Reduction and reconstruction aspects of second-order dynamical systems with symmetry

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Abstract. We examine the reduction process of a system of second-order ordinary differential equations which is invariant under a Lie group action. With the aid of connection theory, we explain why the associated vector field decomposes in three parts and we show how the integral curves of the original system can be reconstructed from the reduced dynamics. An illustrative example confirms the results.

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1 Introduction

There are several different geometric contexts in which we find manifestations of the following behaviour: roughly speaking, there is a system of second-order ordinary differential equations which admits a Lie group of symmetries; this system is reduced to a coupled pair of sets of equations, one of second order and one of first order. For example:

- the geodesics of a manifold with a Kaluza-Klein metric, and the Wong equations [12];
- systems of mechanical type with symmetry, including control systems [3] (for the case of a single symmetry);
- Lagrange-Poincaré equations and reduction by stages [4, 11];
- non-abelian Routh reduction [9];

- Chaplygin systems [2, 5].

In different cases different auxiliary geometrical structures are required to carry out the reduction. In the present paper we shall explain the nature of this reduction process for a second-order dynamical system, that is, a dynamical system governed by second-order differential equations, which is invariant under a Lie group action, using the simplest type of auxiliary machinery imaginable.

When we come to discuss the second-order case we shall represent a system of second-order ordinary differential equations by a special kind of vector field. There is therefore a lot to be said for starting off by considering the question for arbitrary vector fields, or in other words for first-order dynamical systems.

Suppose given a dynamical system, represented by a vector field X on a manifold M, which admits a Lie group G of symmetries. Suppose further that G acts freely and effectively on M so that M is a principal bundle over a manifold B with group G; let $\pi^M : M \to B$ be the projection. Since X is invariant under the action of G there is a vector field $\overline{X}$ on B which is π^M-related to X; this is the reduced dynamical system.

As a process of reduction, however, this is clearly incomplete in the sense that there is no way of reconstructing the original dynamical system from the reduced one; any two G-invariant dynamical systems on M which differ by a π^M-vertical vector field (which is necessarily also G-invariant) have the same reduced dynamics. To see what is at stake, let us introduce coordinates (x^i, x^a) on M such that the orbits of G, or in other words the fibres of $M \to B$, are given by x^i = constant; the x^i may therefore be regarded as coordinates on B. Let us denote by $\tilde{E}_a$ a basis for the fundamental vector fields on M generated by the G-action; then

$$\tilde{E}_a = K_a^b \frac{\partial}{\partial x^b}$$

for some non-singular matrix-valued function (K_a^b). Suppose further that we have at our disposal a distribution on M which is transverse to the fibres and G-invariant. Such a distribution will be spanned by vector fields X_i of the form

$$X_i = \frac{\partial}{\partial x^i} - \Lambda_i^a \tilde{E}_a$$

for certain functions Λ_i^a. We may then write

$$X = Y^i X_i + Z^a \tilde{E}_a = Y^i \frac{\partial}{\partial x^i} + (Z^a - \Lambda_i^a Y^i)\tilde{E}_a.$$

The necessary and sufficient conditions for X to be G-invariant are that $Y^i X_i$ and $Z^a \tilde{E}_a$ are separately G-invariant. In particular, the Y^i are independent of the x^a, so that $Y^i \partial/\partial x^i$ may be regarded as a vector field on the base manifold B: this is the reduced dynamical system $\overline{X}$, of course. The integral curves of X are solutions of the differential equations

$$\dot{x}^i = Y^i, \qquad \dot{x}^a = (Z^b - \Lambda_i^b \dot{x}^i) K_b^a.$$

The equations of the first set define the integral curves of the reduced dynamical system. The remainder can in principle be used to reconstruct an integral curve of the original dynamical system from a known integral curve of the reduced one.

This description of the process is somewhat disingenuous: a fibre-transverse G-invariant distribution on a principal G-bundle is of course just a connection, or more accurately a principal connection. This observation gives us the opportunity to describe the reduction in a coordinate-independent way, as is clearly desirable; when we do so, moreover, the reconstruction step acquires a more transparent geometrical interpretation than is apparent from the description above. Our basic contention is that the simplest additional machinery that is required to give a geometrically coherent account of the reduction and reconstruction of dynamical systems with symmetry is a connection; and we aim to show how these processes work in that context for second-order dynamical systems.

A second-order dynamical system can be represented by a vector field Γ on the tangent bundle TM of a differentiable manifold M, of the form

$$\Gamma = v^\alpha \frac{\partial}{\partial x^\alpha} + \Gamma^\alpha \frac{\partial}{\partial v^\alpha},$$

where the v^α are the fibre coordinates. Given a vector field X on M, let us denote by X^{C} its complete, or tangent, lift to TM and by X^{V} its vertical lift. Then in terms of the structure described above in the first-order case, we may express a second-order differential equation field Γ on TP in the form

$$\Gamma = v^i X_i^{\mathrm{C}} + v^a \tilde{E}_a^{\mathrm{C}} + D^i X_i^{\mathrm{V}} + D^a \tilde{E}_a^{\mathrm{V}}.$$

It turns out that in order for Γ to be invariant under the action of G on TM induced from its action on M, each of the three components $v^a \tilde{E}_a^{\mathrm{C}}$, $D^a \tilde{E}_a^{\mathrm{V}}$ and $v^i X_i^{\mathrm{C}} + D^i X_i^{\mathrm{V}}$ must be invariant. The last of these represents a second-order dynamical system, albeit in a generalized sense; the last two define the coupled first- and second-order equations which constitute the reduced system mentioned earlier. It is our aim to explain how this decomposition arises, with the help of connection theory; and to discuss the processes of reduction and reconstruction from this standpoint.

In the following section we discuss the first-order case in greater detail. In Section 3 we deal with the connection theory required for the analysis of the reduction and reconstruction of second-order systems, and in Section 4 we carry out that analysis. Section 5 is devoted to consideration of an example.

It will become apparent that a particular kind of Lie algebroid, the Atiyah algebroid of a principal bundle, plays an important role in the theory. In fact one can locate the case discussed here in a more general framework consisting of Lie algebroids and anchored vector bundles. Investigation of this aspect of the matter continues.

2 First-order systems

As before, we suppose that M is a manifold on which a Lie group G acts freely and effectively to the right; we denote the action by $\psi^M : G \times M \to M$. Then $\pi^M : M \to$

$M/G = B$ is a principal fibre bundle and $\pi^M \circ \psi^M_g = \pi^M$ for all $g \in G$. For $\xi \in \mathfrak{g}$, the Lie algebra of G, we denote by ξ_M the fundamental vector field corresponding to $\xi \in \mathfrak{g}$, that is, the infinitesimal generator of the 1-parameter group $\psi^M_{\exp(t\xi)}$ of transformations of M.

The G-action on M can be extended to a G-action $\psi^{TM} : G \times TM \to TM$ on the tangent manifold $\tau : TM \to M$, given by $(g, v_m) \mapsto T_m\psi^M_g(v_m)$, for $m \in M$, $v_m \in T_mM$. This action equips TM with the structure of a principal fibre bundle over TM/G, with projection π^{TM}. Then, of course, $\pi^{TM} \circ \psi^{TM}_g = \pi^{TM}$ and $T\pi^M \circ \psi^{TM}_g = T\pi^M$, where $T\pi^M : TM \to T(M/G)$; on the other hand, $\tau \circ \psi^{TM}_g = \psi^M_g$. We also have an action $\psi^{\mathcal{X}}$ on the space of sections of $TM \to M$, that is, on $\mathcal{X}(M)$, the space of vector fields on M, given by

$$\psi^{\mathcal{X}}(X)(m) = \psi^{TM}_g(X(\psi^M_{g^{-1}}(m))).$$

A vector field X on M is G-invariant if for all $g \in G$

$$X(\psi^M_g(m)) = \psi^{TM}_g(X(m)), \quad \text{or equivalently} \quad \psi^{\mathcal{X}}_g(X) = X.$$

If X is G-invariant then $[\xi_M, X] = 0$ for all $\xi \in \mathfrak{g}$. If G is connected, as we shall generally assume to be the case, this is a sufficient as well as a necessary condition for invariance.

The fundamental vector fields satisfy $\psi^{\mathcal{X}}_g(\xi_M) = (\mathrm{ad}_{g^{-1}}\,\xi)_M$, where ad is the adjoint action of G on $\mathfrak{g}$.

For all $m \in M$, π^{TM}_m induces an isomorphism $T_mM \to (TM/G)_{\pi^M(m)}$, the fibre of TM/G over $\pi^M(m) \in M/G$, and thus also an isomorphism $TM \to (\pi^M)^*TM/G$. As a consequence of this property there is a 1-1 correspondence between invariant vector fields on M and sections of the vector bundle $\overline{\tau} : TM/G \to M/G$ (see e.g. [10]). The vector bundle $\overline{\tau}$ has the structure of a Lie algebroid: the anchor map $\varrho : TM/G \to T(M/G)$ is given by $[\![v]\!] \mapsto T\pi^M(v)$ (here and below $[\![\cdot]\!]$ represents the equivalence class of the argument under G-equivalence, or in other words its G-orbit), which is independent of the choice of $v \in [\![v]\!]$ because of the property $T\pi^M \circ \psi^{TM}_g = T\pi^M$; the bracket of two sections of TM/G is given by the bracket of the associated invariant vector fields. With this Lie algebroid structure $\overline{\tau}$ is called the Atiyah algebroid of the principal G-bundle π^M [7].

The fibre-linear map $T\pi^M : TM \to T(M/G)$ is surjective on the fibres. The kernel of the induced map $TM \to (\pi^M)^*T(M/G)$ is isomorphic to the bundle $M \times \mathfrak{g} \to M$; the identification of $M \times \mathfrak{g}$ as a subbundle of TM is given by $(m, \xi) \mapsto \xi_M(m)$.

A connection on π^M is a right splitting γ of the short exact sequence

$$0 \to M \times \mathfrak{g} \to TM \stackrel{T\pi^M}{\to} (\pi^M)^*T(M/G) \to 0 \qquad (1)$$

of vector bundles over M. The corresponding left splitting $TM \to M \times \mathfrak{g}$ will be denoted by ω. We will write ϖ for its projection on $\mathfrak{g}$. The distinction between ω and ϖ can be made clear as follows. If we identify $M \times \mathfrak{g}$ with a subbundle of TM then ω may be thought of as a type $(1,1)$ tensor field on M; we have $\omega(\xi_M) = \xi_M$, while $\varpi(\xi_M) = \xi$. Needless to say, both ω and ϖ vanish on $\mathrm{im}(\gamma)$. The map γ may be thought of as the horizontal lift, the $\mathfrak{g}$-valued 1-form ϖ as the connection form.

If ϖ satisfies $\varpi(\psi^{TM}_g v) = \mathrm{ad}_{g^{-1}}\,\varpi(v)$ the connection is said to be principal. Equivalently, principal connections are right splittings γ with the property $\gamma(\psi^M_g m, \overline{v}) = \psi^{TM}_g \gamma(m, \overline{v})$ for all $\overline{v} \in T_{\pi^M(m)}(M/G)$. The condition for the connection to be principal when expressed in terms of ω is simply that it is invariant under the G-action on TM, that is, that $\omega \circ \psi_g^{TM} = \psi_g^{TM} \circ \omega$.

The manifold $M \times \mathfrak{g}$ comes equipped with the right action $g \mapsto (\psi_g^M, \mathrm{ad}_{g^{-1}})$; we denote by $\overline{\mathfrak{g}} = (M \times \mathfrak{g})/G$ its quotient under this action. We remark that $\overline{\mathfrak{g}}$ is the (total space of) the vector bundle associated with the principal G-bundle π^M by the adjoint action of G on $\mathfrak{g}$; it is often called the adjoint bundle. When we take the quotient of the exact sequence (1) under the action of G we obtain the following short exact sequence of vector bundles over M/G:

$$0 \to \overline{\mathfrak{g}} \to TM/G \xrightarrow{\varrho} T(M/G) \to 0, \tag{2}$$

which is called the Atiyah sequence [7]. If γ is a principal connection on π^M then $\pi^{TM}(\gamma(m, \overline{v}))$ is independent of the choice of $m \in [\![m]\!] = \overline{\tau}(\overline{v})$ because of the invariance of γ; if we set $\overline{\gamma}(\overline{v}) = \pi^{TM}(\gamma(m, \overline{v}))$ then $\overline{\gamma} : T(M/G) \to TM/G$ is well-defined and satisfies $\varrho \circ \overline{\gamma} = \mathrm{id}$, and is therefore a right splitting of the Atiyah sequence. This establishes a correspondence between principal connections on π^M and splittings of the Atiyah sequence, which is actually 1-1. If $\overline{\gamma}$ is a right splitting of the Atiyah sequence, the corresponding left splitting will be denoted by $\overline{\omega}$.

If ϖ is the connection form of a principal connection on π^M and X a G-invariant vector field on M then $\varpi(X)$ is a $\mathfrak{g}$-valued function on M which satisfies $\varpi(X) \circ \psi_g^M = \mathrm{ad}_{g^{-1}} \varpi(X)$. So the map $m \mapsto (m, \varpi(X)(m)) \in M \times \mathfrak{g}$ is constant on the orbits of the G-action, and therefore defines a section of $\overline{\mathfrak{g}} \to M/G$.

