Bi-quasi-Hamiltonian systems

M. Crampin
Department of Applied Mathematics, The Open University,
Walton Hall, Milton Keynes MK7 6AA, UK;

W. Sarlet
Department of Mathematical Physics and Astronomy, Ghent University,
Krijgslaan 281, B–9000 Gent, Belgium

January 3, 2002

Abstract. A general notion of bi-quasi-Hamiltonian systems is introduced and is related to previous work on various special cases of such systems.

1 Introduction

In a previous paper (Crampin 2001), we have generalized work by Lundmark (2001a, b) on a class of integrable systems, from Euclidean spaces to (pseudo) Riemannian manifolds. In doing so, we came across an interesting class of conformal Killing tensors, called 'special conformal Killing tensors', and the integrable systems of interest live on spaces whose metric allows the coexistence of two such tensors. Physically speaking, the integrable systems in question represent Lagrangian systems with non-conservative forces, but whose force form has a kind of double 'generalized potential' representation. Whereas most classical (finite dimensional) integrable systems are known to have a bi-Hamiltonian formulation, the ones referred to here have a double quasi-Hamiltonian representation and thus could be called bi-quasi-Hamiltonian systems. But they do lead to a bi-Hamiltonian system on an extended space.

In the present paper, we will introduce a general concept of bi-quasi-Hamiltonian systems and explore under what circumstances they can have the property of complete integrability. Bi-quasi-Hamiltonian systems should not be confused with quasi-bi-Hamiltonian systems, as introduced by Brouzet *et al.* (1996) and developed by Morosi and Tondo (1997, 1998). The similarity of the two terms is unfortunate, but the name 'bi-quasi-Hamiltonian system' describes what we have in mind so well that after some reflection we decided to use it even so. The two concepts are in fact related, but the relation is not entirely straightforward: we discuss it in Section 7.

Following the description of the general concept in Sections 2 and 3, we specialize in Section 4 to the case where the two Poisson structures involved in the bi-quasi-Hamiltonian representation each come from a Poisson-Nijenhuis structure. Most of the discussion in such a case is transferred from the Poisson structures involved to the type $(1,1)$ tensor fields which generate them. We then make a further specialization, to the case where the manifold is a cotangent bundle and the $(1,1)$ tensors are the complete lifts of tensor fields on the base manifold. This brings us back to the kind of system studied in Crampin (2001), and we take this opportunity to complement our previous work with some results about the possible generation of families of such bi-quasi-Hamiltonian systems. The final section contains an explicit example which illustrates the theory.

We carry out coordinate calculations at various points in the paper; where we do so, we use the Einstein summation convention.

2 Quasi-Hamiltonian systems

We begin by recalling some generalities about Poisson structures.

A Poisson structure on a manifold M is a bivector field Π which satisfies $[\Pi,\Pi]=0$, where $[\cdot,\cdot]$ is the Schouten bracket. The associated Poisson bracket of functions f, g is given by $\{f,g\}=\Pi(df,dg)$; the vanishing of the Schouten bracket entails the Jacobi identity for the Poisson bracket. Also associated with such a bivector field is a map P of 1-forms to vector fields on M, given by $\langle P(\alpha),\beta\rangle=\Pi(\alpha,\beta)$ for any pair of 1-forms α, β. The Poisson bracket can be extended to 1-forms with the aid of P, as follows (Magri (1985)): for any 1-forms α, β,

$$\{\alpha,\beta\}_P=\mathcal{L}_{P(\alpha)}\beta-\mathcal{L}_{P(\beta)}\alpha-d(\Pi(\alpha,\beta));$$

then $\{df,dg\}=d\{f,g\}$. The Schouten bracket condition on Π can be stated equivalently in terms of the Poisson map and the bracket of 1-forms:

$$[P(\alpha),P(\beta)]=P(\{\alpha,\beta\}_P).$$

The Poisson structure is non-singular if its Poisson map is.

The vector field $P(dH)$ is the Hamiltonian vector field corresponding to the Hamiltonian function H. If Z is a Hamiltonian vector field then $\mathcal{L}_Z\Pi=0$, or equivalently $\mathcal{L}_ZP=0$. Conversely, when Π is non-singular, if $\mathcal{L}_Z\Pi=0$ then Z is locally Hamiltonian, in the sense that in a neighbourhood of any point one can find a function H such that $Z=P(dH)$ on that neighbourhood. In fact the condition $\mathcal{L}_Z\Pi=0$ states that Z is a cocycle in the Lichnerowicz-Poisson complex corresponding to Π, and Z is Hamiltonian if and only if it is a coboundary. The Poisson map P, when it is non-singular, induces an isomorphism of the Lichnerowicz-Poisson and the de Rham cohomology groups (Vaisman (1994)), and so the vanishing of the first de Rham cohomology group of M is a sufficient condition for a vector field which satisfies $\mathcal{L}_Z\Pi = 0$ to be Hamiltonian. We shall deal mostly with non-singular Poisson structures, and we shall generally assume that $\mathcal{L}_Z\Pi = 0$ is sufficient as well as necessary for Z to be Hamiltonian.

A vector field Z on a Poisson manifold is said to be quasi-Hamiltonian if there is a nowhere-vanishing function F such that FZ is Hamiltonian. Thus $FZ = P(dH)$ for some function H. Note that if M is connected F must be everywhere positive or everywhere negative; we can assume the former without loss of generality, by absorbing a negative sign in H if necessary. We assume henceforth that M is connected and F is positive. Now let $V = F^{-1}P(dF) = P(d\log F)$; then $\mathcal{L}_Z\Pi = Z \wedge V$, and of course $\mathcal{L}_V\Pi = 0$. Conversely, if $\mathcal{L}_Z\Pi = Z \wedge V$ and $V = P(d\Phi)$ say then

$$\mathcal{L}_{e^\Phi Z}\Pi = e^\Phi \mathcal{L}_Z\Pi - Z \wedge P(d(e^\Phi)) = 0.$$

Thus if $\mathcal{L}_Z\Pi = Z \wedge V$ where $\mathcal{L}_V\Pi = 0$, and Π is non-singular, then Z is quasi-Hamiltonian, at least locally.

The function H is a first integral of Z.

