A Lie algebroid approach to Lagrangian systems with symmetry

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Abstract. The Euler-Lagrange equations of a Lagrangian which is invariant under a Lie group action can be reduced to the so-called Lagrange-Poincaré equations. This set of equations can be seen to fall apart into a 'horizontal' and a 'vertical' equation. In this paper, we will obtain a new intrinsic characterization of both equations, based on a decomposition of the Lie algebroid structure of the carrying space. We show that also in the Hamiltonian framework horizontal and vertical equations appear.

Keywords. Lie algebroids, connections, reduction, systems with symmetry, Lagrange-Poincaré equations, Hamilton-Poincaré equations.

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1 Introduction

Consider a mechanical system on a manifold Q whose Lagrangian l is invariant under an action of a Lie group G, i.e. $l(gv_q) = l(v_q)$. l gives rise to a Lagrangian function $\bar{l}$ on TQ/G, defined by means of $\bar{l}([v_q]) = l(v_q)$ and the Euler-Lagrange equations of the system can be reduced to the so-called Lagrange-Poincaré equations[1] for the reduced Lagrangian (see e.g. [5]). For any principal connection A on $\pi_G : Q \to M = Q/G$, it is possible to construct an isomorphism α_A between the spaces TQ/G and $TM \oplus \tilde{\mathfrak{g}}$, where $\tilde{\mathfrak{g}}$ is the total manifold of the adjoint bundle $\tau : \tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G \to M$. The reduced Lagrangian can then be regarded as a function L on $TM \oplus \tilde{\mathfrak{g}}$, defined by $L(\dot{x}, \mathsf{v}) = \bar{l}(\alpha_A{}^{-1}(\dot{x}, \mathsf{v}))$. The Lagrange-Poincaré equations take then the form

$$\left\{\begin{array}{rcl} \dfrac{d}{dt}\dfrac{\partial L}{\partial \mathsf{v}^a} & = & -\dfrac{\partial L}{\partial \mathsf{v}^b}\left(C^b_{ad}\mathsf{v}^d - \Gamma^b_{ja}\dot{x}^j\right), \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}^i} - \dfrac{\partial L}{\partial x^i} & = & -\dfrac{\partial L}{\partial \mathsf{v}^b}\left(\Gamma^b_{ic}\mathsf{v}^c - \omega^b_{ij}\dot{x}^j\right). \end{array}\right. \tag{1}$$

where C^c_{ab} are the structure constants of the Lie algebra $\mathfrak{g}$, ω^a_{ij} are the curvature coefficients of A and Γ^b_{ia} are the connection coefficients of the associated connection (for more details, see the next section).

One of the advantages of a description of the Lagrange-Poincaré equations on $TM \oplus \tilde{\mathfrak{g}}$ is that they can be seen to fall apart into two distinguished sets of equations. Indeed, the above separation of (1) does not depend on the choice of the coordinates and the *vertical* and *horizontal* equations are, respectively, the first and second equations. In [5], Cendra *et al.* gave an intrinsic formulation of

[1] We will use the terminology of e.g. [3, 4, 5].

these two equations. However, in the explicit expression of the horizontal and vertical equations, an additional linear connection ∇^M on M needed to be invoked, even though the equations are in fact independent of the choice of such a connection. In this paper, we will present a new intrinsic description of the horizontal and the vertical Lagrange-Poincaré equations for which no extra connection ∇^M on M is required.

The framework in which we will situate the Lagrange-Poincaré equations will be different from the one in [5]. Both TQ/G and $TM \oplus \tilde{\mathfrak{g}}$ can be given the structure of vector bundles over M with projections τ_Q/G and π, respectively. This paper aims to fully exploit the observation that the bundles τ_Q/G and π carry also a Lie algebroid structure. The idea that the carrying space of Lagrangian systems with symmetry is a Lie algebroid is not new. Weinstein showed (Corollary 4.6 in [15]) that the reduced equations fall in the category of so-called *Lagrange equations on a Lie algebroid* (see also Theorem 9.7 in [6]). In general, if $\pi : V \to M$ is a Lie algebroid with structure functions ρ^i_α and $D^\gamma_{\alpha\beta}$ and if $L(x, y) \in C^\infty(V)$, then a dynamical system of the form

$$\begin{cases} \dot{x}^i = \rho^i_\alpha(x) y^\alpha, \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial y^\alpha}\right) = \rho^i_\alpha \dfrac{\partial L}{\partial x^i} - D^\gamma_{\alpha\beta} y^\beta \dfrac{\partial L}{\partial y^\gamma}. \end{cases} \tag{2}$$

is called a *Lagrangian system on the Lie algebroid* π. Due to Martínez [10], we know that it is convenient to *extend* the Lie algebroid structure to a certain *prolongation* bundle and to look at Lagrangian systems as sections of the prolongation bundle.

The Lie algebroid structure on $\tau_Q/G : TQ/G \to M$ (or on $\pi : TM \oplus \tilde{\mathfrak{g}} \to M$) which turns (2) into (1) is the so-called Atiyah algebroid. We will show that the horizontal and vertical equations appear, within the framework of [10], due to a natural decomposition of the Lie algebroid structure on π. At the end of the paper, we will investigate the Hamiltonian counterpart of the above method. In [4], Hamilton-Poincaré equations have been introduced. We will show that also horizontal and vertical equations can be identified.

2 Natural constructions on principal fibre bundles

In this section we recall some basic facts about Atiyah algebroids. For proofs and detailed calculations we refer to [1, 5, 9].

Definition 1. *A Lie algebroid is a vector bundle $\pi : V \to M$, which comes equipped with a bracket operation $[\cdot,\cdot] : \mathrm{Sec}(\pi) \times \mathrm{Sec}(\pi) \to \mathrm{Sec}(\pi)$ and a linear bundle map $\rho : V \to TM$ (and its extension $\rho : \mathrm{Sec}(\pi) \to \mathcal{X}(M)$), which are related in such a way that (i) $[\cdot,\cdot]$ is a real Lie algebra bracket on the vector space* $\mathrm{Sec}(\pi)$*; (ii) ρ satisfies for all $s, r \in \mathrm{Sec}(\pi)$, $f \in C^\infty(M)$:*

$$[s, fr] = f[s, r] + \rho(s)(f)\, r.$$

Locally, if $\{e_\alpha\}$ is a basis for $\mathrm{Sec}(\pi)$ with adapted coordinates $(x^i, y^\alpha) \in V$, then the structure functions ρ^i_α are the coefficients of the anchor map ρ. The structure functions $D^\gamma_{\alpha\beta}$ are given by $[e_\alpha, e_\beta] = D^\gamma_{\alpha\beta} e_\gamma$.