We next describe the reduction of a G-invariant vector field. As we pointed out earlier, a G-invariant vector field X can be identified with a section $\tilde{X}$ of TM/G, given by $\tilde{X}(\pi^M(m)) = \pi^{TM}(X(m))$. If γ is a principal connection then $\tilde{X}$ can in turn be decomposed into a vector field $\overline{X} = \varrho\tilde{X}$ on M/G and a section $\overline{\omega}(\tilde{X})$ of $\overline{\mathfrak{g}}$. The relation between X and $\overline{X}$ is $\overline{X}(\pi^M(m)) = T\pi^M(X(m))$; $\overline{X}$ is the reduced vector field of X.

The decomposition may be described in a slightly different way. Given a connection γ, any $X \in \mathcal{X}(M)$ can be decomposed into its horizontal and vertical components with respect to γ; the horizontal component is determined by a section of $(\pi^M)^*T(M/G)$, and the vertical component can be identified with $\omega(X)$ and hence with the $\mathfrak{g}$-valued function $\varpi(X)$. When X is G-invariant and γ is a principal connection, the horizontal component is the horizontal lift of a vector field on M/G, namely $\overline{X}$; and the section of $\overline{\mathfrak{g}}$ that $\varpi(X)$ defines is just $\overline{\omega}(\tilde{X})$.

The following fact about the integral curves of an invariant vector field is well-known. Suppose that $t \mapsto c(t)$ is an integral curve of X, so that $\dot{c} = X \circ c$. Then the curve $t \mapsto \overline{c}(t) = \pi^M(c(t))$ in M/G is an integral curve of $\overline{X}$, that is,

$$\dot{\overline{c}} = \overline{X} \circ \overline{c}. \tag{3}$$

Indeed, $\dot{\overline{c}} = T\pi^M \circ \dot{c} = T\pi^M \circ (X \circ c) = \overline{X} \circ (\pi^M \circ c)$. In fact an integral curve c of X is completely determined by the underlying integral curve $\overline{c}$ of $\overline{X}$ and a curve $t \mapsto g(t)$ in G. To see this, note that there is a unique curve $\overline{c}^\gamma$ in M, the horizontal lift of $\overline{c}$ through $c(0)$, such that

- $\overline{c}^\gamma$ projects onto $\overline{c}$
- $\overline{c}^\gamma(0) = c(0)$
- the tangent $\dot{\overline{c}}^\gamma$ to $\overline{c}^\gamma$ is everywhere horizontal (so that $\overline{c}^\gamma$ satisfies $\dot{\overline{c}}^\gamma = \gamma(\overline{c}^\gamma, \dot{\overline{c}})$).

Then since c also projects onto $\overline{c}$ there is a curve $t \mapsto g(t) \in G$, with $g(0) = e$ (the identity element of G), such that $c(t) = \psi^M_{g(t)}\overline{c}^\gamma(t)$. Now let θ be the Maurer-Cartan form of G: then $t \mapsto \theta(\dot{g}(t))$ is a curve in $\mathfrak{g}$. By differentiating the equation $c = \psi^M_g\overline{c}^\gamma$ we see that $\dot{g}$ must satisfy

$$\dot{c} = \psi^{TM}_g \left((\theta(\dot{g}))_M \circ \overline{c}^\gamma + \dot{\overline{c}}^\gamma \right). \tag{4}$$

But $\dot{c} = X \circ c = X \circ (\psi^M_g\overline{c}^\gamma) = \psi^{TM}_g(X \circ \overline{c}^\gamma)$, whence

$$(\theta(\dot{g}))_M \circ \overline{c}^\gamma + \dot{\overline{c}}^\gamma = X \circ \overline{c}^\gamma.$$

The first term on the left-hand side (which when evaluated at t is the value at $\overline{c}^\gamma(t)$ of the fundamental vector field corresponding to $\theta(\dot{g}(t)) \in \mathfrak{g}$) is vertical, the second horizontal, so this equation is simply the decomposition of X into its horizontal and vertical components, at any point of $\overline{c}^\gamma$. In particular,

$$\theta(\dot{g}) = \varpi(X \circ \overline{c}^\gamma), \tag{5}$$

the right-hand side being of course a curve in $\mathfrak{g}$. This is a differential equation for the curve g, and has a unique solution with specified initial value. (When G is a matrix group $\theta(\dot{g}) = g^{-1}\dot{g}$; and the equation $\theta(\dot{g}) = \xi$, where $t \mapsto \xi(t)$ is a curve in $\mathfrak{g}$, can be written $\dot{g} = g\xi$, from which the assertion is obvious. See for example [13] for the general case.) Thus the curve g is uniquely determined by equation (5) and the initial condition $g(0) = e$.

We can conclude that one can reconstruct the integral curves of the G-invariant vector field X from those of the reduced vector field $\overline{X}$, always assuming that we have at our disposal a principal connection γ. In order to carry out the reconstruction we need to solve successively

$$\left\{ \begin{array}{lcll} \dot{\overline{c}} & = & \overline{X}(\overline{c}) & \text{for } \overline{c} \\ \dot{\overline{c}}^\gamma & = & \gamma(\overline{c}^\gamma, \dot{\overline{c}}) & \text{for } \overline{c}^\gamma \\ \theta(\dot{g}) & = & \varpi(X \circ \overline{c}^\gamma) & \text{for } g, \end{array} \right.$$

to obtain finally the integral curve $c = \psi^M_g\overline{c}^\gamma$ of X.

We now give some explicit expressions for the decomposition. We consider first the construction of a basis of vertical vector fields, that is, vector fields tangent to the orbits of the G-action. There are in fact two possible choices, at least locally, corresponding to what are sometimes called, as in [2], the 'moving basis' and the 'body-fixed basis'. The reference is to rigid body dynamics; the point is that the body-fixed basis is invariant.

Let $\{E_a\}$ be a basis for $\mathfrak{g}$, and C^c_{ab} the corresponding structure constants. The moving basis consists of the fundamental vector fields $(E_a)_M$. These vector fields are not of course invariant: in fact for any fundamental vector field ξ_M, $\psi^{\mathcal{X}}_g(\xi_M) = (\mathrm{ad}_{g^{-1}}\,\xi)_M$, as we pointed out earlier. We will usually write $\tilde{E}_a$ instead of $(E_a)_M$ for convenience.

The definition of the body-fixed basis depends on a choice of local trivialization of $\pi^M : M \to M/G$. Let $U \subset M/G$ be an open set over which M is locally trivial. The projection π^M is locally given by projection onto the first factor in $U \times G \to U$, and the action by $\psi_g^M(x, h) = (x, hg)$. The maps

$$\overline{E}_a : U \to (M \times \mathfrak{g})/G|_U \quad \text{by} \quad x \mapsto [\![(x, e), E_a]\!]$$

will give a local basis for $\mathrm{Sec}(\overline{\mathfrak{g}}) = \mathrm{Sec}((M \times \mathfrak{g})/G)$ over U. These maps can be considered as sections of $TM/G \to M/G$ by means of the identification

$$\overline{E}_a \in \mathrm{Sec}(\overline{\mathfrak{g}}) \quad \Longleftrightarrow \quad \overline{E}_a : x \mapsto \pi^M\big(\tilde{E}_a(x, e)\big) \in \mathrm{Sec}(TM/G).$$

Recall first that the injection $M \times \mathfrak{g} \to TM$ is given by $(m, \xi) \mapsto \xi_M(m)$. Further, it is clear that the two elements $((x, e), \xi)$ and $((x, g), \mathrm{ad}_{g^{-1}} \xi)$ of $M \times \mathfrak{g}$ belong to the same equivalence class in $\overline{\mathfrak{g}}$. This is in perfect agreement with the above identification, since

$$\pi^{TM}\big(\xi_M(x, e)\big) = \pi^{TM}\big((\mathrm{ad}_{g^{-1}} \xi)_M(x, g)\big).$$

Now sections of TM/G can be lifted to invariant vector fields on M. For the above sections, the invariant vector fields are

$$\hat{E}_a : (x, g) \mapsto (\mathrm{ad}_{g^{-1}} E_a)_M(x, g) = \psi_g^{TM}\big((E_a)_M(x, e)\big).$$

Then, indeed, $\psi_g^{\mathcal{X}}(\hat{E}_a) = \hat{E}_a$. The corresponding basis for $\mathrm{Sec}(M \times \mathfrak{g})$ is given by the sections $(x, g) \mapsto ((x, g), \mathrm{ad}_{g^{-1}} E_a)$. By contrast the fundamental vector fields $\tilde{E}_a$, identified as sections of $M \times \mathfrak{g} \to M$, are given by $(x, g) \mapsto ((x, g), E_a)$. The relation between the two sets of vector fields can be expressed as $\hat{E}_a(x, g) = A_a^b(g)\tilde{E}_b(x, g)$ where $(A_a^b(g))$ is the matrix representing $\mathrm{ad}_{g^{-1}}$ with respect to the basis $\{E_a\}$ of $\mathfrak{g}$.

In fact the body-fixed basis $\{\hat{E}_a\}$ associated with a local trivialization $(\pi^M)^{-1}U \equiv U \times G$ is obtained just by transferring to $(\pi^M)^{-1}U$ the right-invariant vector fields on G associated with the basis $\{E_a\}$ of $\mathfrak{g}$. The moving basis, on the other hand, corresponds to the left-invariant vector fields on G associated with the basis $\{E_a\}$.

Let us take coordinates (x^i, x^a) on M such that (x^i) are coordinates on U, (x^a) coordinates on the fibre. Then there are 'action functions' (so-called in [2]) such that $\tilde{E}_a = K_a^b(x^c)\partial/\partial x^b$. The relation $[\tilde{E}_a, \tilde{E}_b] = C_{ab}^c \tilde{E}_c$ leads to the property

$$K_a^c \frac{\partial K_b^d}{\partial x^c} - K_b^c \frac{\partial K_a^d}{\partial x^c} = C_{ab}^e K_e^d.$$

The invariance of the vector fields $\hat{E}_a$ can be expressed as

$$[\tilde{E}_b, \hat{E}_a] = 0 \quad \Longleftrightarrow \quad \tilde{E}_b(A_a^c) + A_a^d C_{bd}^c = 0.$$

We can use these differential equations as another way of constructing a body-fixed basis, as follows. We seek local vector fields $\{\hat{E}_a\}$, given in terms of the moving basis $\{\tilde{E}_a\}$ by $\hat{E}_b = A_b^a \tilde{E}_a$ where (A_b^a) is a locally defined non-singular matrix-valued function on M, which are G-invariant, which is to say that $[\tilde{E}_a, \hat{E}_b] = 0$ for all a and b. Thus, as above, the A^a_b must satisfy

$$\tilde{E}_a(A^c_b) + C^c_{ad}A^d_b = 0. \tag{6}$$

This is a system of linear partial differential equations for the unknowns A^a_b. The integrability conditions

$$[\tilde{E}_a, \tilde{E}_b](A^d_c) + C^d_{be}\tilde{E}_a(A^e_c) - C^d_{ae}\tilde{E}_b(A^e_c) = 0$$

are identically satisfied by virtue of the Jacobi identity. The equations therefore have solutions locally on M, and a solution can be specified by choosing a local cross-section of the G action and specifying the value of (A^a_b) on it; the natural choice, which we make, is to take it to be the identity matrix. The A^a_b will then be independent of the x^i.

A simple calculation shows that

$$[\hat{E}_a, \hat{E}_b] = -A^d_a A^e_b \bar{A}^c_f C^f_{de}\hat{E}_c,$$

where the $\bar{A}^a_b$ are the components of the matrix inverse to (A^a_b). On the other hand, if we write $[\hat{E}_a, \hat{E}_b] = -\hat{C}^c_{ab}\hat{E}_c$ then the coefficients $\hat{C}^c_{ab}$ must be G-invariant, since everything else in the equation is. It follows that the value of $\hat{C}^c_{ab}$ along any fibre of $(\pi^M)^{-1}U \to U$ is the same as its value on the section which determines the local trivialization, that is, where $g = e$; if we take (A^a_b) to be the identity there we obtain $\hat{C}^c_{ab} = C^c_{ab}$, that is, $[\hat{E}_a, \hat{E}_b] = -C^c_{ab}\hat{E}_c$ (as one would expect).

We now consider the horizontal vector fields. We have at our disposal the local coordinate basis $\{\partial/\partial x^i\}$ of $\mathcal{X}(T(M/G))$; we put

$$\overline{X}_i = \overline{\gamma}\left(\frac{\partial}{\partial x^i}\right) \in \mathrm{Sec}(TM/G).$$

The sections $\{\overline{X}_i, \overline{E}_a\}$ form a basis of $\mathrm{Sec}(TM/G)$. They can be lifted to a basis $\{X_i, \hat{E}_a\}$ of $\mathcal{X}(M)$, consisting only of invariant sections. Then

$$X_i(x, g) = \gamma\left((x, g), \left.\frac{\partial}{\partial x^i}\right|_x\right).$$

If we set

$$X_i = \frac{\partial}{\partial x^i} - \gamma^b_i(x^i, x^a)\hat{E}_b = \frac{\partial}{\partial x^i} - \gamma^b_i(x^i, x^a)A^c_b(x^a)\tilde{E}_c,$$

then invariance of X_i amounts to

$$[\tilde{E}_b, X_i] = 0 \quad \Longleftrightarrow \quad \tilde{E}_b(\gamma^c_i) = 0 \quad \Longleftrightarrow \quad \frac{\partial \gamma^c_i}{\partial x^b} = 0.$$

For future use we calculate $[\hat{E}_a, X_i]$ here also. We have

$$[\hat{E}_a, X_i] = -X_i(A^c_a)\bar{A}^b_c\hat{E}_b.$$

Now $\partial A^b_a/\partial x^i = 0$, and so

$$X_i(A^c_a) = -\gamma^d_i A^e_d \tilde{E}_e(A^c_a) = \gamma^d_i C^c_{ef} A^e_d A^f_a = \gamma^d_i C^e_{da} A^c_e, \tag{7}$$

so that

$$[\hat{E}_a, X_i] = \gamma_i^b C_{ab}^c \hat{E}_c, \tag{8}$$

assuming as we may that $[\hat{E}_a, \hat{E}_b] = -C_{ab}^c \hat{E}_c$.

