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An EDS approach to the inverse problem in the calculus of variations

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The inverse problem in the calculus of variations for a given set of second-order ordinary differential equations consists of deciding whether their solutions are those of Euler–Lagrange equations and whether the Lagrangian, if it exists, is unique. This paper discusses the exterior differential systems approach to this problem. In particular, it proposes an algorithmic procedure towards the construction of a certain differential ideal. The emphasis is not so much on obtaining a complete set of integrability conditions for the problem, but rather on producing a minimal set to expedite the differential ideal process. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2358000]

I. INTRODUCTION: THE INVERSE PROBLEM IN THE CALCULUS OF VARIATIONS

The inverse problem in the calculus of variations involves deciding whether for a given system of second-order ordinary differential equations,

$$\ddot{x}^a = F^a(t, x^b \dot{x}^b), \quad a,b = 1, \ldots, n,$$

a so-called multiplier matrix $g_{ab}(t, x^c, \dot{x}^c)$ can be found, such that

$$g_{ab}(\ddot{x}^b - F^b) \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial \dot{x}^a},$$

for some Lagrangian function $L(t, x^b, \dot{x}^b)$, and to what extent such a multiplier, if it exists, is unique. Necessary and sufficient conditions for the existence of a Lagrangian are generally referred to as the *Helmholtz conditions*, but have been formulated in the literature in a variety of different ways. When regarded as conditions that a nonsingular multiplier must satisfy, a concise description of the Helmholtz conditions was derived by Douglas[6] and later recast in the following form by Sarlet:[14]

$$g_{ab} = g_{ba}, \quad \Gamma(g_{ab}) = g_{ac}\Gamma^c_b + g_{bc}\Gamma^c_a, \quad g_{ac}\Phi^c_b = g_{bc}\Phi^c_a, \quad \frac{\partial g_{ab}}{\partial \dot{x}^c} = \frac{\partial g_{ac}}{\partial \dot{x}^b}, \tag{1}$$

where

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$$\Gamma^a_b := -\frac{1}{2}\frac{\partial F^a}{\partial \dot{x}^b}, \quad \Phi^a_b := -\frac{\partial F^a}{\partial x^b} - \Gamma^c_b\Gamma^a_c - \Gamma(\Gamma^a_b),$$

and where

$$\Gamma := \frac{\partial}{\partial t} + \dot{x}^a\frac{\partial}{\partial x^a} + F^a\frac{\partial}{\partial \dot{x}^a}.$$

Douglas solved this problem for $n=2$ in the sense that he exhaustively classified all second-order ODEs according to the existence and multiplicity of solutions of the Helmholtz conditions. He did this essentially via the Jordan normal forms of the matrix Φ^a_b. The corresponding solution for $n=3$ remains unavailable although various subcases for arbitrary n have been elaborated.[5,15] Our principal purpose in this paper is to explore one of the key aspects in the analysis of the inverse problem using Exterior Differential Systems theory (EDS). The general structure of the EDS approach was set out in Anderson and Thompson,[2] although these authors only examined the case of arbitrary n when Φ is a multiple of the identity. As we will demonstrate, progress using EDS almost certainly relies on explicit use of the Jordan normal forms of Φ. This is the approach taken in the thesis.[1] The aspect that we will examine in detail is the so-called differential ideal step, in which there is an algorithmic search for the largest submodule of a certain module of 2-forms generating a differential ideal. We will explore the relation of this step to the hierarchies of integrability conditions for the Helmholtz conditions known in the literature. Importantly, we will expose the details of this step in the case where Φ is diagonalizable with distinct eigenfunctions. We give a nontrivial, three-dimensional example of the step and the way in which it leads directly to a solution of the Helmholtz conditions. A subsequent paper will deal with the remaining steps in the EDS process, and we will give a complete solution of a whole class of equations in the sense of Douglas.

We now outline the geometrical framework upon which progress over the last two or three decades has depended. In geometric terms, Γ is a second-order vector field (SODE) on the first-jet extension J^1E of a bundle $E\to\mathbb{R}$. For all practical purposes, E can be identified (choosing a "trivialization") with a product manifold $\mathbb{R}\times M$, and then $J^1E\equiv\mathbb{R}\times TM$. We shall denote adapted coordinates on $\mathbb{R}\times TM$ by (t,x^a,u^a) from now on, and use π for the projection $\mathbb{R}\times TM\to\mathbb{R}\times M$.

Every SODE equips $\mathbb{R}\times TM$ with a (nonlinear) connection, the connection coefficients being the functions Γ^a_b just introduced. As a result, an adapted local frame for decomposing arbitrary vector fields on $\mathbb{R}\times TM$ into their "horizontal" and "vertical" parts is given by $\{\Gamma, H_a, V_a\}$, where

$$H_a := \frac{\partial}{\partial x^a} - \Gamma^b_a\frac{\partial}{\partial u^b}, \quad V_a := \frac{\partial}{\partial u^a}.$$

The dual basis of 1-forms is given by $\{dt,\theta^a,\psi^a\}$, with

$$\theta^a := dx^a - u^a\,dt, \quad \psi^a := du^a - F^a\,dt + \Gamma^a_b\theta^b.$$

For a given regular Lagrangian function $L\in C^\infty(J^1E)$, we define the Poincaré-Cartan 1-form θ_L by

$$\theta_{\mathrm{L}} := L\,dt + dL\circ S = L\,dt + \frac{\partial L}{\partial u^a}\theta^a,$$

where $S=V_a\otimes dx^a$ is the vertical endomorphism, and the Euler-Lagrange equations come from the unique SODE, determined by (see, e.g., Goldschmidt and Sternberg[8])

$$i_\Gamma d\theta_{\mathrm{L}} = 0 \quad \text{and} \quad dt(\Gamma) = 1.$$

Inspired by the properties of the Poincaré-Cartan 2-form $d\theta_{\mathrm{L}}$, the following theorem from Ref. 4 gives a geometric version of the Helmholtz conditions.

Theorem 1.1: *Given a* SODE Γ, *the necessary and sufficient conditions for there to be a*

Lagrangian for which Γ *is the Euler-Lagrange field is that there should exist a 2-form* Ω, *of maximal rank, which further has the following properties:* Ω *vanishes on any two vertical vector fields,* $\Gamma \lrcorner \Omega = 0$ *and* $d\Omega = 0$.

Observe that the third and fourth conditions imply that $\mathcal{L}_\Gamma \Omega = 0$, and the first condition means that Ω has a one-dimensional kernel, which, as shown by the third condition, is spanned by Γ. Another important observation is that Ω, which if it exists, will become the Poincaré-Cartan 2-form $d\theta_{\mathrm{L}}$ of the corresponding Lagrangian, then has the following particularly simple representation in the adapted frame $\{dt, \theta^a, \psi^a\}$,

$$d\theta_{\mathrm{L}} = g_{ab}\psi^a \wedge \theta^b, \quad \text{with } g_{ab} = \frac{\partial^2 L}{\partial u^a \, \partial u^b}.$$

There are a number of geometrical ways of expressing this feature, one of which requires a brief discussion of the calculus of vector fields and forms along the projection π (see Ref. 17, or 11 for a slightly different approach).