Let $\pi_G : Q \to M$ be a principal fibre bundle with structure group G. Elements of the manifold $(TQ)/G$ are equivalent classes under the induced action of G on TQ. The map $\tau_Q/G : TQ/G \to M, [v_q] \mapsto [q]$ gives TQ/G the structure of a vector bundle, where, for $v_q, u_q \in T_qQ$ and $a \in \mathbb{R}$,

$$a[v_q] = [av_q] \qquad \text{and} \qquad [v_q] + [u_q] = [v_q + u_q]. \tag{3}$$

We will show next that $TQ/G \to M$ carries in fact a Lie algebroid structure. The anchor map $\rho : TQ/G \to TM$ of this Lie algebroid is given by $\rho([v_q]) = T\pi_G(v_q)$. We will follow here the approach of [5] to define the bracket (for slightly different approaches see e.g. [1, 9, 15]). Let's look at the projection $\Pi : TQ \to TQ/G$ over π_G. Expression (3) is essentially saying that the restriction $\Pi_q : \tau_Q^{-1}(q) \to (\tau_Q/G)^{-1}([q])$ is a linear isomorphism for each $q \in Q$ (with inverse Π_q^{-1}). Therefore

$$\Pi^*(\sigma)(q) = \Pi_q^{-1}(\sigma(\pi_G(q))), \qquad \sigma \in \mathrm{Sec}(\tau_Q/G)$$

defines an invariant vector field (i.e. $X \in \mathcal{X}^I(Q)$ if $X(gq) = gX(q)$, for the tangent lift of the action of G on Q). In fact, $\Pi^* : \mathrm{Sec}(\tau_Q/G) \to \mathcal{X}^I(Q)$, is a linear isomorphism. It is even possible to define the Lie algebroid bracket for sections σ_i on $\tau_Q/G : TQ/G \to M$ as the pullback, under the isomorphism Π^*, of the natural bracket of vector fields on Q, i.e.

$$[\sigma_1, \sigma_2] = (\Pi^*)^{-1}[\Pi^*(\sigma_1), \Pi^*(\sigma_2)] \tag{4}$$

The above Lie algebroid structure is the *Atiyah algebroid*.

The adjoint action of G on $\mathfrak{g}$ leads to an induced action of G on $Q \times \mathfrak{g}$, so it makes sense to speak of the equivalence class $[q \cdot \xi]$ of a $(q, \xi) \in Q \times \mathfrak{g}$. The projection $\tau : \tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G \to M$, given by $\tau([q \cdot \xi]) = \pi_G(q)$ defines a surjective submersion which gives τ the structure of a vector bundle: let $[q \cdot \xi_1]$ and $[q \cdot \xi_2]$ be elements of the same fibre $\tau^{-1}([q])$, then we define

$$a[q \cdot \xi_1] = [q \cdot a\xi_1] \qquad \text{and} \qquad [q \cdot \xi_1] + [q \cdot \xi_2] = [q \cdot \xi_1 + \xi_2].$$

The bundle τ is often called *the adjoint bundle*. It can be given the structure of a *Lie algebra bundle* (for a definition of a Lie algebra bundle, see [9]). The Lie algebra structure on a fibre $\tilde{\mathfrak{g}}_x$ is given by

$$[[q \cdot \xi_1], [q \cdot \xi_2]] = [q \cdot [\xi_1, \xi_2]] \qquad \pi_G(q) = x.$$

In particular, a Lie algebra bundle is a Lie algebroid with zero anchor map.

In the introduction we mentioned an isomorphism between TQ/G and $TM \oplus \tilde{\mathfrak{g}}$. Let $A : TQ \to \mathfrak{g}$ be a principal connection (for a definition see e.g. [7]). The map $\alpha_A : TQ/G \to TM \oplus \tilde{\mathfrak{g}}$ given by

$$\alpha_A([v_q]) = T\pi_G(v_q) \oplus [q \cdot A(v_q)]$$

is a well defined vector bundle isomorphism (see e.g. [5, 9]). This observation is the key ingredient in the approach of [5]. Any available principal connection can now be used to transform the Lie algebroid (4) on τ_Q/G into a Lie algebroid on the vector bundle $\pi : TM \oplus \tilde{\mathfrak{g}} \to M$: the new bracket is given by

$$[s_1, s_2] = \alpha_A\big([\alpha_A^{-1}(s_1), \alpha_A^{-1}(s_2)]\big), \qquad s_1, s_2 \in \mathrm{Sec}(\pi) \tag{5}$$

and the anchor map is the projection $\rho : TM \oplus \tilde{\mathfrak{g}} \to TM$. It is possible to give an explicit expression for the bracket (5).

Locally π_G is of the form $X \times G \to X$, with $X \subset \mathbb{R}^n$ open. Choose maps $e_a : X \to \mathfrak{g}$, such that for each x, $\{e_a(x)\}$ is a basis for $\mathfrak{g}$ and let $A(\frac{\partial}{\partial x^i}|_{(x,e)}) = A_i^a(x) e_a(x)$. $\tau : \tilde{\mathfrak{g}} \to M$ is locally $X \times \mathfrak{g} \to X$. Then, the sections e_a defined by $\mathsf{e}_a(x) = [(x, e) \cdot e_a(x)] = (x, e_a(x))$ form a local basis of $\mathrm{Sec}(\tau)$. In this basis, the coefficients of the Lie algebra bundle are exactly the coefficients C_{ab}^c of the Lie algebra $\mathfrak{g}$. We further define a $\tilde{\mathfrak{g}}$-valued two-form ω on M

$$\omega(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})(x) = \omega_{ij}^a \mathsf{e}_a(x) \qquad \text{with} \qquad \omega_{ij}^a = \frac{\partial A_j^a}{\partial x^i} - \frac{\partial A_i^a}{\partial x^j} + C_{bc}^a A_j^b A_i^c,$$

which is clearly related to the curvature of A. Finally, the so-called *associated linear connection* ∇ *on* τ can be defined locally by

$$\nabla_{\frac{\partial}{\partial x^i}} \mathsf{e}_a = \Gamma^b_{ia} \mathsf{e}_b = C^b_{ad} A^d_i \mathsf{e}_b. \tag{6}$$

Intrinsic definitions of the above objects can be found in e.g. [5, 9]. On $\pi : TM \oplus \tilde{\mathfrak{g}} \to M$, sections are of the form $s = X \oplus \mathsf{s}$, with $X \in \mathcal{X}(M)$ and $\mathsf{s} \in \mathrm{Sec}(\tau)$. We can now state Theorem 5.2.4. of [5].

Proposition 1. *An explicit expression of the bracket (5) is*

$$[X_1 \oplus \mathsf{s}_1, X_2 \oplus \mathsf{s}_2] = [X_1, X_2] \oplus \big(\nabla_{X_1}\mathsf{s}_2 - \nabla_{X_2}\mathsf{s}_1 - \omega(X_1, X_2) + [\mathsf{s}_1, \mathsf{s}_2]\big), \tag{7}$$

(for $X_i \in \mathcal{X}(M)$ and $\mathsf{s}_i \in \mathrm{Sec}(\tau)$).

The coefficients of the anchor map ρ are thus $\rho^i_j = \delta^i_j$ and $\rho^i_a = 0$. The bracket (7) is locally given by

$$[\mathsf{e}_i, \mathsf{e}_j] = -\omega^c_{ij}\mathsf{e}_c, \qquad [\mathsf{e}_i, \mathsf{e}_a] = \Gamma^c_{ia}\mathsf{e}_c, \qquad [\mathsf{e}_a, \mathsf{e}_b] = C^c_{ab}\mathsf{e}_c. \tag{8}$$

From these expressions it is clear that the system (1) is a Lagrangian system (2) on the Lie algebroid (7).

3 Almost-Lie algebroids

In what follows, we will use geometric structures that are a little more general than Lie algebroids.

Definition 2. *An almost-Lie algebroid on $\pi : V \to M$ has all the properties of a Lie algebroid, except that the Jacobiator*

$$J(s, t, r) = [s, [t, r]] + [t, [r, s]] + [r, [s, t]], \qquad s, r, t \in \mathrm{Sec}(\pi),$$

is $C^\infty(M)$-linear, but not necessarily zero.

