Structural equations for a special class of conformal Killing tensors

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Abstract

Special conformal Killing tensors have appeared recently in several differential geometric contexts. In this article solutions of the special conformal Killing tensor equation on a pseudo-Riemannian manifold are studied from Wolf's structural equations approach. As a result it follows that the solutions of the special conformal Killing tensor equation are determined by the second order jet. It is shown that the space of solutions is of maximal dimension if and only the metric is of constant curvature. By means of taking a trace the special conformal Killing tensor equation is solved in complete generality for spaces of non-zero constant curvature. Finally the case of two-dimensional spaces is considered in which the special conformal Killing tensor equation is equivalent to the usual Killing tensor equation of valence two.

Keywords: Special conformal Killing tensor, structural equations, spaces of constant curvature.

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1 Introduction

This paper is concerned with symmetric valence 2 tensors L_{ij} on a Riemannian or pseudo-Riemannian manifold which satisfy the equations

$$L_{ij|k} = \frac{1}{2}(g_{jk}\lambda_i + g_{ik}\lambda_j),$$

where λ_i is a covariant vector, which is determined by the equations: in fact $\lambda_i = \lambda_{|i}$ where $\lambda = (g^{jk}L_{jk})_{|i}$ is the trace of L_{ij}. For any such tensor

$$L_{ij|k} + L_{jk|i} + L_{ki|j} = g_{ij}\lambda_k + g_{jk}\lambda_k + g_{ki}\lambda_j,$$

so these tensors are conformal Killing tensors, and since λ_i is a gradient, conformal Killing tensors of gradient type; since not all conformal Killing tensors (even those of gradient type) are of this form, they may be called special conformal Killing tensors.

Special conformal Killing tensors have some very interesting properties, especially when the metric is Riemannian, as we temporarily assume. In the first place, the Nijenhuis torsion of any special conformal Killing tensor vanishes, as it is easy to show by a direct calculation (properly speaking, we should refer here to the type $(1,1)$ tensor obtained by raising an index with the metric, but we leave this to be understood). Since by assumption a special conformal Killing tensor is symmetric it has real eigenvalues, and the eigenvectors may be taken to be pairwise orthogonal; if the eigenvalues are everywhere simple, coordinates may be found with respect to which the metric is orthogonal and the special conformal Killing tensor is diagonal. Conversely, a conformal Killing tensor of valence 2 whose Nijenhuis torsion vanishes and which has simple eigenvalues must be a special conformal Killing tensor (see [7, 10]).

Furthermore, the cofactor tensor of a special conformal Killing tensor is always a Killing tensor. The special conformal Killing tensor equations above are linear, and evidently have the solution $L_{ij} = kg_{ij}$ for any constant k; so if L_{ij} is a special conformal Killing tensor, so is $L_{ij} + kg_{ij}$. By taking the coefficients of powers of k in the cofactor tensor of this special conformal Killing tensor, we obtain n valence 2 Killing tensors, simultaneously diagonal with L_{ij}, one of which is the metric itself (here n is the dimension of the underlying manifold). These Killing tensors may be shown to have pairwise vanishing Schouten brackets, and when the special conformal Killing tensor has simple eigenvalues they are independent; thus in this case the geodesic flow of the metric is completely integrable in the sense of Liouville; furthermore, Eisenhart's theorem holds and the Hamilton-Jacobi equation is separable in the orthogonal coordinates with respect to which the special conformal Killing tensor is diagonal. Special conformal Killing tensors which have simple eigenvalues are sometimes called Benenti tensors, after Benenti who first studied them in the context of the Hamilton-Jacobi separability problem (see for example [1, 2]). Though not all separable Riemannian manifolds admit Benenti tensors (for example, the Liouville spaces do not in general), those that do form an important subclass of separable systems, and have been studied by a number of authors in addition to Benenti himself (see for example [3, 7, 10, 14]). In particular, it has been pointed out recently [4] that the Hamiltonian dynamical systems associated with Benenti tensors in spaces of constant curvature are maximally superintegrable.

Special conformal Killing tensors also occur in other contexts. One example is the study of projectively equivalent metrics. It may be shown that if g_{ij} and h_{ij} are projectively equivalent (so that they have the same geodesics up to reparametrization) then the tensor

$$L_{ij} = \left(\frac{\det h}{\det g}\right)^{1/(n+1)} g_{ik}g_{jl}h^{kl},$$

where $h^{ik}h_{jk} = \delta^i_j$, is a special conformal Killing tensor of g_{ij}; and conversely, given any non-singular special conformal Killing tensor of a metric g_{ij}, the tensor h_{ij} defined by

$$h_{ij} = (\det L)^{-1} g_{ik}g_{jl}\bar{L}^{kl},$$

where $\bar{L}^{ik}L_{jk} = \delta^i_j$, defines a metric projectively equivalent to the first. For more information on this topic see for example [5, 8, 16] (the title of the first of these papers notwithstanding, in this case it is not necessary for the eigenvalues of L_{ij} to be simple to obtain interesting results).

As a third example of the interest of special conformal Killing tensors we cite a class of nonconservative Lagrangian systems studied in [9, 15, 17]. Each such system is based on a kinetic energy, and has first integrals quadratic in velocities formed from special conformal Killing tensors of the metric defining the kinetic energy. When there are two independent first integrals of this form the system is completely integrable.