Vector fields along π are sections of the pullback bundle $\pi^* TE$ over J^1E, and we let $\mathfrak{X}(\pi)$ denote the $C^\infty(J^1E)$ module of such vector fields. Similarly, $\wedge(\pi)$ denotes the graded algebra of forms along π. There is a canonical vector field along π, given by

$$\mathbf{T} := \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a}.$$

The natural bases for $\mathfrak{X}(\pi)$ and $\mathfrak{X}^*(\pi)$ are then $\{\mathbf{T}, \partial/\partial x^a\}$ and $\{dt, \theta^a\}$. We further write $\bar{\mathfrak{X}}(\pi)$ for the elements of $\mathfrak{X}(\pi)$ that have no time component, i.e., $\bar{\mathfrak{X}}(\pi) = \mathrm{Sp}\{\partial/\partial x^a\}$. Then, if

$$X := X^0 \mathbf{T} + X^a \frac{\partial}{\partial x^a}$$

is an arbitrary vector field along π, its horizontal and vertical lift give vector fields on J^1E, respectively, given by

$$X^H := X^0 \Gamma + X^a H_a, \quad X^V := X^a V_a.$$

In what follows, we will almost exclusively have to deal with horizontal and vertical lifts of vector fields along π that belong to the submodule $\bar{\mathfrak{X}}(\pi)$. So, in referring to vector fields on J^1E of the form $X^V, Y^H, \ldots$, it will be understood that $X, Y, \ldots$, belong to $\bar{\mathfrak{X}}(\pi)$. This is important, because it means that all the essential formulas are formally those of the time-independent calculus developed in Refs. 12 and 13, rather than the corresponding ones in Ref. 17. In particular, we will frequently use the commutator relations:

$$[X^V, Y^V] = (\mathrm{D}^V_X Y - \mathrm{D}^V_Y X)^V,$$

$$[X^H, Y^V] = (\mathrm{D}^H_X Y)^V - (\mathrm{D}^V_Y X)^H,$$

$$[X^H, Y^H] = (\mathrm{D}^H_X Y - \mathrm{D}^H_Y X)^H + R(X,Y)^V.$$

Here, D^V_X and D^H_X, the vertical and horizontal covariant derivative operators, are degree zero derivations on scalar and vector-valued forms along π, determined by $\mathrm{D}^H_X F = X^H(F)$, $\mathrm{D}^V_X F = X^V(F)$ for their action on functions $F \in C^\infty(J^1E)$:

$$\mathrm{D}^H_X \frac{\partial}{\partial x^a} = X^b \Gamma^c_{ba} \frac{\partial}{\partial x^c}, \quad \mathrm{D}^V_X \frac{\partial}{\partial x^a} = 0$$

(with $\Gamma^c_{ba} = \partial \Gamma^c_a / \partial u^b$), gives the action on $\bar{\mathfrak{X}}(\pi)$ and the standard duality rules give the action on 1-forms along π. The vector-valued 2-form R along π represents the curvature of the SODE connection with coordinate form,

$$R = \tfrac{1}{2} R^a_{bc} \theta^b \wedge \theta^c \otimes \frac{\partial}{\partial x^a}, \quad R^a_{bc} := H_c(\Gamma^a_b) - H_b(\Gamma^a_c).$$

We do not distinguish notationally the contact forms θ^a, as forms on J^1E from their counterparts along π. In fact, there is a dual process of lifting 1-forms along π giving (with an obvious slight abuse of notation),

$$\theta^{aH} = \theta^a, \quad \theta^{aV} = \psi^a.$$

The dynamical covariant derivative $\boldsymbol{\nabla}$ and the Jacobi endomorphism Φ that appear in (1) arise naturally through the following formulas:

$$[\Gamma, X^V] = -X^H + (\boldsymbol{\nabla} X)^V, \quad [\Gamma, X^H] = (\boldsymbol{\nabla} X)^H + \Phi(X)^V.$$

In coordinates,

$$\Phi = \Phi^a_b \frac{\partial}{\partial x^a} \otimes \theta^b,$$

with Φ^a_b as defined before, whereas $\boldsymbol{\nabla}$, defined to vanish on $\mathbf{T}$ and dually on dt, acts on $\bar{\mathfrak{X}}(\pi)$ (with dual action on contact forms) by

$$\boldsymbol{\nabla} F = \Gamma(F) \text{ on functions}, \quad \boldsymbol{\nabla} \frac{\partial}{\partial x^a} = \Gamma^b_a \frac{\partial}{\partial x^b}, \quad \boldsymbol{\nabla} \theta^a = -\Gamma^a_b \theta^b.$$

Now we can give the link we wanted between the geometric Helmholtz conditions of Theorem 1.1 with their coordinate form in (1). The observation we made about the simple structure of $d\theta_L$ in the adapted coframe means that the 2-form Ω on J^1E of Theorem 1.1 that we seek is completely determined by a symmetric type (0,2) tensor along π, of the form $g = g_{ab} \theta^a \otimes \theta^b$ (i.e., g vanishes on $\mathbf{T}$). To be precise, Ω is the so-called Kähler lift of g, $\Omega = g^K$, which vanishes on Γ and further is defined by

$$g^K(X^V, Y^V) = g^K(X^H, Y^H) = 0, \quad g^K(X^V, Y^H) = g(X, Y).$$

The intrinsic formulation of the conditions (1) (see Ref. 17, or 13 for the autonomous case) then reads as

$$\boldsymbol{\nabla} g = 0, \quad g(\Phi X, Y) = g(X, \Phi Y), \quad \mathrm{D}^V_X g(Y, Z) = \mathrm{D}^V_Y g(X, Z). \tag{2}$$

In the next section we briefly sketch the ideas of the exterior differential systems approach, specifically in the context of the inverse problem, and we identify our objectives concerning the construction of a differential ideal containing all possible two forms Ω.

II. EDS AND THE INVERSE PROBLEM

The inverse problem involves the search for a closed 2-form and so lends itself to analysis by EDS. For a general reference to EDS, we refer to Ref. 3. A thorough analysis of the inverse problem by means of such techniques (at least for autonomous differential equations) can be found in the work of Grifone and Muzsnay,[9,10] where the approach, however, starts from the partial differential equations that the Lagrangian itself has to satisfy, rather than equations such as (1) for the multiplier.