The linearity of the Jacobiator J is equivalent with the property that the anchor map ρ is a Lie algebra homomorphism. A k-form on $\mathrm{Sec}(\pi)$ is a skew-symmetric, $C^\infty(M)$-linear map $\theta : \mathrm{Sec}(\pi) \times \cdots \times \mathrm{Sec}(\pi) \to C^\infty(M)$ (with k-arguments). A k-form ω on π is locally of the form

$$\omega = \omega_{\alpha_1 \ldots \alpha_k} e^{\alpha_1} \wedge \ldots \wedge e^{\alpha_k} \in \bigwedge^k(\pi)$$

($\{e^\alpha\}$ being the dual basis of $\{e_\alpha\}$). For any k-form, the expression

$$\begin{aligned} d\theta(s_1, \ldots, s_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i-1} \rho(s_i)\Big(\theta(s_1, \ldots, \hat{s}_i, \ldots, s_{k+1})\Big) \\ &\quad + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \theta([s_i, s_j], s_1, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_{k+1}) \end{aligned} \tag{9}$$

defines a $(k+1)$-form. The operator d will be called the *exterior derivative.* d is completely determined by its action on functions and 1-forms. It is easy to see that

$$dx^i = \rho^i_\alpha e^\alpha, \qquad de^\gamma = -\frac{1}{2} D^\gamma_{\alpha\beta} e^\alpha \wedge e^\beta.$$

Remark that $d^2 = 0$ is valid only for functions. The property that $d^2 = 0$ for 1-forms (and by induction then also for forms of arbitrary order) is equivalent with the vanishing of the Jacobiator J of the bracket, and therefore π is a Lie algebroid iff $d^2 = 0$.

Let's now come back to the Lie algebroid (7). It is easy to see that the bracket can be decomposed in three almost-Lie algebroid brackets (with the same anchor map ρ)

$$[X_1 \oplus \mathsf{s}_1, X_2 \oplus \mathsf{s}_2] = [X_1 \oplus \mathsf{s}_1, X_2 \oplus \mathsf{s}_2]^1 + [X_1 \oplus \mathsf{s}_1, X_2 \oplus \mathsf{s}_2]^2 - [X_1 \oplus \mathsf{s}_1, X_2 \oplus \mathsf{s}_2]^3$$

where

$$\begin{aligned}
[X_1 \oplus \mathsf{s}_1, X_2 \oplus \mathsf{s}_2]^1 &= [X_1, X_2] \oplus \big(\nabla_{X_1}\mathsf{s}_2 - \nabla_{X_2}\mathsf{s}_1 + [\mathsf{s}_1, \mathsf{s}_2]\big),\\
[X_1 \oplus \mathsf{s}_1, X_2 \oplus \mathsf{s}_2]^2 &= [X_1, X_2] \oplus \big(\nabla_{X_1}\mathsf{s}_2 - \nabla_{X_2}\mathsf{s}_1 - \omega(X_1, X_2)\big),\\
[X_1 \oplus \mathsf{s}_1, X_2 \oplus \mathsf{s}_2]^3 &= [X_1, X_2] \oplus \big(\nabla_{X_1}\mathsf{s}_2 - \nabla_{X_2}\mathsf{s}_1\big).
\end{aligned}$$

Although $[.,.]$ is a Lie algebroid, it is not true that also the brackets $[.,.]^i$ are Lie algebroids: their Jacobiator fails to vanish. Let R^∇ and d^∇ be, respectively, the curvature and the covariant exterior derivative of ∇ (see [8]). Let the adjoint operator of the Lie algebra bundle τ be given by $ad(\mathsf{r})\mathsf{s} = [\mathsf{r}, \mathsf{s}]$.

Proposition 2. $[.,.]^3$ *is a Lie algebroid iff the connection* ∇ *is flat, i.e.* $R^\nabla = 0$. $[.,.]^1$ *is a Lie algebroid iff* ∇ *is flat and ad is parallel, i.e.* $\nabla ad = 0$. $[.,.]^2$ *is a Lie algebroid iff* ∇ *is flat and* ω *is* ∇*-closed, i.e.* $d^\nabla\omega = 0$. *If* $[.,.]$ *is a Lie algebroid and moreover* $i_\omega ad = 0$*, then all brackets are Lie algebroids. Conversely, if* $[.,.]^1, [.,.]^2$ *and* $[.,.]^3$ *are Lie algebroids and* $i_\omega ad = 0$*, then also* $[.,.]$ *is a Lie algebroid.*

In local coordinates, the brackets take the form

$$\begin{array}{lll}
[e_i, e_j]^1 = 0, & [e_i, e_a]^1 = \Gamma^c_{ia} e_c, & [e_a, e_b]^1 = C^c_{ab} e_c,\\
[e_i, e_j]^2 = -\omega^c_{ij} e_c, & [e_i, e_a]^2 = \Gamma^c_{ia} e_c, & [e_a, e_b]^2 = 0,\\
[e_i, e_j]^3 = 0, & [e_i, e_a]^3 = \Gamma^c_{ia} e_c, & [e_a, e_b]^3 = 0.
\end{array}$$

Of course, each of the above almost-Lie algebroids induces its own exterior derivative.

Proposition 3. *For every form* θ *on* $\pi : TM \oplus \tilde{\mathfrak{g}} \to M$*, we can write*

$$d\theta = d^1\theta + d^2\theta - d^3\theta,$$

where d^i *stands for the exterior derivative of the almost-Lie algebroid* $[.,.]^i$.

4 Horizontal and vertical Lagrange-Poincaré equations

The core idea of Martínez's approach to Lagrangian systems (2) on Lie algebroids is that the dynamics should be thought of as a section of a certain *prolongation bundle*. Let $\pi : V \to M$ be an (almost-) Lie algebroid and $\mu : W \to M$ an arbitrary fibre bundle. The *ρ-prolongation of* W is the bundle $\mu^\rho : T^\rho W \to W$, where $T^\rho W = \rho^* TW$, i.e. $(v, X_w) \in T^\rho W$ if $v \in V$ and $X_w \in T_w W$ are such that $T\mu(X_w) = \rho(v)$. The projection μ^ρ is then given by $\mu^\rho(v, X_w) = \tau_W(X_w) = w$ (see also [12]). In the diagram in Figure 1 also the projections μ^2 and ρ^μ on the composing parts of $T^\rho P$ have been drawn.

$$
\begin{tikzcd}
 & TW \arrow[rd, "\tau_W"] \arrow[dd] & \\
T^\rho W \arrow[ru, "\rho^\mu"] \arrow[rr, "\mu^\rho"'] \arrow[dd, "\mu^2"'] & & W \arrow[dd, "\mu"] \\
 & TM \arrow[rd, "\tau_M"] & \\
V \arrow[ru, "\rho"] \arrow[rr, "\pi"'] & & M
\end{tikzcd}
$$

Figure 1: The ρ-prolongation of μ.

The following theorem defines the extension of the almost-Lie algebroid π to the prolongation μ^ρ. A section $\mathcal{Z}$ of μ^ρ is said to be projectable if there exists a section $s \in \mathrm{Sec}(\pi)$ such that $\mu^2 \circ \mathcal{Z} = s \circ \mu$. Remark that $\mathrm{Sec}(\mu^\rho)$ is (locally) finitely generated, over the ring $C^\infty(W)$, by projectable sections.