A vector field X on M can be written as $X = Y^j X_j + Y^b \hat{E}_b$. If X is invariant then

$$\frac{\partial Y^j}{\partial x^c} = 0 \quad \text{and} \quad \frac{\partial Y^b}{\partial x^c} = 0.$$

If alternatively we set $X = Y^j X_j + Z^c \tilde{E}_c$, where $Z^c = A_b^c Y^b$, then the second invariance condition becomes

$$\tilde{E}_d(Z^c) + C_{de}^c Z^e = 0. \tag{9}$$

An invariant vector field X projects onto the section $\tilde{X} : (x^i) \mapsto Y^j(x^i) X_j + Y^a(x^i)\overline{E}_a$ of TM/G and the vector field $\overline{X} : (x^i) \mapsto Y^j(x^i)\partial/\partial x^i$ on M/G. Finally, we have

- $\omega(X) = Y^a \hat{E}_a = Z^a \tilde{E}_a \in TM$;
- $\overline{\varpi}(\tilde{X}) = Y^a \overline{E}_a \in \mathrm{Sec}(\overline{\mathfrak{g}})$;
- $\varpi(X) = Z^a E_a \in C^\infty(M, \mathfrak{g})$.

In fact, as we pointed out before, when X is invariant $\varpi(X)$ defines a section of $\overline{\mathfrak{g}} \to M/G$. We now wish to explain how one can recognise a section of $\overline{\mathfrak{g}}$ in terms of coordinates. Recall that a section of $\overline{\mathfrak{g}}$ can be thought of as a function $M \to \mathfrak{g}$ which is constant on the equivalence classes of the equivalence relation defining the associated bundle structure; that is, a $\mathfrak{g}$-valued function s on M such that $s \circ \psi_g^M = \mathrm{ad}_{g^{-1}} s$. Assuming as always that G is connected, we may equivalently write this condition as $\xi_M(s) + [\xi, s] = 0$ for any $\xi \in \mathfrak{g}$, where the bracket is the Lie algebra bracket of $\mathfrak{g}$. We may express s as $s = s^a E_a$ with respect to a basis $\{E_a\}$ of $\mathfrak{g}$; in terms of the components s^a of s the condition for s to define a section is

$$\tilde{E}_b(s^a) + C_{bc}^a s^c = 0. \tag{10}$$

This makes clear the significance of equation (9).

3 Second-order diagrams and connections

In this section we discuss the connection theory relevant to second-order dynamical systems. Before we do so, however, it will be convenient to make some remarks about splittings of short exact sequences in general; these remarks will be useful later.

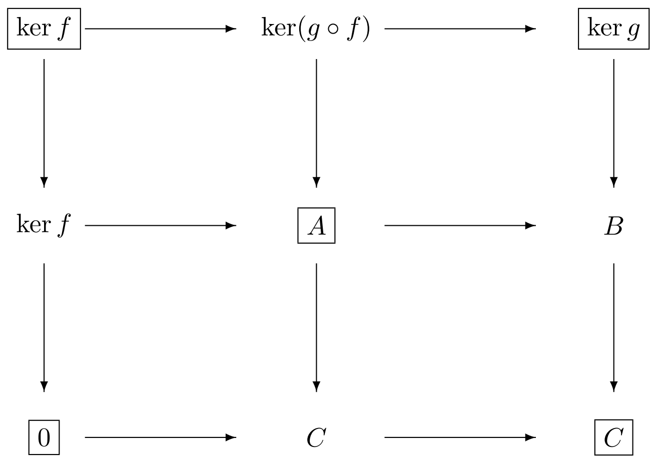
If $0 \to \ker f \to A \xrightarrow{f} B \to 0$ and $0 \to \ker g \to B \xrightarrow{g} C \to 0$ are two short exact sequences of vector bundles over the same manifold, then the sequence

$$0 \to \ker(g \circ f) \to A \xrightarrow{g \circ f} C \to 0$$

is also exact. Moreover the restriction of f to $\ker(g \circ f)$ gives rise to a fourth short exact sequence

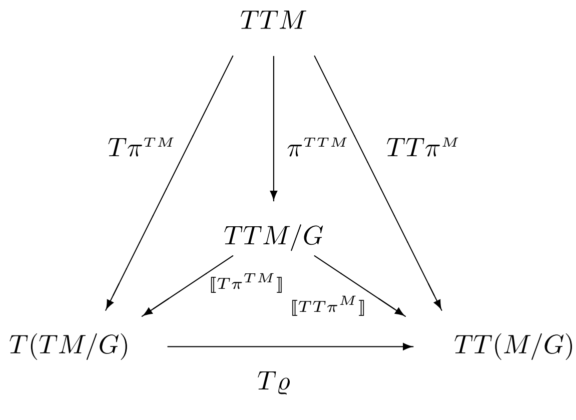
$$0 \to \ker f \to \ker(g \circ f) \xrightarrow{f} \ker g \to 0.$$

In summary, we can draw the following commutative diagram:



The following facts are immediate. Suppose given splittings $\gamma_1 : B \to A$ and $\gamma_2 : C \to A$. If we set $\gamma_3 = f \circ \gamma_2 : C \to B$ then γ_3 is also a splitting. If, in addition, $\gamma_2(C) \subset \gamma_1(B)$ then $\gamma_2 = \gamma_1 \circ \gamma_3$. Furthermore, γ_1 restricts to a splitting $\ker g \to \ker(g \circ f)$. Therefore, given γ_1 and γ_2, each element of A can be uniquely decomposed into three parts, one in C, one in $\ker f$ and one in $\ker g$. We will use this observation for our decomposition of second-order differential equation fields.

We can now turn to the principal matter in hand. The actions of G on M and TM induce also a G-action on TTM: $\psi^{TTM} : G \times TTM \to TTM, (g, X_v) \mapsto T_v\psi^{TM}_g(X_v)$. As before, this means that there exists a principal fibre bundle structure $\pi^{TTM} : TTM \to TTM/G$ with the properties $\pi^{TTM} \circ \psi^{TTM}_g = \pi^{TTM}$ and $T\pi^{TM} \circ \psi^{TTM}_g = T\pi^{TM}$. Again, the fibres of TTM and TTM/G are isomorphic, so $TTM \simeq (\pi^{TM})^* TTM/G$. Therefore, the maps $[\![T\pi^{TM}]\!] : TTM/G \to T(TM/G), [\![X]\!] \mapsto T\pi^{TM}(X)$ and $[\![TT\pi^{M}]\!] : TTM/G \to TT(M/G), [\![X]\!] \mapsto TT\pi^{M}(X)$ are well-defined and lead to the following commutative diagram:



The above diagram contains bundles over TM, TM/G and $T(M/G)$. All of them are Lie algebroids. For example, TTM/G is the Atiyah algebroid of the manifold TM. In addition, all the maps in the diagram are Lie algebroid morphisms.

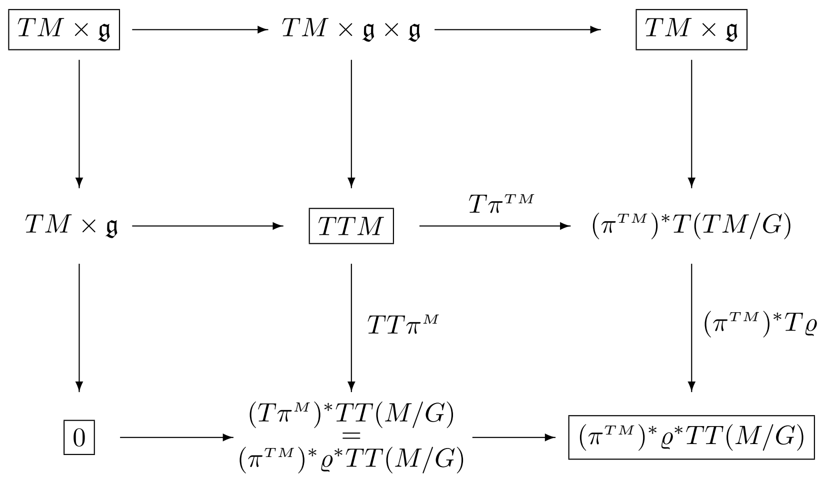
First, we will consider the outer triangle consisting of the spaces TTM, $T(TM/G)$ and $TT(M/G)$. Since π^{TM} is a principal fibre bundle, the kernel of $T\pi^{TM}$ can be identified with $TM \times \mathfrak{g}$, by means of the identification $(v, \xi) \mapsto \xi_{TM}(v) = \xi^{\mathrm{C}}_M(v)$. The two bases $\{X_i, \tilde{E}_a\}$ and $\{X_i, \hat{E}_a\}$ of $\mathcal{X}(M)$, based on the moving and the body-fixed basis, can be used to construct the basis $\{X_i^{\mathrm{C}}, \tilde{E}_a^{\mathrm{C}}, X_i^{\mathrm{V}}, \tilde{E}_a^{\mathrm{V}}\}$ of $\mathcal{X}(TM)$, which we call the standard basis, and also the basis $\{X_i^{\mathrm{C}}, \tilde{E}_a^{\mathrm{C}}, X_i^{\mathrm{V}}, \hat{E}_a^{\mathrm{V}}\}$, which we call the mixed basis. Since $\xi_{TM} = \xi_M^{\mathrm{C}}$, it is clear that the vector fields $\tilde{E}_a^{\mathrm{C}}$ span the vertical subbundle of the projection π^{TM}. The advantage of the mixed basis over the standard basis is that the vector fields X_i^{C}, X_i^{V} and $\hat{E}_a^{\mathrm{V}}$ are all invariant:

$$
\begin{aligned}
&[\tilde{E}_a^{\mathrm{C}}, X_i^{\mathrm{C}}] = [\tilde{E}_a, X_i]^{\mathrm{C}} = 0, \qquad &&[\tilde{E}_a^{\mathrm{C}}, X_i^{\mathrm{V}}] = [\tilde{E}_a, X_i]^{\mathrm{V}} = 0 \\
&[\tilde{E}_a^{\mathrm{C}}, \hat{E}_b^{\mathrm{V}}] = [\tilde{E}_a, \hat{E}_b]^{\mathrm{V}} = 0, \qquad &&[\tilde{E}_a^{\mathrm{C}}, \tilde{E}_b^{\mathrm{C}}] = [\tilde{E}_a, \tilde{E}_b]^{\mathrm{C}} = C^c_{ab}\tilde{E}_c^{\mathrm{C}}.
\end{aligned}
$$

So the vector fields $X_i^{\mathrm{C}}, X_i^{\mathrm{V}}$ and $\hat{E}_a^{\mathrm{V}}$ can be projected to sections of TTM/G (by means of π^{TTM}) and also to vector fields on TM/G (by means of $T\pi^{TM}$); the latter, denoted by $\overline{X_i^{\mathrm{C}}}$, $\overline{X_i^{\mathrm{V}}}$ and $\overline{E_a^{\mathrm{V}}}$, form a basis of $\mathcal{X}(TM/G)$. The following remark may be of some interest. Observe that $TM/G \to M/G$ is a vector bundle; one can therefore define a vertical lift operation taking sections of $TM/G \to M/G$ to vertical vector fields on TM/G. The vector fields $\overline{X_i^{\mathrm{V}}}$ and $\overline{E_a^{\mathrm{V}}}$ on TM/G are in fact the vertical lifts of the sections $\overline{X}_i, \overline{E}_a \in \mathrm{Sec}(TM/G)$; so we could write $\overline{X_i^{\mathrm{V}}} = \overline{X}_i^{\mathrm{V}}$ and $\overline{E_a^{\mathrm{V}}} = \overline{E}_a^{\mathrm{V}}$.

The vector fields $\{\tilde{E}_a^{\mathrm{C}}, \tilde{E}_a^{\mathrm{V}}\}$ span the vertical subbundle of the projection $T\pi^M$; so the kernel of $TT\pi^M$ is isomorphic to $TM \times T\mathfrak{g} \simeq TM \times \mathfrak{g} \times \mathfrak{g}$, the isomorphism being $X^a\tilde{E}_a^{\mathrm{C}} + Z^a\tilde{E}_a^{\mathrm{V}} \mapsto (X^aE_a, Z^aE_a)$. The vector fields $\{\tilde{E}_a^{\mathrm{V}}\}$ span the kernel of the projection ϱ; so $\ker T\varrho$ is isomorphic to $TM \times \mathfrak{g}$, by $Z^a\tilde{E}_a^{\mathrm{V}} \mapsto Z^aE_a$.