Let Z be a vector field on a manifold M which is quasi-Hamiltonian with respect to a Poisson bivector Π, so that $FZ = P(dH)$. Then, as shown in Crampin (2001), the bivector $\hat{\Pi}$ on $M \times \mathbb{R}$ given by

$$\hat{\Pi} = \Pi + (Z + zV) \wedge \frac{\partial}{\partial z},$$

where z is the coordinate on $\mathbb{R}$ and $V = F^{-1}P(dF)$, is a Poisson bivector which projects onto Π. Here Π is extended to $M \times \mathbb{R}$ simply by ignoring z. The Hamiltonian vector field with respect to $\hat{\Pi}$ corresponding to $-z$ is $Z + zV$; its restriction to the zero section is Z. Furthermore, $H + zF$ is a Casimir of $\hat{\Pi}$. In fact the conditions for $\hat{\Pi}$ to be Poisson are again

$$\mathcal{L}_Z\Pi = Z \wedge V, \quad \mathcal{L}_V\Pi = 0,$$

given that Π is Poisson.

3 Bi-quasi-Hamiltonian systems

We now set up a general framework for the study of systems with a dual quasi-Hamiltonian representation, within which a notion of complete integrability occurs naturally, and which covers as a special case the class of cofactor pair systems studied in Lundmark (2001a, b) and Crampin (2001).

Definition A vector field Z on M is said to be bi-quasi-Hamiltonian if

(i) M is equipped with two compatible Poisson structures Π_a, $a = 1, 2$;

(ii) Z is quasi-Hamiltonian with respect to both Poisson structures, i.e. $F_a Z = P_a(dH_a)$ for some functions H_a and (nowhere vanishing) F_a;

(iii) $\mathcal{L}_{V_1}\Pi_2 + \mathcal{L}_{V_2}\Pi_1 = 0$, where $V_a = (F_a)^{-1}P_a(dF_a)$.

Condition (iii), which is a form of compatibility condition (and will be referred to as such below), can be motivated as follows.

Suppose that Z is quasi-Hamiltonian with respect to two Poisson structures Π_1 and Π_2, with $F_a Z = P_a(dH_a)$, $a = 1, 2$. Then we can form the two extended Poisson structures

$$\hat{\Pi}_a = \Pi_a + (Z + zV_a) \wedge \frac{\partial}{\partial z}.$$

We now ask for the conditions for these Poisson structures to be compatible, that is, for $[\hat{\Pi}_1, \hat{\Pi}_2] = 0$. Now

$$\begin{aligned}
&\left[\Pi_1 + (Z + zV_1) \wedge \frac{\partial}{\partial z}, \Pi_2 + (Z + zV_2) \wedge \frac{\partial}{\partial z}\right] \\
&= [\Pi_1, \Pi_2] + (\mathcal{L}_Z\Pi_1 + z\mathcal{L}_{V_2}\Pi_1) \wedge \frac{\partial}{\partial z} + (\mathcal{L}_Z\Pi_2 + z\mathcal{L}_{V_1}\Pi_2) \wedge \frac{\partial}{\partial z} \\
&\quad + (V_1 + V_2) \wedge Z \wedge \frac{\partial}{\partial z} \\
&= [\Pi_1, \Pi_2] + z\,(\mathcal{L}_{V_1}\Pi_2 + \mathcal{L}_{V_2}\Pi_1) \wedge \frac{\partial}{\partial z}
\end{aligned}$$

since $\mathcal{L}_Z\Pi_a = Z \wedge V_a$ by assumption. Thus $\mathcal{L}_{V_1}\Pi_2 + \mathcal{L}_{V_2}\Pi_1 = 0$ is the necessary and sufficient condition for the Poisson structures $\hat{\Pi}_1$ and $\hat{\Pi}_2$ to be compatible, assuming that Π_1 and Π_2 are compatible.

We now establish the existence of involutive first integrals of a bi-quasi-Hamiltonian system.

It follows from the compatibility condition $\mathcal{L}_{V_1}\Pi_2 + \mathcal{L}_{V_2}\Pi_1 = 0$, together with the conditions $\mathcal{L}_{V_a}\Pi_a = 0$, that $\mathcal{L}_{V_1 - tV_2}(\Pi_1 - t\Pi_2) = 0$ for all t. Now if Π_1 is non-singular so is $\Pi_1 - t\Pi_2$ for t sufficiently close to zero. Thus there is some function $\Phi(t)$ such that $V_1 - tV_2 = (P_1 - tP_2)(d\Phi(t))$. Since further $\mathcal{L}_Z(\Pi_1 - t\Pi_2) = Z \wedge (V_1 - tV_2)$, there is some function $H(t)$ such that $F(t)Z = (P_1 - tP_2)(dH(t))$, where $F(t) = e^{\Phi(t)}$; moreover $H(t)$ is a first integral of Z for all t. On setting $t = 0$ we see that $F(0)Z = P_1(dH(0))$. Now for $k = 0, 1, 2, \ldots$ set

$$H_{(k)} = \left.\frac{1}{k!}\frac{\partial^k H}{\partial t^k}\right|_{t=0},$$

and define $F_{(k)}$ similarly. Then each $H_{(k)}$ is a first integral of Z, and

$$F_{(k+1)}Z = P_1(dH_{(k+1)}) - P_2(dH_{(k)}).$$

It follows that for every $j = 0, 1, 2, \dots$,

$$\{H_{(j)}, H_{(k+1)}\}_1 = \{H_{(j)}, H_{(k)}\}_2,$$

where $\{\cdot, \cdot\}_a$ is the Poisson bracket of functions defined by Π_a. Thus

$$\{H_{(j)}, H_{(k+1)}\}_1 = \{H_{(j+1)}, H_{(k)}\}_1.$$

It follows that if l and m differ by an even integer then $\{H_{(l)}, H_{(m)}\}_1 = 0$, since

$$\{H_{(l)}, H_{(l+2n)}\}_1 = \{H_{(l+n)}, H_{(l+n)}\}_1.$$

If l and m differ by an odd integer, on the other hand,

$$\{H_{(l)}, H_{(m)}\}_1 = \{H_{(l)}, H_{(m-1)}\}_2;$$

since l and $m-1$ differ by an even integer, $\{H_{(l)}, H_{(m-1)}\}_2 = 0$ by the same argument. Thus $\{H_{(l)}, H_{(m)}\}_1 = 0$ for all l, m, and $\{H_{(l)}, H_{(m)}\}_2 = 0$ likewise. We summarize the results in the following statement.

