On the geometry of generalized metrics*

Tom Mestdag[•], József Szilasi[••] and Viktória Tóth[••]

[•]Department of Mathematical Physics and Astronomy
Ghent University, Krijgslaan 281, S9, B-9000 Ghent, Belgium

[••]Institute of Mathematics and Informatics
University of Debrecen, H-4010 Debrecen P.O.B. 12, Hungary

Abstract. We study metrics on the pullback bundle of a tangent bundle by its own projection. We investigate the circumstances under which an arbitrary metric admits a regular Lagrangian and thus an associated semispray. We present a simple coordinate-free formulation for all metric derivatives.

Mathematics Subject Classification (2000): 35C60, 53C60, 70H03.

Keywords: generalized metrics, Lagrangian vector fields, semi-Finsler metrics, metric derivatives.

Introduction

The concept of a Finsler manifold is as old as the concept of a Riemannian manifold, since it was Riemann himself who suggested the investigation of more general 'non-Riemannian' metrics in his *Habilitationsvortrag* of 1854 (see e.g. [21, 4]). A class of these more general metrics, called now the class of Finsler metrics, was first investigated by Finsler in his thesis (1918). However, Finsler geometers usually do not refer to a metric tensor as the cornerstone of their theory: traditionally Finsler geometry is cast in terms of a 1-homogeneous function, called the *fundamental function*, or a 2-homogeneous function, the so-called *energy*, and only secondary, is the 'Finsler metric' introduced as the Hessian of the energy. In contrast with this point of view, in our present paper, we intend to treat Finsler metrics (and more general structures) as being prior to the energy.

Speaking in coordinate terms, the most striking difference between a Riemannian metric and a Finsler metric is that the local components of the latter typically depend also on the fibre coordinates of the tangent manifold. In the past, a lot of models have been proposed

*We would like to dedicate this paper to Prof. L. Tamássy on the occasion of his 80th birthday.

to describe Finsler geometry. In our experience it turns out to be convenient to think about Finsler metrics as a special subset in the class of symmetric and non-degenerate (0,2) tensor fields of a certain pullback bundle. In the following, we will refer to all such tensor fields as *metrics*. In [13], M. Hashiguchi gave a necessary and sufficient condition for a metric to be the Hessian of some Finsler energy and used for the first time the adjective *normal* to distinguish Finsler metrics from all others.

Although Finsler geometry has proved its merit in a lot of domains in physics, biology, ecology, etc. (see e.g. [2]), there remain a number of theories that need a less restricted class of metrics (see e.g. [19] Ch. XI, Ch. XII and the references therein). In this paper we will study metrics in a broader context, meaning that they need not necessarily be the Hessian of some energy. There exists a long history of attempts to generalize Finsler geometry, mainly written in the language of classical tensor calculus. Here, we will mention only two papers which have a direct link with the current paper. In [23] and [18], the two authors considered a subclass of metrics that is more general than the class of normal metrics. These two subclasses are different from each other: J.R. Vanstone, building upon earlier work of A. Moór [20], studied certain aspects of homogeneous metrics, while the metrics of R. Miron satisfy a weaker condition than M. Hashiguchi's. We will come back to the precise characterization of these two classes in the fourth section. A lot of applications in metric geometry involve the use of a metric derivative. Both papers have in common that the authors were able to provide a local coordinate formulation for a 'canonical' metric derivative in their subclass. The main reason why they could find such a formulation is related to the ability of their subclass to generate a *regular Lagrangian*. The regularity of this Lagrangian implies the existence of a canonical semispray, which in turn leads to an associated horizontal distribution. In general, such a horizontal distribution is not 'canonically' available for an arbitrary metric. Its presence makes it possible to simplify the problem of metric derivatives to the search for two appropriate tensor fields on the pullback bundle. Later, in [19], R. Miron and M. Anastasiei recognized this idea in a theorem that gives an explicit coordinate formulation for all 'metrical connections' when the availability of a horizontal distribution is assumed (which is a priori not related to the metric).

After elaborating some elementary tools, we give a review on Hessians, Lagrangians and metrics in the third section. Next, we investigate the circumstances under which a metric admits a regular Lagrangian and thus an associated semispray. Section 3 ends with a survey in the form of a tabular with the most interesting cases. In the last section we present a coordinate-free description of a metric derivative associated to a given horizontal distribution and we supply a simple coordinate-free formulation for all other metric derivatives.

1 Basic setup

In order to keep our paper more or less self-contained, we shall start this section with a brief overview of the elementary terminology and fix some notational conventions. A recent reference for a detailed survey of this background material is the study [22].

We work over an n-dimensional smooth manifold M and assume that its topology is Hausdorff, second countable and connected. $C^\infty(M)$ denotes the ring of real-valued smooth functions on M. $\mathcal{X}(M)$ and $\mathcal{A}^k(M)$ $(1 \leq k \leq n)$ stand for the $C^\infty(M)$-modules of vector fields and differential k-forms on M, respectively. $\mathcal{A}^0(M) := C^\infty(M)$; $\mathcal{A}(M) := \bigoplus_{k=0}^{n} \mathcal{A}^k(M)$ is the exterior algebra of M. The familiar wedge product $\wedge$ makes $\mathcal{A}(M)$ into a graded algebra over the ring $C^\infty(M)$. A *vector k-form* on M is a $C^\infty(M)$-multilinear skew-symmetric map $(\mathcal{X}(M))^k \to \mathcal{X}(M)$ $(1 \leq k \leq n)$. The $C^\infty(M)$-module of vector k-forms will be denoted by $\mathcal{B}^k(M)$. We agree that $\mathcal{B}^0(M) := \mathcal{X}(M)$ and we denote the direct sum $\bigoplus_{k=0}^{n} \mathcal{B}^k(M)$ by $\mathcal{B}(M)$.

$\tau : TM \to M$ is the tangent bundle of M; $\overset{\circ}{T}M \subset TM$ is the (open) set of all nonzero tangent vectors. The natural projection $\overset{\circ}{T}M \to M$ is denoted by $\overset{\circ}{\tau}$. We shall remain in the smooth category, however, in Finsler geometry, the smoothness of some objects living on the tangent bundle is guaranteed only over $\overset{\circ}{T}M$. The elements of the kernel of the tangent map $T\tau : TTM \to TM$ of the tangent bundle projection τ form the *vertical submanifold* VTM of TTM; VTM is the total manifold of the *vertical bundle* $V\tau : VTM \to TM$ to τ. The $C^\infty(TM)$-module $\mathcal{X}^v(TM)$ of the sections of the vertical bundle is called the module of *vertical vector fields* on TM. The *vertical lift* of a smooth function f on M is the function $f^v := f \circ \tau \in C^\infty(TM)$, the *complete lift* of f is the function $f^c : TM \to \mathbb{R}$, $v \mapsto f^c(v) := v(f)$. *Any vector field on TM is uniquely determined by its action on the complete lifts of smooth functions on M*, so, given a vector field X on M, there exist unique vector fields X^v and X^c on TM, such that $X^v f^c = (Xf)^v$ and $X^c f^c = (Xf)^c$ for all $X \in \mathcal{X}(M)$. X^v and X^c are said to be the *vertical* and the *complete lift of X*, respectively. Now we can formulate our first local basis principle: *if $(X_1, \dots, X_n)$ is a local basis of vector fields on M, then $(X_1^c, \dots, X_n^c, X_1^v, \dots, X_n^v)$ is a local basis of vector fields on TM.*

The majority of our concepts will live on the pullback bundle $\tau^*\tau$ of the tangent bundle by its own projection. It is a vector bundle over TM with total manifold $\tau^*TM = TM \times_M TM := \{(v, w) \in TM \times TM \mid \tau(v) = \tau(w)\}$. The fibres of $\tau^*\tau$ are the n-dimensional real vector spaces $\{v\} \times T_{\tau(v)}M \cong T_{\tau(v)}M$, $v \in TM$. Any section of $\tau^*\tau$ is of the form

$$\tilde{X} : v \in TM \mapsto \tilde{X}(v) = (v, \underline{X}(v)) \in TM \times_M TM,$$

where $\underline{X} : TM \to TM$ is a smooth map such that $\tau \circ \underline{X} = \tau$. In particular, we have the specific section

$$\delta : v \in TM \mapsto \delta(v) := (v, v) \in TM \times_M TM$$

of $\tau^*\tau$, called the *canonical vector field along τ*. In the following we shall identify the sections of $\tau^*\tau$ with the smooth maps $\underline{X} : TM \to TM$ that satisfy the requirement $\tau \circ \underline{X} = \tau$. The $C^\infty(TM)$-module of such maps is denoted by $\mathcal{X}(\tau)$, and an element of this module is said to be a *vector field along the tangent bundle projection*. Under this identification the canonical section δ corresponds to the identity map 1_{TM}. A special class of vector fields along the projection is formed by the sections of the form $\hat{X} := X \circ \tau$, where X is a vector field on M. For obvious reasons, $\hat{X}$ will be called the *lift of X*

into $\mathcal{X}(\tau)$, or a *basic vector field* along τ. If $(X_1, \dots, X_n)$ is a local basis of $\mathcal{X}(M)$, then $(\hat{X}_1, \dots, \hat{X}_n)$ is a local basis for $\mathcal{X}(\tau)$. This observation, referred to as the *second local basis principle*, will simplify a great deal of our calculations.

By a *one-form along* τ we mean an element of the dual module of $\mathcal{X}(\tau)$. As in the case of vector fields along τ, any one-form $\tilde{\alpha}$ along τ may be regarded as a smooth map of TM into $T^*M := \bigcup_{p \in M} (T_pM)^*$ that satisfies the condition $\tau^* \circ \tilde{\alpha} = \tau$, where τ^* is the natural projection $T^*M \to M$. We denote the $C^\infty(TM)$-module of these maps by $\mathcal{A}^1(\tau)$. For any one-form α on M, the map $\hat{\alpha} := \alpha \circ \tau$ is a one-form along τ, called the *lift of* α *into* $\mathcal{A}^1(\tau)$, or a *basic one-form* along τ. By a *k-fold contravariant, l-fold covariant tensor field,* briefly a *type* (k, l) *tensor field along* τ, we mean a $C^\infty(TM)$-multilinear map $\big(\mathcal{A}^1(\tau)\big)^k \times \big(\mathcal{X}(\tau)\big)^l \to C^\infty(TM)$. The $C^\infty(TM)$-module of these tensor fields will be denoted by $\mathcal{T}^k_l(\tau)$. We agree, as usual, that $\mathcal{T}^0_0(\tau) := C^\infty(TM)$. The elements of $\mathcal{T}^k_l(\tau)$ may indeed be regarded as 'fields' which smoothly assign to each element v of the base manifold TM a type (k, l) tensor on the fibre $\{v\} \times T_{\tau(v)}M \cong T_{\tau(v)}M$ over v. For example, if $g \in \mathcal{T}^0_2(\tau)$, then g may be interpreted as a smooth map $v \in TM \mapsto g_v$, where $g_v : T_{\tau(v)}M \times T_{\tau(v)}M \to \mathbb{R}$ is a bilinear form. Notice that any tensor field A on M induces a *basic tensor field* $\hat{A} := A \circ \tau$ along τ. To end this brief summary on vector fields, forms and tensor fields along τ, we mention that we will also use $\tau^*\tau$*-valued vector forms on* TM. By a $\tau^*\tau$-valued k-form on TM we mean a skew-symmetric $C^\infty(TM)$-multilinear map of $(\mathcal{X}(TM))^k$ into $\mathcal{X}(\tau)$.

Most of our canonical objects may be identified from the short exact sequence

$$0 \to \tau^*TM \xrightarrow{\mathbf{i}} TTM \xrightarrow{\mathbf{j}} \tau^*TM \to 0 \tag{1}$$

of vector bundles over TM. Here the map $\mathbf{j}$ is defined by $\mathbf{j}(z) := (v, T\tau(z))$, for all $v \in TM, z \in T_vTM$, while the simplest description of $\mathbf{i}$ uses local coordinates. Let $\big(U, (u)^n_{i=1}\big)$ be a chart on M, and let us consider the *induced chart*

$$\big(\tau^{-1}(U), (x^i)^n_{i=1}, (y^i)^n_{i=1}\big); \quad x^i := (u^i)^v,\ y^i := (u^i)^c \qquad (1 \leq i \leq n). \tag{2}$$

on TM. Then for any vectors $v, w \in T_{\tau(v)}M$,

$$\mathbf{i}(v, w) = \sum_{i=1}^n y^i(w) \Big(\frac{\partial}{\partial y^i}\Big)_v =: y^i(w) \Big(\frac{\partial}{\partial y^i}\Big)_v. \tag{3}$$

The composite of $\mathbf{i}$ and δ yields another canonical object, the *Liouville vector field* $C := \mathbf{i} \circ \delta$ on TM. The short exact sequence (1) gives rise to a short exact sequence

$$0 \to \mathcal{X}(\tau) \xrightarrow{\mathbf{i}} \mathcal{X}(TM) \xrightarrow{\mathbf{j}} \mathcal{X}(\tau) \to 0.$$

of modules over $C^\infty(TM)$, where, for simplicity, we also denote by $\mathbf{i}$ and $\mathbf{j}$ the induced maps between the modules of sections. Notice that

$$\mathbf{i}\hat{X} = X^v, \qquad \mathbf{j}X^c = \hat{X} \qquad \text{for all } X \in \mathcal{X}(M). \tag{4}$$

The map $\mathbf{i}$ is an isomorphism between $\mathcal{X}(\tau)$ and $\mathcal{X}^v(TM)$, so any vertical vector field on TM can be represented uniquely in the form $\mathbf{i}\tilde{X}$ ($\tilde{X} \in \mathcal{X}(\tau)$). The map $\mathbf{j}$ is surjective, therefore any vector field along τ is of the form $\mathbf{j}\xi$, $\xi \in \mathcal{X}(TM)$. $\mathbf{i}$ and $\mathbf{j}$ enable us to introduce our next canonical object, the *vertical endomorphism* $J := \mathbf{i} \circ \mathbf{j}$. J is a type $(1,1)$ tensor field on TM. Using (4) and the relation $\mathbf{j} \circ \mathbf{i} = 0$, it follows that

$$(5) \qquad JX^c = X^v, \qquad JX^v = 0 \qquad \text{for all } X \in \mathcal{X}(M),$$

therefore $ImJ = KerJ = \mathcal{X}^v(TM)$ and $J^2 = 0$.