Anderson and Thompson in Ref. 2 describe the three components of the EDS process: finding a differential ideal, setting up a Pfaffian system for finding the closed 2-forms within that ideal, and finally analyzing this system, following the Cartan-Kähler theory to determine the generality of the solution (if any). We limit ourselves to a brief synopsis of the reasoning that underlies the first two steps here, with particular emphasis on the new elements involving the eigenspectrum of Φ that we want to bring to the differential ideal construction.

Given a SODE Γ, we know that the 2-form Ω we are looking for is going to be the Kähler lift of a symmetric $(0,2)$-tensor field g along the projection, i.e., $g=g_{ab}\theta^a\otimes\theta^b$, $g_{ab}=g_{ba}$. So we start by considering on J^1E the module Σ^0 of such Kähler lifts: intrinsically, ω belongs to Σ^0, if and only if

$$i_\Gamma\omega=0, \tag{3}$$

$$\omega(X^V,Y^V)=\omega(X^H,Y^H)=0, \tag{4}$$

$$\omega(X^V,Y^H)=\omega(Y^V,X^H). \tag{5}$$

(During the EDS process we will use lower case ω as the generic name for our 2-forms, reserving Ω for the 2-form of Theorem 1.1.) In coordinates, Σ^0 is spanned by the 2-forms

$$\omega^{ab}:=\frac{1}{2}(\psi^a\wedge\theta^b+\psi^b\wedge\theta^a). \tag{6}$$

Note however, that $\{dt,\theta^a,\psi^a\}$ may as well be any basis comprising of dt, n horizontal forms, and n vertical forms. In fact later we will use such a basis made up of the eigenforms of Φ.

The first step in the EDS approach produces from Σ^0 a sequence of submodules $\Sigma^0\supset\Sigma^1\supset\Sigma^2\supset\cdots$, arriving finally (or not at all) at a nontrivial submodule $\Sigma^l=\mathrm{Sp}\{\omega^k\}$ generating a differential ideal. Obtaining Σ^l consists of computing at each stage the exterior derivative of forms belonging to the submodule under consideration, and verifying whether they belong to the ideal generated by that submodule. In principle, whenever an obstruction is found, it is translated into a further restriction on the admissible 2-forms and the process is restarted from there. We shall be more specific about this in a moment. But, for the time being, assume that we have found $\Sigma^l=\mathrm{Sp}\{\omega^1,\ldots,\omega^d\}$, so that

$$d\omega^k=\xi^k_h\wedge\omega^h,\quad k=1,\ldots,d,$$

for some 1-forms ξ^k_h. In order to construct a closed 2-form in Σ^l we first identify a basis of d-tuples of 1-forms ρ^A_h such that $\rho^A_h\wedge\omega^h=0$. Then, if $r_k\omega^k\in\Sigma^l$ is required to be closed, the functions r_k must solve the Pfaffian system of equations (the notations are taken to conform with those in Ref. 2),

$$dr_k+r_h\xi^h_k+p_A\rho^A_k=0, \tag{7}$$

for some as yet arbitrary functions p_A. The freedom in the choice of p_A must then be exploited in the final part of the EDS procedure. This last part is by no means a straightforward matter; in fact, it is fair to say that it consists of several steps still and may even, if the involutivity test fails, lead to prolonging the system and starting again (see, e.g., Ref. 2 for a brief survey). We will argue in the final section that it may be better, therefore, to address the partial differential equations of the inverse problem in a more direct way, once the differential ideal procedure is complete. Our approach to specific examples should be contrasted with that of Anderson and Thompson,[2] who follow the formal EDS process.

Here are the details of the differential ideal process. At each step Σ^i, say, we will first identify the requirements for a 3-form ρ to be in $\langle\Sigma^i\rangle$, the ideal generated by Σ^i, because a large part of such an analysis does not depend on whether or not $\rho=d\omega\in\langle\Sigma^i\rangle$. Once we apply these requirements to such $d\omega$, the general formula,

$$d\omega(U,V,W) = \sum_{U,V,W} [U(\omega(V,W)) - \omega([U,V],W)], \tag{8}$$

(where a notation like $\Sigma_{\xi,\eta,\zeta}$ will always refer to a cyclic sum over the indicated arguments), produces algebraic restrictions on admissible 2-forms. If these are different from those already implemented, they are used to define the next submodule Σ^{i+1} in the sequence. But a couple of important remarks are in order here. First of all, the issue of algebraic conditions in the inverse problem is quite delicate. There are several infinite hierarchies of such conditions (see, e.g., Refs. 9 and 10), but it is impossible to tell in all generality which of these are more important, from which others possibly might follow, or simply which are more efficient in determining the existence or nonexistence of a multiplier. We therefore propose to integrate the decision about the usefulness of algebraic restrictions as much as possible into this differential ideal algorithm. That is to say, we shall attempt to obtain at each step in restricting to a submodule Σ^i, conditions for a 3-form ρ to belong to $\langle\Sigma^i\rangle$ which are both necessary and sufficient. It is, to some extent, the degree to which fresh conditions tend to be sufficient, which will guide the decision about the selection of further restrictions for defining the next submodule. It is precisely in this way that we will be able to push the EDS procedure beyond the level of results obtained in Ref. 2. However, as has always been the case in the study of the multiplier problem, there is no possibility of a general solution for an arbitrary dimension. At some point further progress relies on a classification of cases and subcases. Most notorious in this respect is the paper by Douglas[6] (see Ref. 16 for a geometrical account of Douglas' analysis). We should expect this to occur also in our current attempt.

III. THE FIRST STEP IN THE EDS ALGORITHM

Consider the module Σ^0, defined above, and let ρ be a 3-form in $\langle\Sigma^0\rangle$, so that $\rho=\beta_k\wedge\omega^k$ for some 1-forms β_k and $\omega^k\in\Sigma^0$. Then $i_\Gamma\rho=\beta_k(\Gamma)\omega^k$, so that $i_\Gamma\rho$ belongs to Σ^0 *if and only if*

$$\rho(\Gamma,X^V,Y^V) = 0, \tag{9}$$

$$\rho(\Gamma,X^H,Y^H) = 0, \tag{10}$$

$$\rho(\Gamma,X^V,Y^H) = \rho(\Gamma,Y^V,X^H). \tag{11}$$

Next, starting from the contraction of ρ with a vertical vector field,

$$i_{X^V}\rho = \beta_k(X^V)\omega^k - \beta_k\wedge i_{X^V}\omega^k,$$

we must have

$$\rho(X^V,Y^V,Z^V) = 0, \tag{12}$$

but how can we take other restrictions on Σ^0 into account, when further combinations of horizontal and vertical vector fields are inserted? For example, we can manipulate the right-hand side by using the properties of $\omega^k\in\Sigma^0$, until every appearance of the β_k has been eliminated and a condition about ρ emerges. We have

$$\begin{aligned}\rho(X^V,Y^V,Z^H) &= \beta_k(X^V)\omega^k(Y^V,Z^H) - \beta_k(Y^V)\omega^k(X^V,Z^H) = \beta_k(X^V)\omega^k(Z^V,Y^H) - \beta_k(Y^V)\omega^k(Z^V,X^H)\\ &= \rho(X^V,Z^V,Y^H) + \beta_k(Z^V)\omega^k(X^V,Y^H) - \beta_k(Y^V)\omega^k(Z^V,X^H) = \rho(X^V,Z^V,Y^H)\\ &\quad + \beta_k(Z^V)\omega^k(Y^V,X^H) - \beta_k(Y^V)\omega^k(Z^V,X^H) = \rho(X^V,Z^V,Y^H) + \rho(Z^V,Y^V,X^H).\end{aligned}$$