Theorem Let Z be bi-quasi-Hamiltonian. Then there exist functions $H_{(k)}$ ($k = 0, 1, 2, \dots$), such that

$$\begin{aligned} F_{(0)}Z &= P_1(dH_{(0)}), \\ F_{(k+1)}Z &= P_1(dH_{(k+1)}) - P_2(dH_{(k)}), \quad k = 0, 1, 2, \dots \end{aligned}$$

for some functions $F_{(k)}$. The $H_{(k)}$ are in involution with respect to both Poisson brackets. In particular, if the manifold M is $2n$-dimensional and the functions $H_{(k)}$, $k = 0, 1, \dots, n-1$, are functionally independent then Z is completely integrable in the sense of Liouville.

4 Poisson-Nijenhuis pencils

We now specialize the results of the preceding section to the case where each of the two Poisson structures giving rise to a bi-quasi-Hamiltonian system actually comes from a Poisson-Nijenhuis structure.

A Poisson-Nijenhuis structure (Π, J) on a manifold M consists of a Poisson structure Π and a type $(1,1)$ tensor field J such that $PJ^* = JP$, the Magri-Morosi concomitant $\mu_{P,J}$ of P and J is zero and the torsion, or Nijenhuis tensor, N_J of J is zero. (Here J^* denotes the adjoint of J.)

The Magri-Morosi concomitant is defined as follows (Magri (1984), Nunes da Costa (1996)): for any 1-form α and vector field X on M,

$$\mu_{P,J}(\alpha, X) = (\mathcal{L}_{P(\alpha)}J)(X) - P(\mathcal{L}_X(J^*\alpha)) + P(\mathcal{L}_{J(X)}\alpha).$$

When $PJ^* = JP$, $\mu_{J,P}$ is a type $(1,2)$ tensor field on M.

These conditions are sufficient for JP to define a second Poisson structure on M (Magri (1984, 1985), Kosmann-Schwarzbach (1990)). We denote the Poisson bivector by Π_J and the associated Poisson map by $P_J = JP = PJ^*$.

Note that when $N_J = 0$ we have available the differential d_J of Frölicher-Nijenhuis theory (Frölicher (1956)), which satisfies $dd_J + d_J d = 0$, ${d_J}^2 = 0$; it is determined essentially by these properties and by its action on functions, which is given by $d_J f = J^*(df)$.

We shall be interested in vector fields Z that are quasi-Hamiltonian with respect to Π_J, with $FZ = P_J(dH)$. Of particular interest are those for which $d_J d\Phi = 0$, where $F = e^\Phi$. Alternatively, we then have $dd_J\Phi = 0$, so $d_J\Phi = d\phi$ for some function ϕ (at least locally), whence $d_J F = F d\phi$. One important case covered by this occurs when $\Phi = \log\det J$, so that $F = \det J$ (assuming, without essential loss of generality, that $\det J > 0$).

Proposition For any J such that $N_J = 0$, we have $d_J(\det J) = (\det J)d(\operatorname{tr} J)$.

Proof This follows from the formula

$$d(\det J) = J^i_{j|k} C^j_i dx^k,$$

where C is the cofactor tensor of J, and the calculation is carried out using any symmetric connection (or indeed partial differentiation). Thus

$$\begin{aligned} d_J(\det J) &= J^i_{j|k} J^k_l C^j_i dx^l \\ &= (J^i_{l|k} J^k_j + J^k_{j|l} J^i_k - J^k_{l|j} J^i_k) C^j_i dx^l \\ &= (J^i_{l|k} \delta^k_i + J^k_{j|l} \delta^j_k - J^k_{l|j} \delta^j_k)(\det J) dx^l \\ &= J^k_{k|l} (\det J) dx^l \end{aligned}$$

as asserted. □

It follows that $d_J\Phi = d(\operatorname{tr} J)$, and so $d_J d\Phi = 0$. Slightly more generally, if $F = (\det J)^r$ for some power r then $d_J\Phi = d(r \operatorname{tr} J)$ and $d_J d\Phi = 0$ also.

Definition A quasi-Hamiltonian vector field Z such that $(\det J)^r Z = P_J(dH)$ will be called a Pfaffian quasi-Hamiltonian vector field.

This is a slight extension of the terminology used in Morosi (1997). All of our examples of bi-quasi-Hamiltonian systems, later in the paper, will actually be of the Pfaffian type.

Now suppose that we have two Poisson-Nijenhuis structures with the same initial Poisson bivector Π, and type $(1,1)$ tensors J and K, which both commute with P, have zero Magri-Morosi concomitants with respect to P, and satisfy $N_J = N_K = 0$. Let the corresponding bivectors be Π_J, Π_K.

Definition The Poisson-Nijenhuis structures (Π, J) and (Π, K) are said to be compatible if the Poisson bivectors Π_J and Π_K are compatible with each other (each is compatible with Π by construction).

Proposition For Π_J and Π_K to be compatible, it is sufficient that the Nijenhuis bracket $[J, K] = 0$. If P, J and K are invertible, the condition is also necessary.

Proof When $[J, K] = 0$, we have $N_{(J+K)} = 0$. Obviously, $J + K$ commutes with P and $\mu_{P,J+K} = \mu_{P,J} + \mu_{P,K} = 0$. Hence,

$$[\Pi_{(J+K)}, \Pi_{(J+K)}] = [\Pi_J + \Pi_K, \Pi_J + \Pi_K] = 2[\Pi_J, \Pi_K] = 0.$$

Conversely, if P, J and K are invertible, the compatibility of Π_J and Π_K implies that $[J, K] = 0$. Indeed, the condition for the compatibility of two Poisson structures, expressed in terms of their Poisson maps Q, R, say, is

$$[Q(\alpha), R(\beta)] + [R(\alpha), Q(\beta)] = Q(\{\alpha, \beta\}_R) + R(\{\alpha, \beta\}_Q).$$

If we take $Q = P_J$, $R = P_K$, and set $\alpha = P^{-1}(X)$, $\beta = P^{-1}(Y)$ for vector fields X, Y, we obtain

$$[J(X), K(Y)] + [K(X), J(Y)] = JK^{-1}([K(X), K(Y)]) + KJ^{-1}([J(X), J(Y)]).$$

When the facts that $N_J = N_K = 0$ are used to substitute for the terms on the right-hand side, this becomes $[J, K](X, Y) = 0$. □

We consider, therefore, two Poisson-Nijenhuis structures (Π, J), (Π, K) such that $[J, K] = 0$. Then $(\Pi, J - tK)$ is a Poisson-Nijenhuis structure for every constant t. We then have a pencil of Poisson bivectors $\Pi_{J-tK} = \Pi_J - t\Pi_K$, so we call $(\Pi, J - tK)$ a Poisson-Nijenhuis pencil. We shall be mostly interested in the case in which P, J and K are all invertible.