Though special conformal Killing tensors of pseudo-Riemannian metrics have not been studied to the same extent as those of Riemannian metrics, it is clear that a number of these results will carry over, perhaps with some modifications.

Since special conformal Killing tensors have these interesting applications, it seems important to establish the basic properties of the solutions of the special conformal Killing tensor equations. In particular, since the equations are linear the solution space is a vector space over the reals, which we denote by V. It is known that in n-dimensional Euclidean space the dimension of V is $\frac{1}{2}(n+1)(n+2)$ [7, 10, 15]. The main purpose of this note is to show that this is the maximal dimension that V can have, and that if the maximum is achieved then the space is a space of constant curvature. In [4] the authors, having stated that spaces of constant curvature are maximally superintegrable, say that 'it seems plausible that they are multiseparable too'. We show here exactly the extent to which this is the case.

Our method is to derive a system of structural equations, in the sense of Hauser and Malhiot [12, 13] and Wolf [19], for special conformal Killing tensors. That is to say, we find a set of tensorial quantities F_A which satisfy a system of equations of the form $F_{A|i} = \Gamma^B_{Ai} F_B$ (sum over B intended), among which are to be found the special conformal Killing tensor equations. The equations of this extended set are the structural equations. The F_A consist of the symmetric tensor L_{ij} and tensors constructed from it and its covariant derivatives; the coefficients Γ^B_{Ai} are tensorial quantities which are independent of the F_A and in fact are built out of the metric, and the curvature and its covariant derivatives. The structural equations are equivalent to the original special conformal Killing tensor equations, in the sense that given any solution L_{ij} of the special conformal Killing tensor equations, the corresponding F_A satisfy the structural equations, and conversely given any solution of the structural equations the L_{ij} component of F_A satisfies the special conformal Killing tensor equations.

The advantage of expressing the problem of finding special conformal Killing tensors in the form of solving the structural equations derives from the distinctive nature of these equations: each covariant derivative $F_{A|i}$ is a linear combination of the F_A. It follows that given any point x of the underlying manifold, the linear map sending a solution L_{ij} of the special conformal Killing tensor equations to $F_A(x)$ is injective, so that the largest value the dimension of V can have is the number of variables F_A in the structural equations. Moreover, the integrability conditions of the structural equations are easily found by covariantly differentiating the equations, using the Ricci identities to eliminate second covariant derivatives, and substituting for the first derivatives introduced by using the original equations. The resulting conditions are

$$(\Gamma^B_{Ai|j} - \Gamma^B_{Aj|i} + \Gamma^C_{Ai}\Gamma^B_{Cj} - \Gamma^C_{Aj}\Gamma^B_{Ci} - R^B{}_{Aij})F_B = 0,$$

where the $R^B{}_{Aij}$ are appropriate combinations of components of the curvature tensor. When V has maximal dimension the values of the F_A may be chosen arbitrarily at each point of the underlying manifold, so

$$\Gamma^B_{Ai|j} - \Gamma^B_{Aj|i} + \Gamma^C_{Ai}\Gamma^B_{Cj} - \Gamma^C_{Aj}\Gamma^B_{Ci} - R^B{}_{Aij}$$

must vanish everywhere, and this gives algebraic conditions on the curvature and its covariant derivatives from which the properties of the spaces for which V has maximal dimension can be determined.

The relevant features of structural equations, including those described above, seem to be treated as common knowledge rather than derived in the literature; we give a brief discussion with proofs in an appendix.

In tensor calculations we follow the sign conventions of Eisenhart [11], so that the Ricci identities are (for example) $K_{i|jk} - K_{i|kj} = R^l{}_{ijk}K_l$, and the Ricci tensor is given by $R_{ij} = R^k{}_{ijk}$. The Einstein summation convention is in force almost throughout.

2 The structural equations

For any symmetric tensor L_{ij} we set $\lambda_i = (g^{jk}L_{jk})_{|i}$ and $\mu = g^{ij}\lambda_{i|j}$; λ_i is a covariant vector and μ a scalar.

Theorem The equations

$$\begin{aligned}
L_{ij|k} &= \tfrac{1}{2}(g_{jk}\lambda_i + g_{ik}\lambda_j) \\
\lambda_{i|j} &= \frac{1}{n}\left(2R^k_j L_{ik} - 2g^{kl}R^m{}_{ijk}L_{lm} + g_{ij}\mu\right) \\
\mu_{|i} &= \frac{2}{n-1}\left(g^{jl}(2R^k_{i|l} - R^k_{l|i})L_{jk} + (n+1)R^j_i\lambda_j\right)
\end{aligned}$$

are structural equations for special conformal Killing tensors (with $F_A = \{L_{ij}, \lambda_i, \mu\}$).