Although we have now explained the most important canonical objects, sometimes it will be convenient to assume the presence of one additional ingredient. A *horizontal map for* τ is a (right) splitting $\mathcal{H} : \tau^*TM \to TTM$ of the short exact sequence (1), i.e. a strong bundle map such that $\mathbf{j} \circ \mathcal{H} = 1_{\tau^*TM}$. The existence of a horizontal map is guaranteed by the second countability of the topology of the base manifold. Let $\mathcal{H}_v := \mathcal{H} \upharpoonright \{v\} \times T_{\tau(v)}M$ ($v \in TM$), $HTM := \bigcup_{v \in TM} Im\mathcal{H}_v$, and let $H\tau$ be the natural projection of HTM onto TM. There is a unique smooth manifold structure on HTM which makes $H\tau : HTM \to TM$ into a vector bundle. This vector bundle is said to be the *horizontal bundle* induced by $\mathcal{H}$ and denoted by $H\tau$. Then $TTM = HTM \oplus VTM$; fibrewise $T_vTM = Im\mathcal{H}_v \oplus V_vTM$ ($V_vTM := KerT_v\tau$) for all $v \in TM$. The sections of $H\tau$ are called *($\mathcal{H}$-)horizontal vector fields* on TM. For the $C^\infty(TM)$-module of horizontal vector fields we use the notation $\mathcal{X}^h(TM)$, then $\mathcal{X}(TM) = \mathcal{X}^h(TM) \oplus \mathcal{X}^v(TM)$. Any horizontal map $\mathcal{H}$ makes it possible to define a lifting process of vector fields on M to vector fields on TM. The *horizontal lift* of $X \in \mathcal{X}(M)$ is the horizontal vector field X^h given by $X^h(v) = \mathcal{H}(v, X(\tau(v)))$ for all $v \in TM$). Evidently, $X^h = \mathcal{H} \circ \hat{X}$. Any right splitting $\mathcal{H}$ of (1) induces a left splitting $\mathcal{V} : TTM \to \tau^*TM$ of (1) such that $\mathcal{V} \circ \mathbf{i} = 1_{\tau^*TM}$; $Ker\mathcal{V} = Im\mathcal{H}$ and hence $\mathcal{V} \circ \mathcal{H} = 0$. Thus, specifying a horizontal map for τ, we arrive at the fundamental 'double exact' sequence

$$0 \rightleftarrows \tau^*TM \underset{\mathcal{V}}{\overset{\mathbf{i}}{\rightleftarrows}} TTM \underset{\mathcal{H}}{\overset{\mathbf{j}}{\rightleftarrows}} \tau^*TM \rightleftarrows 0.$$

$\mathcal{V}$ is called the *vertical map* belonging to $\mathcal{H}$. The maps $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$ and $\mathbf{v} = 1_{TTM} - \mathbf{h}$ are said to be (respectively) the *horizontal* and the *vertical projectors* determined by $\mathcal{H}$. $\mathbf{h}$ and $\mathbf{v}$ are obviously $(1,1)$ tensor fields on TM, or equivalently, vector one-forms on TM, i.e. $\mathbf{h}, \mathbf{v} \in \mathcal{B}^1(TM)$. We have the relations $\mathbf{h}^2 = \mathbf{h}$, $Im\mathbf{h} = HTM$, $Ker\mathbf{h} = VTM$; $\mathbf{v} = \mathbf{i} \circ \mathcal{V}$, $\mathbf{v}^2 = \mathbf{v}$, $Im\mathbf{v} = VTM$, $Ker\mathbf{v} = HTM$. Thus $\mathbf{h}$ and $\mathbf{v}$ are indeed projection operators; $\mathbf{h}$ projects TTM onto HTM along VTM, $\mathbf{v}$ projects TTM onto VTM along HTM. Concerning these technical tools, we collect here some useful formulae:

$$(6) \qquad J \circ \mathbf{h} = J, \qquad \mathbf{h} \circ J = 0, \qquad J \circ \mathbf{v} = 0, \qquad \mathbf{v} \circ J = J.$$

$$(7) \qquad JX^h = X^v, \quad J[X^h, Y^h] = [X, Y]^v, \quad \mathbf{h}X^c = X^h, \quad \mathbf{h}[X^h, Y^h] = [X, Y]^h.$$

In general there is no canonical way to specify a horizontal map. However, in the presence of some additional structure, a horizontal map may be given canonically. We recall here a well-known and very important construction. Suppose that ξ is a *semispray* on M, i.e. $\xi : TM \to TTM$ is a C^1 vector field which is smooth on TM and has the property

$J\xi = C$. Then the map

$$X \in \mathcal{X}(M) \mapsto X^h := \frac{1}{2}(X^c + [X^v, \xi]) \in \mathcal{X}(\overset{\circ}{T}M) \tag{8}$$

defines a horizontal lifting and, consequently, a horizontal map $\mathcal{H}$ for τ. $\mathcal{H}$ will be mentioned as the *horizontal map generated by the semispray* ξ. If ξ is a *spray*, i.e. $[C, \xi] = \xi$, then the horizontal map $\mathcal{H}$ is *homogeneous* in the sense that $[X^h, C] = 0$ for all $X \in \mathcal{X}(M)$.

2 Derivations and Berwald derivatives

Let r be an integer. By a *graded derivation* of degree r of the exterior algebra $\mathcal{A}(M)$ we mean an $\mathbb{R}$-linear map $\mathcal{D} : \mathcal{A}(M) \to \mathcal{A}(M)$ such that $\mathcal{D}\big(\mathcal{A}^k(M)\big) \subset \mathcal{A}^{k+r}(M)$ for all $k \in \{0, \dots, n\}$ and $\mathcal{D}(\alpha \wedge \beta) = (\mathcal{D}\alpha) \wedge \beta + (-1)^{rk} \alpha \wedge \mathcal{D}\beta$ if $\alpha \in \mathcal{A}^k(M)$, $\beta \in \mathcal{A}(M)$. The *graded commutator* of two graded derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ is given by $[\mathcal{D}_1, \mathcal{D}_2] := \mathcal{D}_1 \circ \mathcal{D}_2 - (-1)^{r_1 r_2} \mathcal{D}_2 \circ \mathcal{D}_1$ where r_1 and r_2 are the degrees of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. $[\mathcal{D}_1, \mathcal{D}_2]$ is also a graded derivation whose degree is $r_1 + r_2$. The classical graded derivations of $\mathcal{A}(M)$ are the *substitution operator* i_X (induced by $X \in \mathcal{X}(M)$), the Lie derivative d_X (with respect to $X \in \mathcal{X}(M)$), and the *exterior derivative* d; their degrees are $-1, 0$, and 1, respectively. i_X, d_X and d are related by H. Cartan's 'magic' formula

$$d_X = i_X \circ d + d \circ i_X = [i_X, d]. \tag{9}$$

In the *Frölicher-Nijenhuis theory* of derivations, two graded derivations of $\mathcal{A}(M)$ are associated to any vector k-form $K \in \mathcal{B}^k(M)$: the derivation i_K of degree $k-1$ defined by $i_K \restriction C^\infty(M) := 0$ and $i_K \alpha := \alpha \circ K$ for $\alpha \in \mathcal{A}^1(M)$, and the derivation $d_K := [i_K, d] = i_K \circ d - (-1)^{k-1} d \circ i_K$ of degree k. As an immediate consequence, we obtain

(10) *if $f \in C^\infty(M)$ and $K \in \mathcal{B}^k(M)$, then $d_K f = i_K df = df \circ K$.*

A characteristic property of d_K is expressed by

$$[d, d_K] := d \circ d_K - (-1)^k d_K \circ d = 0. \tag{11}$$

For any vector k-form K and vector l-form L on M there is a unique vector $(k+l)$-form on M, denoted by $[K, L]$, such that $d_{[K,L]} = [d_K, d_L]$. $[K, L]$ is said to be the *Frölicher–Nijenhuis bracket* of K and L. If K and L are 0-forms, i.e. vector fields on M, then $[K, L]$ is the usual bracket of vector fields. In the case $L := Y \in \mathcal{X}(M) = \mathcal{B}^0(M)$ and $K \in \mathcal{B}^1(M)$ we have

$$[K, Y]X = [KX, Y] - K[X, Y] \qquad \text{for all } X \in \mathcal{X}(M). \tag{12}$$

If K and L are both vector one-forms, or in other words type $(1,1)$ tensor fields on M, then their brackets acts by

$$\begin{aligned}[K, L](X, Y) \;=\; & [KX, LY] + [LX, KY] + (K \circ L + L \circ K)[X, Y] \\ & - K[LX, Y] - K[X, LY] - L[KX, Y] - L[X, KY].\end{aligned} \tag{13}$$

In our calculations over TM the operators i_J and $d_J = i_J \circ d - d \circ i_J$ play a distinguished role. Notice that over the induced chart (2) we have the coordinate expression

$$d_J F \restriction \tau^{-1}(U) = \frac{\partial F}{\partial y^i} dx^i \qquad \text{for all } F \in C^\infty(TM).$$

We shall need the following relations:

$$(14) \quad [i_C, i_J] = 0,\ [i_C, d_J] = i_J,\ [d_J, d_C] = d_J,\ [i_J, d_J] = 0,\ [d_J, d_J] = 0,\ [i_\xi, i_J] = i_{J\xi}.$$

A more or less analogous theory of derivations of $\mathcal{A}(\tau)$ was elaborated by E. Martínez, J.F. Cariñena and W. Sarlet [16, 17], see also [22]. We shall borrow only one ingredient from this theory, the *v-exterior derivative* d^v defined by

$$(15) \qquad (d^v F)(\tilde{X}) := dF(\mathbf{i}\tilde{X}) = (\mathbf{i}\tilde{X})F \qquad \text{for all } F \in C^\infty(TM),$$

$$(16) \qquad d^v \hat{\alpha} := 0 \qquad \text{for all } \alpha \in \mathcal{A}^1(M).$$

Observe that $d^v F(\mathbf{j}\xi) = (dF \circ J)(\xi) \overset{(10)}{=} d_J F(\xi)$ for any vector field ξ on TM. In coordinates we have

$$d^v F \restriction \tau^{-1}(U) = \frac{\partial F}{\partial y^i} \widehat{du^i}.$$

We can easily deduce the important relation $d^v \circ d^v = 0$.

Lemma 1. *A k-form $\tilde{\alpha}$ $(1 \leq k \leq n)$ along τ is d^v-exact, i.e. there is a $(k-1)$-form $\tilde{\beta}$ along τ such that $\tilde{\alpha} = d^v \tilde{\beta}$ if, and only if, $\tilde{\alpha}$ is d^v-closed, i.e. $d^v \tilde{\alpha} = 0$.*

For a sketchy proof the reader is referred to [16]. A detailed account of d_J-cohomology is presented in K. Ayassou's thesis [3].

To complete this section about our main technical tools, first we define two partial differentials, the *canonical v-covariant differential* ∇^v and the *h-covariant differential* ∇^h, next we piece these together to obtain a covariant derivative operator ∇ (depending on a horizontal map), called the *Berwald derivative* in $\tau^*\tau$. The constructions goes as follows. Consider first, for a given vector field $\tilde{X}$ along τ, the map $\nabla^v_{\tilde{X}}$ whose action on functions, vector fields and one-forms along the projection is given by

$$(17) \quad \nabla^v_{\tilde{X}} F := (d^v F)(\tilde{X}), \quad \nabla^v_{\tilde{X}} \tilde{Y} := \mathbf{j}[\mathbf{i}\tilde{X}, \mathcal{H}\tilde{Y}], \quad (\nabla^v_{\tilde{X}} \tilde{\alpha})(\tilde{Y}) := \nabla^v_{\tilde{X}} \big(\tilde{\alpha}(\tilde{Y})\big) - \tilde{\alpha}(\nabla^v_{\tilde{X}} \tilde{Y}).$$

Although it may look as if we used a horizontal map in the definition of $\nabla^v_{\tilde{X}} \tilde{Y}$, it can easily be checked that the operator ∇^v is in fact independent of the choice of $\mathcal{H}$. Indeed, represent $\tilde{Y}$ in the form $\mathbf{j}\eta$, $\eta \in \mathcal{X}(TM)$. Then we get $\nabla^v_{\tilde{X}} \mathbf{j}\eta = \mathbf{j}[\mathbf{i}\tilde{X}, \mathbf{h}\eta] = \mathbf{j}[\mathbf{i}\tilde{X}, \eta]$, since $[\mathbf{i}\tilde{X}, \mathbf{v}\eta]$ is vertical. In particular, we have

$$(18) \qquad \nabla^v_{\hat{X}} \hat{Y} = 0, \qquad \nabla^v_{\hat{X}} \delta = \hat{X}, \qquad \nabla^v_{\delta} \hat{X} = 0 \qquad \text{for all } X, Y \in \mathcal{X}(M).$$

Now the formula

$$(\nabla^v_{\tilde{X}} \tilde{A})(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k, \tilde{X}_1, \dots, \tilde{X}_l) := (\mathbf{i}\tilde{X})\tilde{A}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k, \tilde{X}_1, \dots, \tilde{X}_l)$$
$$- \sum_{i=1}^{k} \tilde{A}(\tilde{\alpha}_1, \dots, \nabla^v_{\tilde{X}} \tilde{\alpha}_i, \dots, \tilde{\alpha}_k, \tilde{X}_1, \dots, \tilde{X}_l) - \sum_{j=1}^{l} \tilde{A}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k, \tilde{X}_1, \dots, \nabla^v_{\tilde{X}} \tilde{X}_j, \dots, \tilde{X}_l)$$

extends the action of $\nabla^v_{\tilde{X}}$ to a general type (k,l) tensor field along the projection. The canonical v-covariant differential is then the operator ∇^v that associates to a (k,l) tensor field $\tilde{A}$ along τ a $(k,l+1)$ tensor field $\nabla^v\tilde{A}$ along τ given by

$$(\nabla^v\tilde{A})(\tilde{X},\tilde{\alpha}_1,\dots,\tilde{\alpha}_k,\tilde{X}_1,\dots,\tilde{X}_l) := (\nabla^v_{\tilde{X}}\tilde{A})(\tilde{\alpha}_1,\dots,\tilde{\alpha}_k,\tilde{X}_1,\dots,\tilde{X}_l).$$