We next discuss the compatibility condition for a system to be bi-quasi-Hamiltonian with respect to a Poisson-Nijenhuis pencil.

Suppose given a Poisson-Nijenhuis pencil, and a vector field Z such that

$$\mathcal{L}_Z\Pi_J = Z \wedge V, \quad \mathcal{L}_V\Pi_J = 0,$$
$$\mathcal{L}_Z\Pi_K = Z \wedge W, \quad \mathcal{L}_W\Pi_K = 0;$$

thus (still assuming that Π_J and Π_K are non-singular) Z is quasi-Hamiltonian with respect to both Π_J and Π_K and we can put $V = P_J(d\Phi) = P(d_J\Phi)$, $W = P_K(d\Psi) = P(d_K\Psi)$ for some Φ, Ψ. In order for Z to be bi-quasi-Hamiltonian the compatibility condition

$$\mathcal{L}_W\Pi_J + \mathcal{L}_V\Pi_K = 0,$$

or equivalently

$$\mathcal{L}_W P_J + \mathcal{L}_V P_K = 0,$$

must hold. When we express P_J and P_K in terms of P we find that

$$\mathcal{L}_W P_J + \mathcal{L}_V P_K = (\mathcal{L}_W J + \mathcal{L}_V K)P + J\mathcal{L}_W P + K\mathcal{L}_V P.$$

So a particular case of interest occurs when $\mathcal{L}_W P = \mathcal{L}_V P = 0$, that is, when V and W are Hamiltonian with respect to the standard Poisson structure P as well as the derived ones P_J, P_K. In such a case there will be functions ϕ, ψ such that

$$\begin{aligned} V = P(d\phi) \quad &\Longrightarrow \quad d\phi = d_J\Phi \quad \Longrightarrow \quad dd_J\phi = -d_J d\phi = 0; \\ W = P(d\psi) \quad &\Longrightarrow \quad d\psi = d_K\Psi \quad \Longrightarrow \quad dd_K\psi = -d_K d\psi = 0. \end{aligned}$$

Note that these conditions hold in the Pfaffian case, with $\phi = r \operatorname{tr} J$, $\Phi = r \log\det J$ etc.

Under these assumptions, the compatibility condition reduces to

$$\mathcal{L}_W J + \mathcal{L}_V K = 0.$$

Now from the vanishing of the Magri-Morosi concomitant,

$$(\mathcal{L}_W J)(X) = (\mathcal{L}_{P(d\psi)} J)(X) = P\left(\mathcal{L}_X(d_J\psi) - \mathcal{L}_{JX} d\psi\right)$$

for all X, whence the compatibility condition becomes

$$\mathcal{L}_X\left(d_J\psi + d_K\phi\right) - \mathcal{L}_{JX} d\psi - \mathcal{L}_{KX} d\phi = 0.$$

When the homotopy formula is used to express the Lie derivatives, this reduces simply to

$$X \lrcorner \left(dd_J\psi + dd_K\phi\right) = 0.$$

Thus the compatibility condition can be written

$$d_J\beta + d_K\alpha = 0, \quad \alpha = d\phi, \quad \beta = d\psi,$$

where $d_J\alpha = d_J d\phi = 0$, $d_K\beta = d_K d\psi = 0$.

In particular, it follows easily from the coordinate representation of the Nijenhuis bracket that if $[J, K] = 0$ then $d_J(\operatorname{tr} K) + d_K(\operatorname{tr} J) = d(\operatorname{tr} JK)$; so the requisite conditions are satisfied in the Pfaffian case, and we have the following result.

Proposition Suppose that Z is Pfaffian quasi-Hamiltonian with respect to both Π_J and Π_K, so that $(\det J)^r Z = P_J(dG)$ and $(\det K)^r Z = P_K(dH)$ (same power): then the necessary and sufficient condition for Z to be bi-quasi-Hamiltonian is that $[J, K] = 0$.

When this condition holds, or more generally for any system that is bi-quasi-Hamiltonian with respect to a Poisson-Nijenhuis pencil, the results of the previous section concerning the existence of first integrals in involution apply.

5 Bi-differential calculi

We now investigate the conditions for a bi-quasi-Hamiltonian system whose Poisson structures form a Poisson-Nijenhuis pencil, to give rise to a scalar gauged bi-differential calculus, according to the definition given by Dimakis and Müller-Hoissen (2000a), and discussed further in Crampin (2000).

We consider two Poisson-Nijenhuis structures (Π, J), (Π, K) such that $[J, K] = 0$. Then the operators d_J, d_K satisfy

$$d_J{}^2 = d_K{}^2 = d_J d_K + d_K d_J = 0,$$

and therefore form a simple bi-differential calculus (Dimakis 2000a) or bicomplex (Dimakis 2000b).

Let α be a 1-form, and denote by D_J the operator on forms given by

$$D_J\theta = d_J\theta + \alpha \wedge \theta,$$

or $D_J = d_J + \alpha$ for short. Two such operators $D_J = d_J + \alpha$, $D_K = d_K + \beta$ form a scalar gauged bi-differential calculus if

$$D_J{}^2 = D_K{}^2 = D_J D_K + D_K D_J = 0;$$

these conditions hold if and only if

$$d_J\alpha = d_K\beta = d_J\beta + d_K\alpha = 0.$$

But as we showed in the previous section, the compatibility condition for a system which is quasi-Hamiltonian with respect to a Poisson-Nijenhuis pencil and is such that $\mathcal{L}_V P = \mathcal{L}_W P = 0$ is that $d_J\beta + d_K\alpha = 0$, where the 1-forms α and β satisfy $d_J\alpha = d_K\beta = 0$. Thus any bi-quasi-Hamiltonian system of this type, and in particular any Pfaffian bi-quasi-Hamiltonian system, has associated with it a scalar gauged bi-differential calculus.