So it follows that ρ should satisfy

$$\sum_{X,Y,Z} \rho(X^V,Y^V,Z^H) = 0, \tag{13}$$

and in exactly the same way also

$$\sum_{X,Y,Z} \rho(X^V, Y^H, Z^H) = 0. \tag{14}$$

There remains the condition

$$\rho(X^H, Y^H, Z^H) = 0. \tag{15}$$

Details such as the way the cyclic sum condition (13) is obtained will not be repeated further on.

In total, we have obtained seven necessary conditions and, following the strategy deployed in the previous section, we now explore their sufficiency. But before we proceed, we remark that a number of the conditions to be encountered here, and in the subsequent sections, only have an effect when the number n of degrees of freedom of the system is at least 3. This is the case, for example, with the conditions (12) to (15), which are clearly void when $n=2$. As a result, each time a question of the sufficiency of conditions arises, we will say a few words about the case $n=2$. Besides, the case $n=2$ has been extensively studied already (see Refs. 6, 10, and 16) and therefore is not of prime interest for this paper.

Proposition 3.1: For an arbitrary 3-form ρ to belong to the ideal $\langle \Sigma^0 \rangle$, it is necessary and sufficient that ρ satisfies the conditions (9)–(15), where X, Y, Z are arbitrary vector fields along π.

Proof: It is easy to see that conditions (9)–(11) imply that ρ is of the form

$$\rho = dt \wedge \sigma + \bar{\rho}, \quad \text{with } \sigma \in \Sigma^0 \quad \text{and} \quad i_\Gamma \bar{\rho} = 0.$$

For $n \geq 3$, the remaining conditions then indicate that $\bar{\rho}$ is of the form

$$\bar{\rho} = \tfrac{1}{2}\mathcal{A}_{abc}\theta^a \wedge \psi^b \wedge \psi^c + \tfrac{1}{2}\mathcal{B}_{abc}\psi^a \wedge \theta^b \wedge \theta^c,$$

where $\mathcal{A}_{abc}$ and $\mathcal{B}_{abc}$ are skew-symmetric in their last two indices and satisfy [in view of (13) and (14)]

$$\sum_{abc} \mathcal{A}_{abc} = \sum_{abc} \mathcal{B}_{abc} = 0.$$

For $n=2$, $i_\Gamma \bar{\rho}=0$ is already enough to ensure that $\bar{\rho}$ is of the above form, but with two of the three indices the same. The skew symmetry of the coefficients in their last two indices then implies that they formally can be regarded as having this cyclic sum property as well.

The skew symmetry $\mathcal{A}_{abc}=-\mathcal{A}_{acb}$ and the cyclic sum property $\Sigma_{abc}\mathcal{A}_{abc}=0$ imply that

$$\begin{aligned}
\mathcal{A}_{abc}\theta^a \wedge \psi^b \wedge \psi^c &= \tfrac{1}{3}(\mathcal{A}_{abc}\theta^a \wedge \psi^b \wedge \psi^c + \mathcal{A}_{bca}\theta^b \wedge \psi^c \wedge \psi^a + \mathcal{A}_{cab}\theta^c \wedge \psi^a \wedge \psi^b)\\
&= \tfrac{1}{3}\mathcal{A}_{abc}(\psi^c \wedge \theta^a + \psi^a \wedge \theta^c) \wedge \psi^b - \tfrac{1}{3}\mathcal{A}_{bca}(\psi^c \wedge \theta^b + \psi^b \wedge \theta^c) \wedge \psi^a\\
&= -\tfrac{4}{3}\mathcal{A}_{abc}\omega^{ab} \wedge \psi^c,
\end{aligned}$$

where we have used the skew symmetry in the first and the cyclic sum property in the last term for the transition from the first to the second line. The same is also true for the other term in $\bar{\rho}$, so

$$\rho = dt \wedge \sigma - \tfrac{2}{3}\mathcal{A}_{abc}\omega^{ab} \wedge \psi^c + \tfrac{2}{3}\mathcal{B}_{abc}\omega^{ab} \wedge \theta^c, \tag{16}$$

or putting

$$A_{abc} = \mathcal{A}_{abc} + \mathcal{A}_{bac}, \quad B_{abc} = \mathcal{B}_{abc} + \mathcal{B}_{bac},$$

$$\rho = dt \wedge \sigma - \tfrac{1}{3}A_{abc}\omega^{ab} \wedge \psi^c + \tfrac{1}{3}B_{abc}\omega^{ab} \wedge \theta^c, \tag{17}$$

where the new coefficients now are symmetric in their first two indices and still have the cyclic sum property $\Sigma_{abc}A_{abc}=\Sigma_{abc}B_{abc}=0$. This manifestly exhibits that ρ belongs to $\langle \Sigma^0 \rangle$. □

Remark: An expression like $A_{abc}\omega^{ab} \wedge \psi^c$ with $A_{abc}=A_{bac}$ clearly belongs to $\langle \Sigma^0 \rangle$ without any further requirements. That the above necessary and sufficient conditions are not contradictory, however, follows from the fact that one can assume that the coefficients further have the cyclic sum property without loss of generality. Verifying this is left to the reader.

From now on we will use the representation (17) of ρ.

To terminate the first step now, we apply the *necessary and sufficient conditions* (9)–(15) to an exact 3-form $d\omega$, for any $\omega \in \Sigma^0$ and use thereby the familiar identity (8). In doing that, only the second part that involves the Lie brackets can contribute; the list of bracket relations that are frequently used in these calculations has been given in Sec. I. It easily follows then that the conditions (9) and (11)–(14) are identically satisfied, whereas the remaining conditions (10) and (15) give rise to the following extra restrictions on admissible 2-forms (which are, in one form or another, well known in the literature):

$$\omega((\Phi X)^V, Y^H) = \omega((\Phi Y)^V, X^H), \tag{18}$$

$$\sum_{X,Y,Z} \omega(R(X,Y)^V, Z^H) = 0. \tag{19}$$

We could implement both of these new requirements to define the next submodule Σ^1, but it will be more convenient to continue in stages and start by implementing the Φ condition (18) only.