In the same way as above, specifying a horizontal map $\mathcal{H}$ for τ, and starting from

$$\nabla^h_{\tilde{X}}F := (\mathcal{H}\tilde{X})F, \quad \nabla^h_{\tilde{X}}\tilde{Y} := \mathcal{V}[\mathcal{H}\tilde{X},\mathbf{i}\tilde{Y}], \quad (\nabla^h_{\tilde{X}}\tilde{\alpha})(\tilde{Y}) := \nabla^h_{\tilde{X}}\big(\tilde{\alpha}(\tilde{Y})\big) - \tilde{\alpha}(\nabla^h_{\tilde{X}}\tilde{Y}) \tag{19}$$

($F\in C^\infty(TM)$, $\tilde{Y}\in\mathcal{X}(\tau)$, $\tilde{\alpha}\in\mathcal{A}^1(\tau)$) we can introduce the *h-covariant differential* ∇^h. Having these partial differentials, the map

$$\nabla : (\xi,\tilde{Y})\in\mathcal{X}(TM)\times\mathcal{X}(\tau)\mapsto \nabla_\xi\tilde{Y} := \nabla^v_{\mathcal{V}\xi}\tilde{Y} + \nabla^h_{\mathbf{j}\xi}\tilde{Y}\in\mathcal{X}(\tau)$$

is a *covariant derivative operator* in the vector bundle $\tau^*\tau$ in the sense that ∇ is an $\mathbb{R}$-bilinear map satisfying the conditions $\nabla_{F\xi}\tilde{Y} = F\nabla_\xi\tilde{Y}$ and $\nabla_\xi F\tilde{Y} = (\xi F)\tilde{Y} + F\nabla_\xi\tilde{Y}$ concerning the multiplication with a smooth function F on TM. The covariant derivative operator ∇ is said to be the *Berwald derivative in* $\tau^*\tau$ induced by $\mathcal{H}$. Explicitly,

$$\nabla_\xi\tilde{Y} = \mathbf{j}[\mathbf{v}\xi,\mathcal{H}\tilde{Y}] + \mathcal{V}[\mathbf{h}\xi,\mathbf{i}\tilde{Y}] \qquad \text{for all } \xi\in\mathcal{X}(TM) \text{ and } \tilde{Y}\in\mathcal{X}(\tau). \tag{20}$$

The canonical v-covariant differential ∇^v is intimately related to the v-exterior derivative d^v: we have

$$d^v\tilde{\alpha} = (k+1)Alt\nabla^v\tilde{\alpha} \qquad \text{for all } \tilde{\alpha}\in\mathcal{A}^k(\tau), \tag{21}$$

where Alt is the alternator in $\mathcal{A}^k(\tau)$. For a proof the reader is referred to [22]. Applying this observation, we exhibit now a convenient expression for the d^v-exactness of a one-form in terms of the operator ∇^v. We say that a one-form $\tilde{\alpha}$ along τ is ∇^v-exact if there exists a function $F\in C^\infty(TM)$ such that $\nabla^v F = \tilde{\alpha}$.

Lemma 2. *A one-form $\tilde{\alpha}$ along τ is ∇^v-exact if, and only if, $\nabla^v\tilde{\alpha}$ is symmetric, i.e.*

$$(\nabla^v\tilde{\alpha})(\tilde{X},\tilde{Y}) = (\nabla^v\tilde{\alpha})(\tilde{Y},\tilde{X}) \qquad \textit{for all } \tilde{X},\tilde{Y}\in\mathcal{X}(\tau).$$

Proof: $\nabla^v \restriction C^\infty(TM) = d^v \restriction C^\infty(TM)$ by (17), so ∇^v-exactness and d^v-exactness are equivalent conditions for one-forms. In view of Lemma 1, a one-form $\tilde{\alpha}$ is d^v-exact if, and only if, it is d^v-closed. Since $(d^v\tilde{\alpha})(\tilde{X},\tilde{Y}) \overset{(21)}{=} 2Alt(\nabla^v\tilde{\alpha})(\tilde{X},\tilde{Y}) = (\nabla^v\tilde{\alpha})(\tilde{X},\tilde{Y}) - (\nabla^v\tilde{\alpha})(\tilde{Y},\tilde{X})$, the assertion follows. □

Finally, we point out that the Frölicher-Nijenhuis formalism provides a concise and extremely elegant way to define the basic geometric data of a horizontal map. Namely, let $\mathcal{H}$ be a horizontal map on M, and let $\mathbf{h}$ be the horizontal projector determined by $\mathcal{H}$. Then the vector forms

$$\mathbf{t} := [\mathbf{h},C]\in\mathcal{B}^1(TM), \quad \mathbf{T} := [J,\mathbf{h}]\in\mathcal{B}^2(TM), \quad \Omega := -\frac{1}{2}[\mathbf{h},\mathbf{h}]\in\mathcal{B}^2(TM) \tag{22}$$

are said to be the *tension*, the *torsion* and the *curvature* of $\mathcal{H}$, respectively. It is known that a *horizontal map is generated by a semispray according to* (8) *if, and only if, its torsion vanishes* (a theorem of M. Crampin).

3 Hessians, Lagrangians, and metrics

By the *Hessian* of a smooth function F on TM we mean the type $(0,2)$ tensor field $\nabla^v\nabla^v F := \nabla^v(\nabla^v F) = \nabla^v(d^v F)$ along τ. We have

$$\nabla^v\nabla^v F(\hat{X},\hat{Y}) = X^v(Y^v F) \qquad \text{for all } X,Y \in \mathcal{X}(M). \tag{23}$$

Indeed, $\nabla^v\nabla^v F(\hat{X},\hat{Y}) = \big(\nabla^v_{\hat{X}}(\nabla^v F)\big)(\hat{Y}) \overset{(17)}{=} \nabla^v_{\hat{X}}\big(\nabla^v F(\hat{Y})\big) - \nabla^v F(\nabla^v_{\hat{X}}\hat{Y}) \overset{(15)(18)}{=} X^v(Y^v F)$. Observe that $X^v(Y^v F) = [X^v,Y^v]F + Y^v(X^v F) = Y^v(X^v F)$ whence $\nabla^v\nabla^v F$ is symmetric, at least when its arguments are basic vector fields along τ. Since the basic vector fields form a local basis for $\mathcal{X}(\tau)$ and $\nabla^v(\nabla^v F)$ is tensorial, we conclude that *the Hessian of a smooth function $F : TM \to \mathbb{R}$ is a symmetric type $(0,2)$ tensor field along τ.* We obtain by an analogous reasoning that $\nabla^v(\nabla^v\nabla^v F)$ *is a totally symmetric type $(0,3)$ tensor field along τ*, and so forth.

A *Lagrangian* is a smooth function $L : TM \to \mathbb{R}$. The one-form $\theta_L := d_J L$ and the two-form $\omega_L := d\theta_L = dd_J L$ are said to be the *Lagrange one-form* and the *Lagrange two-form*, respectively, while the function $E_L := CL - L$ is called the *energy* associated to L. In this context, we shall use the notation g_L for the Hessian $\nabla^v\nabla^v L$. We have the relations

$$i_C\omega_L = d_J E_L, \qquad i_J\omega_L = 0. \tag{24}$$

Indeed, $i_C\omega_L = i_C dd_J L \overset{(11)}{=} -i_C d_J dL \overset{(14)}{=} d_J i_C dL - i_J dL = d_J\big(dL(C) - L\big) = d_J E_L$ and $i_J\omega_L = i_J dd_J L \overset{(11)}{=} -i_J d_J dL \overset{(14)}{=} -d_J i_J dL \overset{(10)}{=} -d_J d_J L = -\frac{1}{2}[d_J,d_J]L \overset{(14)}{=} 0$.

Lemma 3. *The Lagrange two-form ω_L and the Hessian g_L are related by*

$$\omega_L(J\xi,\eta) = g_L(\mathbf{j}\xi,\mathbf{j}\eta) \qquad \textit{for all } \xi,\eta \in \mathcal{X}(TM). \tag{25}$$

Proof: For any vector fields X, Y on M we have $\omega_L(JX^c,Y^c) \overset{(5)}{=} \omega_L(X^v,Y^c) = d(d_J L)(X^v,Y^c) = X^v dL(JY^c) - Y^c dL(JX^v) - dL(J[X^v,Y^c]) \overset{(5)}{=} X^v(Y^v L) \overset{(23)}{=} g_L(\hat{X},\hat{Y})$ (taking into account that $[X^v,Y^c]$ is vertical). Remark that $\omega_L(X^v,Y^v)$ always vanishes. Applying the first local basis principle, we can conclude the desired relation. □

In order to avoid any confusion, before proceeding we emphasize that by the *non-degeneracy* of a type $(0,2)$ tensor field we mean the usual pointwise property (see e.g. [1], 3.1.4 Definition). This leads to a corresponding property at the level of vector fields, but not vice versa.

Returning to our Lagrangian $L \in C^\infty(TM)$: it is called a *regular Lagrangian* if the Lagrange two-form ω_L is non-degenerate. Due to Lemma 3, it is possible to cast the regularity condition in terms of the Hessian g_L.

Corollary 1. *The Lagrange two-form ω_L is non-degenerate if, and only if, the Hessian g_L is non-degenerate.*

The proof is easy and is omitted. The regularity condition prescribed on L plays a prominent role in the proof of the following well-known and fundamental fact [1], [8]: *For a regular Lagrangian L, there exists a unique semispray ξ_L such that*

$$i_{\xi_L}\omega_L = -dE_L. \tag{26}$$

To verify this, let us first observe that $dE_L \neq 0$, since $i_J dE_L = d_J E_L \overset{(24)}{=} i_C\omega_L \neq 0$ by the non-degeneracy of ω_L. Thus, also for this reason, there exists a unique, non-zero ξ_L satisfying (26). Since $i_{J\xi_L}\omega_L \overset{(14)}{=} i_{\xi_L} i_J\omega_L - i_J i_{\xi_L}\omega_L \overset{(24)}{=} -i_J i_{\xi_L}\omega_L \overset{(26)}{=} i_J dE_L = d_J E_L \overset{(24)}{=} i_C\omega_L$, we conclude, again by non-degeneracy, the desired relation $J\xi_L = C$. The semispray ξ_L is called the *Lagrangian vector field for L.*

Tensor fields of the form $\nabla^v\nabla^v L$ will be crucial in our argumentation. Therefore it is important to know exactly when a type $(0,2)$ tensor field along τ is the Hessian of a smooth function on TM. For a systematic treatment of this problem we need some preparation.

Definition 1. By a *generalized metric*, briefly a *metric*, we mean a symmetric and non-degenerate type $(0,2)$ tensor field along τ. The function $E := \frac{1}{2}g(\delta,\delta)$ is the *(absolute) energy* of g. The canonical v-covariant derivative $\mathcal{C}_\flat := \nabla^v g$ of g is said to be the *lowered Cartan tensor* of g, the type $(1,2)$ tensor field $\mathcal{C}$ along τ determined by

$$g\big(\mathcal{C}(\tilde{X},\tilde{Y}),\tilde{Z}\big) := (\nabla^v g)(\tilde{X},\tilde{Y},\tilde{Z}) = \mathcal{C}_\flat(\tilde{X},\tilde{Y},\tilde{Z}) \tag{27}$$

is the *first Cartan tensor* of g. The one-form θ_g on TM given by

$$\theta_g(\xi) := g(\mathbf{j}\xi,\delta), \qquad \xi \in \mathcal{X}(TM), \tag{28}$$

or the one-form $\tilde{\theta}_g : \tilde{X} \in \mathcal{X}(\tau) \mapsto \tilde{\theta}_g(\tilde{X}) := g(\tilde{X},\delta)$ along τ, is called the *Lagrange one-form* associated to g; $\omega_g := d\theta_g$ is the *Lagrange two-form* associated to g.

From Corollary 1 we can immediately conclude that if L is a regular Lagrangian, then $g_L = \nabla^v\nabla^v L$ is a metric. Its absolute energy is $E = \frac{1}{2}\big(\nabla^v(\nabla^v L)\big)(\delta,\delta) = \frac{1}{2}\big(\nabla^v_\delta(\nabla^v L(\delta)) - \nabla^v L(\nabla^v_\delta\delta)\big) \overset{(18)}{=} \frac{1}{2}\big(\nabla^v_\delta((\mathbf{i}\delta)L) - \nabla^v L(\delta)\big) = \frac{1}{2}\big(C(CL) - CL\big) = \frac{1}{2}CE_L$.

Now we list some elementary properties of the first Cartan tensor associated to a metric g.

$$\mathcal{C}_\flat(\hat{X},\hat{Y},\hat{Z}) = X^v\big(g(\hat{Y},\hat{Z})\big) \qquad \textit{for all } X,Y,Z \in \mathcal{X}(M); \tag{29}$$

therefore *$\mathcal{C}_\flat$ is symmetric in its last two variables.*

$$\mathcal{C} \textit{ vanishes if, and only if, } g = \hat{g}_M := g_M \circ \tau, \tag{30}$$

where g_M is a pseudo-Riemannian metric on M.

$$\textit{If } g = \nabla^v\nabla^v F \ \big(F \in C^\infty(TM)\big), \textit{ then } \mathcal{C}_\flat(\hat{X},\hat{Y},\hat{Z}) = X^v\big(Y^v(Z^v F)\big), \tag{31}$$

therefore *if g is a Hessian, then the first Cartan tensor associated to g is symmetric, the lowered first Cartan tensor is totally symmetric.*

Lemma 4. *The Hessian $g_E := \nabla^v\nabla^v E = \frac{1}{2}\nabla^v\nabla^v\big(g(\delta,\delta)\big)$ of the absolute energy of g can be expressed in terms of g and the first Cartan tensor of g as follows:*

$$(32)\qquad \begin{aligned} g_E(\hat{X},\hat{Y}) \;=\; & \frac{1}{2}g\big(\mathcal{C}(\hat{X},\delta),\mathcal{C}(\hat{Y},\delta)\big)+\frac{1}{2}g\big((\nabla^v_{\hat{X}}\mathcal{C})(\hat{Y},\delta),\delta\big)+g\big(\mathcal{C}(\hat{Y},\delta),\hat{X}\big) \\ & +g\big(\mathcal{C}(\hat{X},\hat{Y}),\delta\big)+g(\hat{X},\hat{Y}), \qquad \textit{for all } X,Y\in\mathcal{X}(M). \end{aligned}$$