Furthermore, there are functions ϕ and Φ such that $\alpha = d\phi = d_J\Phi$, and functions ψ and Ψ such that $\beta = d\psi = d_K\Psi$. We can express D_J in the form

$$D_J\theta = d_J\theta + d\phi \wedge \theta = e^{-\Phi} d_J(e^{\Phi}\theta),$$

and similarly for D_K.

Now consider the dynamics Z. We can write

$$\begin{aligned} Z &= e^{-\Phi} P_J(dG) = P(e^{-\Phi} d_J G) = P(D_J(e^{-\Phi} G)) \\ &= e^{-\Psi} P_K(dH) = P(e^{-\Psi} d_K H) = P(D_K(e^{-\Psi} H)), \end{aligned}$$

for Hamiltonian functions G, H. These Hamiltonians are related by $D_J(e^{-\Phi}G) = D_K(e^{-\Psi}H)$; thus the functions $U = e^{-\Phi}G$ and $U = e^{-\Psi}H$ both satisfy the equation $D_J D_K U = 0$, a generalization of what was called the fundamental equation in Rauch-Wojciechowski (1999) and Lundmark (2001b). If we take the view that the vector fields V and W, or the functions Φ and Ψ, are the givens, then this equation provides a way of generating dynamical vector fields Z which fit the corresponding bi-quasi-Hamiltonian structure; in particular, we can take the Pfaffian set-up with $\Phi = r \log \det J$, $\Psi = r \log \det K$.

In the Poisson-Nijenhuis case the recurrence relation

$$P_J(dH_{(k+1)}) - P_K(dH_{(k)}) = F_{(k+1)}Z = \frac{F_{(k+1)}}{F_{(0)}} P_J(dH_{(0)})$$

can be expressed in terms of differential operators as follows:

$$d_J H_{(k+1)} = d_K H_{(k)} + \frac{F_{(k+1)}}{F_{(0)}} d_J H_{(0)}.$$

Furthermore, the general formula $V_1 - tV_2 = (P_1 - tP_2)(d\Phi(t))$ from Section 3 here becomes $V - tW = P_{J-tK}(d\Phi(t))$, from which it follows that $d_J\Phi - t d_K\Psi = d_{J-tK}\Phi(t)$. For $F(t) = e^{\Phi(t)}$, there results: $d_{J-tK}F(t) = F(t)(d_J\Phi - t d_K\Psi) = F(t)(d\phi - t d\psi)$. Hence, the $F_{(k)}$ satisfy the recurrence relation

$$d_J F_{(k+1)} - d_K F_{(k)} = F_{(k+1)} d\phi - F_{(k)} d\psi.$$

That is to say, we have

$$\begin{aligned} d_J F_{(k+1)} - (d\phi)F_{(k+1)} &= d_K F_{(k)} - (d\psi)F_{(k)} \\ d_J H_{(k+1)} - \vartheta F_{(k+1)} &= d_K H_{(k)} \end{aligned}$$

where

$$\vartheta = \frac{1}{F_{(0)}} d_J H_{(0)}.$$

Note that $P(\vartheta) = Z$, whence $\vartheta = e^{-\Phi} d_J G = e^{-\Psi} d_K H$; it follows that $d_J\vartheta = \vartheta \wedge d\phi$ and $d_K\vartheta = \vartheta \wedge d\psi$.

Define the 2×2 matrix differential operators

$$\Delta_J = d_J + \begin{bmatrix} -d\phi & 0 \\ -\vartheta & 0 \end{bmatrix}, \quad \Delta_K = d_K + \begin{bmatrix} -d\psi & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows from the formulas for $d_J\vartheta$ and $d_K\vartheta$ just obtained that ${\Delta_J}^2 = {\Delta_K}^2 = 0$ and $\Delta_J\Delta_K + \Delta_K\Delta_J = 0$. Thus Δ_J and Δ_K are the differential operators of a gauged bi-differential calculus operating on two-component column vectors, and

$$\Delta_J \begin{bmatrix} F_{(k+1)} \\ H_{(k+1)} \end{bmatrix} = \Delta_K \begin{bmatrix} F_{(k)} \\ H_{(k)} \end{bmatrix}.$$

A special case of this construction was discussed in Crampin (2000).

6 Cofactor and cofactor pair systems

In the preceding two sections, the discussion centred on a general Poisson manifold (M, P), with additional structure coming from some type $(1,1)$ tensor fields on M. An interesting particular case occurs when M is a cotangent bundle T^*Q say, equipped with its standard Poisson structure Π, and the type $(1,1)$ tensors on M are complete lifts of torsionless tensors L on the base manifold Q. For the complete lift $\tilde{L}$ of L, we have $\det \tilde{L} = (\det L)^2$. Moreover, for any two such tensors L_1, L_2,

$$[\tilde{L}_1, \tilde{L}_2] = \widetilde{[L_1, L_2]};$$

so a vector field Z on T^*Q which is Pfaffian quasi-Hamiltonian with respect to $\tilde{L}_1$ and $\tilde{L}_2$ will be bi-quasi-Hamiltonian if and only if $[L_1, L_2] = 0$. This result, which we have deduced from the more general analysis of Section 4, can also be proved easily by an explicit computation.

The following class of quasi-Hamiltonian systems has been studied in Marciniak (1998), Rauch-Wojciechowski (1999), Lundmark (2001a, b), and Crampin (2001). Let g be a (pseudo)-Riemannian metric on Q and L a symmetric type $(0,2)$ tensor field such that

$$(\nabla_X L)(Y, Z) = \tfrac{1}{2}(g(X, Y)\langle Z, \alpha\rangle + g(X, Z)\langle Y, \alpha\rangle).$$

Such a tensor is a conformal Killing tensor of g, and the type $(1,1)$ tensor obtained by raising an index on L with g automatically has vanishing torsion. The 1-form α is given by $d(\operatorname{tr} L)$. Such a tensor L is called a special conformal Killing tensor. We shall generally be concerned with special conformal Killing tensors which are non-singular. We shall denote such a tensor by the same letter whatever the position of its indices. Let A be the cofactor tensor of a special conformal Killing tensor, so that $AL = (\det L)I$, and let μ be a 1-form on Q which satisfies $D_L\mu = d_L\mu + d(\operatorname{tr} L) \wedge \mu = 0$. Let H be the function on T^*Q given by

$$H(q^i, p_i) = \tfrac{1}{2}A^{ij}p_ip_j + V(q)$$

where V is a function such that $D_L((\det L)^{-1}V) = \mu$. Then the quasi-Hamiltonian vector field Z where $(\det L)Z = P_{\tilde{L}}(dH)$ is called a cofactor system. Since $\det \tilde{L} = (\det L)^2$ it is Pfaffian, with $r = \frac{1}{2}$. The vector field Γ on TQ obtained via the diffeomorphism $(q^i, p_i) \mapsto (q^i, g^{ij}p_j)$ takes the form

$$\Gamma = \Gamma_0 - M^V,$$

where Γ_0 is the geodesic field of the metric and M is the vector field obtained by raising the index on μ. It is an example of a non-conservative Lagrangian system, and μ represents a generalized force.