Proposition 4. *[12] Let π be an almost-Lie algebroid. The vector bundle μ^ρ inherits an almost-Lie algebroid structure from the one on V and the standard Lie algebroid structure on TW. The anchor map is $\rho^\mu : T^\rho W \to TW, \rho^1(v, X_w) = X_w$, and the bracket can be defined in terms of projectable sections as follows. If $\mathcal{Z}_1$, $\mathcal{Z}_2$ are two projectable sections of μ^ρ given by $\mathcal{Z}_k(w) = (s_k(m), X_k(w))$, $k = 1, 2$ for some sections s_k of τ and X_k of τ_W, then the bracket $[\mathcal{Z}_1, \mathcal{Z}_2]$ is the section given by*

$$[\mathcal{Z}_1, \mathcal{Z}_2](w) = ([s_1, s_2](m), [X_1, X_2](w)). \tag{10}$$

By construction, if π is, in particular, a Lie algebroid, then so is also the prolongation. Let's use coordinates (x^i, w^A) for W. The bases $\{e_\alpha\}$ of $\mathrm{Sec}(\pi)$ and $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial w^A}\}$ of $\mathcal{X}(W)$ induce a basis $\{\mathcal{X}_\alpha, \mathcal{V}_A\}$ for $\mathrm{Sec}(\mu^\rho)$, where

$$\mathcal{X}_\alpha(w) = \left(e_\alpha(\mu(w)), \rho^i_\alpha \frac{\partial}{\partial x^i}\bigg|_w \right) \qquad \text{and} \qquad \mathcal{V}_A(w) = \left(0(\mu(w)), \frac{\partial}{\partial w^A}\bigg|_w \right). \tag{11}$$

Sections in the span of $\{\mathcal{V}_A\}$ are called *vertical sections*. The set of all *vertical elements* of $T^\rho W$, i.e. those whose projection on V, via μ^2, vanishes, will be denoted by $\mathcal{V}^\rho V$. W.r.t. the basis (11), the anchor map ρ^μ of the Lie algebroid structure on μ^ρ is given by

$$\rho^\mu(\mathcal{X}_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}, \qquad \text{and} \qquad \rho^\mu(\mathcal{V}_A) = \frac{\partial}{\partial w^A},$$

while the extended bracket of the Lie algebroid structure on μ^ρ is of the form

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = D^\gamma_{\alpha\beta} \mathcal{X}_\gamma, \qquad [\mathcal{X}_\alpha, \mathcal{V}_A] = 0 \quad \textit{and} \quad [\mathcal{V}_A, \mathcal{V}_B] = 0. \tag{12}$$

Proposition 5. *Consider the Lie algebroid structure (7) on $\pi : V = TM \oplus \tilde{\mathfrak{g}} \to M$ and an arbitrary fibre bundle $\mu : W \to M$. The extended Lie algebroid $[.,.]$ on $\mu^\rho : T^\rho W \to W$ is made up from the extensions of the almost-Lie algebroids $[.,.]^1$,$[.,.]^2$ and $[.,.]^3$, i.e.*

$$[\mathcal{Z}_1, \mathcal{Z}_2] = [\mathcal{Z}_1, \mathcal{Z}_2]^1 + [\mathcal{Z}_1, \mathcal{Z}_2]^2 - [\mathcal{Z}_1, \mathcal{Z}_2]^3$$

for $\mathcal{Z}_i \in \mathrm{Sec}(\pi^\rho)$.

From the expression (11) for the projectable sections, the proof is obvious. For the remaining part of this section, we will assume that also μ is $\pi : V = TM \oplus \tilde{\mathfrak{g}} \to M$. The elements of the basis (11) are then $\{\mathcal{X}_i, \mathcal{X}_a, \mathcal{V}_i, \mathcal{V}_a\}$. If $\{\mathcal{X}^i, \mathcal{X}^a, \mathcal{V}^i, \mathcal{V}^a\}$ is the dual basis of 1-forms, then local expressions for the exterior derivative d on $\bigwedge(\pi^\rho)$ (see expression (9)) are given by

$$
\begin{aligned}
&dx^i = \mathcal{X}^i, \qquad d\dot{x}^i = \mathcal{V}^i, \qquad d\mathsf{v}^a = \mathcal{V}^a, \\
&d\mathcal{X}^i = 0, \qquad d\mathcal{X}^a = \frac{1}{2}\omega_{ij}\mathcal{X}^i \wedge \mathcal{X}^j - \Gamma^a_{ib}\mathcal{X}^i \wedge \mathcal{X}^b - \frac{1}{2}C^a_{bc}\mathcal{X}^b \wedge \mathcal{X}^c, \\
&d\mathcal{V}^i = 0, \qquad d\mathcal{V}^a = 0.
\end{aligned}
$$

Corollary 1. *For every form θ on π^ρ, we can write*

$$d\theta = d^1\theta + d^2\theta - d^3\theta,$$

where d^i stands for the exterior derivative of the extended almost-Lie algebroids on π^ρ.

All d^i have the same action on functions. For 1-forms,

$$
\begin{aligned}
&d^1\mathcal{X}^i = 0, \quad d^1\mathcal{X}^a = -\Gamma^a_{ib}\mathcal{X}^i \wedge \mathcal{X}^b + \frac{1}{2}C^a_{bc}\mathcal{X}^b \wedge \mathcal{X}^c, \quad d^1\mathcal{V}^i = 0, \quad d^1\mathcal{V}^a = 0, \\
&d^2\mathcal{X}^i = 0, \quad d^2\mathcal{X}^a = \frac{1}{2}\omega_{ij}\mathcal{X}^i \wedge \mathcal{X}^j - \Gamma^a_{ib}\mathcal{X}^i \wedge \mathcal{X}^b, \quad d^2\mathcal{V}^i = 0, \quad d^2\mathcal{V}^a = 0. \\
&d^3\mathcal{X}^i = 0, \quad d^3\mathcal{X}^a = -\Gamma^a_{ib}\mathcal{X}^i \wedge \mathcal{X}^b, \quad d^3\mathcal{V}^i = 0, \quad d^3\mathcal{V}^a = 0.
\end{aligned}
$$

We now introduce some useful canonical objects that live on $T^\rho V$. Let's look first at the set of vertical sections. There exists a naturally defined *vertical lift* ${}^V : \pi^*V \to \mathcal{V}^\rho V \subset T^\rho V$. Indeed, if $(v_0, v) \in \pi^*V$, then we can define an element $(v_{v_0})^v \in T_{v_0}V$ by means of its action on functions $f \in C^\infty(V)$,

$$(v_{v_0})^v(f) = \frac{d}{dt} f(v_0 + tv)\bigg|_{t=0}.$$

The required $(v_0, v)^V \in \mathcal{V}^\rho V$ is then $(0, (v_{v_0})^v)$. The definition, of course, extends to the level of sections. We won't make a notational difference, between $X \in \mathcal{X}(M)$ and $X \oplus 0 \in \mathrm{Sec}(\pi)$, so the meaning of X^V as a section of the prolongation bundle π^ρ should be clear. Then, $e_i^V = \mathcal{V}_i$ and $e_a^V = \mathcal{V}_a$. It is well-known that there exists a *canonical section* on $\pi^*\pi$, $\mathbf{T} = v^\alpha e_\alpha = \dot{x}^i e_i + \mathsf{v}^a e_a$. It can be decomposed into two sections $\mathbf{T}_{TM} = \dot{x}^i e_i$ and $\mathbf{T}_{\tilde{\mathfrak{g}}} = \mathsf{v}^a e_a$. Their two vertical lifts, $\mathcal{C}_{TM} = \dot{x}^i \mathcal{V}_i$ and $\mathcal{C}_{\tilde{\mathfrak{g}}} = \mathsf{v}^a \mathcal{V}_a$, add up to the Liouville section $\mathcal{C} = v^\alpha \mathcal{V}_\alpha \in \mathrm{Sec}(\pi^\rho)$.