The first step has identified the structure of 3-forms in $\langle \Sigma^0 \rangle$. When we apply the differential ideal algorithm to Σ^0 now, conditions on ρ that have Γ as one of the arguments will be easy to handle and merely impose that the 2-form σ in (17) belongs to the submodule under consideration. Other conditions must clearly come in pairs: for each condition that has two vertical and one horizontal argument and thus has an effect on the A part in ρ only, there will be a corresponding condition that has the "mirror" effect on the B part. The underlying reason for that is the basic symmetry property (5) of 2-forms in Σ^0, which makes that a condition like (18), for example, can equivalently be written in the form

$$\omega((\Phi X)^H, Y^V) = \omega((\Phi Y)^H, X^V). \tag{20}$$

Bearing this in mind avoids unnecessary duplications of conditions later on.

Finally, we examine the necessary and sufficient conditions that $\langle \Sigma^0 \rangle$ is itself a differential ideal.

Theorem 3.2: *The module* Σ^0 *generates a differential ideal if and only* Φ *is a multiple of the identity*.

Proof: From the immediately preceding discussion, it follows that $\langle \Sigma^0 \rangle$ is a differential ideal if and only if (18) and (19) are satisfied by all $\omega \in \Sigma^0$. If Φ is a multiple of the identity, the first of these is satisfied because of (5), and the second is true by virtue of the identity $V_a(\Phi^c_b) - V_b(\Phi^c_a) = 3R^c_{ab}$. Conversely, if (18) is true for all $\omega \in \Sigma^0$, then it is true for all ω^{ab} given by (6): using $X = \partial / \partial x^c$, $Y = \partial / \partial x^d$ yields

$$\Phi^a_c \delta^b_d + \Phi^b_c \delta^a_d = \Phi^a_d \delta^b_c + \Phi^b_d \delta^a_c.$$

Hence, $\Phi = \mu I$ and (19) follows automatically, again because of the stated identity. □

This is a stronger result than that obtained by Anderson and Thompson in Ref. 2, where it was shown that if Φ is a multiple of the identity then $\langle \Sigma^0 \rangle$ is a differential ideal. Anderson and Thompson demonstrated that in this case the system is variational. However, in general there remain obstructions to variationality after a differential ideal has been obtained.

IV. A SECOND STEP IN THE PROCESS

We define Σ^1 to be the submodule of Σ^0 whose elements satisfy condition (18).

Let $\rho = \beta_k \wedge \omega^k$ be a 3-form in $\langle \Sigma^1 \rangle$. Then, a contraction with Γ leads to the further restriction

$$\rho(\Gamma,(\Phi X)^V,Y^H)=\rho(\Gamma,(\Phi Y)^V,X^H), \tag{21}$$

which, as indicated before, says that σ in (17) must be in Σ^1 and plays further no role. When we take X^V as the first argument and follow the procedure that led to (13), but this time with $(\Phi Y)^V,Z^H$ as a further argument, we obtain

$$\sum_{X,Y,Z}(\rho(X^V,(\Phi Y)^V,Z^H)-\rho(X^V,(\Phi Z)^V,Y^H))=0.$$

But the left-hand side is $\Sigma_{X,Y,Z}(\rho(X^V,(\Phi Y)^V,Z^H)+\rho((\Phi Y)^V,Z^V,X^H))$ and then (13) gives the more transparent version

$$\sum_{X,Y,Z}\rho(X^V,Y^V,(\Phi Z)^H)=0. \tag{22}$$

Immediately we conclude that the "mirror" condition, which can, of course, independently be derived, will read as

$$\sum_{X,Y,Z}\rho(X^H,Y^H,(\Phi Z)^V)=0. \tag{23}$$

Unfortunately, this is not the end of the line, as there are other possible combinations of terms. For example, with $(\Phi X)^V$ as a first argument, rather than X^V,

$$i_{(\Phi X)^V}\rho=\beta_k((\Phi X)^V)\omega^k-\beta_k\wedge i_{(\Phi X)^V}\omega^k,$$

we can choose $(\Phi Y)^V,Z^H$ as second and third arguments. The by now familiar procedure of eliminating all terms involving the 1-forms β_k then leads to the new requirement

$$\sum_{X,Y,Z}\rho((\Phi X)^V,(\Phi Y)^V,Z^H)=0, \tag{24}$$

and its counterpart

$$\sum_{X,Y,Z}\rho((\Phi X)^H,(\Phi Y)^H,Z^V)=0. \tag{25}$$

We already reach a point here where it is difficult to say whether all such necessary conditions will generically be independent or whether perhaps there are still other ways of producing more conditions; hence our strategy to approximate, as best as possible, conditions that are also sufficient and demonstrate their utility in this way. So let us address the sufficiency question here.

Let ρ be of the form (17), where the A and B coefficients are symmetric in their first two indices and can, as argued before, without loss of generality be assumed to have the cyclic sum property $\Sigma_{abc}A_{abc}=\Sigma_{abc}B_{abc}=0$. The two conditions (22) and (24) affect only the A-term (23) and (25) will have the same sort of effect on the B term. Using a basis of horizontal and vertical vector fields, (H_i,V_j) say, we can compute the A-term of ρ acting on the triple $(V_r,V_t,\Phi^u_sH_u)$ and then take a cyclic sum over the indices (r,s,t). What remains (leaving out a numerical factor) is the following condition:

$$\sum_{rst}(A_{rut}-A_{tur})\Phi^u_s=0, \tag{26}$$

which, by recombining terms, can equivalently be written in the perhaps more appealing form

$$\sum_{rst}(A_{rut}\Phi^u_s-A_{sut}\Phi^u_r)=0. \tag{27}$$

Similarly, evaluating $\rho((\Phi X)^V,(\Phi Y)^V,Z^H)$ on $(\Phi^u_sV_u,\Phi^v_tV_v,H_r)$ and then taking the cyclic sum, the condition (24) is found to mean:

$$\sum_{rst} (A_{ruv} - A_{rvu})\Phi_s^u \Phi_t^v = 0. \tag{28}$$

Now, the 2-forms in Σ^1 we are talking about are of the form

$$\omega = h_{ab}\omega^{ab}, \quad \text{with} \quad h_{su}\Phi_r^u - h_{ru}\Phi_s^u = 0 \tag{29}$$

[we purposely avoid using g_{ab}, which we reserve for candidate multipliers satisfying (1)]. So the idea would be to prove that the requirements (26) [or (27)] and (28) force the functions A_{abc} to be of the form

$$A_{abc} = h_{ab}{}^k\, b_{kc}, \tag{30}$$

where, for each k, the $h_{ab}{}^k$ have the property (29) and the b_{kc} are arbitrary functions, representing the components of the 1-forms β_k in an expression like $\rho = \beta_k \wedge \omega^k$. Unfortunately, we were unable to prove that this is true in all generality, but we shall show now that it is a valid statement in an interesting (reasonably generic) special case.