In this way we arrive at the following diagram of short exact sequences (taking into account the fact that $(T\pi^M)^*TT(M/G) = (\pi^{TM})^*\varrho^*TT(M/G)$):



There is a similar diagram for the spaces over M/G:

$$
\begin{array}{ccccc}
\boxed{\overline{\tau}^*\overline{\mathfrak{g}}} & \longrightarrow & \overline{\tau}^*(\overline{\mathfrak{g}}\times\overline{\mathfrak{g}}) & \longrightarrow & \boxed{\overline{\tau}^*\overline{\mathfrak{g}}} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{\tau}^*\overline{\mathfrak{g}} & \longrightarrow & \boxed{TTM/G} & \xrightarrow{[\![T\pi^{TM}]\!]} & T(TM/G) \\
\downarrow & & \downarrow{\scriptstyle [\![TT\pi^{M}]\!]} & & \downarrow{\scriptstyle T\varrho} \\
\boxed{0} & \longrightarrow & \varrho^*TT(M/G) & \longrightarrow & \boxed{\varrho^*TT(M/G)}
\end{array}
$$

Recall that $\overline{\tau}$ is the projection $TM/G \to M/G$. The identification $(TM \times \mathfrak{g})/G \simeq \overline{\tau}^*\overline{\mathfrak{g}}$ is given explicitly by $[\![v_m, \xi]\!] \mapsto ([\![v_m]\!], [\![m, \xi]\!])$.

We will use the basis $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial v^i}\}$ for $\mathcal{X}(T(M/G))$. The basic sections $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial v^i}\}$ can also be used for bases of vector fields along the projection in the case of certain pull-back bundles, that is both for $\mathrm{Sec}((T\pi^M)^*TT(M/G))$ and $\mathrm{Sec}(\varrho^*TT(M/G))$. Finally, we will also use $\{\overline{X_i^{\mathrm{C}}}, \overline{X_i^{\mathrm{V}}}, \overline{E_a^{\mathrm{V}}}\}$ as a basis for $\mathrm{Sec}((\pi^{TM})^*T(TM/G))$.

The two square commutative diagrams above will play the same role as the short exact sequences (1) and (2) in the first-order case. We will show that a principal connection on M induces splittings for all the short exact sequences in the squares. The idea is that connections of the M square which are G-invariant in the appropriate sense automatically give rise to connections for the M/G square and vice versa. We shall explicitly construct connections in both the middle horizontal and vertical sequences in the M square diagram, and use the general results at the beginning of this section to complete the task.

The first induced connection lives on the middle horizontal line of the M square diagram; it is the so-called vertical lift of the principal connection on M.

The construction of the vertical lift of a principal connection can be described as follows. Suppose given a connection on a principal G-bundle $\pi^M : M \to M/G$, specified by its connection form ϖ. We show that the pull-back $\tau^*\varpi$ of ϖ to TM (where $\tau : TM \to M$ is the tangent bundle projection) is the connection form of a principal connection on the principal G-bundle $TM \to TM/G$. Clearly, $\tau^*\varpi$ is a $\mathfrak{g}$-valued 1-form on TM. The action ψ^{TM} of G on TM is τ-related to the action ψ^M on M. Moreover, the fundamental vector fields corresponding to the two actions are related by $\xi_{TM} = (\xi_M)^{\mathrm{C}}$ for any $\xi \in \mathfrak{g}$; and in particular $T\tau(\xi_{TM}) = \xi_M$. Thus

$$\tau^*\varpi(\xi_{TM}) = \varpi(T\tau(\xi_{TM})) = \varpi(\xi_M) = \xi,$$

while

$$\psi_g^{TM\,*}\tau^*\varpi = \tau^*\psi_g^{M\,*}\varpi = \mathrm{ad}_{g^{-1}}\,\tau^*\varpi,$$

as required. The connection defined by $\tau^*\varpi$ is called the vertical lift of the original connection; its right and left splittings are denoted by γ^{V} and ω^{V} (so that the connection form ϖ^{V} is just given by $\varpi^{\mathrm{V}} = \tau^*\varpi$). The right splitting γ^{V} at the level of the M square can be given as follows. Let $\pi^{TM}(v) = \tilde{v}$; then

$$\gamma^{\mathrm{V}} : (\pi^{TM})^* T(TM/G) \to TTM, \qquad (v, X_{\tilde{v}}) \mapsto W,$$

where W is determined by the condition $T\pi^{TM}(W) = X_{\tilde{v}}$ and $T\tau(W) = \gamma(m, T\overline{\tau}(X_{\tilde{v}}))$, where $m = \tau(v)$. The first conditions shows that the above defines a splitting. To see that it is the one that corresponds with the vertical lift connection, we give the actions of γ^{V} and ω^{V} on the basis vector fields. We have $T\tau(X_i^{\mathrm{C}}) = X_i \circ \tau$, $T\tau(X_i^{\mathrm{V}}) = 0$ and $T\tau(\hat{E}_a^{\mathrm{V}}) = 0$. Likewise, $T\overline{\tau}(\overline{X_i^{\mathrm{C}}}) = \frac{\partial}{\partial x^i} \circ \overline{\tau}$, $T\overline{\tau}(\overline{X_i^{\mathrm{V}}}) = 0$ and $T\overline{\tau}(\overline{E_a^{\mathrm{V}}}) = 0$, where $\overline{\tau} : TM/G \to M/G$. It follows that

$$\gamma^{\mathrm{V}}(\overline{X_i^{\mathrm{C}}}) = X_i^{\mathrm{C}}, \quad \gamma^{\mathrm{V}}(\overline{X_i^{\mathrm{V}}}) = X_i^{\mathrm{V}}, \quad \gamma^{\mathrm{V}}(\overline{E_a^{\mathrm{V}}}) = \hat{E}_a^{\mathrm{V}},$$

and from these formulas we get for the associated left splitting

$$\omega^{\mathrm{V}}(X_i^{\mathrm{C}}) = 0, \quad \omega^{\mathrm{V}}(X_i^{\mathrm{V}}) = 0, \quad \omega^{\mathrm{V}}(\tilde{E}_a^{\mathrm{C}}) = \tilde{E}_a^{\mathrm{C}} \quad \text{and} \quad \omega^{\mathrm{V}}(\hat{E}_a^{\mathrm{V}}) = 0 = \omega^{\mathrm{V}}(\tilde{E}_a^{\mathrm{V}}).$$

From the above relations it is clear that $\varpi^{\mathrm{V}} = \tau^*\varpi$, and that therefore γ^{V} is indeed the right splitting corresponding to the vertical lift of the principal connection on M specified by ϖ, as it was defined initially.

The second connection of interest is a connection on the middle vertical line in the M square diagram, that is, it is a connection on the bundle $T\pi^M : TM \to T(M/G)$. It is in fact a particular case of a quite general construction which can be described as follows.

We first make an obvious remark. The complete lift operation $\mathcal{X}(M) \to \mathcal{X}(TM)$, $X \mapsto X^{\mathrm{C}}$, is not $C^\infty(M)$-linear: in fact for a function f on M we have $(fX)^{\mathrm{C}} = fX^{\mathrm{C}} + \dot{f}X^{\mathrm{V}}$, where $\dot{f}$ is the total derivative of f; the point to note is that $(fX)^{\mathrm{C}}$ is a $C^\infty(TM)$-linear combination of X^{C} and X^{V}. Suppose now that M is equipped with a distribution (vector field system) $\mathcal{D}$. Let $\{X_i\}$ be a local vector field basis for $\mathcal{D}$, and consider the local vector fields $\{X_i^{\mathrm{C}}, X_j^{\mathrm{V}}\}$ on TM: they are linearly independent, and there are $2\dim\mathcal{D}$ of them. Furthermore, if $\{Y_i\}$ is another local basis for $\mathcal{D}$ then the span of $\{Y_i^{\mathrm{C}}, Y_j^{\mathrm{V}}\}$ coincides with the span of $\{X_i^{\mathrm{C}}, X_j^{\mathrm{V}}\}$, as follows from the observation above about $(fX)^{\mathrm{C}}$. The span of $\{X_i^{\mathrm{C}}, X_j^{\mathrm{V}}\}$, where $\{X_i\}$ is any local basis of $\mathcal{D}$, accordingly defines a $2\dim\mathcal{D}$-dimensional distribution $\mathcal{D}'$ on TM. Suppose next that ϕ is a diffeomorphism of M and ϕ^{C} is the induced diffeomorphism of TM. Denote by $\phi^{\mathcal{X}}$ the action of ϕ on vector fields on M, $\phi^{\mathrm{C}\mathcal{X}}$ the action of ϕ^{C} on vector fields on TM. Then $\phi^{\mathcal{X}}(X)^{\mathrm{C}} = \phi^{\mathrm{C}\mathcal{X}}(X^{\mathrm{C}})$ and $\phi^{\mathcal{X}}(X)^{\mathrm{V}} = \phi^{\mathrm{C}\mathcal{X}}(X^{\mathrm{V}})$ (these are the integrated versions of two formulas for brackets between complete and vertical lifts which we used earlier). Thus if $\mathcal{D}$ is invariant under the action of some group G on M then $\mathcal{D}'$ is invariant under the induced action of G on TM. Now let $M \to M/G$ be a principal G-bundle and $\mathcal{D}$ the horizontal distribution of a principal connection: then $\mathcal{D}'$ is a G-invariant distribution on TM which is transverse to the fibres of $TM \to T(M/G)$, that is, a connection on $TM \to T(M/G)$, which is G-invariant in the appropriate sense.

It is easy to describe the left splitting of the new connection, as follows.

The complete lift construction can be extended from vector fields to tensor fields, as is shown in [15]. In particular, given a type $(1,1)$ tensor field A on a manifold M, its complete lift A^{C} is a type $(1,1)$ tensor field on TM with the following properties:

$$A^{\mathrm{C}}(X^{\mathrm{V}}) = A(X)^{\mathrm{V}}, \quad A^{\mathrm{C}}(X^{\mathrm{C}}) = A(X)^{\mathrm{C}}, \quad \mathcal{L}_{X^{\mathrm{C}}} A^{\mathrm{C}} = (\mathcal{L}_X A)^{\mathrm{C}},$$

for any vector field X on M. Moreover, for any two type $(1,1)$ tensor fields A, B on M, $A^{\mathrm{C}}B^{\mathrm{C}} = (AB)^{\mathrm{C}}$. The complete lift A^{C} may be described explicitly as follows. Regard A as a fibre-linear map $TM \to TM$, fibred over the identity. Let $\sigma : TTM \to TTM$ denote the canonical involution: then A^{C}, regarded as a fibre-linear map $TTM \to TTM$, is given by $A^{\mathrm{C}} = \sigma \circ TA \circ \sigma$ (where TA is the tangent map, or differential, of the map A).

Consider now the left splitting $\omega : TM \to M \times \mathfrak{g}$ of a principal connection on $\pi^M : M \to M/G$. As we pointed out earlier, it can be considered as a type $(1,1)$ tensor field on M, when we regard $M \times \mathfrak{g}$ as a subbundle of TM; from this point of view, for each $m \in M$, ω_m is the projection onto the vertical subspace of T_mM along the horizontal subspace; since it is a projection operator ω satisfies $\omega^2 = \omega$. The fact that the connection is principal is equivalent to the fact that, as a type $(1,1)$ tensor, ω is G-invariant, which is to say that $\omega \circ \psi^{\mathcal{X}}_g = \psi^{\mathcal{X}}_g \circ \omega$ for all $g \in G$, where $\psi^{\mathcal{X}}$ is the G-action on vector fields. When G is connected the latter condition is equivalent to $\mathcal{L}_{\xi_M}\omega = 0$ for all $\xi \in \mathfrak{g}$. We take the complete lift ω^{C}, to obtain a type $(1,1)$ tensor field on TM. Now $(\omega^{\mathrm{C}})^2 = (\omega^2)^{\mathrm{C}} = \omega^{\mathrm{C}}$, so ω^{C} is a projection operator. From the formulas for the action of ω^{C} on vertical and complete lifts it is clear that it vanishes on vertical and complete lifts of vector fields which are horizontal with respect to ω, that is, on $\mathcal{D}'$. Moreover, for any $\xi \in \mathfrak{g}$ we have

$$\omega^{\mathrm{C}}(\xi_M^{\mathrm{V}}) = (\omega(\xi_M))^{\mathrm{V}} = \xi_M^{\mathrm{V}}, \quad \omega^{\mathrm{C}}(\xi_M^{\mathrm{C}}) = (\omega(\xi_M))^{\mathrm{C}} = \xi_M^{\mathrm{C}},$$

so that $\mathrm{im}(\omega^{\mathrm{C}})$ can be identified with $\mathfrak{g} \times \mathfrak{g}$ in the required manner, and ω^{C} therefore defines a connection. Finally, we have

$$\mathcal{L}_{\xi_{TM}}\omega^{\mathrm{C}} = \mathcal{L}_{\xi_M^{\mathrm{C}}}\omega^{\mathrm{C}} = (\mathcal{L}_{\xi_M}\omega)^{\mathrm{C}} = 0,$$

which expresses the G-invariance of ω^{C}.