The fibre linear map $j : T^\rho V \to \pi^*V : (v, X_{v_0}) \mapsto (v_0, v)$ is surjective and its kernel is exactly the set of vertical elements $\mathcal{V}^\rho V$. The composition of j with the vertical lift gives a second important concept, that of the *vertical endomorphism* $S = {}^V \circ j = \mathcal{X}^\alpha \otimes \mathcal{V}_\alpha$. It is a $\mathrm{Sec}(\pi^\rho)$-valued 1-form on $\mathrm{Sec}(\pi^\rho)$. The elements $j_{TM}(\mathcal{Z})$ and $j_{\tilde{\mathfrak{g}}}(\mathcal{Z})$ are the projections of $j(\mathcal{Z})$ onto $\pi^*(TM \oplus \{0\}) = \pi^*TM$ and $\pi^*(\{0\} \oplus \tilde{\mathfrak{g}}) = \pi^*\tilde{\mathfrak{g}}$, respectively. We then obtain the two composing parts $S_{\tilde{\mathfrak{g}}} = {}^V \circ j_{\tilde{\mathfrak{g}}} = \mathcal{X}^a \otimes \mathcal{V}_a$ and $S_{TM} = {}^V \circ j_{TM} = \mathcal{X}^i \otimes \mathcal{V}_i$ of S.

Next to the vertical lift, there is also a second natural lift on (almost)-Lie algebroids, *the complete lift*. We will only give a coordinate expression here (for more details, see [10]). If $s = s^\alpha e_\alpha$ is a section of π, then

$$s^C = s^\alpha \mathcal{X}_\alpha + (\rho^i_\beta v^\beta \frac{\partial s^\alpha}{\partial x^i} - D^\alpha_{\beta\delta} s^\beta v^\delta)\mathcal{V}^\alpha \ \in \mathrm{Sec}(\pi^\rho).$$

We have shown that for $\pi : TM \oplus \tilde{\mathfrak{g}} \to M$, there are four almost-Lie algebroid structures and therefore we have to choose which one we will use to define the complete lift. In the following, the complete lift will be constructed by means of the easiest bracket, namely $[.,.]^3$. Then, the complete lifts of $X \in \mathcal{X}(M)$ and $\mathsf{r} \in \mathrm{Sec}(\tau)$ are

$$\begin{aligned} X^C &= X^i \mathcal{X}_i + \frac{\partial X^i}{\partial \dot{x}^j} \dot{x}^j \mathcal{V}_i - \Gamma^a_{jb} X^j \mathsf{v}^b \mathcal{V}_a \\ \mathsf{r}^C &= \mathsf{r}^a \mathcal{X}_a + \Big(\frac{\partial \mathsf{r}^a}{\partial x^j} \dot{x}^j + \Gamma^a_{jb} \mathsf{r}^b \dot{x}^j \Big) \mathcal{V}_a. \end{aligned}$$

Remark that, if we would have chosen any other bracket, the complete lift would contain (annoying) additional terms in C^c_{ab} and ω^c_{ij}. If $\{e_i, e_a\}$ is a local basis for $\mathrm{Sec}(\pi)$, then $\{e^C_i = \mathcal{X}_i, e^C_a = \mathcal{X}_a, e^V_i = \mathcal{V}_i, e^V_a = \mathcal{V}_a\}$ is a local basis for $\mathrm{Sec}(\pi^\rho)$.

A special subclass of sections of π^ρ are the so-called *pseudo*-SODE*s* (or just 'SODE', second order differential equation, in [10]). An element Γ of this class is characterized by the property $\pi^2 \circ \Gamma = id$ and it is therefore locally of the form

$$\dot{x}^i \mathcal{X}_i + \mathsf{v}^a \mathcal{X}_a + f^i(x, \dot{x}^i, \mathsf{v}) \mathcal{V}^i + f^a(x, \dot{x}^i, \mathsf{v}) \mathcal{V}^a.$$

One of the main points in the current set-up is that Lagrangian systems on Lie algebroids are represented by a pseudo-SODE Γ. The solutions of the dynamical system (2) are then given by the integral curves of the associated vector field

$$\rho^\pi(\Gamma) = \frac{\partial}{\partial x^i} + f^i(x, \dot{x}^i, \mathsf{v}) \frac{\partial}{\partial \dot{x}^i} + f^a(x, \dot{x}^i, \mathsf{v}) \frac{\partial}{\partial \mathsf{v}^a} \ \in \mathcal{X}(TM \oplus \tilde{\mathfrak{g}}).$$

For a Lagrangian $L \in C^\infty(TM \oplus \tilde{\mathfrak{g}})$, the *Poincaré-Cartan 1-form* is $\theta_L = S(dL) \in \bigwedge(\pi^\rho)$. The *principal energy* is a function on $TM \oplus \tilde{\mathfrak{g}}$ given by $E_L = \rho^\pi(\mathcal{C})L - L$. We will only consider regular Lagrangians, i.e. those whose Hessian is at any point non-degenerate. A Lagrangian system on π is then a pseudo-SODE solution Γ of the equation

$$i_\Gamma d\theta_L = -dE_L. \tag{14}$$

For the algebroid (7), the integral curves of $\rho^\pi(\Gamma) \in \mathcal{X}(TM \oplus \tilde{\mathfrak{g}})$ are the solutions of the equations (1). The operator $\mathcal{L}P(l)$ from [5] can, within our approach, be identified with the 1-form $i_\Gamma d\theta_L + dE_L$ on π^ρ. Recall that we have called the first equation, respectively last equation in (1), the *vertical* and *horizontal* Lagrange-Poincaré equations. As was announced in the introduction, we will show that $i_\Gamma d\theta_L + dE_L$ can be dexomposed into two forms $\mathcal{H}or, \mathcal{V}er \in \bigwedge^1(\pi^\rho)$. First, we will need to decompose the main objects: Let $E_{\tilde{\mathfrak{g}}} = \rho^\pi(\mathcal{C}_{\tilde{\mathfrak{g}}})L$ and $E_{TM} = \rho^\pi(\mathcal{C}_{TM})L - L$. Then $E_L = \rho^\pi(\mathcal{C})L - L = E_{\tilde{\mathfrak{g}}} + E_{TM}$. Further let $\theta_{\tilde{\mathfrak{g}}} = S_{\tilde{\mathfrak{g}}}(dL)$ and $\theta_{TM} = S_{TM}(dL)$. Last but not least, we need to define a 1-form β on $\mathrm{Sec}(\pi^\rho)$ by means of its action on vertical and complete lifts:

Definition 3. *Let β be the 1-form on $\mathrm{Sec}(\pi^1)$ defined by*

$$\beta(\mathsf{r}^V) = dL(\mathsf{r}^V), \quad \beta(\mathsf{r}^C) = dL(\mathsf{r}^C), \quad \beta(X^V) = 0 \quad \textit{and} \quad \beta(X^C) = 0.$$

In coordinates, $\beta = \frac{\partial L}{\partial v^a}(\mathcal{V}^a + \Gamma^a_{ib} \mathsf{v}^b \mathcal{X}^i)$.

Proposition 6. *The 'vertical' Lagrange-Poincaré equation is given by*

$$i_\Gamma d^1 \theta_{\tilde{\mathfrak{g}}} = -dE_{\tilde{\mathfrak{g}}} - \beta. \tag{15}$$

The 'horizontal' Lagrange-Poincaré equation is given by

$$i_\Gamma (d^1 \theta_{TM} + d^2 \theta_{\tilde{\mathfrak{g}}} - d^3 \theta_{\tilde{\mathfrak{g}}}) = -dE_{TM} + \beta. \tag{16}$$

Proof. Since $d^1\theta_{TM} = d^2\theta_{TM} = d^3\theta_{TM}$, it is clear that if Γ satisfies both (15) and (16), it must also be a solution of the equation (14). We will prove now the converse: if Γ is the pseudo-SODE solution of (14), it will also satisfy the vertical equation (15) and the horizontal equation (16) separately.