Proof Suppose first that L_{ij} is a special conformal Killing tensor, so that

$$L_{ij|k} = \tfrac{1}{2}(g_{jk}\lambda_i + g_{ik}\lambda_j).$$

By differentiating again and using the Ricci identity we obtain

$$\tfrac{1}{2}(g_{jk}\lambda_{i|l} + g_{ik}\lambda_{j|l} - g_{jl}\lambda_{i|k} - g_{il}\lambda_{j|k}) = R^m{}_{ikl}L_{mj} + R^m{}_{jkl}L_{im}.$$

It follows, by multiplying by g^{il} (say), summing, and renaming indices, that

$$\lambda_{i|j} = \frac{1}{n}\left(2R^k_j L_{ik} - 2g^{kl}R^m{}_{ijk}L_{lm} + g_{ij}\mu\right).$$

Then

$$\begin{aligned}
n(\lambda_{i|jk} - \lambda_{i|kj}) &= nR^l{}_{ijk}\lambda_l \\
&= 2\left((R^l_{j|k} - R^l_{k|j})L_{il} - g^{lm}R^n{}_{ijk|l}L_{mn}\right) \\
&\quad + \left(g_{ik}R^l_j - g_{ij}R^l_k - 3R^l{}_{ijk}\right)\lambda_l + g_{ij}\mu_{|k} - g_{ik}\mu_{|j},
\end{aligned}$$

where we have substituted for $L_{ij|k}$ from the special conformal Killing tensor equations, and used the cyclic and Bianchi identities and the symmetry of the Ricci tensor to simplify various terms. Thus

$$\begin{aligned}
g_{ij}\mu_{|k} - g_{ik}\mu_{|j} &= 2\left(g^{lm}R^n{}_{ijk|l}L_{mn} - (R^l_{j|k} - R^l_{k|j})L_{il}\right) \\
&\quad + \left((n+3)R^l{}_{ijk} - (g_{ik}R^l_j - g_{ij}R^l_k)\right)\lambda_l.
\end{aligned}$$

If one multiplies by g^{ij}, sums, and renames indices, one obtains the stated equation for $\mu_{|i}$.

Conversely, suppose that the given equations are satisfied for some tensor L_{ij}, covariant vector λ_i and scalar μ. Then L_{ij} is evidently a special conformal Killing tensor; it follows from the special conformal Killing tensor equations that $\lambda_i = (g^{jk}L_{jk})_{|i}$; and on multiplying the equation for $\lambda_{i|j}$ by g^{ij} we find that $\mu = g^{ij}\lambda_{i|j}$.

Finally, the given equations have the form required for them to be structural equations. They are thus structural equations for special conformal Killing tensors. □

Corollary 1 The dimension of the space of solutions of the special conformal Killing tensor equations is at most $\frac{1}{2}(n+1)(n+2)$. □

We now prove that for $n \geq 3$ the maximal dimension is attained if and only if the space is a space of constant curvature. Rather than proceeding exactly as described in the introduction by deriving the integrability conditions of the structural equations in all generality, which involves somewhat complicated calculations, we first show that it is a necessary condition that the space has constant curvature, and then show that the integrability conditions are satisfied for a space of constant curvature.

Corollary 2 For $n \geq 3$, the dimension of the space of solutions of the special conformal Killing tensor equations is $\frac{1}{2}(n+1)(n+2)$ if and only if the space has constant curvature.

Proof In the course of the proof of the theorem we showed that

$$\begin{aligned} nR^l{}_{ijk}\lambda_l &= 2\left((R^l_{j|k} - R^l_{k|j})L_{il} - g^{lm}R^n{}_{ijk|l}L_{mn}\right) \\ &\quad + \left(g_{ik}R^l{}_j - g_{ij}R^l_k - 3R^l{}_{ijk}\right)\lambda_l + g_{ij}\mu_{|k} - g_{ik}\mu_{|j}, \end{aligned}$$

where

$$\mu_{|i} = \frac{2}{n-1}\left(g^{jl}(2R^k_{i|l} - R^k_{l|i})L_{jk} + (n+1)R^j_i\lambda_j\right);$$

when $\mu_{|i}$ from the second of these sets of equations is substituted in the first we obtain the integrability conditions for the $\lambda_{i|j}$ equations. When the solution space has the maximum dimension the integrability conditions of the structural equations must be satisfied, at each point of the underlying manifold, with arbitrary choices of the values of L_{ij}, λ_i and μ. In particular we can choose to take $L_{ij} = 0$ at the point, so that we must have

$$nR^l{}_{ijk}\lambda_l = \left(g_{ik}R^l_j - g_{ij}R^l_k - 3R^l{}_{ijk}\right)\lambda_l + g_{ij}\mu_{|k} - g_{ik}\mu_{|j},$$

where

$$\mu_{|i} = \frac{2(n+1)}{n-1}R^j_i\lambda_j,$$

and this for any λ_i. That is to say,

$$\begin{aligned} (n+3)R^l{}_{ijk} &= -\left(g_{ij}R^l_k - g_{ik}R^l_j\right) + \frac{2(n+1)}{n-1}\left(g_{ij}R^l_k - g_{ik}R^l_j\right) \\ &= \frac{n+3}{n-1}\left(g_{ij}R^l_k - g_{ik}R^l_j\right), \end{aligned}$$

which is to say that

$$R^l{}_{ijk} = \frac{1}{n-1}\left(g_{ij}R^l_k - g_{ik}R^l_j\right).$$

Then by taking a trace we obtain

$$R_{ij} = \frac{1}{n} g_{ij} R,$$

where R is the curvature scalar, so that

$$R^l{}_{ijk} = \frac{R}{n(n-1)} \left(g_{ij}\delta^l_k - g_{ik}\delta^l_j \right).$$

Thus the curvature tensor takes the constant curvature form pointwise, whence the result by Schur's Theorem ([11], Chapter II, Section 26) (we require $n \geq 3$ for Schur's Theorem to hold).