The connection determined by ω^{C} was first described, in its essentials, by Vilms [14]. In fact it was shown in [14] that a connection on a vector bundle $E \to X$ induces a connection on the bundle $TE \to TX$. Of course the bundle $\pi^M : M \to M/G$ we are dealing with in the current situation is not a vector bundle; nevertheless, Vilms's result may be extended to cover it. We therefore call this connection the Vilms connection; however, we denote its splittings by γ^{C} and ω^{C} (as before).

Note the important but somewhat subtle difference between the constructions of the two connections: in constructing the vertical lift connection we specify the initial connection by the $\mathfrak{g}$-valued connection form ϖ, but in constructing the Vilms connection we specify it by the type $(1,1)$ tensor field ω defining the right splitting. We mention this because there is a concept of the vertical lift of a type $(1,1)$ tensor field, and it is important to realise that we do not use this concept here.

The right splitting γ^{C} of the Vilms connection is a map $(T\pi^M)^*TT(M/G) \to TTM$, which may be specified as follows. We denote by $\overline{\sigma}$ the canonical involution of $TT(M/G)$. Let $T\pi^M(v) = \overline{v}$. The right splitting of the Vilms connection is given by

$$\gamma^{\mathrm{C}} : (v, Y_{\overline{v}}) \mapsto \sigma\left(T\gamma(v, \overline{\sigma}(Y_{\overline{v}}))\right).$$

The fact that this is a splitting is due to the property $TT\pi^M \circ \sigma = \overline{\sigma} \circ TT\pi^M$, as it is easy to see. Indeed,

$$TT\pi^M \circ \gamma^{\mathrm{C}}(v, Y) = \overline{\sigma} \circ T(T\pi^M \circ \gamma)(v, \overline{\sigma}(Y)) = \overline{\sigma} \circ \overline{\sigma}(Y) = Y.$$

We calculate the corresponding right splitting, and confirm that it is ω^{C}. The right splitting is given by $\mathrm{id} - \gamma^{\mathrm{C}} \circ TT\pi^M$; we have

$$\begin{aligned}
\mathrm{id} - \gamma^{\mathrm{C}} \circ TT\pi^M &= \mathrm{id} - \sigma \circ T\gamma \circ \overline{\sigma} \circ TT\pi^M \\
&= \mathrm{id} - \sigma \circ T\gamma \circ TT\pi^M \circ \sigma = \mathrm{id} - \sigma \circ T(\gamma \circ T\pi^M) \circ \sigma \\
&= \mathrm{id} - \sigma \circ T(\mathrm{id} - \omega) \circ \sigma = \sigma \circ T\omega \circ \sigma = \omega^{\mathrm{C}}.
\end{aligned}$$

In terms of the standard basis $\{X_i^{\mathrm{C}}, \tilde{E}_a^{\mathrm{C}}, X_i^{\mathrm{V}}, \tilde{E}_a^{\mathrm{V}}\}$ we have

$$\omega^{\mathrm{C}}(X_i^{\mathrm{C}}) = 0, \quad \omega^{\mathrm{C}}(X_i^{\mathrm{V}}) = 0, \quad \omega^{\mathrm{C}}(\tilde{E}_a^{\mathrm{C}}) = \tilde{E}_a^{\mathrm{C}} \quad \text{and} \quad \omega^{\mathrm{C}}(\tilde{E}_a^{\mathrm{V}}) = \tilde{E}_a^{\mathrm{V}};$$

equally, $\omega^{\mathrm{C}}(\hat{E}_a^{\mathrm{V}}) = \hat{E}_a^{\mathrm{V}}$. Since $TT\pi^M(X_i^{\mathrm{C}}) = \frac{\partial}{\partial x^i} \circ T\pi^M$ and $TT\pi^M(X_i^{\mathrm{V}}) = \frac{\partial}{\partial v^i} \circ T\pi^M$, it also follows that

$$\gamma^{\mathrm{C}}\left(\frac{\partial}{\partial x^i}\right) = X_i^{\mathrm{C}} - \omega^{\mathrm{C}}(X_i^{\mathrm{C}}) = X_i^{\mathrm{C}} \quad \text{and} \quad \gamma^{\mathrm{C}}\left(\frac{\partial}{\partial v^i}\right) = X_i^{\mathrm{V}} - \omega^{\mathrm{C}}(X_i^{\mathrm{V}}) = X_i^{\mathrm{V}}.$$

We noted above that because of the invariance of ω, the right splitting ω^{C} of the Vilms connection is invariant. It follows that the Vilms connection can be quotiented to give a connection on the M/G-square.

As a consequence of the existence of the two connections described so far, we can deduce for both square diagrams a third connection. Clearly $\mathrm{im}(\gamma^{\mathrm{C}}) \subset \mathrm{im}(\gamma^{\mathrm{V}})$, so there is a connection γ' such that $\gamma^{\mathrm{C}} = \gamma^{\mathrm{V}} \circ \gamma'$, and a connection $\overline{\gamma'}$ such that and $\overline{\gamma^{\mathrm{C}}} = \overline{\gamma^{\mathrm{V}}} \circ \overline{\gamma'}$. For the appropriate bases

$$\overline{\gamma'}(\frac{\partial}{\partial x^i}) = \overline{X_i^{\mathrm{C}}} \qquad \text{and} \qquad \overline{\gamma'}(\frac{\partial}{\partial v^i}) = \overline{X_i^{\mathrm{V}}},$$

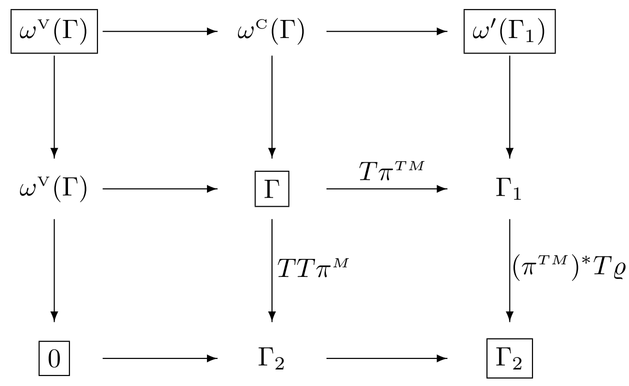
and

$$\overline{\omega'}(\overline{X_i^{\mathrm{C}}}) = 0, \quad \overline{\omega'}(\overline{X_i^{\mathrm{V}}}) = 0 \quad \text{and} \quad \overline{\omega'}(\overline{E_a^{\mathrm{V}}}) = \overline{E_a^{\mathrm{V}}}.$$

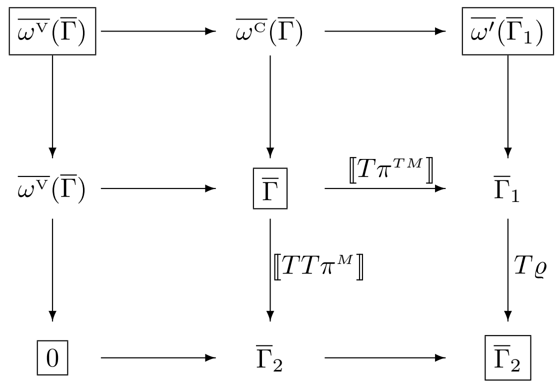
4 Second-order systems

We now come to the consideration of second-order systems. We assume given a second-order differential equation field, that is, a vector field Γ on TM such that $T\tau\Gamma(v) = v$ for all $v \in TM$, where $\tau : TM \to M$ is the tangent bundle projection. Furthermore, we assume that Γ is G-invariant, so that it satisfies $\psi_g^{TTM}\Gamma(v) = \Gamma(\psi_g^{TM}v)$. There is therefore a section $\overline{\Gamma}$ of TTM/G such that $\pi^{TTM} \circ \Gamma = \overline{\Gamma} \circ \pi^{TM}$. Under the appropriate maps Γ projects onto Γ_1 and Γ_2 as shown below, and can be decomposed into elements which are boxed in the diagram. Analogously, $\overline{\Gamma}$ projects on $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$, and has a similar decomposition. Of course, all the elements are related to each other in an appropriate way; for example $\Gamma_1 \in \mathrm{Sec}((\pi^{TM})^*T(TM/G))$, which is given by $\Gamma_1(v) = (v, T\pi^{TM}(\Gamma(v)))$, can be reduced to the vector field $\overline{\Gamma}_1 \in \mathcal{X}(TM/G)$.

Sections of bundles over M:



Sections of bundles over M/G:



In fact the connections give a G-invariant decomposition of Γ into three parts:

$$
\begin{aligned}
\Gamma &= \gamma^{\mathrm{V}}(\Gamma_1) + \omega^{\mathrm{V}}(\Gamma) \\
&= \gamma^{\mathrm{C}}(\Gamma_2) + \omega^{\mathrm{C}}(\Gamma) \\
&= \gamma^{\mathrm{C}}(\Gamma_2) + \gamma^{\mathrm{V}}(\omega'(\Gamma_1)) + \omega^{\mathrm{V}}(\Gamma).
\end{aligned}
$$

Since Γ is a second-order differential equation field the $\mathfrak{g}$-valued function $\varpi^{\mathrm{V}}(\Gamma)$ on TTM is given by

$$\varpi^{\mathrm{V}}(\Gamma(v)) = \varpi(T\tau(\Gamma(v)) = \varpi(v)$$

for all $v \in TM$; thus (for a given choice of ϖ) $\varpi^{\mathrm{V}}(\Gamma)$ is the same for all second-order differential equation fields Γ.

There is an analogous three-way decomposition of $\overline{\Gamma}$, as shown in the right-hand diagram. Here $\overline{\Gamma}_1$ is a vector field on TM/G, $\overline{\Gamma}_2$ is a section of $\varrho^*TT(M/G)$ (i.e. a vector field along ϱ) and $\overline{\omega'}(\overline{\Gamma}_1) \in \mathrm{Sec}(TM/G \times \overline{\mathfrak{g}})$.

As well as being a section of $TTM/G \to TM/G$, $\overline{\Gamma}$ may also be regarded as a section of the so-called prolongation bundle $T^{\varrho}(TM/G) \to TM/G$, whose fibre at $\tilde{v} \in TM/G$ is

$$T^{\varrho}_{\tilde{v}}(TM/G) = \{(\tilde{w}, X_{\tilde{v}}) \in TM/G \times T(TM/G) \mid \varrho(\tilde{w}) = T\overline{\tau}(X_{\tilde{v}})\}.$$

Theorem 9.1 of [6] shows that the identification of the quotient bundle $TTM/G \to TM/G$ with the above bundle is given by the isomorphism

$$TTM/G \to T^{\varrho}(TM/G), \qquad [\![W]\!] \mapsto (\pi^{TM}T\tau(W), T\pi^{TM}(W)).$$

This map is independent of the choice of W since for any other $\psi^{TTM}_g W$ within the same equivalence class, $T\tau(\psi^{TTM}_g W) = T(\tau \circ \psi^{TM}_g)(W) = T\tau(W)$. The second component of the isomorphism is in fact the map $[\![T\pi^{TM}]\!] : TTM/G \to T(TM/G)$. Keeping in mind that Γ is a second-order differential equation field, $\overline{\Gamma} \in \mathrm{Sec}(TTM/G)$ can be identified with the section $\tilde{v} \mapsto (\tilde{v}, \overline{\Gamma}_1(\tilde{v}))$ of the prolongation bundle. As a consequence, the composing parts of this section satisfy $T\overline{\tau}(\overline{\Gamma}_1(\tilde{v})) = \varrho(\tilde{v})$. This property clearly resembles the defining property $T\tau(\Gamma(v)) = v$ of a second-order differential equation field; sections of the prolongation bundle of the above form were therefore called 'pseudo second-order differential equation sections' in e.g. [10] or 'second-order differential equations' in e.g. [6].

Next, we will discuss the reconstruction process. We will use the following notations. Let $v(t) \in TM$ denote an integral curve of Γ and let $c(t)$ be the corresponding base integral curve, that is, $c(t) = \tau(v(t)) \in M$. It follows from the fact that Γ is a second-order differential equation field that $v = \dot{c}$ (when we consider the latter as a curve in TM). We will write $\tilde{v}(t) = \pi^{TM}(v(t)) \in TM/G$ and $\overline{c}(t) = \pi^M(c(t)) \in M/G$. Obviously, $\overline{\tau}(\tilde{v}) = \overline{c}$ and moreover $T\pi^M(v(t)) = \varrho(\tilde{v}(t)) = \dot{\overline{c}}(t) \in T(M/G)$. In a previous section we encountered the horizontal lift of $\overline{c}$ (with respect to the connection on π^M), which we denoted by $\overline{c}^\gamma$.

We first note that the vertical lift connection is a principal connection on the principal fibre bundle $\pi^{TM} : TM \to TM/G$. So just as in the first-order case we can construct the integral curve $t \mapsto v(t)$ of the invariant vector field $\Gamma \in \mathcal{X}(TM)$ from an integral curve $t \mapsto \tilde{v}(t) = \pi^{TM}(v(t))$ of the reduced vector field $\overline{\Gamma}_1 \in \mathcal{X}(TM/G)$, by

- taking the horizontal lift $\tilde{v}^\gamma$ of $\tilde{v}$ through $v(0)$ (with respect to $\gamma^{\rm V}$), and
- finding the solution $t \mapsto g(t) \in G$ of the equation

$$\theta(\dot{g}) = \varpi^{\rm V}(\Gamma \circ \tilde{v}^\gamma) \tag{11}$$

  with $g(0) = e$ (where θ is the Maurer-Cartan form of G);

the required integral curve is given by

$$v(t) = \psi^{TM}_{g(t)} \tilde{v}^\gamma(t).$$

The right-hand side of equation (11) can equally well be written as $\varpi(\dot{\tilde{v}}^\gamma)$.