For any Γ, the 1-form $i_\Gamma d\theta_L + dE_L$ is semi-basic, so it vanishes identically on vertical sections. We will show now that also the 'vertical' form $\mathcal{V}er = i_\Gamma d^1\theta_{\tilde{\mathfrak{g}}} + dE_{\tilde{\mathfrak{g}}} + \beta \in \bigwedge^1(\pi^\rho)$ vanishes on vertical sections. Since all involved objects are tensor fields, we can use in the the proof vertical (and later complete lifts) of *basic* sections, i.e. sections in $\mathrm{Sec}(\tau)$ and $\mathcal{X}(M)$ (and not the more arbitrary sections along π). First of all, we find that

$$\begin{aligned}\mathcal{V}er(\mathsf{r}^V) = (i_\Gamma d^1\theta_{\tilde{\mathfrak{g}}} + dE_{\tilde{\mathfrak{g}}} + \beta)(\mathsf{r}^V) \quad &= \quad \rho^1(\Gamma)(\theta_{\tilde{\mathfrak{g}}}(\mathsf{r}^V)) - \rho^1(\mathsf{r}^V)(\theta_{\tilde{\mathfrak{g}}}(\Gamma)) - \theta_{\tilde{\mathfrak{g}}}([\Gamma, \mathsf{r}^V]^1) \\ &\quad\quad +\rho^1(\mathsf{r}^V)((\rho^1\mathcal{C}_{\tilde{\mathfrak{g}}})L) + \beta(\mathsf{r}^V). \end{aligned} \tag{17}$$

Since $\theta_{\tilde{\mathfrak{g}}}(\mathsf{r}^V) = 0$, $\theta_{\tilde{\mathfrak{g}}}(\Gamma) = dL(S_{\tilde{\mathfrak{g}}}(\Gamma)) = \rho^1(\mathcal{C}_{\tilde{\mathfrak{g}}})(L)$ and $\theta_{\tilde{\mathfrak{g}}}([\Gamma, \mathsf{r}^V]^1) = dL(S_{\tilde{\mathfrak{g}}}([\Gamma, \mathsf{r}^V]^1)) = dL(\mathsf{r}^V) = \beta(\mathsf{r}^V)$, the proposed follows. By interchanging r^V for X^V in expression (17) it is clear that also $\mathcal{V}er(X^V) = 0$, because $\theta_{\tilde{\mathfrak{g}}}(X^V) = 0$, $S_{\tilde{\mathfrak{g}}}([\Gamma, X^V]) = 0$ and $\beta(X^V) = 0$. Therefore, also $\mathcal{V}er$ is semi-basic.

We prove next that $\mathcal{V}er$ also vanishes on complete lifts of the form X^C. Since $\theta_{\tilde{\mathfrak{g}}}(X^C) = 0$, it is easy to see that $\mathcal{V}er(X^C) = -\theta_{\tilde{\mathfrak{g}}}([\Gamma, X^C]^1)$. Essentially, what we have to calculate is $j_{\tilde{\mathfrak{g}}}[\Gamma, X^C]^1$ which is the '$\tilde{\mathfrak{g}}$'-part of $j[\Gamma, X^C]^1$. This vanishes because, for any pseudo-SODE, the bracket $j[\Gamma, X^C]^1$ does not have components in e_a.

Finally, we show that the two 1-forms $i_\Gamma d\theta + dE$ and $\mathcal{V}er$ coincide on complete lifts r^C. It is easy to see that $i_\Gamma(d\theta + dE)(\mathsf{r}^C) = \rho^1(\Gamma)(\theta(\mathsf{r}^C)) - \theta([\Gamma, \mathsf{r}^C]) - \rho^1(\mathsf{r}^C)L$. On the other hand, because $\beta(\mathsf{r}^C) = dL(\mathsf{r}^C)$, we can find that $\mathcal{V}er(\mathsf{r}^C) = \rho^1(\Gamma)(\theta_{\tilde{\mathfrak{g}}}(\mathsf{r}^C)) - \theta_{\tilde{\mathfrak{g}}}([\Gamma, \mathsf{r}^C]^1) - \rho^1(\mathsf{r}^C)L$. We thus have to prove that $\theta([\Gamma, \mathsf{r}^C]) = \theta_{\tilde{\mathfrak{g}}}([\Gamma, \mathsf{r}^C]^1)$, or $j[\Gamma, \mathsf{r}^C] = j_{\tilde{\mathfrak{g}}}[\Gamma, \mathsf{r}^C]^1$ which can easily be verified in coordinates. We can thus conclude that if $i_\Gamma d\theta_L + dE_L = 0$, then also $\mathcal{V}er = 0$.

We can now conclude the proof. If Γ satisfies $i_\Gamma d\theta_L + dE_L = 0$ and the vertical equation (15), then it automatically also satisfies (16). Remark that both $i_\Gamma d\theta + dE$ and the 'horizontal' form $\mathcal{H}or = i_\Gamma(d^1\theta_{TM} + d^2\theta_{\tilde{\mathfrak{g}}} - d^3\theta_{\tilde{\mathfrak{g}}}) + dE_{TM} - \beta \in \bigwedge^1(\pi^\rho)$ will agree on sections X^C, but $\mathcal{H}or$ will vanish identically on sections of the form r^C. □

5 Examples: Wong's equations

Remark that, so far, we have not used a linear connection ∇^M on M. We will recall here an example of [3, 5] where such a connection is easily available and study it within the present framework. In this way we will arrive back at the description of [5].

Let g be a Riemannian metric on M and κ a bi-invariant metric on G. The Lagrangian

$$l(v_q) = \frac{1}{2}\kappa(A(v_q), A(v_q)) + \frac{1}{2}g(\pi_G(q))(T\pi_G(v_q), T\pi_G(v_q))$$

on Q is G-invariant and if we put $k([q\cdot\xi], [q\cdot\eta]) = \kappa(\xi, \eta)$ then k is a fibre metric on $\tilde{\mathfrak{g}}$ and the reduced Lagrangian becomes

$$L(x, \dot{x} \oplus \mathsf{v}) = \frac{1}{2}k(\mathsf{v}, \mathsf{v}) + \frac{1}{2}g_x(\dot{x}, \dot{x}) = \frac{1}{2}\kappa_{ab}\mathsf{v}^a\mathsf{v}^b + \frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j.$$

Remark that bi-invariance of the metric means that

$$k(\mathsf{r},[\mathsf{s},\mathsf{t}])+k(\mathsf{t},[\mathsf{s},\mathsf{r}])=0 \quad \text{or} \quad \kappa_{cd}C^c_{ab}=-\kappa_{cb}C^c_{da}.$$

The Lagrange-Poincaré equations for the above problem are called *Wong's equations* (see [5] and the references therein for applications in physics where such a Lagrangian arises). Next to the the (associated) linear connection ∇ on τ (with connection coefficients $\Gamma^b_{ia}=C^b_{ad}A^d_i$, see expression (6)), there is now a second connection around: the Levi-Civita connection of the metric g (with coefficients N^i_{jk}), which is a linear connection on M, in what follows denoted by ∇^M. We will use this information within our current set-up, that is, that of the prolongation bundle $\pi^\rho : T^\rho V \to V$.