We now have to show that there are no further conditions on the space coming from the remaining integrability conditions for the structural equations; this we do by showing that these conditions are all satisfied for a space of constant curvature. We write the curvature as

$$R^l{}_{ijk} = B \left(g_{ij}\delta^l_k - g_{ik}\delta^l_j \right), \quad B = \frac{R}{n(n-1)};$$

B is of course the constant (sectional) curvature, and $R_{ij} = (n-1)Bg_{ij}$. The Ricci identity for a tensor $A_{i_1 \dots i_p}$ takes the simple form

$$A_{i_1 \dots i_p|jk} - A_{i_1 \dots i_p|kj} = B \sum_{r=1}^{p} \left(g_{i_r j} A_{i_1 \dots k \dots i_p} - g_{i_r k} A_{i_1 \dots j \dots i_p} \right).$$

In a space of constant curvature

$$\begin{aligned} \lambda_{i|j} &= 2BL_{ij} + \frac{1}{n} g_{ij}(\mu - 2B\lambda), \quad \lambda = g^{kl} L_{kl} \\ \mu_{|i} &= 2(n+1)B\lambda_i, \quad \lambda_i = \lambda_{|i}. \end{aligned}$$

From the first equations we see that $\lambda_{i|j}$ is symmetric (as indeed it is in general); but $\mu_{|ij} = 2(n+1)B\lambda_{i|j}$, and therefore $\mu_{|ji} = \mu_{|ij}$, so the integrability conditions for the $\mu_{|i}$ equations are satisfied. For the integrability conditions of the first equations we have

$$\begin{aligned} &\lambda_{i|jk} - \lambda_{i|kj} \\ &= 2B(L_{ij|k} - L_{ik|j}) + \frac{1}{n} \left(g_{ij}(\mu_{|k} - 2B\lambda_k) - g_{ik}(\mu_{|j} - 2B\lambda_j) \right) \\ &= -B(g_{ij}\lambda_k - g_{ik}\lambda_j) + 2B(g_{ij}\lambda_k - g_{ik}\lambda_j) \\ &= B(g_{ij}\lambda_k - g_{ik}\lambda_j) = R^l{}_{ijk}\lambda_l, \end{aligned}$$

so these integrability conditions are satisfied. Finally, we have to consider the integrability conditions for the special conformal Killing tensor equations, which are

$$\tfrac{1}{2}(g_{jk}\lambda_{i|l} + g_{ik}\lambda_{j|l} - g_{jl}\lambda_{i|k} - g_{il}\lambda_{j|k}) = R^m{}_{ikl}L_{mj} + R^m{}_{jkl}L_{im},$$

with $\lambda_{i|j}$ as above. The terms involving λ and μ in $\lambda_{i|j}$ make no contribution to the left-hand side; the integrability conditions are

$$B(g_{jk}L_{il} + g_{ik}L_{jl} - g_{jl}L_{ik} - g_{il}L_{jk}) = R^m{}_{ikl}L_{mj} + R^m{}_{jkl}L_{im},$$

and these are also satisfied.

Thus in any space of constant curvature the integrability conditions of the structural equations are satisfied, the equations are completely integrable, and there is a solution with specified values of L_{ij}, λ_i and μ at any given point. The tensor L_{ij} is a special conformal Killing tensor, and so the linear map from V to the values of L_{ij}, λ_i and μ at the point is surjective; it is therefore an isomorphism, and so the dimension of V in a space of constant curvature is $\frac{1}{2}(n+1)(n+2)$. □

The following consequence of the structural equations, while it is not required in the rest of the paper, is included because it is potentially useful for the determination of the special conformal Killing tensors of any particular metric.

Corollary 3 A special conformal Killing tensor L_{ij} satisfies

$$R^k_i L_{jk} = R^k_j L_{ik}.$$

Proof From the structural equation

$$\lambda_{i|j} = \frac{1}{n}\left(2R^k_j L_{ik} - 2g^{kl}R^m{}_{ijk}L_{lm} + g_{ij}\mu\right)$$

we obtain

$$\lambda_{i|j} - \lambda_{i|j} = \frac{2}{n}\left(R^k_j L_{ik} - R^k_i L_{jk} - g^{kl}(R^m{}_{ijk} - R^m{}_{jik})L_{lm}\right).$$

Now

$$R^m{}_{ijk} - R^m{}_{jik} = R^m{}_{ijk} + R^m{}_{jki} = -R^m{}_{kij},$$

and $g^{kl}R^m{}_{kij}L_{lm} = g^{kl}g^{mn}R_{nkij}L_{lm} = 0$ since R_{nkij} is skew in its first two indices. But since λ_i is a gradient $\lambda_{i|j}$ is symmetric, whence the result. □

3 Special conformal Killing tensors in spaces of non-zero constant curvature

We now obtain explicit formulae for the special conformal Killing tensors in a space of constant curvature. Those in a flat space are easily derived — they are given in Cartesian coordinates by

$$L_{ij} = \alpha x_i x_j + (\beta_i x_j + \beta_j x_i) + \gamma_{ij}$$

where α, β_i and γ_{ij} are constants, with $\gamma_{ji} = \gamma_{ij}$, and we have written x_i for $\eta_{ij}x^j$ where η_{ij} is the (constant) metric — so we confine our attention here to spaces of non-zero constant curvature.