PROOF: $g_E(\hat{X},\hat{Y}) = \frac{1}{2}X^v\big(Y^v g(\delta,\delta)\big) = \frac{1}{2}X^v\Big((\nabla^v_{\hat{Y}}g)(\delta,\delta)+2X^v g(\nabla^v_{\hat{Y}}\delta,\delta)\Big)$
$\overset{(18)}{=} \frac{1}{2}X^v\Big(g\big(\mathcal{C}(\hat{Y},\delta),\delta\big)\Big)+X^v g(\hat{Y},\delta) = \frac{1}{2}(\nabla^v_{\hat{X}}g)\big(\mathcal{C}(\hat{Y},\delta),\delta\big)+\frac{1}{2}g\Big(\nabla^v_{\hat{X}}\big(\mathcal{C}(\hat{Y},\delta)\big),\delta\Big)$
$+\frac{1}{2}g\big(\mathcal{C}(\hat{Y},\delta),\hat{X}\big)+(\nabla^v_{\hat{X}}g)(\hat{Y},\delta)+g(\hat{Y},\hat{X}) = \frac{1}{2}g\Big(\mathcal{C}\big(\hat{X},\mathcal{C}(\hat{Y},\delta)\big),\delta\Big)+\frac{1}{2}g\big((\nabla^v_{\hat{X}}\mathcal{C})(\hat{Y},\delta),\delta\big)+$
$\frac{1}{2}g\big(\mathcal{C}(\hat{Y},\hat{X}),\delta\big)+\frac{1}{2}g\big(\mathcal{C}(\hat{Y},\delta),\hat{X}\big)+g\big(\mathcal{C}(\hat{X},\hat{Y}),\delta\big)+g(\hat{X},\hat{Y}) \overset{(29)}{=} \frac{1}{2}g\big(\mathcal{C}(\hat{X},\delta),\mathcal{C}(\hat{Y},\delta)\big)+$
$\frac{1}{2}g\big((\nabla^v_{\hat{X}}\mathcal{C})(\hat{Y},\delta),\delta\big)+g\big(\mathcal{C}(\hat{Y},\delta),\hat{X}\big)+g\big(\mathcal{C}(\hat{X},\hat{Y}),\delta\big)+g(\hat{X},\hat{Y})$. □

Definition 2. A metric g along τ is said to be *variational*, if the first Cartan tensor associated to g is symmetric (or, equivalently, the lowered first Cartan tensor is totally symmetric); *weakly variational*, if $g\big(\mathcal{C}(\tilde{X},\tilde{Y}),\delta\big)=g\big(\mathcal{C}(\tilde{Y},\tilde{X}),\delta\big)$, or, equivalently $\mathcal{C}_\flat(\tilde{X},\tilde{Y},\delta)=\mathcal{C}_\flat(\tilde{Y},\tilde{X},\delta)$ for all $\tilde{X},\tilde{Y}\in\mathcal{X}(\tau)$; *normal*, if $\mathcal{C}(\tilde{X},\delta)=0$ for all $\tilde{X}\in\mathcal{X}(\tau)$; *weakly normal*, if $\mathcal{C}_\flat(\tilde{X},\delta,\delta)=0$ for all $\tilde{X}\in\mathcal{X}(\tau)$ and *homogeneous*, if $\mathcal{C}_\flat(\delta,\tilde{X},\tilde{Y})=0$ for all $\tilde{X},\tilde{Y}\in\mathcal{X}(\tau)$.

Since $g(\mathcal{C}(\delta,\tilde{X}),\tilde{Y})=(\nabla^v_\delta g)(\tilde{X},\tilde{Y})$, g is homogeneous if, and only if $\nabla^v_\delta g=0$. In coordinates this means that the components of g are homogeneous functions of degree 0. We shall deal systematically with the weak variationality, normality and weak normality in the next section. The meaning of variationality is explained by the next

Result 1. *A metric g is the Hessian of a (necessarily regular) Lagrangian if, and only if, it is variational.*

We sketch a simple proof, whose idea is taken from a paper by O. Krupková [15]. The necessity of the condition is obvious from (31), the regularity of the desired Lagrangian is guaranteed by Corollary 1. To verify the sufficiency, first we construct a Lagrangian for g locally. Choose an induced chart $\big(\tau^{-1}(U),(x^i)_{i=1}^n,(y^i)_{i=1}^n\big)$ on TM according to (2). Let τ_U be the natural projection of $\tau^{-1}(U)$ into M, and let us consider the pull-back bundle $\tau_U^*\tau:\tau^{-1}(U)\times_M TM\to\tau^{-1}(U)$. We denote the restriction of g to $\tau_U^*\tau$ by g_U. If $g_{ij}:=g_U\Big(\frac{\hat{\partial}}{\partial u^i},\frac{\hat{\partial}}{\partial u^j}\Big)$ $(1\le i,j\le n)$, then for the lowered first Cartan tensor $(\mathcal{C}_\flat)_U$ of g_U we have the coordinate expression $(\mathcal{C}_\flat)_U\Big(\frac{\hat{\partial}}{\partial u^i},\frac{\hat{\partial}}{\partial u^j},\frac{\hat{\partial}}{\partial u^k}\Big)=\frac{\partial g_{jk}}{\partial y^i}$ $(1\le i,j,k\le n)$. Thus, locally, the variationality of g means that the expressions $\frac{\partial g_{jk}}{\partial y^i}$ are totally symmetric in i,j,k; in particular,

$$(33)\qquad \frac{\partial g_{ij}}{\partial y^k}=\frac{\partial g_{ik}}{\partial y^j}\qquad (1\le i,j,k\le n).$$

Let us denote by $\imath$ the natural inclusion $t\in[0,1]\mapsto t\in\mathbb{R}$, and for each fixed $v\in\tau^{-1}(U)$, let c_v be the map given by $t\in[0,1]\mapsto tv\in\tau^{-1}(U)$. We define the functions F_{ij} on $\tau^{-1}(U)$

by setting $F_{ij}(v) := \int_0^1 \imath(g_{ij} \circ c_v)$; $1 \leq i, j \leq n$. Let $L_U(v) := y^i(v)y^j(v)\int_0^1 F_{ij} \circ c_v$. Using (33) repeatedly, it is easy to check that $\nabla^v\nabla^v L_U = g_U$. Having local solutions, the desired global Lagrangian can immediately be constructed with the help of a partition of unity on M. Let $(U_k)_{k\in K}$ be an open covering of M, constituted by coordinate neighbourhoods. Choose a (smooth) partition of unity $(f_k)_{k\in K}$ subordinate to (U_k). If $L := \sum_{k\in K}(f_k)^v L_{U_k}$, where $\nabla^v\nabla^v L_{U_k} = g_{U_k}$ for all $k \in K$, then L is a smooth function on TM. Since $\nabla^v(f_k)^v = 0$, it follows that $\nabla^v\nabla^v L = g$.

4 Variationality and regularity

In Definition 1, we have introduced the Lagrange one-form and two-form associated to g. For a variational metric, their interpretation is easy.

Lemma 5. *Let g be a variational metric, $g = \nabla^v\nabla^v L$. Then $\theta_g = \theta_{E_L} = d_J E_L$ and therefore $\omega_g = \omega_{E_L}$.*

Proof: For any vector field X on M we have $\theta_g(X^c) := g(\mathbf{j}X^c, \delta) = g(\hat{X}, \delta) = \nabla^v(\nabla^v L)(\hat{X}, \delta) = \nabla^v_{\hat{X}}\big(\nabla^v L(\delta)\big) - \nabla^v L(\nabla^v_{\hat{X}}\delta) \overset{(17)(18)}{=} X^v(CL) - \nabla^v L(\hat{X}) = X^v(CL - L) = X^v E_L = (d_J E_L)(X^c) = \theta_{E_L}(X^c)$. Since obviously $\theta_g(X^v) = (d_J E_L)(X^v) = 0$, by the appropriate local basis principle we conclude the assertion. $\square$

By far the most important metrics are the variational metrics. In some special cases it is possible to associate a variational metric to a generalized metric g. As a first example, consider the Hessian $g_E := \nabla^v\nabla^v E$ of the absolute energy $E = \frac{1}{2}g(\delta, \delta)$. Then, due to Corollary 1, g_E is non-degenerate (and hence a metric) if, and only if, $\omega_E = d\theta_E = dd_J E$ is non-degenerate, i.e. E is a regular Lagrangian. In what follows we shall call metrics that satisfy this requirement *E-regular*. Using the non-degeneracy of g, it is always possible to introduce a new vector one-form $\tilde{A}$ along the projection, implicity given by

$$g\big(\tilde{A}(\tilde{X}), \tilde{Y}\big) := g_E(\tilde{X}, \tilde{Y}) \qquad \text{for all } \tilde{X}, \tilde{Y} \in \mathcal{X}(\tau). \tag{34}$$

A second way to define $\tilde{A}$ uses relation (25) to connect $\tilde{A}$ with ω_E:

$$g\big(\tilde{A}(\mathbf{j}\xi), \mathbf{j}\eta\big) = \omega_E(J\xi, \eta) \qquad \text{for all } \xi, \eta \in \mathcal{X}(TM). \tag{35}$$

Lemma 6. *A metric g along τ is E-regular if, and only if, the vector one-form $\tilde{A}$, regarded as a type (1,1) tensor field along the projection, is injective.*

Proof: Let us remark first that by injectivity we mean here that the maps $\tilde{A}_v : T_{\tau(v)}M \to T_{\tau(v)}M$ are injective for all $v \in TM$. However, it will be convenient to verify the statement only at vector field level. Suppose that g is E-regular, i.e. g_E is non-degenerate but $\tilde{A}$ is not injective. Then there exist distinct vector fields $\tilde{X}, \tilde{Y}$ along τ such that $\tilde{A}(\tilde{X}) = \tilde{A}(\tilde{Y})$. So, $g\big(\tilde{A}(\tilde{X}) - \tilde{A}(\tilde{Y}), \tilde{Z}\big) = g_E(\tilde{X} - \tilde{Y}, \tilde{Z})$ for all $\tilde{Z} \in \mathcal{X}(\tau)$. It follows that $\tilde{X} - \tilde{Y} = 0$, but this is a contradiction.

Conversely, let $\tilde{A}$ be injective. If $0 = g_E(\tilde{X}, \tilde{Y}) = g\big(\tilde{A}(\tilde{X}), \tilde{Y}\big)$ for all $\tilde{Y}$ in $\mathcal{X}(\tau)$, then, due to the non-degeneracy of g, $\tilde{A}(\tilde{X}) = 0$, and consequently $\tilde{X} = 0$. This concludes the proof. $\square$

Proposition 1. *If g is a variational metric along τ, then we have for the vector one-form $\tilde{A}$ the formula*

$$\tilde{A}(\tilde{X}) = \tilde{X} + 2\mathcal{C}(\tilde{X}, \delta) + \frac{1}{2}(\nabla^v_{\tilde{X}}\mathcal{C})(\delta, \delta) + \frac{1}{2}\mathcal{C}\big(\tilde{X}, \mathcal{C}(\delta, \delta)\big). \tag{36}$$

PROOF: We may restrict ourselves to basic vector fields. By (34), Lemma 4, and the total symmetry of $\mathcal{C}_\flat$, for any vector fields X, Y on M we have

$$g(\tilde{A}(\hat{X}), \hat{Y}) = g(\hat{X}, \hat{Y}) + 2g(\mathcal{C}(\hat{X}, \delta), \hat{Y}) + \frac{1}{2}g\big(\mathcal{C}(\hat{X}, \delta), \mathcal{C}(\hat{Y}, \delta)\big) + \frac{1}{2}g\big((\nabla^v_{\hat{X}}\mathcal{C})(\hat{Y}, \delta), \delta\big).$$

Next, taking the X^v-derivative of the identity $g\big(\mathcal{C}(\hat{Y}, \delta), \delta\big) = g\big(\mathcal{C}(\delta, \delta), \hat{Y}\big)$, it can be easily shown that

$$g\big((\nabla^v_{\hat{X}}\mathcal{C})(\hat{Y}, \delta), \delta\big) = g\big((\nabla^v_{\hat{X}}\mathcal{C})(\delta, \delta), \hat{Y}\big) + g\big(\mathcal{C}\big(\hat{X}, \mathcal{C}(\delta, \delta)\big), \hat{Y}\big) - g\big(\mathcal{C}(\hat{X}, \delta), \mathcal{C}(\hat{Y}, \delta)\big).$$

Replacing this expression of $g\big((\nabla^v_{\hat{X}}\mathcal{C})(\hat{Y}, \delta), \delta\big)$ into the above relation, the non-degeneracy of g leads to the desired formula. $\square$

Corollary 2. *A variational metric g is variational with respect to its own absolute energy E if $\tilde{A} = 1_{\mathcal{X}(\tau)}$, or*

$$2\mathcal{C}(\tilde{X}, \delta) + \frac{1}{2}(\nabla^v_{\tilde{X}}\mathcal{C})(\delta, \delta) + \frac{1}{2}\mathcal{C}\big(\mathcal{C}(\delta, \delta), \tilde{X}\big) = 0 \textit{ for all } \tilde{X} \in \mathcal{X}(\tau).$$

As any regular Lagrangian via the Euler-Lagrange equation (26), an E-regular metric also generates a semispray ξ_E by the relation $i_{\xi_E}\omega_E = -d(CE - E)$. The crucial factor here is the presence of a symplectic form (in our case ω_E). Next, we will introduce a new class of metrics, to which a symplectic form can also be associated. First we relate a metric g with its Lagrange two-form ω_g utilizing the first Cartan tensor.

