bundle). That there exist such notions as torsion and curvature for the non-linear case also, becomes clear only if the interest is shifted, as we do, from horizontality on $TP \to P$ to horizontality on $\tilde{\pi}^1 : T^\rho P \to P$.

3.4. Generalised connection associated with a pseudo-Sode

It is time to bring the pseudo-Sodes back into the picture. In the special case that the pseudo-Sode Γ lives on the extended Lie algebroid $\tilde{\pi}^1$ of an affine Lie algebroid π, we can easily define a $\tilde{\rho}$-connection on π that is associated with the pseudo-Sode: we only have to formally copy the formula (6):

$$P_H = \frac{1}{2}\Big(I - d_\Gamma S + \mathcal{X}^0 \otimes \Gamma\Big). \tag{9}$$

Here, the vertical endomorphism is given by (4) and, if d is the exterior derivative of the extended Lie algebroid $\tilde{\pi}^1$, $d_\Gamma = [i_\Gamma, d]$ plays the role of the Lie derivative in the classical theory (see e.g. [9, 13] for a definition). The proof relies (exactly as in the case of Sodes, see e.g. [4]) on the observation that the square of the operator $Q = \mathcal{L}_\Gamma S - \mathcal{X}^0 \otimes \Gamma$ equals the identity, and thus gives rise to two complementary operators $P_H = \frac{1}{2}(I - Q)$ and $P_V = \frac{1}{2}(I + Q)$. We will omit it here. The connections coefficients for the pseudo-SODE (5) are

$$\begin{aligned}
\Gamma_0^\alpha &= -f^\alpha + \frac{1}{2} y^\beta \Big(\frac{\partial f^\alpha}{\partial y^\beta} + C^\alpha_{0\beta} \Big) = -f^\alpha - y^\beta \Gamma^\alpha_\beta \qquad \text{and} \\
\Gamma^\alpha_\beta &= -\frac{1}{2} \Big(\frac{\partial f^\alpha}{\partial y^\beta} + y^\gamma C^\alpha_{\gamma\beta} + C^\alpha_{0\beta} \Big).
\end{aligned} \tag{10}$$

PROPOSITION 3. *A $\tilde\rho$-connection on π is associated with a pseudo-*SODE *(by means of (9)) if and only if its torsion vanishes.*

Proof. We give a short coordinate proof. Substituting connection coefficients of the form (10), one finds that indeed $T = 0$ holds. On the other hand, if $T = 0$, then $\frac{\partial \Gamma^\gamma_\alpha}{\partial y^\beta} + \frac{1}{2} C^\gamma_{\beta\alpha} = \frac{\partial \Gamma^\gamma_\beta}{\partial y^\alpha} + \frac{1}{2} C^\gamma_{\alpha\beta}$ and thus $\frac{\partial}{\partial y^\beta}\Big(\Gamma^\gamma_\alpha + \frac{1}{2} y^\mu C^\gamma_{\mu\alpha}\Big) = \frac{\partial}{\partial y^\alpha}\Big(\Gamma^\gamma_\beta + \frac{1}{2} y^\mu C^\gamma_{\mu\beta}\Big)$. This means that there exist functions $g^\gamma \in C^\infty(E)$, such that

$$\Gamma^\gamma_\alpha + \frac{1}{2} y^\mu C^\gamma_{\mu\alpha} = \frac{\partial g^\gamma}{\partial y^\alpha}. \tag{11}$$

We will use this observation in the other part of the (zero) torsion: the coefficient of $\mathcal{X}^0 \wedge \mathcal{X}^\alpha \otimes \mathcal{V}_\gamma$ can be rewritten as $\frac{\partial}{\partial y^\alpha}\Big(\Gamma^\gamma_0 - y^\mu C^\gamma_{0\mu} - 2g^\gamma + y^\beta \frac{\partial g^\gamma}{\partial y^\beta}\Big) = 0$, thus

$$\Gamma^\gamma_0 - y^\mu C^\gamma_{0\mu} - 2g^\gamma + y^\beta \frac{\partial g^\gamma}{\partial y^\beta} = h^\gamma \tag{12}$$

for some functions h^γ on M. Let us introduce now the functions $f^\gamma = -2g^\gamma - h^\gamma - C^\gamma_{0\mu} y^\mu$. Expressions (11,12) for g^γ and h^γ can be rewritten in terms of f^α, leading indeed to the connection coefficients (10). □

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