The structural equations can be used to advantage here also. We write them in terms of $\lambda = g^{kl}L_{kl}$: they are

$$\begin{aligned} L_{ij|k} &= \tfrac{1}{2}(g_{jk}\lambda_{|i} + g_{ik}\lambda_{|j}) \\ \lambda_{|ij} &= 2BL_{ij} + \frac{1}{n}g_{ij}(\mu - 2B\lambda) \\ \mu_{|i} &= 2(n+1)B\lambda_{|i}, \end{aligned}$$

with $B \neq 0$ by assumption. By the third of these, $\mu - 2(n+1)B\lambda$ is a constant, say $2nBk$ for convenience; then from the second

$$L_{ij} = \frac{1}{2B}\lambda_{|ij} - g_{ij}(\lambda + k).$$

From the first we find that λ must satisfy the third-order differential equations

$$\lambda_{|ijk} = B(g_{jk}\lambda_{|i} + g_{ik}\lambda_{|j} + 2g_{ij}\lambda_{|k}).$$

Theorem In a space of non-zero constant curvature B, the map $L_{ij} \to g^{ij}L_{ij} = \lambda$ determines an isomorphism between the space of special conformal Killing tensors and the space of solutions of the equations

$$\lambda_{|ijk} = B(g_{jk}\lambda_{|i} + g_{ik}\lambda_{|j} + 2g_{ij}\lambda_{|k})$$

for the scalar λ.

Proof We have already shown that if L_{ij} is a special conformal Killing tensor then $\lambda = g^{ij}L_{ij}$ satisfies the differential equations. Suppose conversely that λ satisfies these differential equations. Note first that if $\mu = g^{ij}\lambda_{|ij}$

then from the equations $\mu_{|k} = 2(n+1)B\lambda_{|k}$, so $\mu - 2(n+1)B\lambda$ is a constant, which we write as $2nBk$ as before. Set

$$L_{ij} = \frac{1}{2B}\lambda_{|ij} - g_{ij}(\lambda + k).$$

Then

$$\begin{aligned} L_{ij|k} &= \frac{1}{2B}\lambda_{|ijk} - g_{ij}\lambda_{|k} \\ &= \tfrac{1}{2}(g_{jk}\lambda_{|i} + g_{ik}\lambda_{|j} + 2g_{ij}\lambda_{|k}) - g_{ij}\lambda_{|k}, \end{aligned}$$

so L_{ij} is a special conformal Killing tensor; and

$$g^{ij}L_{ij} = \frac{1}{2B}\mu - n(\lambda + k) = \lambda. \quad \square$$

Note in passing that this result shows that the dimension of the solution space of the third-order equations is again $\frac{1}{2}(n+1)(n+2)$. This fact could equally well have been established by noticing that the equations are structural equations — or more precisely, if we take variables λ, λ_i and λ_{ij} with the latter symmetric, the equations

$$\begin{aligned} \lambda_{|i} &= \lambda_i \\ \lambda_{i|j} &= \lambda_{ij} \\ \lambda_{ij|k} &= B(g_{jk}\lambda_i + g_{ik}\lambda_j + 2g_{ij}\lambda_k) \end{aligned}$$

constitute a structural system for the third-order equations.

We could therefore approach the problem of finding the special conformal Killing tensors in a space of non-zero constant curvature by attempting to solve the third-order equations. As it happens, we may find sufficient solutions indirectly, due to the following convenient fact: the traces of valence 2 Killing tensors (Killing 2-tensors) in a space of constant curvature satisfy the same equations, and all such Killing tensors are explicitly known. We next derive this result about Killing tensors.

Theorem The trace κ of any Killing 2-tensor K_{ij} in a space of constant curvature B satisfies

$$\kappa_{|ijk} = B(g_{jk}\kappa_{|i} + g_{ik}\kappa_{|j} + 2g_{ij}\kappa_{|k}).$$

Proof The Killing tensor condition gives $\kappa_{|i} = -2g^{jk}K_{ij|k}$. We differentiate this twice covariantly, and use the Ricci identity to reorder indices. First,

$$\begin{aligned}\kappa_{|ij} &= -2g^{lm}K_{il|mj} \\ &= -2g^{lm}\left(K_{il|jm} + B(g_{im}K_{jl} + g_{lm}K_{ij} - g_{ij}K_{ml} - g_{lj}K_{im})\right) \\ &= -2\left(g^{lm}K_{il|jm} + B(nK_{ij} - g_{ij}\kappa)\right).\end{aligned}$$

Likewise, $\kappa_{|ji} = -2(g^{lm}K_{jl|im} + B(nK_{ij} - g_{ij})\kappa)$; but $\kappa_{|ji} = \kappa_{|ij}$, so

$$\begin{aligned}\kappa_{|ij} &= -g^{lm}(K_{il|jm} + K_{jl|im}) - 2B(nK_{ij} - g_{ij}\kappa) \\ &= g^{lm}K_{ij|lm} - 2B(nK_{ij} - g_{ij}\kappa).\end{aligned}$$

Thus

$$\kappa_{|ijk} = g^{lm}K_{ij|lmk} - 2B(nK_{ij|k} - g_{ij}\kappa_{|k}).$$

We now work the k to the left in the first term on the right-hand side, using the Ricci identity, in two stages, obtaining

$$\begin{aligned}g^{lm}K_{ij|lmk} &= g^{lm}K_{ij|lkm} + B\left((n-2)K_{ij|k} + \tfrac{1}{2}(g_{jk}\kappa_{|i} + g_{ik}\kappa_{|j})\right) \\ &= g^{lm}K_{ij|klm} + B\left((n-3)K_{ij|k} + g_{jk}\kappa_{|i} + g_{ik}\kappa_{|j}\right).\end{aligned}$$