Lemma 7. *For any vector fields ξ, η on TM we have*

$$\omega_g(J\xi, \eta) = g(\mathbf{j}\xi, \mathbf{j}\eta) + g\big(\mathcal{C}(\mathbf{j}\xi, \delta), \mathbf{j}\eta\big). \tag{37}$$

PROOF: Due to our first local basis principle, we can assume that $\xi = X^c$, $X \in \mathcal{X}(M)$; then $J\xi = X^v$, $\mathbf{j}\xi = \hat{X}$. If $\eta = Y^v$, $Y \in \mathcal{X}(M)$, then the right-hand side of (37) vanishes automatically, while $\omega_g(X^v, Y^v) = d\theta_g(X^v, Y^v) = X^v\theta_g(Y^v) - Y^v\theta_g(X^v) - \theta_g([X^v, Y^v]) = X^v g(\mathbf{j}Y^v, \delta) - Y^v g(\mathbf{j}X^v, \delta) = 0$. So it remains to check the statement for an η of the form Y^c. A straightforward calculation leads to the result: $\omega_g(X^v, Y^c) = X^v\theta_g(Y^c) - Y^c\theta_g(X^v) - \theta_g([X^v, Y^c]) = X^v g(\hat{Y}, \delta) - Y^c g(\mathbf{j}X^v, \delta) - g(\mathbf{j}[X, Y]^v, \delta) = X^v g(\hat{Y}, \delta) = (\nabla^v_{\hat{X}}g)(\hat{Y}, \delta) + g(\hat{Y}, \hat{X}) = g\big(\mathcal{C}(\hat{X}, \hat{Y}), \delta\big) + g(\hat{Y}, \hat{X}) = g(\hat{X}, \hat{Y}) + g\big(\mathcal{C}(\hat{X}, \delta), \hat{Y}\big)$. $\square$

Compared with (35), Lemma 7 brings the map

$$\tilde{B} : \tilde{X} \in \mathcal{X}(\tau) \mapsto \tilde{B}(\tilde{X}) := \tilde{X} + \mathcal{C}(\tilde{X}, \delta) \tag{38}$$

into the spotlight. Obviously, $\tilde{B}$ is a type (1,1) tensor field, or, equivalently, a vector one-form along τ. Using our new ingredient we have

$$\omega_g(J\xi, \eta) = g(\tilde{B}(\mathbf{j}\xi), \mathbf{j}\eta) \qquad \text{for all } \xi, \eta \in \mathcal{X}(TM). \tag{39}$$

Proposition 2. *The Lagrange two-form ω_g of a metric g is non-degenerate if, and only if, the* (1,1) *tensor field $\tilde{B}$ defined by* (38) *is injective.*

Proof: For the sake of simplicity, we shall again give a proof only at the level of vector fields. The reasoning at the level of fibres is analogous and left to the reader. Suppose that ω_g is non-degenerate, but $\tilde{B}$ is not injective. Then there exists a non-zero vector field $\tilde{X}_0$ along τ such that $\tilde{B}(\tilde{X}_0) = 0$. $\tilde{X}_0$ may be represented in the form $\mathbf{j}\xi_0$, $\xi_0 \in \mathcal{X}(TM)$. So for all $\eta \in \mathcal{X}(TM)$ we have $0 = g\big(\tilde{B}(\tilde{X}_0), \mathbf{j}\eta\big) \overset{(39)}{=} \omega_g(J\xi_0, \eta)$. Thus $J\xi_0 = \mathbf{ij}\xi_0 = 0$, which is a contradiction, since $\mathbf{j}\xi_0 = \tilde{X}_0 \neq 0$.

Conversely, let $\tilde{B}$ be injective. Then $\tilde{B}$ is surjective as well. Therefore, if $0 = \omega_g(\xi_0, J\eta) \overset{(39)}{=} -g\big(\tilde{B}(\mathbf{j}\eta), \mathbf{j}\xi_0\big)$ for all $\eta \in \mathcal{X}(TM)$, then also $g(\tilde{X}, \mathbf{j}\xi_0) = 0$ for all $\tilde{X} \in \mathcal{X}(\tau)$. Non-degeneracy of g implies that ξ_0 must be vertical, and hence of the form $J\xi_1$, for a certain $\xi_1 \in \mathcal{X}(TM)$. Thus, finally, $0 = \omega_g(J\xi_1, \eta) = g(\mathbf{j}\xi_1, \mathbf{j}\eta)$ for all $\eta \in \mathcal{X}(TM)$, which implies $\mathbf{j}\xi_1 = 0$ and so $\xi_0 = \mathbf{ij}\xi_1 = 0$. This concludes the proof of the converse statement. □

It can be checked immediately that $\tilde{B}$ is injective (and therefore surjective) if, and only if, in any induced chart of type (2) we have

$$det\Big((\delta_i^j) + \Big(y^l \frac{\partial g_{lk}}{\partial y^i} g^{jk}\Big)\Big) \neq 0 \qquad \Big((g_{ij}) = \Big(g\Big(\frac{\hat{\partial}}{\partial u^i}, \frac{\hat{\partial}}{\partial u^j}\Big)\Big), (g^{ij}) = (g_{ij})^{-1}\Big).$$

This local condition for a generalized metric was introduced by R. Miron in [18], so we shall use the following terminology:

Definition 3. A metric g along τ is said to be *Miron regular*, if the map $\tilde{B} : \tilde{X} \in \mathcal{X}(\tau) \mapsto \tilde{B}(\tilde{X}) := \tilde{X} + \mathcal{C}(\tilde{X}, \delta)$ is injective (in the sense that the linear maps $\tilde{B}_v : T_{\tau(v)}M \to T_{\tau(v)}M$ are injective for all $v \in TM$).

Corollary 3. *For a variational metric $g = \nabla^v\nabla^v L$ the condition of Miron-regularity coincides with the condition that the energy E_L associated to L is a regular Lagrangian.*

Indeed, this is an immediate consequence of Lemma 5 and Proposition 2. We can push this idea a little bit further: *given a metric g along τ, when does there exist a smooth function L_g on TM, such that $\theta_{L_g} = \theta_g$?* In such a case, Miron-regularity will turn out to be equivalent with the regularity of the Lagrangian L_g. If the Lagrangian L_g is regular, then we can associate a semispray ξ_g to it, by means of (26). Our next answer clarifies the meaning of weak variationality introduced in Definition 2.

Proposition 3. *For a metric g there exists a Lagrangian L_g on TM such that $\theta_g = \theta_{L_g}$ if, and only if, g is weakly variational, i.e. $g\big(\mathcal{C}(\tilde{X},\tilde{Y}),\delta\big) = g\big(\mathcal{C}(\tilde{Y},\tilde{X}),\delta\big)$ for all vector fields $\tilde{X},\tilde{Y}$ along τ.*

Proof: Observe first that for any function F in $C^\infty(TM)$ we have $\theta_F(\eta) := (d_J F)(\eta) \overset{(10)}{=} (J\eta)F = \mathbf{i}(\mathbf{j}\eta)F = \nabla^v F(\mathbf{j}\eta)$ ($\eta \in \mathcal{X}(TM)$). Thus the question 'when is $\theta_g = \theta_{Lg}$ for a Lagrangian L_g?' is equivalent with the question 'when does a Lagrangian L_g exist, such that $\tilde{\theta}_g = \nabla^v L_g$?'. But the latter question has been answered in Lemma 2: if, and only if,

$$(40) \qquad (\nabla^v\tilde{\theta}_g)(\tilde{X},\tilde{Y}) = (\nabla^v\tilde{\theta}_g)(\tilde{Y},\tilde{X}) \qquad \text{for all } \tilde{X},\tilde{Y} \in \mathcal{X}(\tau).$$

Since $(\nabla^v\tilde{\theta}_g)(\tilde{X},\tilde{Y}) = (\nabla^v_{\tilde{X}}\tilde{\theta}_g)(\tilde{Y}) = (\mathbf{i}\tilde{X})[\tilde{\theta}_g(\tilde{Y})] - \tilde{\theta}_g(\nabla^v_{\tilde{X}}\tilde{Y}) = (\mathbf{i}\tilde{X})[g(\tilde{Y},\delta)] - g(\nabla^v_{\tilde{X}}\tilde{Y},\delta) \overset{(18)}{=} (\nabla^v_{\tilde{X}}g)(\tilde{Y},\delta) + g(\tilde{Y},\tilde{X}) = g\big(\mathcal{C}(\tilde{X},\tilde{Y}),\delta\big) + g(\tilde{X},\tilde{Y})$, and in the same way $(\nabla^v\tilde{\theta}_g)(\tilde{Y},\tilde{X}) = g\big(\mathcal{C}(\tilde{Y},\tilde{X}),\delta\big) + g(\tilde{X},\tilde{Y})$, it follows that (40) holds if, and only if, g is weakly variational. □

From our reasoning we infer immediately

Corollary 4. *A generalized metric is weakly variational if, and only if, its Lagrange one-form is ∇^v-exact.*

Notice that if g is weakly variational and hence $\theta_g = \theta_{L_g}$ ($L_g \in C^\infty(TM)$), then we also have $\omega_g = \omega_{L_g}$. By Lemma 5 any variational metric $g = \nabla^v\nabla^v L$ is weakly variational with the Lagrange one-form $\theta_g = \theta_{E_L}$. It is easy to recognize where the idea of the Lagrangian L_g is coming from. The definition (39) of the tensor $\tilde{B}$ strongly resembles the definition (35) of the tensor $\tilde{A}$. Inspired by this analogy, let us define a type (0,2) tensor field γ_g along τ, associated to $\tilde{B}$ in such a way that it plays the same role as g_E in (34). To make a long story short, let

$$(41) \qquad \gamma_g(\tilde{X},\tilde{Y}) := g\big(\tilde{B}(\tilde{X}),\tilde{Y}\big) \qquad \text{for all } \tilde{X},\tilde{Y} \in \mathcal{X}(\tau).$$

When does γ_g become a metric? The answer is conveyed in the next

Proposition 4. *For a generalized metric g along τ the following conditions are equivalent:*

(i) The type (0,2) tensor field γ_g given by (41) is a metric.

(ii) g is Miron-regular and weakly variational.

(iii) γ_g is the Hessian of a regular Lagrangian.

Proof: (i)⇒(ii) If γ_g is a metric, then its symmetry implies immediately that g is weakly variational. Following the same line of reasoning as in the proof of Lemma 6, we can conclude that non-degeneracy of γ_g implies that $\tilde{B}$ is injective, i.e. g is Miron-regular.
(ii)⇒(iii) If g is weakly variational, then $\tilde{\theta}_g$ is ∇^v-exact by Corollary 4, i.e. $\tilde{\theta}_g = \nabla^v L_g$, $L_g \in C^\infty(TM)$. Thus for any basic vector fields $\hat{X},\hat{Y}$ we have $\gamma_g(\hat{X},\hat{Y}) = g\big(B(\hat{X}),\hat{Y}\big) \overset{(39)}{=} \omega_g(X^v,Y^c) = \omega_{L_g}(X^v,Y^c) \overset{(25)}{=} \nabla^v\nabla^v L_g(\hat{X},\hat{Y})$, so γ_g is the Hessian of the Lagrangian L_g. Since g is Miron-regular, $\omega_g = \omega_{L_g}$ is non-degenerate, therefore L_g is a regular Lagrangian.
(iii)⇒(i) This is clear by Corollary 1. □

Corollary 5. *For a weakly variational metric the tensor $\tilde{A}$ also takes the form (36).*

Indeed, analyzing the proof of Proposition 1, we find that it relies only on the weak variationality of the metric.

Now we have a look at the class of variational *and* Miron-regular metrics. Metrics enjoying these two properties at the same time will be called *semi-Finsler metrics.* If g is a semi-Finsler metric, then the function $\tilde{E}_L := CE_L - E_L$ is said to be the *principal energy* associated to g. Notice, that $d_J\tilde{E}_L \neq 0$, and hence $d\tilde{E}_L \neq 0$. Indeed, since g is Miron-regular, the Lagrange two-form ω_g is non-degenerate by Proposition 2, therefore $0 \neq i_C\omega_g = i_C d\theta_g \overset{\text{Lemma 5}}{=} i_C dd_J E_L = -i_C d_J dE \overset{(14)}{=} -i_J dE_L + d_J i_C dE_L = -d_J E_L + d_J(CE_L) = d_J\tilde{E}_L$.

Proposition 5. *Let $g = \nabla^v\nabla^v L$ be a semi-Finsler metric along τ. There exists a unique semispray ξ_g on M such that*

$$i_{\xi_g}\omega_g = -d\tilde{E}_L, \qquad \tilde{E}_L := CE_L - E_L.$$

ξ_g is just the Lagrangian vector field for the Lagrangian E_L, i.e.

$$i_{\xi_g}\omega_{E_L} = -d\tilde{E}_L, \qquad E_L := CL - L. \tag{42}$$

Proof: By Corollary 3, E_L is indeed a regular Lagrangian, so the general result quoted in section 3 guarantees the existence and uniqueness of the Lagrangian vector field ξ_{E_L} satisfying $i_{\xi_{E_L}}\omega_{E_L} = -d\tilde{E}_L$ (cf. (26)). We have remarked above (after Corollary 4) that $\omega_{E_L} = \omega_g$; this concludes the proof. □

ξ_g is called the *canonical semispray* of the semi-Finsler metric g. If $g = \nabla^v\nabla^v L$ is a semi-Finsler metric, then, as a consequence of Corollary 1, L is also a regular Lagrangian. So we have another semispray on M, the Lagrangian vector field ξ_L for L. Next we establish an important relation between ξ_L and canonical semispray ξ_g.

Proposition 6. *Let $g = \nabla^v\nabla^v L$ be a semi-Finsler metric, ξ_g the canonical semispray for g, ξ_L the Lagrangian vector field for L. The difference $\xi := \xi_g - \xi_L$ is the unique (necessarily vertical) vector field on TM such that*

$$i_\xi\omega_g = i_{[C,\xi_L]-\xi_L}\omega_L, \tag{43}$$

where ω_g and ω_L are the Lagrange two-forms associated to g and to L, respectively.

Proof: On the one hand we have $i_{[C,\xi_L]}\omega_L = d_C i_{\xi_L}\omega_L - i_{\xi_L}d_C\omega_L \overset{(26)}{=} -d_C dE_L - i_{\xi_L}(i_C d\omega_L + di_C\omega_L) \overset{(24)}{=} -d_C dE_L - i_{\xi_L} dd_J E_L \overset{\text{Lemma 5}}{=} -d_C dE_L - i_{\xi_L}\omega_g$. On the other hand, $i_{\xi_g}\omega_g \overset{(42)}{=} -d(CE_L - E_L) = -dd_C E_L + dE_L \overset{(26)}{=} -d_C dE_L - i_{\xi_L}\omega_L$ and therefore $i_{[C,\xi_L]}\omega_L - i_{\xi_g}\omega_g = i_{\xi_L}\omega_L - i_{\xi_L}\omega_g$, i.e. $i_{\xi_g-\xi_L}\omega_g = i_{[C,\xi_L]-\xi_L}\omega_L$, as was to be shown. □

Using formulae (25), (39) and (43), it is clear that if, $\xi = \mathbf{i}\tilde{X}$, $\tilde{X} \in \mathcal{X}(\tau)$, then (due to the injectivity of $\tilde{B}$) this $\tilde{X}$ is the unique vector field along τ such that $\tilde{B}(\tilde{X}) = \mathbf{j}[C, \xi_L] - \delta$.