If L_1 and L_2 are two special conformal Killing tensors then clearly $L_1 + L_2$ is also a special conformal Killing tensor. The torsion of $L_1 + L_2$ therefore vanishes; it follows that $[L_1, L_2] = 0$, and so the corresponding Poisson-Nijenhuis structures are compatible. If Z is a cofactor system with respect to both special conformal Killing tensors then it is Pfaffian quasi-Hamiltonian with respect to each Poisson-Nijenhuis structure and is therefore bi-quasi-Hamiltonian. The force form μ must satisfy $D_{L_1}\mu = D_{L_2}\mu = 0$.

The functions $H(t)$ and $F(t)$ in this case are polynomials in t, and the system is completely integrable; $H(t)$ takes the form

$$H(t) = \tfrac{1}{2}A(t)^{ij}p_i p_j + V(t),$$

where $A(t)$ is the cofactor tensor of $L_1 - tL_2$, and $V(t)$ is a polynomial of degree $n-1$ such that

$$D_{(L_1-tL_2)}(\det(L_1 - tL_2)^{-1}V(t)) = \mu.$$

For a given g, μ there are functions V_1, V_2 such that

$$D_{L_1}((\det L_1)^{-1}V_1) = D_{L_2}((\det L_2)^{-1}V_2) = \mu;$$

these are the 'generalized potentials' of μ with respect to L_1 and L_2 referred to in the Introduction.

Now consider $\nu = D_{L_2}((\det L_1)^{-1}V_1)$. We have

$$D_{L_1}\nu = -D_{L_2}D_{L_1}((\det L_1)^{-1}V_1) = -D_{L_2}\mu = 0,$$

and of course $D_{L_2}\nu = 0$: so ν defines a new bi-cofactor system, with the same metric but a different force form. The quadratic part of the new first integral function is unchanged; the 'potential' part $W(t)$ satisfies

$$D_{(L_1-tL_2)}(\det(L_1 - tL_2)^{-1}W(t)) = \nu.$$

In fact $W(t)$ is given in terms of $V(t)$ by

$$tW(t) = V(t) - \frac{\det(L_1 - tL_2)}{\det L_1}V(0).$$

Note that $V(0) = V_1$. The right hand side is a polynomial of degree n in t, which takes the value 0 when $t = 0$; so $W(t)$ is a polynomial of degree $n-1$. Moreover

$$\begin{aligned} &D_{(L_1-tL_2)}(\det(L_1 - tL_2)^{-1}\left(V(t) - \frac{\det(L_1 - tL_2)}{\det L_1}V(0)\right) \\ &\quad = \mu - D_{L_1}(\det(L_1)^{-1}V_1) + tD_{L_2}((\det L_1)^{-1}V_1) = t\nu \end{aligned}$$

as required. With these considerations, we have supplemented the analysis in Crampin (2001) with the appropriate generalization of Lundmark's construction (Lundmark (2001b)) of a hierarchy of cofactor pair systems.

7 Quasi-bi-Hamiltonian systems

The theory we have described above covers, of course, the special case in which the vector field Z is Hamiltonian, not just quasi-Hamiltonian, with respect to one of the two Poisson structures, so that (say) $F_1 = 1$ in the notation of Section 3. This case corresponds roughly to that of a quasi-bi-Hamiltonian system as defined and discussed in Brouzet (1996), Morosi (1997, 1998); however, the compatibility condition for a bi-quasi-Hamiltonian system, which reduces to $\mathcal{L}_{V_2}\Pi_1 = 0$, is not required to hold in the definition of a quasi-bi-Hamiltonian system given in those papers. But to the best of our knowledge, all of the examples of quasi-bi-Hamiltonian systems in the literature do fall within our framework: in fact they all appear to be cases of cofactor systems, for which the compatibility condition holds automatically anyway. We therefore feel justified in discussing briefly some of the results in the literature on quasi-bi-Hamiltonian systems in the light of our approach.

We wish first to compare our main integrability result with the following proposition, due to Tondo (Tondo (1995), Morosi (1997)). (The statement of the proposition has been edited to fit in with the notation and terminology of the present paper.)

Proposition Let M be a $2n$-dimensional manifold equipped with an invertible Poisson tensor P_1, and let Z be a Hamiltonian vector field with Hamiltonian H: $Z = P_1(dH)$. Let a tensor $J: TM \to TM$ exist such that the tensor $P_2: T^*M \to TM$ defined by $P_2 = JP_1$ is skew-symmetric. Denote by $\alpha_{(k)} = (J^*)^k dH$ $(k = 0, 1, 2, \ldots)$ the 1-forms obtained by the iterated action of J^*. If there exist $n-1$ independent functions $H_{(l)}$ $(l = 1, 2, \ldots, n)$ and $\frac{1}{2}n(n+1)$ functions $\rho_{(kl)}$ $(k = 0, 1, \ldots, n-1;\ 0 \le l \le k)$ with $\rho_{(00)} = 1$ and $\rho_{(kk)} \neq 0$ $(k \neq 0)$, such that the 1-forms $\alpha_{(k)}$ can be written as $\alpha_{(k)} = \sum_{l=0}^{k} \rho_{(kl)} dH_{(l)}$ $(k = 0, 1, \ldots, n-1)$, then the functions $H_{(k)}$ are in involution with respect to the Poisson bracket defined by P_1 and are first integrals of Z. The Hamiltonian system is Liouville integrable. Moreover, if P_2 is a Poisson tensor then the functions $H_{(k)}$ are in involution also with respect to the Poisson bracket defined by P_2.