So-called *ρ-connections on* π [2, 14] are direct complements $\mathcal{H}^\rho V$ of $\mathcal{V}^\rho V$ in $T^\rho V$. Locally, such a connection is determined by certain connection coefficients $\tilde\Gamma^\alpha_\beta \in C^\infty(V)$. A section $\mathcal{Z}\in \mathrm{Sec}(\pi^\rho)$ is horizontal if it is of the form

$$\mathcal{Z}=Z^\alpha\mathcal{H}_\alpha=Z^\alpha\big(\mathcal{X}_\alpha-\tilde\Gamma^\beta_\alpha\mathcal{V}^\beta\big) \quad \in \mathrm{Sec}(\pi^\rho).$$

If $\{e_\alpha\}$ is a basis of $\mathrm{Sec}(\pi)$, then the horizontal lift of $s=s_\alpha e_\alpha\in\mathrm{Sec}(\pi^*\pi)$ is $s^H=s^\alpha\mathcal{H}^\alpha$. Special cases of such connections are those where the connection coefficients are linear, i.e. $\tilde\Gamma^\alpha_\beta=\tilde\Gamma^\alpha_{\beta\gamma}y^\gamma$. Then, a ρ-connection can be represented by an operator $\nabla^\rho : \mathrm{Sec}(\pi)\times\mathrm{Sec}(\pi)\to\mathrm{Sec}(\pi)$, where

$$\nabla^\rho_{e_\beta}e_\gamma=\tilde\Gamma^\alpha_{\beta\gamma}e_\alpha.$$

Such an operator has the following properties w.r.t. multiplications of functions $f\in C^\infty(M)$,

$$\nabla^\rho_{fs}r=f\nabla^\rho_s r \qquad \text{and} \qquad \nabla^\rho_s fr=f\nabla_s r+\rho(s)fr, \qquad \forall s,r\in\mathrm{Sec}(\pi). \tag{18}$$

In the case of Wong's equations, the following can immediately be verified.

Proposition 7. *The operator ∇^ρ, defined by*

$$\nabla^\rho_{X\oplus\mathsf{s}}Y\oplus\mathsf{r}=\nabla^M_X Y\oplus\nabla_X\mathsf{r},$$

is a linear ρ-connection on π.

The connection coefficients of this connection are

$$\tilde\Gamma^a_i=\Gamma^a_{ib}\mathsf{v}^b, \qquad \tilde\Gamma^a_b=0, \qquad \tilde\Gamma^i_j=N^i_{jk}\dot x^k, \qquad \tilde\Gamma^i_b=0.$$

and the horizontal space is spanned by $\{H_a=\mathcal{X}_a, H_i=\mathcal{X}_i-\tilde\Gamma^a_i\mathcal{V}_a-\tilde\Gamma^j_i\mathcal{V}_j\}$. The horizontal lift of the canonical section $\mathbf{T}$ is a pseudo-SODE and will be denoted by $\tilde\Gamma$. In coordinates,

$$\tilde\Gamma=\dot x^i\mathcal{X}^i+\mathsf{v}^a\mathcal{X}^a-\Gamma^a_{ib}\dot x^i\mathsf{v}^b\mathcal{V}_a-N^i_{jk}\dot x^j\dot x^k\mathcal{V}^i.$$

From the proof of Proposition 6 it has become clear that the essential part of the vertical equation is given by its action on r^C. Since the difference between r^C and r^H is a vertical section and since $\mathcal{V}er$ vanishes on vertical sections, we may as well look at the action of $\mathcal{V}er$ on horizontal sections r^H. It is easy to see that $\beta(\mathsf{r}^H)=0$. We therefore easily find that

$$0=\mathcal{V}er(\mathsf{r}^H)=\rho^1(\Gamma)(\theta_{\tilde{\mathfrak{g}}}(\mathsf{r}^H))-\theta_{\tilde{\mathfrak{g}}}([\Gamma,\mathsf{r}^H]^1). \tag{19}$$

The 1-form $\theta_{\tilde{\mathfrak{g}}}$ on $\mathrm{Sec}(\pi^\rho)$ is semibasic and therefore leads to a 1-form θ_k on $\tau : \tilde{\mathfrak{g}} \to M$ along π, given by $\theta_k(\mathsf{r}) = \theta_{\tilde{\mathfrak{g}}}(\mathsf{r}^H) = k(\mathbf{T}_{\tilde{\mathfrak{g}}}, \mathsf{r})$. We thus need to know $\theta_k(j_{\tilde{\mathfrak{g}}}[\Gamma, \mathsf{r}^H]^1)$, or, essentially, the '$\tilde{\mathfrak{g}}$'-part of $j[\Gamma, \mathsf{r}^H]^1$. The difference between the pseudo-SODE Γ and the pseudo-SODE $\tilde{\Gamma}$ is a vertical section $\mathcal{W}$. Since we assume r to be a basic section, all the brackets $[\mathcal{W}, \mathsf{r}^H]^i$ will be vertical and therefore not contribute to the bracket we want to compute. What is left is therefore nothing but $(j[\tilde{\Gamma}, \mathsf{r}^H]^1)_{\tilde{\mathfrak{g}}}$. To express this term, it will be convenient to introduce a new linear connection D which has the advantage that its action on forms along π (such as θ_k) makes sense. D will be a linear ρ^π-connection on $\pi^*\pi$, the so-called Berwald-type connection D (see also [13]). It is an operator $\mathrm{D} : \mathrm{Sec}(\pi^\rho) \times \mathrm{Sec}(\pi^*\pi) \to \mathrm{Sec}(\pi^*\pi)$ with properties that are analogous to those of (18), but with ρ replaced by ρ^π. In this case, we can define this connection by means of its action on *basic* sections, that is: let $\mathsf{r}, \mathsf{s} \in \mathrm{Sec}(\tau)$ and $X, Y \in \mathcal{X}(M)$, then

$$\mathrm{D}_{\mathsf{r}^H}\mathsf{s} = 0, \qquad \mathrm{D}_{X^H}\mathsf{s} = \nabla_X \mathsf{s}, \qquad \mathrm{D}_{\mathsf{r}^V}\mathsf{s} = 0, \qquad \mathrm{D}_{X^V}\mathsf{s} = 0,$$
$$\mathrm{D}_{\mathsf{r}^H}Y = 0, \qquad \mathrm{D}_{X^H}Y = \nabla^M_X Y, \qquad \mathrm{D}_{\mathsf{r}^V}Y = 0, \qquad \mathrm{D}_{X^V}\mathsf{s} = 0.$$

On basic sections, $\mathrm{D}_\Gamma = \mathrm{D}_{\tilde{\Gamma}} + \mathrm{D}_{\mathcal{W}} = \mathrm{D}_{\tilde{\Gamma}}$. It can easily be calculated that $\mathrm{D}_{\tilde{\Gamma}}\mathsf{r} = (j[\tilde{\Gamma}, \mathsf{r}^H]^1)_{\tilde{\mathfrak{g}}} - \mathsf{r}^a \mathsf{v}^b C^c_{ba} e_c$. Taking all this into account (19) is then

$$0 = \rho^1(\Gamma)(\theta_k(\mathsf{r})) - \theta_k(\mathrm{D}_\Gamma \mathsf{r}) - \kappa_{cd} C^c_{ba} \mathsf{v}^d \mathsf{v}^b \mathsf{r}^a e_c.$$