Corollary 6. *If $g = \nabla^v\nabla^v L$ is a semi-Finsler metric, then the Lagrangian vector field for L is a spray if, and only if, it coincides with the canonical semispray for g.* $\square$

Next we turn to those special weakly variational metrics which are distinguished by the property $\tilde{\theta}_g = \nabla^v E = \frac{1}{2}\nabla^v\big(g(\delta,\delta)\big)$, or, equivalently, for which $\theta_g = \theta_E$.

Lemma 8. $(\theta_E - \theta_g)(X^c) = \frac{1}{2}\mathcal{C}_\flat(\hat{X},\delta,\delta)$ *for all $X \in \mathcal{X}(M)$, therefore $\theta_g = \theta_E$ if, and only if, $\mathcal{C}_\flat(.,\delta,\delta)$, i.e. (see Definition 2) if g is weakly normal.*

Proof: $(\theta_E - \theta_g)(X^c) = d_J E(X^c) - g(\mathbf{j}X^c,\delta) = X^v E - g(\hat{X},\delta) = \frac{1}{2}X^v\big(g(\delta,\delta)\big) - g(\nabla^v_{\hat{X}}\delta,\delta) = \frac{1}{2}(\nabla^v_{\hat{X}}g)(\delta,\delta) = \frac{1}{2}\mathcal{C}_\flat(\hat{X},\delta,\delta)$. $\square$

Proposition 7. *Suppose that g is a weakly normal metric. Then*

$$\gamma_g = g_E, \text{ therefore } \tilde{A} = \tilde{B}; \tag{44}$$

$$g_E(\tilde{X},\delta) = g(\tilde{X},\delta) = \tilde{\theta}_g(\tilde{X}) \qquad \text{for all } \tilde{X} \in \mathcal{X}(\tau), \tag{45}$$

and the absolute energy $E = \frac{1}{2}g(\delta,\delta)$ is homogeneous of degree 2, i.e. $CE = 2E$.

Proof: Since $\theta_g = \theta_E$ by Lemma 8, we also have $\omega_g = d\theta_g = d\theta_E = \omega_E$. Thus, for all vector fields X, Y on M, we obtain $\gamma_g(\hat{X},\hat{Y}) := g\big(\tilde{B}(\hat{X}),\hat{Y}\big) \overset{(39)}{=} \omega_g(X^v,Y^c) = \omega_E(X^v,Y^c) = X^v(Y^v E) \overset{(23)}{=} (\nabla^v\nabla^v E)(\hat{X},\hat{Y}) = g_E(\hat{X},\hat{Y})$. Hence $\gamma_g = g_E$, which implies by (34) and (41) that $\tilde{A} = \tilde{B}$. The verification of (45) is also easy. Let ξ_0 be a semispray on M. Then $g_E(\hat{X},\delta) = g_E(\mathbf{j}X^c,\mathbf{j}\xi_0) \overset{Lemma\,3}{=} \omega_E(X^v,\xi_0) = \omega_g(X^v,\xi_0) \overset{(39)}{=} g\big(\tilde{B}(\hat{X}),\delta\big) = g(\hat{X},\delta) + g\big(\mathcal{C}(\hat{X},\delta),\delta\big) = g(\hat{X},\delta) =: \tilde{\theta}_g(\hat{X})$, since $g\big(\mathcal{C}(\hat{X},\delta),\delta\big) = \mathcal{C}_\flat(\hat{X},\delta,\delta) = 0$ by our assumption. This proves (45). Finally, $CE = \frac{1}{2}\mathbf{i}\delta\big(g(\delta,\delta)\big) = \frac{1}{2}\big((\nabla^v_\delta g)(\delta,\delta) + 2g(\nabla^v_\delta\delta)\big) = \frac{1}{2}\mathcal{C}_\flat(\delta,\delta,\delta) + g(\delta,\delta) = 2E$. $\square$

R. Miron studied (in our terminology) Miron-regular and weakly normal metrics. In [18], he gives a coordinate expression of a horizontal map. This horizontal map is clearly generated by the semispray ξ_E, in this case defined by $i_{\xi_E}\omega_E = -dE$. In fact, we can prove that ξ_E is a spray. Let us calculate first $d_C\theta_E = i_C d\theta_E + d i_C\theta_E = i_C\omega_E \overset{(24),Prop.\,7}{=} d_J E = \theta_E$ and $d_C\omega_E \overset{(11)}{=} dd_C\theta_E = d\theta_E = \omega_E$. It follows that $i_{[C,\xi_E]}\omega_E = d_C i_{\xi_E}\omega_E - i_{\xi_E}d_C\omega_E = -d_C dE - i_{\xi_E}\omega_E = -d(CE) - i_{\xi_E}\omega_E = 2i_{\xi_E}\omega_E - i_{\xi_E}\omega_E = i_{\xi_E}\omega_E$, and thus $\xi_E = [C,\xi_E]$.

Corollary 7. *If g is both variational ($g = \nabla^v\nabla^v L$) and weakly normal, then E and E_L differ only in a vertical lift.*

Proof: For a variational metric $g = \nabla^v\nabla^v L$ we have $\theta_g = d_J E_L$ by Lemma 5. Since $\theta_g = \theta_E = d_J E$ by Lemma 8, we get $d_J(E - E_L) = 0$. This implies easily (see e.g. [22], 2.31) that $E - E_L$ is the vertical lift of a smooth function on M. $\square$

In the cases of a variational metric w.r.t. L, a weakly variational and Miron-regular metric w.r.t. L_g and an E-regular metric, respectively, equation (26) guarantees the existence of a unique 'canonical' semispray, which we denote by ξ_L, ξ_g and ξ_E. As can be seen from

(8), these semisprays generate a corresponding horizontal map, that we will denote by $\mathcal{H}$ (in all cases).

It is well-known in the study of semisprays that an important role is played by the *Jacobi endomorphism*, Φ and the *Lie derivative with respect to* ξ, $\mathcal{L}^h_\xi$. The first is a (1,1) tensor field along τ: if ξ is a fixed semispray, then $\Phi(\tilde X) := \mathcal{V}[\xi, \mathcal{H}\tilde X]$. The second object is a derivation. In the same way as e.g. the definitions for $\nabla^v_{\tilde X}$ and $\nabla^h_{\tilde X}$ were cast (for $\tilde X \in \mathcal{X}(\tau)$), it is only necessary to prescribe the action of $\mathcal{L}^h_\xi$ on functions, vector fields and one-forms:

$$\mathcal{L}^h_\xi f = \xi(f), \qquad \mathcal{L}^h_\xi \tilde X = \mathbf{j}[\xi, \mathcal{H}\tilde X], \qquad \mathcal{L}^h_\xi \tilde\alpha(\tilde X) = \mathcal{L}^h_\xi(\tilde\alpha(\tilde X)) - \tilde\alpha(\mathcal{L}^h_\xi \tilde X)$$

for any $f \in C^\infty(TM)$, $\tilde X \in \mathcal{X}(\tau)$ and $\tilde\alpha \in \mathcal{A}^1(\tau)$. More details about this horizontal Lie derivative are available in [22].

Suppose now that a variational metric $g = \nabla^v\nabla^v L$ is also given. It is possible to verify directly when a given semispray ξ is the Lagrangian vector field of L.

Result 2. *[5, 17] Let g be a variational metric with respect to L and let a semispray ξ be given. ξ is the semispray associated to L by means of (26) if, and only if,*

$$g(\Phi(\tilde X), \tilde Y) = g(\tilde X, \Phi(\tilde Y)) \qquad \textit{and} \qquad \mathcal{L}^h_\xi g = 0, \tag{46}$$

for all $\tilde X, \tilde Y \in \mathcal{X}(\tau)$.

The requirement that g is variational, together with the conditions (46) are called *the Helmholtz conditions*. A similar version of this proposition can be found in [5], although it is cast there in the setup of time-dependent Lagrangians. However, only minor modifications on that proof lead to the statement of our proposition. It is possible to find an analogue of this statement for an E-regular metric or for a weakly variational and Miron-regular metric.

Corollary 8. *Let g be an E-regular metric and let a semispray ξ be given. ξ is the semispray associated to E if, and only if,*

$$g(\tilde A(\Phi(\tilde X)), \tilde Y) = g(\tilde X, \tilde A(\Phi(\tilde Y))) \quad \textit{and} \quad (\mathcal{L}^h_\xi g)(\tilde A(\tilde X), \tilde Y) = -g((\mathcal{L}^h_\xi \tilde A)(\tilde X), \tilde Y), \tag{47}$$

for all $\tilde X, \tilde Y \in \mathcal{X}(\tau)$.

Proof: In this case ξ must obey the conditions of the previous result with respect to the variational metric g_E. Using (34), the first condition in (47) follows immediately, while also $0 = (\mathcal{L}^h_\xi g_E)(\tilde X, \tilde Y) = \mathcal{L}^h_\xi\big(g_E(\tilde X, \tilde Y)\big) - g_E(\mathcal{L}^h_\xi \tilde X, \tilde Y) - g_E(\tilde X, \mathcal{L}^h_\xi \tilde Y) \overset{(34)}{=} \mathcal{L}^h_\xi\big(g(\tilde A(\tilde X), \tilde Y)\big) - g(\tilde A(\mathcal{L}^h_\xi \tilde X), \tilde Y) - g(\tilde A(\tilde X), \mathcal{L}^h_\xi \tilde Y) = (\mathcal{L}^h_\xi g)(\tilde A(\tilde X), \tilde Y) - g(\tilde A(\mathcal{L}^h_\xi \tilde X), \tilde Y) + g(\mathcal{L}^h_\xi\big(\tilde A(\tilde X)\big), \tilde Y) = (\mathcal{L}^h_\xi g)(\tilde A(\tilde X), \tilde Y) + g((\mathcal{L}^h_\xi \tilde A)(\tilde X), \tilde Y)$. □

Corollary 9. *Let g be a Miron-regular and weakly variational metric w.r.t. L_g and let a semispray ξ be given. ξ is the semispray associated to L_g if, and only if,*

$$g(\tilde B(\Phi(\tilde X)), \tilde Y) = g(\tilde X, \tilde B(\Phi(\tilde Y))) \quad \textit{and} \quad (\mathcal{L}^h_\xi g)(\tilde B(\tilde X), \tilde Y) = -g((\mathcal{L}^h_\xi \tilde B)(\tilde X), \tilde Y), \tag{48}$$

for all $\tilde X, \tilde Y \in \mathcal{X}(\tau)$.

Proof: We can follow the same line of reasoning as in the previous corollary, if we substitute γ_g for g_E and $\tilde{B}$ for $\tilde{A}$. $\square$

The last corollary can be useful because, even if we do not know exactly the Lagrangian L_g, we can always check the conditions (48) on a given semispray to conclude if it is indeed the canonical semispray of the given weakly variational and Miron-regular metric.

We are now ready to recover Finsler geometry.

Proposition 8. *A metric is variational with respect to a homogeneous function of degree 2 if, and only if, it is normal, i.e.* $\mathcal{C}(\tilde{X}, \delta) = 0$ *for all* $\tilde{X} \in \mathcal{X}(\tau)$.

Proof: Suppose first that $g = \nabla^v \nabla^v L$, where $CL = 2L$. Then for all vector fields X, Y on M we have $2X^v(Y^v L) = X^v\big(Y^v(CL)\big) = X^v\big([Y^v, C]L + C(Y^v L)\big) = X^v(Y^v L) + [X^v, C](Y^v L) + C\big(X^v(Y^v L)\big) = 2X^v(Y^v L) + C\big(X^v(Y^v L)\big)$, hence $0 = C\big(X^v(Y^v L)\big) = C\big(\nabla^v\nabla^v L(\hat{X}, \hat{Y})\big) = C\big(g(\hat{X}, \hat{Y})\big) = (\nabla^v_\delta g)(\hat{X}, \hat{Y}) = g\big(\mathcal{C}(\delta, \hat{X}), \hat{Y}\big) \overset{Res.1}{=} g\big(\mathcal{C}(\hat{X}, \delta), \hat{Y}\big)$; so we obtain $\mathcal{C}(., \delta) = 0$. Conversely, if g is normal, then $\tilde{B} = 1_{\mathcal{X}(\tau)}$ by (38), and hence $\tilde{A} = 1_{\mathcal{X}(\tau)}$ by Proposition 7. Thus $g = \gamma_g = g_E = \nabla^v\nabla^v E$, proving that g is variational. Here the energy E is homogeneous of degree two, also by Proposition 7. $\square$

Corollary 10. *For a homogeneous metric, variationality and normality are equivalent conditions.*

Before going on, let us see an immediate way to obtain a normal metric.

Corollary 11. *If g is a weakly normal and E-regular (or, equivalently, Miron-regular) metric, then g_E is normal.*

Proof: Since g is weakly normal, we have $CE = 2E$. The E-regularity of g implies, that g_E is a variational metric, namely $g_E = \nabla^v\nabla^v E$, so Proposition 8 leads to the desired conclusion. $\square$

As stated in the Introduction, a *Finsler metric* is the Hessian of a Lagrangian E. To be more precise: E is assumed to be of class C^1 on TM, smooth on $\overset{\circ}{T}M$, positive-homogeneous of degree 2, and, of course, it is required that the two-form $dd_J E$ is non-degenerate. As is well-known, the weakening of the differentiability is necessary: if E is (at least) of class C^2 on the whole tangent manifold TM, then the first Cartan tensor of g_E vanishes, and g_E comes from a pseudo-Riemannian metric on M (see (30)). It follows from our above considerations that if g is a metric in the pull-back bundle of τ by the map $\overset{\circ}{\tau} : \overset{\circ}{T}M \to M$, and satisfies the normality condition $\mathcal{C}(., \delta) = 0$, then g is a Finsler metric, and (M, E), $E := \frac{1}{2}g(\delta, \delta)$, is a *Finsler manifold* in the usual sense. (The only, minor difficulty is to check that E has a (unique) C^1-extension into TM; for this technicality we refer to [22].)

Thus, roughly speaking, *normal metrics lead to the territory of Finsler geometry.* According to our preceding remark, when we speak of a normal metric g, we shall always assume (at least tacitly) that g lives in the pull-back bundle $\overset{\circ}{\tau}{}^*\tau$.