In our analysis, we have assumed the existence of two non-singular Poisson tensors from the first; we may define J as $P_2{P_1}^{-1}$. On the other hand, we assume only that Z is quasi-Hamiltonian. To relate our results to Tondo's proposition we must show that the functions $H_{(k)}$ defined in Section 3 are related to the 1-forms $\alpha_{(k)} = (J^*)^k dH_{(0)}$ in the manner specified in the proposition. Now we can write

$$F_{(k+1)}Z = P_1(dH_{(k+1)}) - P_1(J^* dH_{(k)}),$$

and $F_{(0)}Z = P_1(dH_{(0)})$. Thus

$$dH_{(k+1)} = J^* dH_{(k)} + \hat{F}_{(k+1)} dH_{(0)},$$

where $\hat{F}_{(k)} = F_{(k)}/F_{(0)}$, from which it follows that

$$dH_{(k)} = \sum_{l=0}^{k} \hat{F}_{(k-l)}\alpha_{(l)}.$$

Note that the matrix of coefficients on the right-hand side is lower triangular with 1s on the diagonal. When this system of equations is solved for the $\alpha_{(k)}$ we obtain a relation of the required form. In fact we have $\rho_{(kk)} = 1$ for all k, not just for $k = 0$; and moreover $\rho_{(kl)} = \rho_{(ij)}$ when $k - l = i - j$. In Morosi (1997), Tondo's proposition is applied to a couple of examples of quasi-bi-Hamiltonian systems. It is a feature of these examples, not remarked on let alone explained in this paper, that the $\rho_{(ij)}$ follow the pattern identified above. While Tondo's proposition is rather more general than our result, so far as we know it has been applied only to quasi-bi-Hamiltonian systems, and indeed only to systems of cofactor type (as we have already remarked); so this greater generality is in practice somewhat illusory.

When one considers the examples in Morosi (1997) more carefully, one realises that it is not at all clear what role the fact that the systems in question are quasi-bi-Hamiltonian plays, since the expressions for the $H_{(k)}$ and $\rho_{(kl)}$ are merely quoted and no indication is given as to how they were derived. To clarify what is going on, we shall consider one of these examples, and analyse it using our techniques. We have chosen the first example from the paper for this purpose. It is the Hamiltonian system $Z = P_0(dH)$ where P_0 is the standard Poisson structure on $T^*\mathbb{R}^3$ and

$$H = \tfrac{1}{2}(2p_1p_2 + p_3^2) - \tfrac{5}{8}q_1^4 + \tfrac{5}{2}q_1^2q_2 + \tfrac{1}{2}q_1q_3^2 - \tfrac{1}{2}q_2^2.$$

It is stated that Z is also quasi-Hamiltonian with respect to the Poisson tensor P_1 given by

$$P_1 = \begin{bmatrix} 0 & A \\ -A^T & B \end{bmatrix}, \quad \text{where } A = -\begin{bmatrix} q_1 & -1 & 0 \\ 2q_2 & q_1 & q_3 \\ q_3 & 0 & 0 \end{bmatrix}, \quad B = -\begin{bmatrix} 0 & -p_2 & -p_3 \\ p_2 & p_3 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

in fact $q_3^2 Z = P_1(dH_{(2)})$ where

$$H_{(2)} = \tfrac{1}{2}q_3^2p_2^2 + \left(\tfrac{1}{2}q_1^2 + q_2\right)p_3^2 - q_3p_1p_3 - q_1q_3p_2p_3 + \tfrac{1}{2}q_1^3q_3^2 - q_1q_2q_3^2 - \tfrac{1}{8}q_3^4.$$

Note first that the terms quadratic in momenta in H are derived from the flat metric whose matrix representation is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is of normal hyperbolic type. When this metric is used to lower an index on A we obtain the symmetric matrix

$$-\begin{bmatrix} -1 & q_1 & 0 \\ q_1 & 2q_2 & q_3 \\ 0 & q_3 & 0 \end{bmatrix},$$

which is easily seen to be a special conformal Killing tensor. The tensor P_1 is, apart from sign, the Poisson-Nijenhuis tensor associated with the complete lift of the type $(1,1)$ tensor A. The determinant of A is $-q_3^2$, and its cofactor tensor, in type $(2,0)$ form, is

$$-\begin{bmatrix} 0 & 0 & -q_3 \\ 0 & q_3^2 & -q_1 q_3 \\ -q_3 & -q_1 q_3 & q_1^2 + 2q_2 \end{bmatrix};$$

it is clear that this determines the terms in $H_{(2)}$ quadratic in momenta. Moreover, it is easy to check that if V represents the potential of $H = H_{(0)}$ and W the potential of $H_{(2)}$ then $d_A W = (\det A) dV$. This is therefore an example of a cofactor system, and the preceding theory applies, explaining for example why the relevant functions $\rho_{(ij)}$ satisfy $\rho_{(11)} = \rho_{(22)} = 1$, $\rho_{(21)} = \rho_{(32)}$.

8 An example

We finish with an example of a true bi-quasi-Hamiltonian system. It is a cofactor pair system on the sphere and as such illustrates our generalization Crampin (2001) of the work on Euclidean spaces of Lundmark (Lundmark (2001a, b)). Starting from a known example of a kinetic energy Lagrangian with two further quadratic integrals, we will use a constructive approach to find all force forms which determine a non-conservative system of cofactor pair type.

Take the kinetic energy to be

$$T = \tfrac{1}{2} g_{ij}(q)\dot{q}_i \dot{q}_j = \tfrac{1}{2}\left(\dot{q}_1^2 + \sin^2 q_1 \dot{q}_2^2\right).$$

It is straightforward to verify that the symmetric type (0,2) tensors with matrix representation

$$L_1 = \begin{bmatrix} \sin 2q_1 \cos q_2 & -\sin^2 q_1 \sin q_2 \\ -\sin^2 q_1 \sin q_2 & 0 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} \sin 2q_1 \sin q_2 & \sin^2 q_1 \cos q_2 \\ \sin^2 q_1 \cos q_2 & 0 \end{bmatrix},$$

are special conformal Killing tensors with respect to the given metric. The corresponding type (1,1) tensors which one needs to set up the differential operators $D_{L_a} = d_{L_a} + \alpha_a$ read

$$\begin{aligned} L_1 &= \sin 2q_1 \cos q_2 \frac{\partial}{\partial q_1} \otimes dq_1 - \sin q_2 \left(\frac{\partial}{\partial q_1} \otimes dq_2 + \sin^2 q_1 \frac{\partial}{\partial q_2} \otimes dq_1 \right), \\ L_2 &= \sin 2q_1 \sin q_2 \frac{\partial}{\partial q_1} \otimes dq_1 + \cos q_2 \left(\frac{\partial}{\partial q_1} \otimes dq_2 + \sin^2 q_1 \frac{\partial}{\partial q_2} \otimes dq_1 \right). \end{aligned}$$