Let us look at the relation that determines the horizontal lift $\tilde{v}^\gamma$ of $\tilde{v}$ to TM. Let $\tilde{v}$ be a given curve in TM/G (not necessarily an integral curve of $\overline{\Gamma}_1$). By definition, $\tilde{v}^\gamma$ projects onto $\tilde{v}$ and is a solution of

$$\dot{\tilde{v}}^\gamma = \gamma^{\rm V}(\tilde{v}^\gamma, \dot{\tilde{v}}).$$

From the conditions that determine $\gamma^{\rm V}$, we know that this is equivalent with the properties $T\pi^{TM} \circ \dot{\tilde{v}}^\gamma = \dot{\tilde{v}}$, and $T\tau \circ \dot{\tilde{v}}^\gamma = \gamma(T\overline{\tau} \circ \dot{\tilde{v}})$. The first property simply recalls that $\pi^{TM}(\tilde{v}^\gamma) = \tilde{v}$. If we denote as before $\overline{\tau} \circ \tilde{v} = \overline{c}$, then $T\overline{\tau} \circ \dot{\tilde{v}} = \dot{\overline{c}}$. So, we can deduce from the second property that $T\tau \circ \dot{\tilde{v}}^\gamma = \dot{\overline{c}}^\gamma$. To conclude, the curve $\tilde{v}^\gamma$ is completely determined by the properties $\tau \circ \tilde{v}^\gamma = \overline{c}^\gamma$ and $\pi^{TM} \circ \tilde{v}^\gamma = \tilde{v}$.

Any element of the vector bundle $TM/G \to M/G$ can be written as a sum of two parts via the splitting $\overline{\gamma}$ of the Atiyah sequence (2):

$$\tilde{v} = \overline{\xi} + \overline{\gamma}(\dot{\overline{c}}),$$

where $\overline{\xi} = \overline{\omega}(\tilde{v}) \in \overline{\mathfrak{g}}$. We can use this to give a more explicit formulation of $\tilde{v}^\gamma$. Let $\xi(t) \in \mathfrak{g}$ be such that $\overline{\xi} = [\![\overline{c}^\gamma, \xi]\!]$. Then the curve $\xi_M \circ \overline{c}^\gamma + \dot{\overline{c}}^\gamma$ in TM projects onto $\overline{c}^\gamma$

by means of τ and projects onto $\tilde{v} = \overline{\xi} + \overline{\gamma}(\dot{\overline{c}})$ by means of π^{TM}; so it can only be $\tilde{v}^\gamma$. Therefore, $\varpi(\tilde{v}^\gamma) = \xi$ and the horizontal part of $\tilde{v}^\gamma$ is $\dot{\overline{c}}^\gamma$.

Suppose now again that $\tilde{v}(t)$ is an integral curve of $\overline{\Gamma}_1$:

$$\dot{\tilde{v}} = \overline{\Gamma}_1 \circ \tilde{v}; \tag{12}$$

and that $v(t) = \psi^{TM}_{g(t)}\tilde{v}^\gamma(t)$ is an integral curve of Γ. Notice that

$$c = \tau \circ v = \tau \circ \psi^{TM}_g \tilde{v}^\gamma = \psi^M_g(\tau \circ \tilde{v}^\gamma) = \psi^M_g \overline{c}^\gamma;$$

that is to say, the curve in G required to bring $\tilde{v}^\gamma$ to v in TM is the same as the curve in G required to bring $\overline{c}^\gamma$ to $c = \tau \circ v$ in M. In fact from equation (4), since Γ is a second-order differential equation field,

$$v = \dot{c} = \psi^{TM}_g \left((\theta(\dot{g}))_M \circ \overline{c}^\gamma + \dot{\overline{c}}^\gamma \right). \tag{13}$$

So, in the case that $\tilde{v}$ is an integral curve of $\overline{\Gamma}_1$, the curve $\xi(t) = \varpi(\tilde{v}^\gamma(t)) \in \mathfrak{g}$ must equal $\theta(\dot{g})$, which agrees with equation (11).

We turn finally to the integral curves $\tilde{v}$ of $\overline{\Gamma}_1$. We will use the connection $\overline{\omega'}$ to decompose equation (12) into two coupled equations for the two curves $\overline{\xi} \in \overline{\mathfrak{g}}$ and $\overline{c} \in M/G$ that constitute $\tilde{v}$. The first equation is related to $\overline{\Gamma}_2 \in \mathrm{Sec}(\varrho^* TT(M/G))$, which can be considered as a map $\overline{\Gamma}_2 : TM/G \to TT(M/G)$; thus $\overline{\Gamma}_2 \circ \tilde{v}$ is a curve in $TT(M/G)$. The second equation is related to $\overline{\omega'}(\overline{\Gamma}_1)$, which is a vertical vector field on TM/G. This vector field can be regarded as a section of $(\overline{\tau})^*\overline{\mathfrak{g}}$, thus as a map $TM/G \to \overline{\mathfrak{g}}$; so $\overline{\omega'}(\overline{\Gamma}_1 \circ \tilde{v})$ can be regarded as a curve in $\overline{\mathfrak{g}}$. The projection of this curve onto M/G is obviously $\overline{c}$.

We need to introduce one more concept: that of the associated linear connection on the associated bundle $\overline{\mathfrak{g}} \to M/G$ (see also [4, 7]). In fact, we will only need its covariant derivative operator $\frac{D^A}{Dt}$ which acts on curves $\overline{\xi}$ in $\overline{\mathfrak{g}}$. Let $\overline{c}(t)$ be the projection of $\overline{\xi}(t)$ on M/G and let $c(t)$ be any curve in M that projects on $\overline{c}(t)$. Let $\xi(t) \in \mathfrak{g}$ be such that $\overline{\xi} = [\![c, \xi]\!]$. Then, the covariant derivative of $\overline{\xi}$ can be defined as

$$\frac{D^A\overline{\xi}}{Dt} = [\![c, \dot{\xi} - [\varpi \circ \dot{c}, \xi]]\!]$$

($\dot{\xi}$ stands here for the projection on the second argument of this curve in $T\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$). To see that this definition is independent of the choice of the representative in the equivalence class, take any other $d(t) \in M$ with $d(t) = \psi^M_{h(t)}c(t)$. The corresponding curve $\xi^d(t)$ in $\mathfrak{g}$ such that $\overline{\xi} = [\![d, \xi^d]\!]$ is then equal to $ad_{h^{-1}}\xi$. Moreover, $\dot{d} = \psi^{TM}_h\big(\dot{c} + (\theta(\dot{h}))(c)\big)$ and $\dot{\xi}_d = ad_{h^{-1}}\big(\dot{\xi} + [\theta(\dot{h}), \xi]\big)$. So, indeed,

$$\begin{aligned} [\![d, \dot{\xi}^d - [\varpi \circ \dot{d}, \xi^d]]\!] &= [\![d, ad_{h^{-1}}\big(\dot{\xi} + [\theta(\dot{h}), \xi]\big) - [ad_{h^{-1}}\big(\varpi \circ \dot{c} + \theta(\dot{h})\big), ad_{h^{-1}}\xi]]\!] \\ &= [\![\psi^M_h c, ad_{h^{-1}}\big(\dot{\xi} - [\varpi \circ \dot{c}, \xi]\big)]\!] = [\![c, \dot{\xi} - [\varpi \circ \dot{c}, \xi]]\!]. \end{aligned}$$

Remark that in the particular case of the horizontal lift, $\frac{D^A}{Dt}[\![\overline{c}^\gamma, \xi]\!] = [\![\overline{c}^\gamma, \dot{\xi}]\!]$.

One can show that the explicit formula for the associated linear connection is

$$\nabla^A : \mathcal{X}(M/G) \times \mathrm{Sec}(\bar{\mathfrak{g}}) \to \mathrm{Sec}(\bar{\mathfrak{g}}) : (\overline{X}, \bar{\xi}) \mapsto \nabla^A_{\overline{X}} \bar{\xi} = [\bar{\gamma}(\overline{X}), \bar{\xi}].$$

Here $[\cdot,\cdot]$ stands for the above mentioned Lie algebroid bracket of the Atiyah algebroid TM/G. For more details, see e.g. [7].

Theorem. *Let $\bar{\xi}(t) \in \bar{\mathfrak{g}}$, $\bar{c}(t) \in M/G$ and put $\tilde{v} = \bar{\xi} + \bar{\gamma}(\dot{\bar{c}})$. If $\bar{c}$ and $\bar{\xi}$ are solutions of*

$$\left\{ \begin{array}{lcl} \ddot{\bar{c}} & = & \overline{\Gamma}_2 \circ \tilde{v}, \\ \dfrac{D^A \bar{\xi}}{Dt} & = & \overline{\omega'}(\overline{\Gamma}_1 \circ \tilde{v}), \end{array} \right. \tag{14}$$

then $\tilde{v}$ is an integral curve of $\overline{\Gamma}_1$. Solve $\dot{\bar{c}}^\gamma = \gamma(\bar{c}^\gamma, \dot{\bar{c}})$ for $\bar{c}^\gamma(t) \in M$ and let $\xi(t) \in \mathfrak{g}$ be such that $\bar{\xi} = [\![\bar{c}^\gamma, \xi]\!]$. If $g(t) \in G$ is a solution of

$$\theta(\dot{g}) = \xi \tag{15}$$

then the curve $v = \psi_g^{TM}(\xi_M \circ \bar{c}^\gamma + \dot{\bar{c}}^\gamma) = \psi_g^{TM} \tilde{v}^\gamma$ is an integral curve of Γ.

Conversely, suppose that v is an integral curve of Γ. Let $\tilde{v} = \pi^{TM} \circ v$, $\bar{c} = \pi^M \circ \tau \circ v$ and $\bar{\xi} = \bar{\varpi} \circ \tilde{v}$. Then $\tilde{v}$ is an integral curve of $\overline{\Gamma}_1$ and $\bar{c}$ and $\bar{\xi}$ satisfy (14). Compute $\bar{c}^\gamma$ from $\dot{\bar{c}}^\gamma = \gamma(\bar{c}^\gamma, \dot{\bar{c}})$. Let $g \in G$ be such that $c = \psi_g^{TM} \bar{c}^\gamma$ and let $\xi \in \mathfrak{g}$ be such that $\bar{\xi} = [\![\bar{c}^\gamma, \xi]\!]$. Then g satisfies equation (15).

If d is a curve in M such that $\pi^M(d) = \bar{c}$ and if we denote by ξ^d the curve in $\mathfrak{g}$ which is such that $\bar{\xi} = [\![d, \xi]\!]$, then the last equation of (14) could equivalently be written as $\dot{\xi} - [\varpi(\dot{d}), \xi] = \omega'(\Gamma_1 \circ v^d)$, where $v^d = \xi_M \circ d + \gamma(d, \dot{\bar{c}})$ is the unique curve on TM that projects on both $\tilde{v}$ and d. Indeed, the relation between $\omega'(\Gamma_1)$ and $\overline{\omega}'(\overline{\Gamma}_1)$ is

$$\overline{\omega}'(\overline{\Gamma}_1)(\pi^{TM}(v_m)) = [\![m, \omega'(\Gamma_1)(v_m)]\!],$$

and therefore, $\overline{\omega}'(\overline{\Gamma}_1 \circ \tilde{v}) = [\![d, \omega'(\Gamma_1 \circ \tilde{v}^d)]\!]$.

If one is interested only in the coordinates on M/G ('shape' variables in [2]), it is necessary only to solve equations (14) where the symmetry has already been cancelled out. If the whole motion on M is required one will have to solve the whole system.

The proof of theorem will follow from the considerations of the coordinate version of the reduced second-order equations in the following paragraphs.

We begin our description of the coordinate expression of the equations with a general remark. If we take any local basis $\{X_\alpha\}$ of vector fields on some manifold M, not necessarily a coordinate basis, and express any tangent vector v at $m \in M$ in terms of this basis so that $v = v^\alpha X_\alpha|_m$, then the v^α will serve as fibre coordinates on TM. A vector field Γ on TM will be a second-order differential equation field if and only if it takes the form $\Gamma = v^\alpha X_\alpha^{\mathrm{C}} + F^\alpha X_\alpha^{\mathrm{V}}$, and its integral curves will satisfy $\dot{v}^\alpha = F^\alpha$. In terms of a new basis $\{Y_\alpha\}$, where $Y_\alpha = A_\alpha^\beta X_\beta$, we have $v = v^\alpha X_\alpha|_m = w^\alpha Y_\alpha|_m$ where $w^\beta A_\beta^\alpha = v^\alpha$. Moreover, $\Gamma = w^\alpha Y_\alpha^{\mathrm{C}} + G^\alpha Y_\alpha^{\mathrm{V}}$ where $F^\alpha = A_\beta^\alpha G^\beta + \dot{A}_\beta^\alpha w^\beta$, the overdot here indicating the total derivative.