The last term vanishes because of the bi-invariance of the metric and the skew-symmetry of the Lie algebra bracket. To conclude, the vertical Wong equation is

$$\mathrm{D}_\Gamma \theta_k = 0. \tag{20}$$

Similar as above, one can define a 1-form θ_g of TM along π by means of $\theta_g(X) = \theta_{TM}(X^H) = g(\mathbf{T}_{TM}, X)$. The essential part of the horizontal Wong equation is

$$\begin{aligned} 0 = \mathcal{H}or(X^H) \;&=\; \rho^1(\Gamma)(g(\mathbf{T}_{TM}, X)) - g(\mathbf{T}_{TM}, \mathrm{D}_\Gamma X) \\ &\quad -\theta_k((j[\Gamma, X^H]^2 - j[\Gamma, X^H]^3)_{\tilde{\mathfrak{g}}}) - dL(X^H). \end{aligned} \tag{21}$$

A small coordinate calculation shows that the third term is in fact $\theta_k(\omega(\mathbf{T}_{TM}, X))$. The last term is given by

$$X^i\Big(\frac{\partial L}{\partial x^i} - \frac{\partial L}{\partial \dot{x}^k} N^k_{ij}\dot{x}^j - \frac{\partial L}{\partial \mathsf{v}^c}\Gamma^c_{ib}\mathsf{v}^b\Big). \tag{22}$$

Using the explicit expression of the Lagrangian and the Levi-Civita connection coefficients, it is easy to see that the first two terms in (22) cancel out and that the remaining term is $\kappa_{cd}X^i\Gamma^c_{ib}\mathsf{v}^b\mathsf{v}^d$. Also this term vanishes, due to the explicit expression of the connection coefficients $\Gamma^c_{ib} = A^d_i C^c_{db}$ and due to the assumed bi-invariance of κ and the skew-symmetry of the Lie algebra. Finally, we will rewrite the first two terms in (21). Remark first that

$$(\mathrm{D}_{X^H} g)(Y, Z) = (\nabla^M g)(X, Y) = 0 \quad \text{and} \quad (\mathrm{D}_{\mathsf{r}^h} g)(Y, Z) = \rho^\pi(\mathsf{r}^H)(g(Y, Z)) = 0,$$

since $\rho^\pi(\mathsf{r}^H) = \rho^\pi(\mathsf{r}^a \mathcal{X}^a) = 0$. So, in particular for $\tilde{\Gamma} = \mathbf{T}^H$, also $\mathrm{D}_{\tilde{\Gamma}} g = 0$. Moreover, since g is basic also $\mathrm{D}_{\sigma^V} g = 0$ for all $\sigma \in \mathrm{Sec}(\pi^*\pi)$ and thus also $\mathrm{D}_\Gamma g = \mathrm{D}_{\tilde{\Gamma}} + \mathrm{D}_{\mathcal{W}} g = 0$. In particular,

$$0 = (\mathrm{D}_\Gamma g)(\mathbf{T}_{TM}, X) = \rho^1(\Gamma)(g(\mathbf{T}_{TM}, X)) - g(\mathrm{D}_\Gamma \mathbf{T}_{TM}, X) - g(\mathbf{T}_{TM}, \mathrm{D}_\Gamma X),$$

so we can rewrite the horizontal Wong equation (21) as

$$g(\mathrm{D}_\Gamma \mathbf{T}_{TM}, X) + \theta_k(\omega(\mathbf{T}_{TM}, X)) = 0. \tag{23}$$

(20) and (23) can also be found in [5] (in a somehow different style).

6 Hamilton-Poincaré equations

In [4], a Hamiltonian version of the Lagrange-Poincaré equations has been developed. The corresponding equations, the so-called Hamilton-Poincaré equations can also be seen to fit in our approach. Let's come back first to the most general idea of a prolongation in Section 4. In fact, let μ now be $\pi^* : V^* \to M$, the dual bundle of a Lie algebroid $\pi : V \to M$. Then, in [11] (see also [6]) it has been shown that the Lie algebroid $(\pi^*)^\rho : T^\rho V^* \to V^*$ is the ideal arena to host the Hamiltonian formalism on a Lie algebroid. We only need to consider the canonical 1-form θ^0 on $(\pi^*)^\rho$, defined by $\theta^0(w)(v, W) = w(v)$ $(w \in V^*, (v, W) \in T^\rho V^*)$ and its exterior derivative. Let $H \in C^\infty(V^*)$ be a Hamiltonian function. Hamilton's equations are then given by the integral curves of $\rho^\pi(X_H)$ where X_H is a section of $(\pi^*)^\rho$ which satisfies

$$i_{X_H} d\theta_0 = -dH. \tag{24}$$

Locally, if (x^i, p_α) are coordinates on V^*, then $\theta^0 = p_\alpha \mathcal{X}^\alpha$ and the Hamilton equations on a Lie algebroid are

$$\begin{cases} \dot{x}^i &= \rho^i_\alpha(x) \dfrac{\partial H}{\partial p_\alpha}, \\[2ex] \dot{p}_\alpha &= -\rho^i_\alpha \dfrac{\partial H}{\partial x^i} - D^\gamma_{\alpha\beta} p_\gamma \dfrac{\partial H}{\partial p_\beta}. \end{cases} \tag{25}$$

Coming back to the particular case of systems with symmetry, it is now not difficult to see that the so-called Hamilton-Poincaré equations from [4] are nothing but the equations (25) for the extension of the Lie algebroid (7) to $T^\rho(T^*M \oplus \tilde{\mathfrak{g}}^*)$. Let $\{e^i, e^a\}$ be the basis of $\mathrm{Sec}(\pi^*)$, dual to the basis $\{e_i, e_a\}$ of $\mathrm{Sec}(\pi)$. We will denote $\{\mathcal{X}_i, \mathcal{X}_a, \mathcal{P}^i, \mathcal{P}^a\}$ for the induced basis (11) on $\mathrm{Sec}((\pi^*)^\rho)$. For coordinates (x^i, p_i, p_a) on $T^*M \oplus \tilde{\mathfrak{g}}^*$, the expressions (25) are in this situation:

$$\begin{cases} \dot{x}^i &= \dfrac{\partial H}{\partial p_i}, \\[2ex] \dot{p}_i &= -\dfrac{\partial H}{\partial x^i} + \omega^c_{ij} \mathsf{p}_c \dfrac{\partial H}{\partial p_j} - \Gamma^c_{ib} \mathsf{p}_c \dfrac{\partial H}{\partial \mathsf{p}_b}, \\[2ex] \dot{\mathsf{p}}_a &= \Gamma^c_{ja} \mathsf{p}_c \dfrac{\partial H}{\partial p_j} - C^c_{ab} \mathsf{p}_c \dfrac{\partial H}{\partial \mathsf{p}_b}. \end{cases} \tag{26}$$

The decomposition of the extended Lie algebroid in Proposition 5 will ensure again that the above set of equations can be decomposed into a 'horizontal' and 'vertical' set. Look at the decomposition $\theta^0_{\tilde{\mathfrak{g}}} = \mathsf{p}_a \mathcal{X}^a$ and $\theta^0_{TM} = p_i \mathcal{X}^i$ of the canonical section and let γ be $\frac{\partial H}{\partial \mathsf{p}_a}(\mathcal{P}_a - \Gamma^b_{ia} \mathsf{p}_b \mathcal{X}^i)$.

Proposition 8. *The vertical Hamilton-Poincaré equation is*

$$i_{X_H} d^1 \theta^0_{\tilde{\mathfrak{g}}} = -\gamma. \tag{27}$$

The horizontal Hamilton-Poincaré equation is

$$i_{X_H}(d^1 \theta^0_{TM} + d^2 \theta^0_{\tilde{\mathfrak{g}}} - d^3 \theta^0_{\tilde{\mathfrak{g}}}) = -dH + \gamma. \tag{28}$$

A coordinate calculation shows that (28) gives the first and second equation in (26), while (27) is given by the first and third equation.

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