We end this section with a summarizing tabular.

metric	no condition	weakly variational	weakly normal	variational	variational w.r.t. E	normal
g	-	-	-	$\exists\, L$	$L = E$	$L = E$
γ_g	-	$\exists\, L_g$	$L_g = E$	$L_g = E_L$	$L_g = CE - E$	$L_g = E$
g_E	E	E	$E = L_g$	$E = \frac{1}{2} CE_L$	$E = \frac{1}{2}(CCE - CE)$	$E = E$

Comment. For an arbitrary generalized metric it is possible that its absolute energy E is regular. When we start to impose restrictions on the metric, we get two main subclasses: the class of variational metrics (meaning that there should exist a Lagrangian L) and the class of weakly variational metrics (providing the existence of a Lagrangian L_g via Proposition 4). The regularity of the Lagrangian L is guaranteed; that of L_g has to be imposed (Miron-regularity, see Proposition 4). For each class we can consider a subcategory, consisting of those special metrics where the Lagrangian (L or L_g) is exactly the absolute energy E, so we get metrics that are variational with respect to E (column 6) and weakly normal metrics (column 4). In these cases, the regularity of L and L_g coincides with the regularity of E. The two subclasses have a non-empty intersection with each other, constituted by those metrics that are both variational with respect to E and weakly variational with respect to E. In the intersection, $L_g = CE - E$ should also be E, and thus E is homogeneous of degree 2. In accordance with Proposition 8, we mention here the metrics that are variational with respect to a homogeneous function of degree 2, normal (column 7).

5 The metric derivatives

In this section we will give an elegant formulation for all *metric derivatives*. We assume that g is a metric along τ and that a horizontal map $\mathcal{H} : \mathcal{X}(\tau) \to \mathcal{X}(TM)$ is specified. A covariant derivative operator D in $\tau^*\tau$ is said to be *metric* if $\xi g(\tilde{Y}, \tilde{Z}) = g(D_\xi \tilde{Y}, \tilde{Z}) + g(\tilde{Y}, D_\xi \tilde{Z})$ for all $\xi \in \mathcal{X}(TM)$ and $\tilde{Y}, \tilde{Z} \in \mathcal{X}(\tau)$. Equivalently, once a horizontal map $\mathcal{H}$ is specified, D is metric if it is both v-metric and h-metric, i.e. if we have $\mathbf{i}\tilde{X} g(\tilde{Y}, \tilde{Z}) = g(D^v_{\tilde{X}} \tilde{Y}, \tilde{Z}) + g(\tilde{Y}, D^v_{\tilde{X}} \tilde{Z})$ and $\mathcal{H}\tilde{X} g(\tilde{Y}, \tilde{Z}) = g(D^h_{\tilde{X}} \tilde{Y}, \tilde{Z}) + g(\tilde{Y}, D^h_{\tilde{X}} \tilde{Z})$, where $D^v_{\tilde{X}} \tilde{Y} := D_{\mathbf{i}\tilde{X}} \tilde{Y}$, $D^h_{\tilde{X}} \tilde{Y} = D_{\mathcal{H}\tilde{X}} \tilde{Y}$, $\tilde{X} \in \mathcal{X}(\tau)$. Applying the covariant exterior derivative d^D with respect to D, we now define the *torsions* of D. If $K : \mathcal{X}(TM) \to \mathcal{X}(\tau)$ is a $\tau^*\tau$-valued one-form on TM, then

$$(49) \qquad d^D K(\xi, \eta) := D_\xi (K\eta) - D_\eta (K\xi) - K[\xi, \eta]; \qquad \xi, \eta \in \mathcal{X}(TM).$$

(A general definition of d^D in the context of vector bundles can be found in [10]). The canonical map $\mathbf{j}$ and the vertical map $\mathcal{V}$ belonging to $\mathcal{H}$ may be regarded as $\tau^*\tau$-valued one-forms on TM; their covariant exterior derivatives $T^v(D) := d^D \mathcal{V}$ and $T^h(D) := d^D \mathbf{j}$ are said to be the *vertical* and the *horizontal torsion* of D, respectively (although $T^h(D)$ does not depend on any horizontal structure). With the help of $T^h(D)$, we define the *h-horizontal torsion* $\mathcal{T}$ and the *h-mixed torsion* $\mathcal{S}$ of D by

$$(50)\qquad \mathcal{T}(\tilde{X},\tilde{Y}) := T^h(D)(\mathcal{H}\tilde{X},\mathcal{H}\tilde{Y}) \overset{(49)}{=} D_{\mathcal{H}\tilde{X}}\tilde{Y} - D_{\mathcal{H}\tilde{Y}}\tilde{X} - \mathbf{j}[\mathcal{H}\tilde{X},\mathcal{H}\tilde{Y}] \qquad \text{and}$$

$$(51)\quad \mathcal{S}(\tilde{X},\tilde{Y}) := T^h(D)(\mathcal{H}\tilde{X},\mathbf{i}\tilde{Y}) \overset{(49)}{=} -D_{\mathbf{i}\tilde{Y}}\tilde{X} - \mathbf{j}[\mathcal{H}\tilde{X},\mathbf{i}\tilde{Y}] \overset{(17)}{=} -D_{\mathbf{i}\tilde{Y}}\tilde{X} + \nabla_{\mathbf{i}\tilde{Y}}\tilde{X}.$$

Following [9], a covariant derivative D in $\tau^*\tau$ is called *symmetric*, if $\mathcal{T}=0$ and $\mathcal{S}$ is symmetric. Using $T^v(D)$, it is possible to define three more partial torsions of D, the *v-horizontal torsion* $\mathcal{R}^1$, the *v-mixed torsion* $\mathcal{P}^1$, and the *v-vertical torsion* $\mathcal{Q}^1$:

$$(52)\qquad \mathcal{R}^1(\tilde{X},\tilde{Y}) := T^v(D)(\mathcal{H}\tilde{X},\mathcal{H}\tilde{Y}) = -\mathcal{V}[\mathcal{H}\tilde{X},\mathcal{H}\tilde{Y}],$$

$$(53)\qquad \mathcal{P}^1(\tilde{X},\tilde{Y}) := T^v(D)(\mathcal{H}\tilde{X},\mathbf{i}\tilde{Y}) = D_{\mathcal{H}\tilde{X}}\tilde{Y} - \mathcal{V}[\mathcal{H}\tilde{X},\mathbf{i}\tilde{Y}] \overset{(19)}{=} D_{\mathcal{H}\tilde{X}}\tilde{Y} - \nabla_{\mathcal{H}\tilde{X}}\tilde{Y},$$

$$(54)\qquad \mathcal{Q}^1(\tilde{X},\tilde{Y}) := T^v(D)(\mathbf{i}\tilde{X},\mathbf{i}\tilde{Y}) = D_{\mathbf{i}\tilde{X}}\tilde{Y} - D_{\mathbf{i}\tilde{Y}}\tilde{X} - \mathcal{V}[\mathbf{i}\tilde{X},\mathbf{i}\tilde{Y}].$$

Observe that $\mathcal{R}^1$ does not depend on the covariant derivative operator D: since

$$(55)\qquad \mathbf{i}\mathcal{R}^1(\mathbf{j}\xi,\mathbf{j}\eta) = -\mathbf{v}[\mathbf{h}\xi,\mathbf{h}\eta] = \Omega(\xi,\eta) \qquad \text{for all } \xi,\eta\in\mathcal{X}(TM),$$

(see (13) and (23)), $\mathcal{R}^1$ is merely another expression for the curvature of $\mathcal{H}$. It can also readily be seen that a covariant derivative operator in $\tau^*\tau$ is completely determined, once one has given two $\tau^*\tau$-valued two-forms to play the role of the h-mixed and v-mixed torsions. In particular, as it can be seen at once from (51) and (54), the choice $\mathcal{S}=0$, $\mathcal{P}^1=0$ leads to the Berwald derivative ∇ induced by $\mathcal{H}$. We denote the torsions of ∇ by $\mathring{\mathcal{T}}$, $\mathring{\mathcal{S}}$, etc. Another remark will also be appropriate. Let $X,Y\in\mathcal{X}(M)$. Then $\mathbf{i}\mathring{\mathcal{T}}(\mathbf{j}X^h,\mathbf{j}Y^h) = \mathbf{i}\mathring{\mathcal{T}}(\hat{X},\hat{Y}) \overset{(50)}{=} \mathbf{i}\nabla_{X^h}\hat{Y} - \mathbf{i}\nabla_{Y^h}\hat{X} - J[X^h,Y^h] \overset{(19)(7)}{=} [X^h,Y^v]-[Y^h,X^v]-[X,Y]^v$. On the other hand, applying (12), (6) and (7), we get $\mathbf{T}(X^h,Y^h) \overset{(23)}{:=} [J,\mathbf{h}](X^h,Y^h) = [X^h,Y^v] - [Y^h,X^v]-[X,Y]^v$, therefore

$$(56)\qquad \mathbf{i}\mathring{\mathcal{T}}(\mathbf{j}X^h,\mathbf{j}Y^h) = \mathbf{T}(X^h,Y^h) \qquad \text{for all } X,Y\in\mathcal{X}(M).$$

This means that the h-horizontal torsion of the Berwald derivative induced by $\mathcal{H}$ and the torsion of $\mathcal{H}$ contain the same information.

Our next objective is to construct a metric derivative which has as many vanishing partial torsions as possible. To achieve this, we need the *lowered second Cartan tensor* $\mathcal{C}^h_\flat := \nabla^h g$ and its metrical equivalent, the *second Cartan tensor* $\mathcal{C}^h$ defined by $g\big(\mathcal{C}^h(\tilde{X},\tilde{Y}),\tilde{Z}\big) := \mathcal{C}^h_\flat(\tilde{X},\tilde{Y},\tilde{Z}) := (\nabla^h_{\tilde{X}}g)(\tilde{Y},\tilde{Z})$ for all $\tilde{X},\tilde{Y},\tilde{Z}\in\mathcal{X}(\tau)$.

Lemma 9. *The tensors* $\mathring{\mathcal{C}}$ *and* $\mathring{\mathcal{C}}^h$ *given by*

$$\begin{aligned} g(\mathring{\mathcal{C}}(\tilde{X},\tilde{Y}),\tilde{Z}) &:= \mathcal{C}_\flat(\tilde{X},\tilde{Y},\tilde{Z}) + \mathcal{C}_\flat(\tilde{Y},\tilde{Z},\tilde{X}) - \mathcal{C}_\flat(\tilde{Z},\tilde{X},\tilde{Y}) \qquad \textit{and}\\ g(\mathring{\mathcal{C}}^h(\tilde{X},\tilde{Y}),\tilde{Z}) &:= \mathcal{C}^h_\flat(\tilde{X},\tilde{Y},\tilde{Z}) + \mathcal{C}^h_\flat(\tilde{Y},\tilde{Z},\tilde{X}) - \mathcal{C}^h_\flat(\tilde{Z},\tilde{X},\tilde{Y}) \end{aligned}$$

are well-defined, symmetric (1,2) tensor fields along τ.

Indeed, well-definedness is assured by non-degeneracy of g. Since both $\mathcal{C}_\flat$ and $\mathcal{C}^h_\flat$ are symmetric in their last two arguments, it follows that $\overset{\circ}{\mathcal{C}}$ and $\overset{\circ}{\mathcal{C}}{}^h$ are symmetric. Some easy consequences may be inferred immediately from the definition of $\overset{\circ}{\mathcal{C}}$. For example, g is variational if, and only if, $\mathcal{C} = \overset{\circ}{\mathcal{C}}$ and if $\overset{\circ}{\mathcal{C}}(.,\delta) = 0$, then g is weakly normal. We show that $\overset{\circ}{\mathcal{C}}(.,\delta) = 0$ *also implies that g is weakly variational and homogeneous, and vice versa.* Indeed, if $\overset{\circ}{\mathcal{C}}(.,\delta) = 0$, then for all $\tilde{X}, \tilde{Y} \in \mathcal{X}(\tau)$ we have

$$
\begin{aligned}
0 &= g(\overset{\circ}{\mathcal{C}}(\tilde{X},\delta),\tilde{Y}) = \mathcal{C}_\flat(\tilde{X},\delta,\tilde{Y}) + \mathcal{C}_\flat(\delta,\tilde{Y},\tilde{X}) - \mathcal{C}_\flat(\tilde{Y},\tilde{X},\delta) \quad \text{and} \\
0 &= g(\overset{\circ}{\mathcal{C}}(\tilde{Y},\delta),\tilde{X}) = \mathcal{C}_\flat(\tilde{Y},\delta,\tilde{X}) + \mathcal{C}_\flat(\delta,\tilde{X},\tilde{Y}) - \mathcal{C}_\flat(\tilde{X},\tilde{Y},\delta).
\end{aligned}
$$

Since $\mathcal{C}_\flat$ is symmetric in its last two variables, after subtraction, the property $\mathcal{C}_\flat(\tilde{X},\tilde{Y},\delta) = \mathcal{C}_\flat(\tilde{Y},\tilde{X},\delta)$ drops out. Thus g is weakly variational, and hence both of the above relations imply $\mathcal{C}(\delta,.) = 0$, i.e. g is homogeneous. Conversely, if g is weakly variational and homogeneous, then the right-hand sides of our above relations vanish, therefore $\overset{\circ}{\mathcal{C}}(.,\delta) = 0$.

Notice that it is more difficult to handle $\overset{\circ}{\mathcal{C}}{}^h$ (and $\mathcal{C}^h$).