As we know already from Douglas,[6] a classification into different cases where a multiplier g_{ab} for a given dynamics Γ does or does not exist, will largely be governed by properties of the Jacobi endomorphism Φ associated with Γ. A specific assumption about Φ that comes up in several situations (see, e.g., Ref. 5) is that of (algebraic) diagonalizability. So let us assume this, and that the (real) eigenvalues $\lambda_{(a)}$ are distinct, and let $\{\phi^a\}$ denote a complete set of eigenforms of Φ, so that

$$\Phi(\phi^a) = \lambda_{(a)}\phi^a \quad \text{(no sum)}.$$

These ϕ^a can be taken to be combinations of contact forms and are of course still semi-basic forms. We now have a new basis for $\mathfrak{X}(J^1E)$, namely $\{dt, \phi^{aH}, \phi^{aV}\}$ and new spanning 2-forms for Σ^0, namely,

$$\phi^{ab} := \tfrac{1}{2}(\phi^{aV} \wedge \phi^{bH} + \phi^{bV} \wedge \phi^{aH}).$$

It is important to realize that with this change of basis, nothing changes in our considerations of the first differential ideal step. For example, the 2-forms in Σ^0 are now of the form $\bar{h}_{ab}\phi^{ab}$ and 3-forms in the ideal $\langle \Sigma^0 \rangle$ are of the form (17), with the ϕ^{aH} replacing the θ^a, and so on. Explicitly,

$$\rho = dt \wedge \sigma - \tfrac{1}{3}\bar{A}_{abc}\phi^{ab} \wedge \phi^{cV} + \tfrac{1}{3}\bar{B}_{abc}\phi^{ab} \wedge \phi^{cH}.$$

We will freely use the original basis formulas in the eigenform basis (and refer to their equation numbers) just by adding overbars and switching θ^a for ϕ^{aH} etc.

The additional restriction (18) that defines Σ^1, or equivalently (29), reduces to

$$(\lambda_{(b)} - \lambda_{(a)})\bar{h}_{ab} = 0, \tag{31}$$

and hence implies that the elements of Σ^1 must be diagonal in the eigenform basis, i.e., $\bar{h}_{ab} = 0$ for $a \neq b$.

Proportion 4.1: Suppose that Φ is diagonalizable with distinct (real) eigenvalues. Then, the necessary and sufficient conditions for a 3-form ρ to be in the ideal $\langle \Sigma^1 \rangle$ are the conditions to be in $\langle \Sigma^0 \rangle$, supplemented by (21), (22), and (24), together with their counterparts (23) and (25).

Proof: What has to be proved is the sufficiency of the conditions. Using the eigenform basis of Φ, we already know that $\rho \in \langle \Sigma^0 \rangle$ implies that it is of the form (17). The first extra condition (21) requires σ to belong to the smaller module Σ^1 now. As explained before, it suffices to study the effect of (22) and (24), or explicitly (26) and (28), on the A term in ρ. Under the present circumstances, this leads to the conditions

$$\sum_{abc} \bar{A}_{abc}(\lambda_{(a)} - \lambda_{(b)}) = 0, \tag{32}$$

$$\sum_{abc} \bar{A}_{abc}\lambda_{(c)}(\lambda_{(a)} - \lambda_{(b)}) = 0. \tag{33}$$

These conditions are identically satisfied whenever two of the indices are the same, so we begin by considering $n \geq 3$. For each set of three distinct indices, they produce, together with the given cyclic sum property, a homogeneous system of algebraic equations with coefficient matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ \lambda_{(a)} - \lambda_{(b)} & \lambda_{(b)} - \lambda_{(c)} & \lambda_{(c)} - \lambda_{(a)} \\ \lambda_{(c)}(\lambda_{(a)} - \lambda_{(b)}) & \lambda_{(a)}(\lambda_{(b)} - \lambda_{(c)}) & \lambda_{(b)}(\lambda_{(c)} - \lambda_{(a)}) \end{pmatrix}.$$

The determinant of this matrix is proportional to $(\lambda_{(a)}-\lambda_{(b)})(\lambda_{(b)}-\lambda_{(c)})(\lambda_{(c)}-\lambda_{(a)})$ and hence is nonzero. Therefore, all $\bar{A}_{abc}$ with distinct indices must be zero. It further follows from the cyclic sum property that $\bar{A}_{aab}=-2\bar{A}_{baa}$. With these data, the A term of ρ becomes

$$\begin{aligned} -\tfrac{1}{3}\bar{A}_{abc}\phi^{ab} \wedge \phi^{cV} &= -\tfrac{1}{3}\left(\sum_{a\neq b} \bar{A}_{aab}\phi^{aa} \wedge \phi^{bV} + 2\sum_{a\neq b} \bar{A}_{baa}\phi^{ab} \wedge \phi^{aV}\right) \\ &= -\tfrac{1}{3}\left(\sum_{a\neq b} \bar{A}_{aab}\phi^{aa} \wedge \phi^{bV} - \sum_{a\neq b} \bar{A}_{aab}\phi^{ab} \wedge \phi^{aV}\right) = -\tfrac{1}{2}\sum_{a\neq b} \bar{A}_{aab}\phi^{aa} \wedge \phi^{bV}, \end{aligned}$$

where we have used the fact that $\phi^{ab}\wedge\phi^{aV}=-\frac{1}{2}\phi^{aa}\wedge\phi^{bV}$. But we know from (31) that the 2-forms in Σ^1 are diagonal in the basis of eigenforms, i.e., of the form $\bar{h}_{aa}\phi^{aa}$, and the above computation then shows that the A term of ρ is manifestly in $\langle\Sigma^1\rangle$. The effect of (23) and (25) on the B term is similar.