We turn now to the case of interest. We have defined on M two local vector field bases $\{X_i, \tilde{E}_a\}$ and $\{X_i, \hat{E}_a\}$, with

$$X_i = \gamma\left(\frac{\partial}{\partial x^i}\right)$$

where the x^i are local coordinates on M/G, and $\{E_a\}$ is a basis for $\mathfrak{g}$. Both $\{\tilde{E}_a\}$ and $\{\hat{E}_a\}$ are bases of vector fields which are vertical with respect to the projection $\pi^M : M \to M/G$. The first, which consists of fundamental vector fields, we called the moving basis. The body-fixed local basis $\{\hat{E}_a\}$, on the other hand, consists of G-invariant vertical vector fields. We have $\hat{E}_a = A_a^b \tilde{E}_b$ where the coefficients A_a^b satisfy $\tilde{E}_a(A_b^c) + C_{ad}^c A_b^d = 0$ (equation (6)).

For any $v \in T_m M$ we set

$$v = v^i X_i|_m + v^a \tilde{E}_a|_m = v^i X_i|_m + w^a \hat{E}_a|_m; \quad v^a = A_b^a w^b.$$

The v^i may be regarded as the fibre coordinates on $T(M/G)$ corresponding to the base coordinates x^i. We show in the following paragraph that the v^a satisfy

$$\tilde{E}_b^{\mathrm{C}}(v^a) + C_{bc}^a v^c = 0; \tag{16}$$

so from equation (10) we may consider $v \mapsto v^a E_a$ as defining a section of $\overline{\tau}^* \overline{\mathfrak{g}} \to TM/G$; this is just the section $\overline{\omega^{\mathrm{V}}}(\overline{\Gamma})$. On the other hand, since $v^a = A_b^a w^b$, where the A_b^a are functions on M, we have

$$A_c^a \tilde{E}_b^{\mathrm{C}}(w^c) + \tilde{E}_b(A_c^a) w^c + C_{bc}^a v^c = 0;$$

but from equation (6)

$$\tilde{E}_b(A_c^a) w^c = -C_{bd}^a A_c^d w^c = -C_{bc}^a v^c,$$

whence $\tilde{E}_b^{\mathrm{C}}(w^a) = 0$, as one might have expected.

Equation (16) is a consequence of the following general considerations. Let $\{Z_\alpha\}$ be any local basis of vector fields on a manifold M, with dual basis of 1-forms θ^α. Let $\hat{\theta}$ be the fibre-linear function on TM defined by a 1-form θ on M, so that $\hat{\theta}(x, v) = \theta_x(v)$. Then if $v^\alpha = \hat{\theta}^\alpha(x, v)$, $v = v^\alpha Z_\alpha|_x$. For any vector field Z on M, $Z^{\mathrm{C}}(\hat{\theta}) = \widehat{\mathcal{L}_Z \theta}$. But $\mathcal{L}_{Z^\alpha} \theta^\beta(Z_\gamma) = -\theta^\beta([Z_\alpha, Z_\gamma])$, so $\mathcal{L}_{Z^\alpha} \theta^\beta = -C_{\alpha\gamma}^\beta \theta^\gamma$ where $[Z_\alpha, Z_\gamma] = C_{\alpha\gamma}^\beta Z_\beta$. That is to say, $Z_\alpha^{\mathrm{C}}(v^\beta) = -C_{\alpha\gamma}^\beta v^\gamma$. It follows that $[Z_\alpha^{\mathrm{C}}, v^\beta Z_\beta^{\mathrm{C}}] = 0$.

The second-order differential equation field Γ may be written

$$\begin{aligned} \Gamma &= v^i X_i^{\mathrm{C}} + v^a \tilde{E}_a^{\mathrm{C}} + D^i X_i^{\mathrm{V}} + D^a \tilde{E}_a^{\mathrm{V}} \\ &= v^i X_i^{\mathrm{C}} + w^a \hat{E}_a^{\mathrm{C}} + D^i X_i^{\mathrm{V}} + F^a \hat{E}_a^{\mathrm{V}}; \end{aligned}$$

we have

$$D^a = A_b^a F^b + \dot{A}_b^a w^b.$$

By assumption Γ is G-invariant, which is to say that $[\tilde{E}_a^{\mathrm{C}}, \Gamma] = 0$. Now $[\tilde{E}_a^{\mathrm{C}}, v^i X_i^{\mathrm{C}}] = 0$, and $[\tilde{E}_a^{\mathrm{C}}, v^b \tilde{E}_b^{\mathrm{C}}] = 0$, as follows from equation (16) and the argument that establishes it. Moreover $[\tilde{E}_a^{\mathrm{C}}, w^b \hat{E}_b^{\mathrm{C}}] = 0$ since $\tilde{E}_a^{\mathrm{C}}(w^b) = 0$ and $[\tilde{E}_a^{\mathrm{C}}, \hat{E}_b^{\mathrm{C}}] = 0$. Thus $v^a \tilde{E}_a^{\mathrm{C}}$ and

$w^a\hat{E}_a^{\rm C}$ are both G-invariant; they are not however equal, but differ by the vertical vector field $w^b\dot{A}_b^a\tilde{E}_a^{\rm V}$, which accordingly is G-invariant. Next, $[\tilde{E}_a^{\rm C}, D^iX_i^{\rm V}] = \tilde{E}_a^{\rm C}(D^i)X_i^{\rm V}$ since $[\tilde{E}_a^{\rm C}, X_i^{\rm V}] = [\tilde{E}_a, X_i]^{\rm V} = 0$. On the other hand,

$$\begin{aligned}
[\tilde{E}_a^{\rm C}, D^b\tilde{E}_b^{\rm V}] &= \tilde{E}_a^{\rm C}(D^b)\tilde{E}_b^{\rm V} + D^b[\tilde{E}_a^{\rm C}, \tilde{E}_b^{\rm V}] \\
&= \tilde{E}_a^{\rm C}(D^b)\tilde{E}_b^{\rm V} + D^b[\tilde{E}_a, \tilde{E}_b]^{\rm V} \\
&= \left(\tilde{E}_a^{\rm C}(D^b) + D^cC_{ac}^b\right)\tilde{E}_b^{\rm V}.
\end{aligned}$$

Finally, $[\tilde{E}_a^{\rm C}, F^b\hat{E}_b^{\rm V}] = \tilde{E}_a^{\rm C}(F^b)\hat{E}_b^{\rm V}$. The remaining coefficients of Γ must therefore satisfy

$$\tilde{E}_a^{\rm C}(D^i) = 0, \quad \tilde{E}_a^{\rm C}(D^b) + D^cC_{ac}^b = 0, \quad \tilde{E}_a^{\rm C}(F^b) = 0;$$

that is to say, D^i and F^b are G-invariant, while the D^a may be regarded as the components of a section of $\overline{\tau}^*\overline{\mathfrak{g}} \to TM/G$.

Observe that

$$\omega^{\rm V}(\Gamma) = v^a\tilde{E}_a^{\rm C} \in \mathcal{X}(TM).$$

The corresponding $\mathfrak{g}$-valued function $\varpi^{\rm V}(\Gamma)$ is given by $\varpi^{\rm V}(\Gamma) = v^aE_a$; it is independent of the choice of Γ, as we remarked before, and as we showed above it in fact determines a section of $\overline{\tau}^*\overline{\mathfrak{g}}$.

We may also express Γ in terms of the mixed basis:

$$\Gamma = v^iX_i^{\rm C} + v^a\tilde{E}_a^{\rm C} + D^iX_i^{\rm V} + G^a\hat{E}_a^{\rm V},$$

where

$$G^a = \bar{A}_b^aD^b = F^a + \bar{A}_c^a\dot{A}_b^cw^b,$$

the $\bar{A}_b^a$ being the components of the inverse of the matrix (A_b^a). It is easy to see that $\tilde{E}_a^{\rm C}(G^b) = 0$. We have

$$\Gamma_1 = v^i\overline{X_i^{\rm C}} + D^i\overline{X_i^{\rm V}} + G^a\overline{E_a^{\rm V}}.$$

Then $\omega'(\Gamma_1) = G^a\overline{E_a^{\rm V}} \in \mathrm{Sec}((\pi^{TM})^*T(TM/G))$, and so

$$\gamma^{\rm V}(\omega'(\Gamma_1)) = G^a\hat{E}_a^{\rm V} = D^a\tilde{E}_a^{\rm V} \in \mathcal{X}(TM).$$

The three-way decomposition of Γ at the level of the M square diagram is therefore given by

$$\begin{aligned}
\Gamma &= \gamma^{\rm C}(\Gamma_2) + \gamma^{\rm V}(\omega'(\Gamma_1)) + \omega^{\rm V}(\Gamma) \\
&= (v^iX_i^{\rm C} + D^iX_i^{\rm V}) + D^a\tilde{E}_a^{\rm V} + v^a\tilde{E}_a^{\rm C}.
\end{aligned}$$

Among the equations for the integral curves of Γ we find

$$\left\{\begin{array}{rcl} \ddot{x}^i &=& D^i, \\ \dot{w}^a &=& F^a. \end{array}\right.$$

We can write the latter as

$$\dot{w}^a + \bar{A}_c^a\dot{A}_b^cw^b = G^a.$$

Since the w^a are G-invariant they can be taken, together with x^i and v^i, as coordinates on TM/G. The integral curves of Γ_1 are the solutions of the equations

$$\left\{ \begin{array}{rcl} \ddot{x}^i & = & D^i, \\ \dot{w}^a + \bar{A}^a_c \dot{A}^c_b w^b & = & G^a. \end{array} \right.$$

This latter equation has a familiar structure: one could think of the term $\bar{A}^a_c \dot{A}^c_b$ as representing the 'angular velocity' of the body-fixed frame with respect to the moving frame, and w^a as components of some velocity with respect to the body-fixed frame; the whole term $\bar{A}^a_c \dot{A}^c_b w^b$ is then of Coriolis type.

We can also write the same equation as

$$A^a_b \dot{w}^b + \dot{A}^a_b w^b = A^a_b G^b,$$

which is equivalent to $\dot{v}^a = D^a$.

Finally, if we rewrite $\dot{A}^c_b$ as $\dot{x}^j X_j(A^c_b) + v^d \tilde{E}_d(A^c_b)$ and use the formulae $X_j(A^c_b) = \gamma^d_j C^e_{db} A^c_e$ obtained earlier (equation (7)) and $\tilde{E}_d(A^c_b) = -C^c_{de} A^e_b$ (equation (6)) we find that

$$\begin{array}{rcl} \bar{A}^a_c \dot{A}^c_b w^b & = & \bar{A}^a_c \left(\dot{x}^j \gamma^d_j C^e_{db} A^c_e - v^d C^c_{de} A^e_b \right) w^b \\ & = & \dot{x}^j \gamma^d_j C^a_{db} w^b - \bar{A}^a_c v^d v^e C^c_{de} = \dot{x}^j \gamma^d_j C^a_{db} w^b. \end{array}$$

But $\gamma^d_j C^a_{db} = \Upsilon^a_{jb}$ are the connection coefficients of the adjoint connection; the equation for w^a is therefore equivalent to

$$\dot{w}^a + \Upsilon^a_{ib} \dot{x}^i w^b = G^a,$$

which is in agreement with the second of equations (14) in the theorem.

5 An example

In this final section we determine the reduced equations for an interesting class of second-order differential equation fields.

The case to be discussed is that in which there is a 'kinetic energy' metric k on M, with Levi-Civita covariant derivative ∇, and the equations of motion of the original dynamical system take the form

$$\nabla_{\dot{c}} \dot{c} = F(c, \dot{c})$$

for the curve $t \mapsto c(t)$ on M. Such a system may be called a system of mechanical type, with F representing a force field.

There is great potential for confusion here, since we will now have two connections of fundamental importance to deal with, the Levi-Civita connection and the connection on the principal bundle $\pi^M : M \to M/G$ (when we have defined the group G and its action); we warn the reader to be on guard.

We may write Γ in the form $\Gamma = \Gamma_0 + \Phi$ where Γ_0 is the geodesic spray of the Levi-Civita connection and Φ is the force term on the right-hand side of the equations of motion considered as a vertical vector field on TM. We now examine the possible symmetry conditions. For any vector field Z on M and any affine spray Γ_0, $[Z^{\rm C}, \Gamma_0]$ is vertical and quadratic in the fibre coordinates. On the other hand, $[Z^{\rm C}, \Phi]$ is vertical since Φ is; but in cases of interest (for example, when F is independent of velocities, or linear in them, or a combination of the two) there will be no terms quadratic in the fibre coordinates; so it is natural to consider the situation where $[Z^{\rm C}, \Gamma_0]$ and $[Z^{\rm C}, \Phi]$ vanish separately. Now $[Z^{\rm C}, \Gamma_0]$ vanishes if and only if Z is an infinitesimal affine transformation of the symmetric covariant derivative defined by Γ_0, which in the case under discussion is the Levi-Civita connection of k. Since any infinitesimal isometry is affine, it is natural to assume further that G is a group of isometries of k, whose elements in addition leave invariant the force term, as represented by the vertical vector field Φ. Such a group is always a symmetry group of Γ, and in many cases the maximal symmetry group will be of this form.