Proposition 9. *Let $g = \nabla^v\nabla^v L$ be a variational metric and $\mathcal{H}$ the horizontal map, generated by ξ_L. Then, $\overset{\circ}{\mathcal{C}}{}^h = \mathcal{C}^h$.*

Proof: It is well-known (see e.g. [7]) that one can derive from the properties (46) that

$$
(57) \qquad \nabla^h g(\tilde{X},\tilde{Y},\tilde{Z}) = \nabla^h g(\tilde{Y},\tilde{X},\tilde{Z}) \qquad \text{or} \qquad g(\mathcal{C}^h(\tilde{X},\tilde{Y}),\tilde{Z}) = g(\mathcal{C}^h(\tilde{Y},\tilde{X}),\tilde{Z}),
$$

for all $\tilde{X},\tilde{Y},\tilde{Z} \in \mathcal{X}(\tau)$. Indeed, let $X, Y \in \mathcal{X}(M)$ and take the ξ_L derivative of the integrability condition $X^v\big(g(\hat{Y},\hat{Z})\big) = Y^v\big(g(\hat{X},\hat{Z})\big)$. Interchange now the positions of ξ_L and X^v, using the bracket $[\xi_L, X^v] = -X^h + \mathbf{i}(\mathcal{L}^h_{\xi_L}\hat{X})$. The property $\mathcal{L}^h_{\xi_L} g = 0$ leads to (57) and the statement now easily follows. □

Proposition 10. *Suppose that the horizontal map $\mathcal{H}$ has vanishing torsion and let ∇ be the Berwald derivative induced by $\mathcal{H}$ in $\tau^*\tau$. If $\overset{\circ}{\mathcal{C}}$ and $\overset{\circ}{\mathcal{C}}{}^h$ are the tensors given by Lemma 9, then the rules*

$$
(58) \qquad D_{\mathbf{i}\tilde{X}}\tilde{Y} := \nabla_{\mathbf{i}\tilde{X}}\tilde{Y} + \frac{1}{2}\overset{\circ}{\mathcal{C}}(\tilde{X},\tilde{Y}), \qquad D_{\mathcal{H}\tilde{X}}\tilde{Y} := \nabla_{\mathcal{H}\tilde{X}}\tilde{Y} + \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h(\tilde{X},\tilde{Y})
$$

define a symmetric, metric derivative in $\tau^*\tau$. More explicitly

$$
(59) \qquad D_\xi\tilde{Y} = \mathbf{j}[\mathbf{v}\xi,\mathcal{H}\tilde{Y}] + \mathcal{V}[\mathbf{h}\xi,\mathbf{i}\tilde{Y}] + \frac{1}{2}\Big(\overset{\circ}{\mathcal{C}}(\mathcal{V}\xi,\tilde{Y}) + \overset{\circ}{\mathcal{C}}{}^h(\mathbf{j}\xi,\tilde{Y})\Big).
$$

For the partial torsions of D we have:

$$
\mathcal{T} = 0,\ \mathcal{S} = -\frac{1}{2}\overset{\circ}{\mathcal{C}},\ (\mathcal{R}^1)_0 = \Omega,\ \mathcal{P}^1 = \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h,\ \mathcal{Q}^1 = 0;\ \text{where } (\mathcal{R}^1)_0(\xi,\eta) := \mathbf{i}\mathcal{R}^1(\mathbf{j}\xi,\mathbf{j}\eta).
$$

PROOF: First we check that D is v-metric. For all vector fields X, Y, Z on M we have $(D^v g)(\hat{X}, \hat{Y}, \hat{Z}) = (D_{X^v} g)(\hat{Y}, \hat{Z}) = X^v(g(\hat{Y}, \hat{Z})) - g(D_{X^v}\hat{Y}, \hat{Z}) - g(\hat{Y}, D_{X^v}\hat{Z}) = \mathcal{C}_\flat(\hat{X}, \hat{Y}, \hat{Z}) - \frac{1}{2}g(\overset{\circ}{\mathcal{C}}(\hat{X}, \hat{Y}), \hat{Z}) - \frac{1}{2}g(\hat{Y}, \overset{\circ}{\mathcal{C}}(\hat{X}, \hat{Z})) = \mathcal{C}_\flat(\hat{X}, \hat{Y}, \hat{Z}) - \frac{1}{2}g(\overset{\circ}{\mathcal{C}}(\hat{X}, \hat{Y}), \hat{Z}) - \frac{1}{2}g(\overset{\circ}{\mathcal{C}}(\hat{Z}, \hat{X}), \hat{Y}) = \mathcal{C}_\flat(\hat{X}, \hat{Y}, \hat{Z}) - \mathcal{C}_\flat(\hat{X}, \hat{Y}, \hat{Z}) = 0$, as we claimed. By a completely analogous calculation we obtain that D is h-metric and hence it is metric. Next we calculate the partial torsions of D.

$\mathcal{T}(\tilde{X}, \tilde{Y}) \overset{(50)}{=} D_{\mathcal{H}\tilde{X}}\tilde{Y} - D_{\mathcal{H}\tilde{Y}}\tilde{X} - \mathbf{j}[\mathcal{H}\tilde{X}, \mathcal{H}\tilde{Y}] \overset{(58)}{=} \nabla_{\mathcal{H}\tilde{X}}\tilde{Y} - \nabla_{\mathcal{H}\tilde{Y}} - \mathbf{j}[\mathcal{H}\tilde{X}, \mathcal{H}\tilde{Y}] + \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h(\tilde{X}, \tilde{Y}) - \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h(\tilde{Y}, \tilde{X}) = \overset{\circ}{\mathcal{T}}(\tilde{X}, \tilde{Y}) \overset{(56)}{=} 0$, since $\mathbf{T} = 0$ by our assumption. From (51) and (58), $\mathcal{S} = -\frac{1}{2}\overset{\circ}{\mathcal{C}}$. Thus, by Lemma 9, $\mathcal{S}$ is symmetric, therefore the covariant derivative operator D is also symmetric. (55) yields immediately $(\mathcal{R}^1)_0 = \Omega$, while (52) and (58) lead to $\mathcal{P}^1 = \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h$. Finally, $\mathcal{Q}^1(\hat{X}, \hat{Y}) \overset{(54)}{=} D_{X^v}\hat{Y} - D_{Y^v}\hat{X} - \mathcal{V}[X^v, Y^v] \overset{(58)}{=} \frac{1}{2}\overset{\circ}{\mathcal{C}}(\hat{X}, \hat{Y}) - \frac{1}{2}\overset{\circ}{\mathcal{C}}(\hat{Y}, \hat{X}) = 0$, concluding the proof of the proposition. □

In [18], R. Miron gave a coordinate formulation for this metric derivative in the case that the metric is weakly normal and Miron-regular and the horizontal map is generated by ξ_E. A coordinate expression of the general case (58) can also be found in [19]. Now we are in a position to describe *all* metric derivatives (depending on a horizontal map). We begin by a simple remark. The *difference* of two covariant derivative operators D^1 and D^2 can be characterized by means of a $C^\infty(TM)$-bilinear map $\varrho : \mathcal{X}(TM) \times \mathcal{X}(\tau) \to \mathcal{X}(\tau)$ such that

$$D^1_\xi \tilde{X} - D^2_\xi \tilde{X} = \varrho(\xi, \tilde{X}) \text{ for all } \xi \in \mathcal{X}(TM),\ \tilde{X} \in \mathcal{X}(\tau).$$

ϱ can be decomposed into a v-part ϱ^v and a h-part ϱ^h given by

$$\varrho^v(\tilde{X}, \tilde{Y}) := \varrho(\mathbf{i}\tilde{X}, \tilde{Y}) \text{ and } \varrho^h(\tilde{X}, \tilde{Y}) := \varrho(\mathcal{H}\tilde{X}, \tilde{Y});$$

then $\varrho(\xi, \tilde{X}) = \varrho^v(\mathcal{V}\xi, \tilde{X}) + \varrho^h(\mathbf{j}\xi, \tilde{X})$.

Proposition 11. *Let D be the metric derivative described in Proposition 10. Then any other metric derivative $\bar{D}$ can be uniquely determined with the help of two (1,2)-tensor fields Φ and Ψ which are related to the difference ϱ of D and $\bar{D}$ by means of*

$$g(\varrho^v(\tilde{X}, \tilde{Y}), \tilde{Z}) = \frac{1}{2}\Big(g(\Phi(\tilde{X}, \tilde{Y}), \tilde{Z}) - g(\Phi(\tilde{X}, \tilde{Z}), \tilde{Y})\Big) \qquad \text{and} \tag{60}$$

$$g(\varrho^h(\tilde{X}, \tilde{Y}), \tilde{Z}) = \frac{1}{2}\Big(g(\Psi(\tilde{X}, \tilde{Y}), \tilde{Z}) - g(\Psi(\tilde{X}, \tilde{Z}), \tilde{Y})\Big) \tag{61}$$

for all vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$ along τ.

PROOF: First, we will show that for a given Φ and Ψ, the above construction gives indeed a metric derivative. For example, if we use the fact that D is metric, then $\big(\bar{D}_{\mathcal{H}\tilde{X}} g\big)(\tilde{Y}, \tilde{Z}) = \mathcal{H}\tilde{X}\big(g(\tilde{Y}, \tilde{Z})\big) - g(\bar{D}_{\mathcal{H}\tilde{X}}\tilde{Y}, \tilde{Z}) - g(\tilde{Y}, \bar{D}_{\mathcal{H}\tilde{X}}\tilde{Z}) = g(\varrho^h(\tilde{X}, \tilde{Y}), \tilde{Z}) + g(\tilde{Y}, \varrho^h(\tilde{X}, \tilde{Z}) = \frac{1}{2}\big(g(\Psi(\tilde{X}, \tilde{Y}), \tilde{Z}) - g(\Psi(\tilde{X}, \tilde{Z}), \tilde{X}) + g(\tilde{Y}, (\Psi(\tilde{X}, \tilde{Z})) - g(\tilde{Z}, (\Psi(\tilde{X}, \tilde{Y}))\big) = 0$, and analogously for the ϱ^v part. On the other hand, suppose that $\bar{D}$ is a metric derivative. Then $\bar{D}$ is completely determined by the tensor fields $\Phi = \varrho^v$ and $\Psi = \varrho^h$, which are exactly of the type (60) and (61) since (due to $\bar{D}g = 0$) $g(\varrho^v(\tilde{X},\tilde{Y}),\tilde{Z}) + g(\varrho^v(\tilde{X},\tilde{Z}),\tilde{Y}) = 0$ and $g(\varrho^h(\tilde{X},\tilde{Y}),\tilde{Z}) + g(\varrho^h(\tilde{X},\tilde{Z}),\tilde{Y}) = 0$. $\square$

A coordinate version of this proof is mentioned in [18, 19]. In fact, there one made use of the so-called *Obata-operators*. In our context, the first Obata-operator can best be viewed as a map Ob that maps a (1,2)-tensor field onto another (1,2)-tensor field, $\mathrm{Ob} : \Psi \mapsto \mathrm{Ob}_\Psi$, where Ob_Ψ is defined by

$$g(\mathrm{Ob}_\Psi(\tilde{X},\tilde{Y}),\tilde{Z}) := \frac{1}{2}\Big(g(\Psi(\tilde{X},\tilde{Y}),\tilde{Z}) - g(\Psi(\tilde{X},\tilde{Z}),\tilde{Y})\Big).$$

The second Obata-operator Ob^*, has an analogous definition

$$g(\mathrm{Ob}^*_\Psi(\tilde{X},\tilde{Y}),\tilde{Z}) := \frac{1}{2}\Big(g(\Psi(\tilde{X},\tilde{Y}),\tilde{Z}) + g(\Psi(\tilde{X},\tilde{Z}),\tilde{Y})\Big).$$

Therefore, (60) and (61) can be restated as $\varrho^v = \mathrm{Ob}_\Phi$ and $\varrho^h = \mathrm{Ob}_\Psi$.

Acknowledgements. We are indebted to Willy Sarlet for many useful discussions.

References

[1] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, Second Edition, Benjamin-Cummings, Reading, Mass., (1978).

[2] P.L. Antonelli, R.S. Ingarden and M. Matsumoto, *The theory of sprays and Finsler spaces with applications in physics and biology*, Fundamental theories of physics **58**, Kluwer Academic Publishers (1993).

[3] E. K. Ayassou, *Cohomologies définies pour une 1-forme vectorielle plate*, Thèse de Doctorat, Université de Grenoble, (1985).

[4] S.S. Chern, *Finsler geometry is just Riemannian geometry without the quadratic restriction*, Notices Amer. Math. Soc. **43** no. 9, 959–963.

[5] M. Crampin, *Jet bundle techniques in analytical mechanics*, Quaderni del consiglio nazionale delle ricerche, gruppo nazionale di fisica matematica **47** (1995).

[6] M. Crampin, *Connections of Berwald type*, Publ. Math. Debrecen, **57** no. 3-4 (2000), 455–473.

[7] M. Crampin, W. Sarlet, E. Martínez, G.B. Byrnes and G.E. Prince, *Towards a geometrical understanding of Douglas's solution of the inverse problem of the calculus of variations*, Inverse Problems **10** (1994), 245–260.

[8] M. de Léon and P. Rodrigues, *Methods of differential geometry in analytical mechanics*, North-Holland Mathematics studies **158**, North-Holland (1989).

[9] J.G. Diaz and G. Grangier, *Courbure et holonomie des variétés Finslèriennes*, Tensor, N.S. **30** (1976), 95–109.

[10] W. Greub, S. Halperin and R. Vanstone, *Connections, Curvature, and Cohomology*, Vol. II, Pure and Applied Mathematics **47** - II, Academic Press, New York, (1972).

[11] J. Grifone, *Structure presque tangente et connexions I*, Ann. Inst. Fourier, Grenoble **22** no. 1 (1972), 287–334.

[12] J. Grifone, *Structure presque tangente et connexions II*, Ann. Inst. Fourier, Grenoble **22** no. 3 (1972), 291–338.

[13] M. Hashiguchi, *On parallel displacements in Finsler spaces*, J. Math. Soc. Japan **10** no. 4 (1958), 365–379.

[14] I. Kolář, P.W. Michor, J. Slovák, *Natural operations in differential geometry*, Springer-Verlag Berlin Heidelberg (1993).

[15] O. Krupková, *Variational metrics on* $\mathbb{R} \times TM$ *and the geometry of nonconservative mechanics*, Math. Slovaca **44** no. 3 (1994), 315–335.

[16] E. Martínez, J.F. Cariñena and W. Sarlet, *Derivations of differential forms along the tangent bundle projection*, Diff. Geom. Appl. **2** (1992), 17–43.

[17] E. Martínez, J.F. Cariñena and W. Sarlet, *Derivations of differential forms along the tangent bundle projection II*, Diff. Geom. Appl. **3** (1993), 1–29.

[18] R. Miron, *Metrical Finsler structures and metrical Finsler connections*, J. Math. Kyoto. Univ. **23** (1983), 219–224.

[19] R. Miron and M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Fundamental Theories of Physics **59**, Kluwer Academic Publishers (1994).

[20] A. Moór, *Entwicklung einer Geometrie der allgemeinen metrischen Linienelementräume*, Acta Sci. Math. Szeged **17** (1956), 85–120.

[21] B. Riemann, *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, edited by H. Weyl, Springer (1921). See also M. Spivak, *A comprehensive introduction to differential geometry.*, Vol. II., Brandeis Univ., Waltham, Mass. (1970).

[22] J. Szilasi, *A setting for Spray and Finsler Geometry*, in: Handbook of Finsler Geometry (ed. P.L. Antonelli), to be published by Kluwer Academic Publishers, Dordrecht.

[23] J.R. Vanstone, *A generalization of Finsler geometry*, Canad. J. Math. **14** (1962), 87–112.