It follows that

$$\kappa_{|ijk} = g^{lm}K_{ij|klm} + B\left(-(n+3)K_{ij|k} + g_{jk}\kappa_{|i} + g_{ik}\kappa_{|j} + 2g_{ij}\kappa_{|k}\right).$$

We next take the cyclic sum:

$$\kappa_{|ijk} + \kappa_{|jki} + \kappa_{|ikj} = 4B(g_{jk}\kappa_{|i} + g_{ik}\kappa_{|j} + g_{ij}\kappa_{|k}),$$

using the symmetry of $\kappa_{|ijk}$ in its first two indices. Finally, we use the Ricci identity in the second and third terms on the left-hand side:

$$3\kappa_{|ijk} + B(g_{jk}\kappa_{|i} + g_{ik}\kappa_{|j} - 2g_{ij}\kappa_{|k}) = 4B(g_{jk}\kappa_{|i} + g_{ik}\kappa_{|j} + g_{ij}\kappa_{|k}),$$

whence the desired result. □

We use this result to obtain the special conformal Killing tensors in a space with metric of arbitrary signature and with non-zero constant curvature. We take the metric to be

$$g_{ij} = \frac{\eta_{ij}}{F^2}, \quad F = 1 + \tfrac{1}{4}B\eta_{kl}x^k x^l,$$

where η_{ij} is a constant non-singular symmetric matrix and B a non-zero constant; such a metric has the same signature as η_{ij}, and constant curvature B. (The coordinate range must of course be chosen so that F never vanishes in it.) We consider those Killing 2-tensors which are sums of symmetrized products of Killing vectors (in fact all Killing 2-tensors in a space of constant curvature are such [18], though we do not need this fact). The Killing vectors take the form

$$\xi^i = \alpha^i_j x^j + \tfrac{1}{2}B(\eta_{jk}\beta^j x^k)x^i + \bar{F}\beta^i, \quad \bar{F} = 1 - \tfrac{1}{4}B\eta_{kl}x^k x^l$$

for constants α^i_j and β_k, where the α^i_j are skew-symmetric in the sense that $\alpha^k_i \eta_{jk} + \alpha^k_j \eta_{ik} = 0$. It is not difficult to show that the functions

$$1, \quad \frac{x^i x^j}{F^2}, \quad \frac{\bar{F}x^i}{F^2}$$

form a basis for the space of traces of such Killing 2-tensors; by the theorem they must satisfy the third-order differential equation

$$\kappa_{|ijk} = B(g_{jk}\kappa_{|i} + g_{ik}\kappa_{|j} + 2g_{ij}\kappa_{|k}),$$

and there are $\frac{1}{2}(n+1)(n+2)$ of them, as required. We can obtain special conformal Killing tensors from them by using the formula

$$L_{ij} = \frac{1}{2B}\lambda_{|ij} - g_{ij}\lambda;$$

this differs from the formula given earlier by the omission of a constant multiple of g_{ij}: this does not change the fact that L_{ij} is a special conformal Killing tensor, of course (though it does mean that λ is not its trace). The constant function 1 effectively gives the trivial special conformal Killing tensor g_{ij}. If we take

$$\lambda = \frac{a_{ij}x^i x^j}{F^2}$$

for some symmetric constant matrix a_{ij} we find (after some calculation) that (up to an overall constant factor)

$$L_{ij} = F^{-4}\left(4F^2 a_{ij} - 2BF(a_{ik}x_j + a_{jk}x_i)x^k + B^2(a_{kl}x^k x^l)x_i x_j\right),$$

where we have written x_i for $\eta_{ij}x^j$. When

$$\lambda = \frac{\bar{F}b_i x^i}{F^2}$$

for some constant vector b_i we obtain (again, apart from a constant factor)

$$L_{ij} = F^{-3}\left(2(b_i x_j + b_j x_i) + B(b_k x^k)(\eta_{kl}x^k x^l \eta_{ij} - 2x_i x_j)\right).$$

4 Two-dimensional spaces

In this section we shall consider the special conformal Killing tensor equations in dimension two.

We have still to complete the analysis of the structural equations for $n = 2$. We show that the previous conclusion, that the space of solutions of the special conformal Killing tensor equations has maximal dimension if and only if the space has constant curvature, holds in dimension two also.

Theorem The dimension of the space of solutions of the special conformal Killing tensor equations in a two-dimensional manifold is 6 if and only if the space has constant curvature.

Proof In a two-dimensional manifold the curvature is given by

$$R^l{}_{ijk} = \tfrac{1}{2}R\left(g_{ij}\delta^l_k - g_{ik}\delta^l_j\right),$$

where R is the curvature scalar. When this is substituted into the structural equations for $\mu_{|i}$ we obtain

$$\mu_{|i} = 2g^{jk}R_{|k}L_{ij} - R_{|i}\lambda + 3R\lambda_i.$$

It follows that

$$\mu_{|ij} - \mu_{|ji} = 2g^{kl}(R_{|jl}L_{ik} - R_{|il}L_{jk}) + 3(R_{|j}\lambda_i - R_{|i}\lambda_j),$$

since $\lambda_{i|j}$ and $R_{|ij}$ are both symmetric. In order for the integrability conditions of the $\mu_{|i}$ equations to hold, the right-hand side of this expression must vanish, and when the dimension of the solution space of the special conformal Killing tensor equations is maximal this must hold for arbitrary choices of λ_i and L_{jk}, at any point. Thus R must be constant. But we know that the integrability conditions of the structural equations hold for a space of constant curvature (of any dimension), so the result is proved. □

We now deal with some other aspects of the two-dimensional case.