We now turn to the choice of a vector field basis on M adapted to the group action. In this case there is a natural choice for the connection on π^M: take its horizontal subspaces to be the orthogonal complements of the tangent planes to the group orbits; they are G-invariant since the group consists of isometries. The vertical vector fields $\tilde{E}_a$ comprise a basis for the Killing fields or infinitesimal isometries. We shall however work with an invariant, body-fixed basis for the vertical vector fields; that is, we choose a local basis of vector fields of the form $\{X_i, \hat{E}_a\}$. The components of k in this basis are denoted by k_{ab}, k_{ai}, k_{ij} in the obvious fashion. The k_{ab} are evidently G-invariant. By construction, $k_{ai} = 0$. The k_{ij} are also G-invariant, and so define functions $\overline{k}_{ij}$ on M/G which are the components of the reduced metric, say $\overline{k}$, with respect to the local vector field basis there. We may without loss of generality take this basis to consist of coordinate fields, as before; the X_i will not in general commute, but $[X_i, X_j]$ will have components tangent to the group orbits; we set $[X_i, X_j] = K^a_{ij}\hat{E}_a$ (this in effect defines K as the curvature of the connection).

The connection on π^M has now been entirely taken care of; references to a connection henceforth always mean the Levi-Civita connection.

We set $\Phi = \Phi^i X_i^{\rm V} + \Phi^a \hat{E}_a^{\rm V}$; by assumption both Φ^i and Φ^a are G-invariant.

In order to find the reduced system it is necessary to express Γ in terms of the adapted basis. For this purpose we need the Christoffel symbols of the Levi-Civita connection with respect to the basis $\{X_i, \hat{E}_a\}$: we set

$$\nabla_{\hat{E}_a}\hat{E}_b = \Gamma^c_{ab}\hat{E}_c + \Gamma^i_{ab}X_i$$

and so on. The order of indices is important; though the Levi-Civita connection is symmetric, it is represented here with respect to a non-coordinate frame. To calculate the Christoffel symbols we need the brackets of the basis vector fields. Recall from equation (8) that $[\hat{E}_a, X_i] = \gamma^b_i C^c_{ab}\hat{E}_c$, and that the connection coefficients of the adjoint connection are given by $\Upsilon^b_{ia} = \gamma^c_i C^b_{ca}$. We therefore have the following bracket relations:

$$[\hat{E}_a, \hat{E}_b] = -C^c_{ab}\hat{E}_c; \quad [X_i, \hat{E}_a] = \Upsilon^b_{ia}\hat{E}_b; \quad [X_i, X_j] = K^a_{ij}\hat{E}_a.$$

Since all of the vector fields appearing are G-invariant, so are all of the coefficients on the right-hand sides. Furthermore, since all of the brackets are vertical the Christoffel symbols with upper index i will be symmetric in their lower indices.

Using these data in the standard Koszul formulae for the Levi-Civita connection coefficients of k with respect to the basis $\{\hat{E}_a, X_i\}$ we find that

$$\begin{aligned}
\Gamma^a_{bc} &= \tfrac{1}{2}\left(-C^a_{bc} + k^{ad}(k_{be}C^e_{dc} + k_{ce}C^e_{bd})\right)\\
\Gamma^i_{bc} &= \tfrac{1}{2}k^{ij}\left(-X_j(k_{bc}) + k_{bd}\Upsilon^d_{jc} + k_{cd}\Upsilon^d_{jb}\right)\\
\Gamma^a_{jb} &= \tfrac{1}{2}k^{ac}\left(X_j(k_{bc}) - k_{bd}\Upsilon^d_{jc} + k_{cd}\Upsilon^d_{jb}\right)\\
\Gamma^a_{bj} &= \tfrac{1}{2}k^{ac}\left(X_j(k_{bc}) - k_{bd}\Upsilon^d_{jc} - k_{cd}\Upsilon^d_{jb}\right)\\
\Gamma^i_{jb} &= -\tfrac{1}{2}k^{ik}k_{bc}K^c_{jk} = \Gamma^i_{bj}\\
\Gamma^a_{jk} &= \tfrac{1}{2}K^a_{jk}\\
\Gamma^i_{jk} &= \overline{\Gamma}^i_{jk},
\end{aligned}$$

where in the final line the $\overline{\Gamma}^i_{jk}$ are the Christoffel symbols of the reduced metric $\overline{k}_{ij}$.

It follows that

$$\begin{aligned}
\Gamma_0 &= \dot{x}^i X^{\mathrm{C}}_i + w^a\hat{E}^{\mathrm{C}}_a\\
&\quad - \left(\dot{x}^j\dot{x}^k\Gamma^i_{jk} + \dot{x}^j w^b(\Gamma^i_{jb} + \Gamma^i_{bj}) + w^b w^c\Gamma^i_{bc}\right)X^{\mathrm{V}}_i\\
&\quad - \left(\dot{x}^j\dot{x}^k\Gamma^a_{jk} + \dot{x}^j w^b(\Gamma^a_{jb} + \Gamma^a_{bj}) + w^b w^c\Gamma^a_{bc}\right)\hat{E}^{\mathrm{V}}_a\\
&= \dot{x}^i X^{\mathrm{C}}_i + w^a\hat{E}^{\mathrm{C}}_a\\
&\quad - \left(\dot{x}^j\dot{x}^k\overline{\Gamma}^i_{jk} - \dot{x}^j w^b k^{ik}k_{bc}K^c_{jk} + w^b w^c k^{ij}\left(-\tfrac{1}{2}X_j(k_{bc}) + k_{bd}\Upsilon^d_{jc}\right)\right)X^{\mathrm{V}}_i\\
&\quad - \left(\dot{x}^j w^b k^{ac}\left(X_j(k_{bc}) - k_{bd}\Upsilon^d_{jc}\right) + w^b w^c k^{ad}k_{be}C^e_{dc}\right)\hat{E}^{\mathrm{V}}_a.
\end{aligned}$$

The reduced equations are therefore

$$\begin{aligned}
\ddot{x}^i + \overline{\Gamma}^i_{jk}\dot{x}^j\dot{x}^k &= \Phi^i + \dot{x}^j w^b k^{ik}k_{bc}K^c_{jk} + w^b w^c k^{ij}\left(\tfrac{1}{2}X_j(k_{bc}) - k_{bd}\Upsilon^d_{jc}\right)\\
\dot{w}^a + \Upsilon^a_{jb}\dot{x}^j w^b &= \Phi^a - \dot{x}^j w^b k^{ac}\left(X_j(k_{bc}) - k_{bd}\Upsilon^d_{jc} - k_{cd}\Upsilon^d_{jb}\right) - w^b w^c k^{ad}k_{be}C^e_{dc}.
\end{aligned}$$

The first equation can be written

$$\frac{\overline{D}(\overline{k}_{ij}\dot{x}^j)}{Dt} = \Phi_i - \dot{x}^j w^b k_{bc}K^c_{ij} + \tfrac{1}{2}w^b w^c\nabla^A_{\partial/\partial x^i}(k_{bc}),$$

where $\overline{D}/Dt$ is the covariant derivative operator of the Levi-Civita connection of $\overline{k}$, and $\Phi_i = \overline{k}_{ij}\Phi^j$. We can write the equation for w^a in either of the following two forms:

$$\begin{aligned}
\frac{D^A w^a}{Dt} &= \Phi^a - w^b k^{ac}\frac{D^A k_{bc}}{Dt} - w^b w^c k^{ad}k_{be}C^e_{dc}\\
\frac{D^A w_a}{Dt} &= \Phi_a - w_b w_c k^{cd}C^b_{ad};
\end{aligned}$$

to obtain the second we have used k_{ab} to lower indices.

When $F = 0$, that is, when $\Phi^i = \Phi^a = 0$, we obtain Wong's equations [4, 12]. The case in which $F \neq 0$ but G is 1-dimensional is discussed by Bullo and Lewis [3, section 5.5.2]. We shall show that in both cases our equations subsume those of the cited authors.

In the case discussed in [4, 12], in addition to $F = 0$ it is assumed that the vertical part of the metric comes from a bi-invariant metric on the Lie group G. This means in the first place that $\mathcal{L}_{\hat{E}_c} k(\hat{E}_a, \hat{E}_b) = 0$ as well as $\mathcal{L}_{\tilde{E}_c} k(\hat{E}_a, \hat{E}_b) = 0$, and secondly that the k_{ab} must be independent of the x^i. From the first condition we easily find that the k_{ab} must satisfy $k_{ad}C^d_{bc} + k_{bd}C^d_{ac} = 0$, and therefore $k_{ac}\Upsilon^c_{ib} + k_{bc}\Upsilon^c_{ia} = 0$. From both together we see that the k_{ab} must be constants. Thus $\nabla^A_{\partial/\partial x^i}(k_{bc}) = 0$ and $w^b w^c k_{be} C^e_{dc} = -w^b w^c k_{de} C^e_{bc} = 0$, and the reduced equations are

$$\begin{aligned} \frac{\overline{D}(\overline{k}_{ij}\dot{x}^j)}{Dt} &= -\dot{x}^j w_b K^b_{ij} \\ \frac{D^A w_a}{Dt} &= 0. \end{aligned}$$

These are equivalent to the equations given in [4, 12].

In the 1-dimensional case we have a single Killing field $\tilde{E}$; this vector field is also clearly invariant, so we shall simplify the notation by denoting it simply by E (Bullo and Lewis in fact write X for this vector field). There is but one component of k_{ab}, which is $k(E, E) = |E|^2$, and k with notional upper indices is just $|E|^{-2}$. Furthermore, $|E|^2$ is itself invariant, and may therefore be considered as a function on M/G. An arbitrary tangent vector V may be written in the form $V = vE + v^i X_i$ (so v is to be identified with the single component of w^a), and since the X_i are orthogonal to E we have

$$v = \frac{k(V, E)}{|E|^2};$$

Bullo and Lewis call the map $V \mapsto k(V, E)$ the momentum map and denote it by J_X. They also introduce a type $(1, 1)$ tensor field on M/G which they call the gyroscopic tensor, which they denote by C_X. The gyroscopic tensor is given essentially as follows. The covariant differential ∇E is a type $(1, 1)$ tensor field on M. Let us denote by $E^\perp$ the distribution orthogonal to E, that is, the distribution spanned by the vector fields X_i. Then ∇E may be used to define an operator on $E^\perp$, by first restricting its arguments to lie in this distribution, and then perpendicularly projecting its values into it. Now in general we have

$$\nabla_{X_i}\hat{E}_a = \Gamma^j_{ia}X_j + \Gamma^b_{ia}\hat{E}_b;$$

so we are concerned here with $\Gamma^j_{ia} = -\frac{1}{2}k_{ab}k^{jk}K^b_{ik}$, albeit in the 1-dimensional case. In fact if we write $[X_i, X_j] = K_{ij}E$ the gyroscopic tensor in component form is

$$C^j_i = |E|^2 k^{jk} K_{ik} = |E|^2 \overline{k}^{jk} K_{ik}.$$

It is clear from this that C^j_i is invariant and that $C_{ij} = \overline{k}_{ik} C^k_j$ is skew-symmetric. Moreover in the 1-dimensional case $\Upsilon^a_{ib} = 0$. The reduced equations of motion in this case are therefore

$$\begin{aligned}\ddot{x}^i + \overline{\Gamma}^i_{jk}\dot{x}^j\dot{x}^k &= \Phi^i + vC^i_j\dot{x}^j + \tfrac{1}{2}v^2\overline{k}^{ij}\frac{\partial|E|^2}{\partial x^j}\\ \dot{v} &= \Phi^0 - \frac{v}{|E|^2}\dot{x}^j\frac{\partial|E|^2}{\partial x^j}.\end{aligned}$$

Here Φ^0 is the E-component of the force. These equations agree with those given by Bullo and Lewis. These authors deal mainly with the case in which the force is derived from a potential, and the last term on the right-hand side of the first equation is subsumed by them into the so-called effective potential. Bullo and Lewis actually give two versions of the reduced equations: one is in terms of v, and is derived above; the other is in terms of $\mu = |E|^2 v$, and the second of the reduced equations is then simply $\dot{\mu} = 0$. But since $|E|^2$ is the single component of k_{ab}, μ is just the single component of w_a. Furthermore Bullo and Lewis have $\Phi_a = 0$. So the equation $\dot{\mu} = 0$ is just the second reduced equation written in terms of w_a.

There is a simple explicit example which nicely illustrates both of these cases, namely the Kaluza-Klein formulation of the equations of motion of a charged particle in a magnetic field. The Hamiltonian and Lagrangian approaches to this topic are well-known: see for example [8]; here we derive the equations from those given above. For M we take $\mathrm{E}^3 \times \mathrm{S}$, with coordinates (x^i, θ). Let A_i be the components of a covector field on E^3, and define a metric k on M, the Kaluza-Klein metric, by

$$k = \delta_{ij}dx^i \odot dx^j + (A_i dx^i + d\theta)^2$$

where (δ_{ij}) is the Euclidean metric. The Kaluza-Klein metric admits the Killing field $E = \partial/\partial\theta$. The vector fields $X_i = \partial/\partial x^i - A_i\partial/\partial\theta$ are orthogonal to E and invariant; moreover $k_{ij} = k(X_i, X_j) = \delta_{ij}$, while $|E| = 1$. Finally

$$[X_i, X_j] = \left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right)\frac{\partial}{\partial\theta}.$$

Putting these values into the reduced equations above we obtain

$$\ddot{x}^i = v\dot{x}^j\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right), \qquad \dot{v} = 0.$$

These are the equations of motion of a particle of unit mass and charge v in a magnetic field whose vector potential is $A_i dx^i$.

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