We further have $\alpha_a = d(\operatorname{tr} L_a)$, where $\operatorname{tr} L_1 = \sin 2q_1 \cos q_2$, $\operatorname{tr} L_2 = \sin 2q_1 \sin q_2$. All we need now to identify a dynamical system which has a bi-quasi-Hamiltonian representation with respect to the two compatible Poisson-Nijenhuis structures $P_{\tilde{L}_a}$, obtained from the standard Poisson structure P on the cotangent bundle of the sphere, is an admissible force form $\mu = M_1 dq_1 + M_2 dq_2$, that is to say, a solution of the simultaneous equations $D_{L_a}\mu = 0$. In coordinates, each of these conditions gives rise to a single, linear first-order partial differential equation for the M_a. Taking suitable linear combinations with $\sin q_2$ and $\cos q_2$, the set of conditions is equivalent to

$$\begin{aligned}
\frac{\partial M_1}{\partial q_1} &= \sin^2 q_1 \frac{\partial M_2}{\partial q_2}, \\
\frac{\partial M_2}{\partial q_1} &= \left(\tfrac{3}{2}\tan q_1 - \cot q_1\right) M_2.
\end{aligned}$$

These equations are readily solved and give:

$$\mu = \frac{2\rho'(q_2)}{\cos^{1/2} q_1}\, dq_1 + \frac{\rho(q_2)}{\cos^{3/2} q_1 \sin q_1}\, dq_2,$$

where ρ is an arbitrary function of q_2. The 'potentials' V_a for which $\mu = D_{L_a}((\det L_a)^{-1} V_a)$ are most easily obtained by computing $dV_a = A_a \mu$, where A_a is the cofactor matrix of L_a. We find

$$V_1 = \frac{2\rho(q_2)\sin q_2}{\cos^{1/2} q_1}, \qquad V_2 = -\frac{2\rho(q_2)\cos q_2}{\cos^{1/2} q_1}.$$

Since we have actually obtained the most general form of μ, we should not expect here to find new cofactor pair systems by applying the procedure explained in Section 6. In fact, we have

$$\nu = D_{L_2}((\det L_1)^{-1} V_1) = (\det L_2)^{-1} d_{L_2}\left(\frac{\det L_2}{\det L_1} V_1\right),$$

and since the ratio of the two determinants depends on q_2 only, it can be absorbed into the arbitrary function ρ; thus ν is indistinguishable from μ.

References

Brouzet, R., R. Caboz, J. Rabenivo and V. Ravoson, "Two degrees of freedom quasi-Hamiltonian systems", J. Phys. A: Math. Gen. 29, 2069–2076 (1996).

Crampin, M. and W. Sarlet, "A class of non-conservative Lagrangian systems on Riemannian manifolds", J. Math. Phys. 42, 4313–4326 (2001)

Crampin, M., W. Sarlet and G. Thompson, "Bi-differential calculi, bi-Hamiltonian systems and conformal Killing tensors", J. Phys. A: Math. Gen. 33, 8755–8770 (2000)

Dimakis, A. and F. Müller-Hoissen, "Bi-differential calculi and integrable models", J. Phys. A: Math. Gen. 33, 957–974 (2000a).

Dimakis, A. and F. Müller-Hoissen, "Bicomplexes and integrable models", J. Phys. A: Math. Gen. 33, 6579–6591 (2000b).

Frölicher, A. and A. Nijenhuis, "Theory of vector-valued differential forms", Proc. Ned. Acad. Wetensch. Ser. A 59, 338–359 (1956).

Kosmann-Schwarzbach, Y. and F. Magri, "Poisson-Nijenhuis structures", Ann. Inst. Henri Poincaré, Phys. Théor. 5335–5381 (1990).

Lundmark, H., "A new class of integrable Newton systems", J. Nonlinear Math. Phys. 8 (Suppl.), 195–199 (2001a).

Lundmark, H., "Newton Systems of Cofactor Type in Euclidean and Riemannian Spaces", PhD thesis, Linköping University (Linköping Studies in Science and Technology Dissertations No. 719), especially Paper 1 (2001b).

Magri, F. and C. Morosi, "A Geometric Characterization of Integrable Hamiltonian Systems Through the Theory of Poisson-Nijenhuis Manifolds", Università di Milano, Quaderno S 19 (1984).

Magri, F., C. Morosi and O. Ragnisco, "Reduction techniques for infinite-dimensional Hamiltonian systems: some ideas and applications", Comm. Math. Phys. 99, 115–140 (1985).

Marciniak, K. and S. Rauch-Wojciechowski, "Two families of non-standard Poisson structures for Newton equations", J. Math. Phys. 39, 5292–5306 (1998).

Nunes da Costa, J. M. and C.-M. Marle, "Reduction of bihamiltonian manifolds and recursion operators", in "Differential Geometry and its Applications (Proc. Conf. Brno, August 1995)", ed. J. Janyška, I. Kolář and J. Slovák, 523–538 (1996).

Morosi, C. and M. Tondo, "Quasi-bi-Hamiltonian systems and separability", J. Phys. A: Math. Gen. 30, 2799–2806 (1997).

Morosi, C. and M. Tondo, "On a class of dynamical systems both quasi-bi-Hamiltonian and bi-Hamiltonian", Phys. Lett. A 247, 59–64 (1998).

Rauch-Wojciechowski, S., K. Marciniak and H. Lundmark, "Quasi-Lagrangian systems of Newton equations", J. Math. Phys. 40, 6366–6398 (1999).

Tondo, G., "On the integrability of stationary and restricted flows of the KdV hierarchy", J. Phys. A: Math. Gen. 28, 5079–5115 (1995).

Vaisman, I., "Lectures on the Geometry of Poisson Manifolds", Basel: Birkhäuser, p. 64 (1994).