As we pointed out in the introduction, a special conformal Killing tensor L_{ij} is a conformal Killing tensor of gradient type. Hence if we put $K_{ij} = L_{ij} - g_{ij}\lambda$ then K_{ij} is a Killing tensor, that is, it satisfies

$$K_{ij|k} + K_{jk|i} + K_{ki|j} = 0.$$

Of course in arbitrary dimension not every valence 2 Killing tensor can be derived in this way. Passage in the opposite direction is guaranteed only for a symmetric tensor K_{ij}, with trace κ, which satisfies the condition

$$K_{ij|k} = \frac{1}{2(n-1)}(2g_{ij}\kappa_{|k} - g_{jk}\kappa_{|i} - g_{ik}\kappa_{|j});$$

such a tensor is evidently Killing; and if we set

$$L_{ij} = K_{ij} - \frac{1}{n-1}g_{ij}\kappa$$

then L_{ij} is a special conformal Killing tensor.

The Killing tensor $K_{ij} = L_{ij} - g_{ij}\lambda$ is the second of the sequence generated by the cofactor construction described in the introduction, the metric being the first; for dimension greater than two there will be others, but in dimension two the sequence has only the two terms. In dimension two, morever, every Killing 2-tensor satisfies the identity $K_{ij|k} = \frac{1}{2}(2g_{ij}\kappa_{|k} - g_{jk}\kappa_{|i} - g_{ik}\kappa_{|j})$, and therefore comes from a special conformal Killing tensor, as we now show.

The Killing 2-tensor equations in dimension two are

$$\begin{aligned} K_{11|1} &= 0 \\ K_{11|2} &= -2K_{12|1} \\ K_{22|1} &= -2K_{12|2} \\ K_{22|2} &= 0. \end{aligned}$$

The derivatives of the trace are given by

$$\begin{aligned} \kappa_{|1} &= g^{11}K_{11|1} + 2g^{12}K_{12|1} + g^{22}K_{22|1} = 2(g^{12}K_{12|1} - g^{22}K_{12|2}) \\ \kappa_{|2} &= g^{11}K_{11|2} + 2g^{12}K_{12|2} + g^{22}K_{22|2} = 2(-g^{11}K_{12|1} + g^{12}K_{12|2}); \end{aligned}$$

these may be solved to give

$$\begin{aligned} K_{12|1} &= \tfrac{1}{2}(g_{12}\kappa_{|1} - g_{11}\kappa_{|2}) \\ K_{12|2} &= \tfrac{1}{2}(-g_{22}\kappa_{|1} + g_{12}\kappa_{|2}). \end{aligned}$$

But the equations $K_{ij|k} = \frac{1}{2}(2g_{ij}\kappa_{|k} - g_{jk}\kappa_{|i} - g_{ik}\kappa_{|j})$ are just

$$\begin{aligned} K_{11|1} &= 0 \\ K_{11|2} &= -g_{12}\kappa_{|1} + g_{11}\kappa_{|2} \end{aligned}$$

$$\begin{aligned} K_{12|1} &= \tfrac{1}{2}(g_{12}\kappa_{|1} - g_{11}\kappa_{|2}) \\ K_{12|2} &= \tfrac{1}{2}(-g_{22}\kappa_{|1} + g_{12}\kappa_{|2}) \\ K_{22|1} &= g_{22}\kappa_{|1} - g_{12}\kappa_{|2} \\ K_{22|2} &= 0, \end{aligned}$$

confirming the claim. It follows that in a two-dimensional manifold every Killing 2-tensor determines a special conformal Killing tensor, and conversely.

Now if there is a non-trivial special conformal Killing tensor in a two-dimensional space it must have simple eigenvalues almost everywhere, since otherwise it will be a scalar multiple of g_{ij} on an open set, and there are no special conformal Killing tensors of this form other than constant multiples of g_{ij}. The metric will therefore be separable, and as we show in an appendix, must therefore take the Liouville form, which (assuming positive-definiteness) is $(s_1 + s_2)\delta_{ij}$, where s_i is a function of x^i alone, $i = 1, 2$. The corresponding special conformal Killing tensor L_{ij} and Killing 2-tensor K_{ij} are given in matrix representation by

$$L = (s_1 + s_2)\begin{pmatrix} s_1 & 0 \\ 0 & -s_2 \end{pmatrix}, \qquad K = (s_1 + s_2)\begin{pmatrix} s_2 & 0 \\ 0 & -s_1 \end{pmatrix}.$$

Appendix 1

We derive here the results about structural equations we use in the main body of the paper. We interpret such equations as follows. We consider a vector bundle $E \to M$, where M is a pseudo-Riemannian manifold equipped with the Levi-Civita connection (though for the following considerations it would be enough to take M to be a manifold with an arbitrary symmetric affine connection); E is supposed to be the Whitney sum of tensor bundles over M, so that the connection induces a connection on E. We take fibre coordinates u^A on E, and denote by Λ^B_{Ai} the connection coefficients for the induced connection with respect to coordinates (x^i, u^A). We consider the structural equations $F_{A|i} = \Gamma^B_{Ai} F_B$ to be equations for a section of E, given in coordinates by $u^A = F^A(x^i)$.