When $n=2$, only the condition (21) survives and implies as before that σ must belong to Σ^1. Moreover, it is easy to verify explicitly that for $n=2$, the $\bar{\rho}$ part of our 3-form in $\langle\Sigma^0\rangle$ automatically belongs to $\langle\Sigma^1\rangle$ as well, so the general claim is still valid. □

We now return once more to the general case with no assumptions about Φ. The final stage in our step 2 procedure is to apply the conditions on 3-forms again to the special case of exact 3-forms. This will determine possibly new restrictions on admissible 2-forms, which can then be used to identify further submodules. The computation related to condition (21) is straightforward and produces the new requirement,

$$\omega((\nabla\Phi(X))^V, Y^H) = \omega((\nabla\Phi(Y))^V, X^H), \tag{34}$$

When (22) is imposed on a 3-form $d\omega$, it merely reproduces the condition (19) we already have, in view of the general identity [see Ref. 13, remembering always that we are restricting to vector fields in $\bar{\mathfrak{X}}(\pi)$]

$$3R(X,Y) = D_X^V\Phi(Y) - D_Y^V\Phi(X). \tag{35}$$

But its counterpart (23) gives rise to the new condition,

$$\sum_{X,Y,Z} \omega(\nabla R(X,Y)^V, Z^H) = 0, \tag{36}$$

because we also have the identity

$$\nabla R(X,Y) = D_X^H\Phi(Y) - D_Y^H\Phi(X). \tag{37}$$

The computations for (24) and (25) run parallel and produce the following new requirements

$$\sum_{X,Y,Z} \omega((D^V_{\Phi X}\Phi(Y) - D^V_{\Phi Y}\Phi(X))^V, Z^H) = 0, \tag{38}$$

$$\sum_{X,Y,Z} \omega((D^H_{\Phi X}\Phi(Y) - D^H_{\Phi Y}\Phi(X))H, Z^V) = 0. \tag{39}$$

Conditions such as (34) and (36) are well known in the literature. It was pointed out in Ref. 14 that such conditions must hold for arbitrary ∇ derivatives of Φ and R and it is not difficult to see how this double hierarchy will emerge in the EDS process also, simply from the restrictions on ρ which have Γ in their arguments and will be produced step by step.

The conditions (38) and (39), however, have only been reported in the thesis[1] (though they must be related in some sense to requirements involving the Nijenhuis tensor of Φ in the approach adopted in Ref. 10). It is impossible to say in all generality which of the many algebraic restrictions are somehow the "more independent ones," but the message from our current algorithmic analysis is that it is likely to be more efficient in applications to impose the last mentioned requirements on admissible 2-forms first, before extracting all information, for example, from the double infinite hierarchy of $\nabla^k\Phi$ and $\nabla^k R$ conditions, which were the only conditions taken into consideration in Ref. 15.

But we must not yet embark on using any of these conditions to define a further submodule, as we have only dealt with half of the information that came out of the first step so far. We shall study the curvature condition (19) in the next section, but for the sake of applications it is worthwhile summing up what we now know about the termination of the differential ideal process at this level.

Proportion 4.2: Assume that Φ is diagonalizable with distinct (real) eigenvalues. Then the necessary and sufficient conditions for Σ^1 to generate a differential ideal are that all 2-forms of the form (29) *satisfy the algebraic conditions* (19), (34), (36), (38), and (39).

Proof: 2-forms in Σ^1 are characterized by (29). If Φ is diagonalizable, the necessary and sufficient conditions for their exterior derivative to belong to $\langle\Sigma^1\rangle$ (as identified by Proposition 4.1 and the subsequent analysis) are that they satisfy the supplementary restrictions (19), (34), (36), (38), and (39). But saying that Σ^1 generates a differential ideal already is the same as saying *that no further restrictions beyond those defining Σ^1 should be found* and thus that the five conditions just mentioned must hold for all 2-forms in Σ^1. □

Remark: as we observed in the proof of Proposition 4.1, only one of the five extra conditions survives when $n=2$ and this condition translates to the $\nabla\Phi$ condition (34) when applied to the exterior derivative of a 2-form in Σ^1. This is, therefore, the only condition to take into account when applying the above proposition to the case $n=2$.

V. A FURTHER STEP FOR DIAGONALIZABLE Φ

We return again to the general situation without assuming that Φ is diagonalizable. Define Σ^2 to be the module of 2-forms in Σ^1, which further satisfy the condition (19).

As before, if $\rho=\beta_k\wedge\omega^k$ is a 3-form in $\langle\Sigma^2\rangle$, a contraction with Γ, in view of (19), leads to the further restriction

$$\sum_{X,Y,Z} \rho(\Gamma, R(X,Y)^V, Z^H) = 0. \tag{40}$$

Likewise, when we contract first with an arbitrary vertical element and then proceed to eliminate all terms involving the β_k, a procedure that is more involved here (but we leave the details to the reader), the condition we obtain reads as

$$\sum_{X,Y,Z,U}^{e} \rho(X^V, R(Y,Z)^V, U^H) = 0, \tag{41}$$

where the upper index in the summation sign is meant to indicate that the sum extends over all even permutations of the indicated vector fields. Needless to say, there will be a mirror condition that can independently be derived, and reads as

$$\sum_{X,Y,Z,U}^{e} \rho(X^H, R(Y,Z)^H, U^V) = 0. \tag{42}$$

Obviously there are more possibilities as in the preceding section. For example, repeating the above computation, but with $(\Phi X)^V$ as first argument, rather than X^V, gives

$$\sum_{X,Y,Z,U}^{e} \rho((\Phi X)^V, R(Y,Z)^V, U^H) = 0, \tag{43}$$

and its analog with two horizontal and one vertical elements.

Once again, we have to try and find necessary conditions that are also sufficient for 3-forms to belong to $\langle \Sigma^2 \rangle$. Should we, for example, search for a condition also with two R-arguments in it? To begin with, here is a further infinite number of necessary conditions for ρ to be in $\langle \Sigma^2 \rangle$.

Lemma 5.1: Necessary conditions for a 3-form ρ to belong to $\langle \Sigma^2 \rangle$ are that

$$\sum_{X,Y,Z,U}^{e} \rho((\Phi^m X)^V, R(Y,Z)^V, U^H) = 0, \tag{44}$$

for all m.

Proof: Observe first that 2-forms that have the symmetry property (5) and satisfy (18), automatically also satisfy

$$\omega((\Phi^m X)^V, Y^H) = \omega((\Phi^m Y)^V, X^H), \tag{45}$$

for all m. Indeed, using successively the properties (18), (20), and (5), we have

$$\omega((\Phi^2 X)^V, Y^H) = \omega((\Phi Y)^V, (\Phi X)^H) = \omega(X^V, (\Phi^2 Y)^H) = \omega((\Phi^2 Y)^V, X^H).$$

The statement for general m follows by induction.

Replacing now ΦX by $\Phi^m X$ in the considerations that lead to (43), it is fairly straightforward to verify that we will arrive at (44) in view of (45). □

When a 2-form ω is in Σ^1 and so has the symmetry property (18) with respect to Φ, it makes no sense to impose symmetry with respect to powers Φ^m as further restrictions, because ω will automatically have these properties. Likewise, if we already knew that a 3-form ρ satisfying (41) and (43) belongs to the ideal $\langle \Sigma^2 \rangle$, there would be no sense in looking further at (44). But it is just because we do not have yet sufficiency, that extra requirements like (44) can have practical value.