Our approach is motivated by the case in which the Γ^B_{Ai} all vanish. Then a solution of the equations defines a section of E which is covariantly constant; or if we think of the connection as defining and being defined by a horizontal distribution on E, a section which is horizontal (that is, an n-dimensional submanifold of E, transverse to the fibres, to which the horizontal distribution is everywhere tangent). Given any point $u \in E$, if there is a horizontal section through u it is unique. Now the zero section of any vector bundle equipped with a linear connection is a horizontal section, and is the unique horizontal section through any point $(x^i, 0)$. Thus if $F_{A|i} = 0$ and $F_A(x^i) = 0$ anywhere then $F_A(x^i) = 0$ everywhere, so that the linear map from solutions of the equations $F_{A|i} = 0$ to their values at an arbitrary point of M is injective, and the maximum dimension of the solution space is the fibre dimension of E. The necessary and sufficient condition for the equations to be completely integrable, that is for there to be a horizontal section through every point of E, is that the horizontal distribution should be integrable (in the sense of Frobenius), or equivalently that the curvature of the induced connection should vanish.

Theorem Consider a system of structural equations $F_{A|i} = \Gamma^B_{Ai} F_B$, as interpreted above. Then

1. if, for any $x \in M$, there is a solution with prescribed values $F_A(x)$ it is unique;
2. the space of solutions has maximum dimension equal to the fibre dimension of E;
3. the maximum dimension of the solution space is attained if and only if the equations are completely integrable;
4. the necessary and sufficient conditions for the system to be completely integrable are that
$$\Gamma^B_{Ai|j} - \Gamma^B_{Aj|i} + \Gamma^C_{Ai}\Gamma^B_{Cj} - \Gamma^C_{Aj}\Gamma^B_{Ci} = R^B{}_{Aij},$$
where $R^B{}_{Aij}$ are the components of the curvature of the induced connection on E.

Proof It is only necessary to note that since the coefficients Γ^B_{Ai} are assumed to be tensorial, $\Lambda^B_{Ai} - \Gamma^B_{Ai}$ determines a new connection on E, with respect to which the structural equations are equations for covariantly constant sections. Thus items (1), (2) and (3) follow immediately from the earlier discussion. It is easy to see that the curvature of the new connection is

$$R^B{}_{Aij} - \left(\Gamma^B_{Ai|j} - \Gamma^B_{Aj|i} + \Gamma^C_{Ai}\Gamma^B_{Cj} - \Gamma^C_{Aj}\Gamma^B_{Ci}\right),$$

whence item (4). □

Appendix 2

Here we show that in two dimensions every metric which is orthogonally separable is of the Liouville form. We use the Levi-Civita conditions, which for a metric in any dimension, in orthogonal coordinates, are

$$\frac{\partial^2 g^{kk}}{\partial x^i \partial x^j} - g_{jj}\frac{\partial g^{jj}}{\partial x^i}\frac{\partial g^{kk}}{\partial x^j} - g_{ii}\frac{\partial g^{ii}}{\partial x^j}\frac{\partial g^{kk}}{\partial x^i} = 0,$$

$i, j, k = 1, 2, \ldots, n$, $i \neq j$; no sum. In two dimensions, for the metric $ds^2 = Edx^2 + Gdy^2$, the Levi-Civita conditions are

$$\begin{aligned} \frac{\partial^2(1/E)}{\partial x \partial y} &= E\frac{\partial(1/E)}{\partial x}\frac{\partial(1/E)}{\partial y} + G\frac{\partial(1/G)}{\partial x}\frac{\partial(1/E)}{\partial y} \\ \frac{\partial^2(1/G)}{\partial x \partial y} &= E\frac{\partial(1/G)}{\partial x}\frac{\partial(1/E)}{\partial y} + G\frac{\partial(1/G)}{\partial x}\frac{\partial(1/G)}{\partial y}. \end{aligned}$$

As a consequence,

$$\frac{\partial^2 \log E}{\partial x \partial y} = -\frac{\partial}{\partial x}\left(E\frac{\partial(1/E)}{\partial y}\right) = -\frac{\partial}{\partial y}\left(G\frac{\partial(1/G)}{\partial x}\right) = \frac{\partial^2 \log G}{\partial x \partial y},$$

so that there are (necessarily non-vanishing) functions $f(x, y)$, $\phi(x)$ and $\psi(y)$ such that $E = \phi f$ and $G = \psi f$. When the manifold is Riemannian, without loss of generality we may assume that ϕ and ψ are positive (so that f is also positive): then by a change of coordinates $u = u(x)$, $v = v(y)$ we can bring the metric to the form $f(u, v)(du^2 + dv^2)$. This new form of the metric must also satisfy the Levi-Civita conditions, which now both become

$$\frac{\partial^2(1/f)}{\partial u \partial v} = \frac{2}{f}\frac{\partial(1/f)}{\partial u}\frac{\partial(1/f)}{\partial v}, \quad \text{or} \quad \frac{\partial^2 f}{\partial u \partial v} = 0$$

as required. For an alternative derivation see [6].

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