Let us now first look at the impact of the curvature conditions we obtained so far, on the A part of ρ. Referring to the coordinate expression of R that was given in Sec. I, the condition (19) that further defines the module Σ^2, reads as

$$\sum_{abc} h_{ra} R^r_{bc} = 0. \tag{46}$$

One easily verifies that (41) and (43) imply that the A part of ρ must have the properties

$$\sum_{abcd}^{e} R^s_{bc}(A_{das} - A_{dsa}) = 0, \tag{47}$$

$$\sum_{abcd}^{e} \Phi_a^r R_{bc}^s (A_{drs} - A_{dsr}) = 0. \tag{48}$$

And (44), for $m=2$ for example, will require that

$$\sum_{abcd}^{e} \Phi_a^r \Phi_r^t R_{bc}^s (A_{dts} - A_{dst}) = 0. \tag{49}$$

It is of some interest to write out explicitly what such conditions mean. The first one, for example, making use of the skew-symmetry of the R_{bc}^a and the symmetry of the A_{abc} to recombine terms, can be written in the form

$$R_{bc}^s(A_{asd} - A_{dsa}) + R_{cd}^s(A_{asb} - A_{bsa}) + R_{db}^s(A_{asc} - A_{csa}) + R_{ad}^s(A_{bsc} - A_{csb}) + R_{ca}^s(A_{bsd} - A_{dsb}) + R_{ab}^s(A_{csd} - A_{dsc}) = 0. \tag{50}$$

Notice that the left-hand side of this expression is skew-symmetric in any pair of indices [the same can be verified for (48) and (49)]. Hence, these conditions are identically satisfied when the free indices are not distinct. In other words, for them to have any effect, the dimension must be at least 4.

We now return to the interesting case of diagonalizable Φ with distinct eigenvalues, as introduced in the previous section. Any further such assumption on Φ has an effect on curvature-type conditions, since Φ determines R according to (35). Let X_a denote a basis of eigenvectors of Φ, dual to the basis of eigenforms considered before; so we have $\Phi(X_a) = \lambda_{(a)} X_a$. With $\bar{R}_{bc}^a$ now denoting the components of R with respect to this adapted frame, we find from (35) and introducing the structure functions τ_{ab}^c, defined by

$$\mathrm{D}_{X_a}^V X_b = \tau_{ab}^c X_c, \tag{51}$$

that

$$3\bar{R}_{ab}^a = -\left(\mathrm{D}_{X_b}^V \lambda_{(a)} + (\lambda_{(a)} - \lambda_{(b)})\tau_{ab}^a\right), \tag{52}$$

$$3\bar{R}_{ab}^b = \mathrm{D}_{X_a}^V \lambda_{(b)} + (\lambda_{(b)} - \lambda_{(a)})\tau_{ba}^b, \tag{53}$$

$$3\bar{R}_{ab}^s = (\lambda_{(b)} - \lambda_{(s)})\tau_{ab}^s - (\lambda_{(a)} - \lambda_{(s)})\tau_{ba}^s, \quad s \neq a,b. \tag{54}$$

Now, when Φ is diagonalizable with distinct eigenvalues, we already know from the Σ^1-analysis that $\bar{A}_{abc}=0$ when all indices are distinct. Taking the condition (50) with a, b, c, d different, the summation in each of the terms gives rise to only two terms. Further simplifications arise from taking into account that $\Sigma_{abc}\bar{A}_{abc}=0$ implies $\bar{A}_{bba}=-2\bar{A}_{abb}$. Finally, using the R information, we find that only components of the type (54) enter. The condition is then

$$\sum_{abcd}^{e} (\lambda_{(b)} - \lambda_{(c)})(\bar{A}_{abb}\tau_{dc}^b + \bar{A}_{acc}\tau_{db}^c) = 0. \tag{55}$$

The corresponding condition (46) for the 2-forms defining Σ^2 (knowing that $\bar{h}_{ab}$ is diagonal) likewise reduces to

$$\sum_{abc} (\lambda_{(a)} - \lambda_{(b)})(\bar{h}_{aa}\tau^a_{cb} + \bar{h}_{bb}\tau^b_{ca}) = 0. \tag{56}$$

The similarity in structure between (55) and (56) becomes even clearer if we proceed as follows: for dealing with an expression such as (41), we write, for arbitrary $X_1, \ldots, X_4$,

$$\sum_{ijkl}^{e} \rho(X_i^V, R(X_j,X_k)^V, X_l^H) = \sum_{i=1}^{4} (-1)^i \sum_{jkl} \rho(X_i^V, R(X_j,X_k)^V, X_l^H). \tag{57}$$

The notation that is being used here on the right-hand side should be read as follows: for each $i = 1, \ldots, 4$, (i,j,k,l) is a cyclic permutation of (1), (2), (3), (4) and then with i being kept fixed, we perform a cyclic sum over the three other indices. Applying this idea to the explicit form of the condition (41) we are considering, (55) can be recast in the form:

$$\begin{aligned}\sum_{abcd}^{e} (\lambda_{(b)} - \lambda_{(c)})(\bar{A}_{abb}\tau^b_{dc} + \bar{A}_{acc}\tau^c_{db}) \equiv & \sum_{bcd} (\lambda_{(b)} - \lambda_{(c)})(\bar{A}_{abb}\tau^b_{dc} + \bar{A}_{acc}\tau^c_{db}) \\ & - \sum_{cda} (\lambda_{(c)} - \lambda_{(d)})(\bar{A}_{bcc}\tau^c_{ad} + \bar{A}_{bdd}\tau^d_{ac}) \\ & + \sum_{dab} (\lambda_{(d)} - \lambda_{(a)})(\bar{A}_{cdd}\tau^d_{ba} + \bar{A}_{caa}\tau^a_{bd}) \\ & - \sum_{abc} (\lambda_{(a)} - \lambda_{(b)})(\bar{A}_{daa}\tau^a_{cb} + \bar{A}_{dbb}\tau^b_{ca}) = 0.\end{aligned} \tag{58}$$

We can now be precise about what it is we should be able to prove to reach sufficiency of conditions for ρ to be in $\langle \Sigma^2 \rangle$. From the Σ^1 analysis we already know that the A part of ρ will be of the form:

$$\rho_A = \sum_{a \neq d} \bar{A}_{daa}\phi^{aa} \wedge \phi^{dV}. \tag{59}$$

To conclude that such a term "manifestly belongs" to the ideal generated by Σ^2 (or, in fact, to any further submodule of Σ^1) we must be able to show that for each fixed d, there exists a function α_d such that $\Sigma_{a\neq d}\bar{A}_{daa}\phi^{aa} + \alpha_d\phi^{dd}$ belongs to Σ^2 (or to the submodule under consideration). For the case at hand, assuming the dimension is at least 4, this 2-form should, in particular, have the property [see (56)] that for each set of three distinct indices a,b,c that are different from d: $\Sigma_{abc}(\lambda_{(a)} - \lambda_{(b)})(\bar{A}_{daa}\tau^a_{cb} + \bar{A}_{dbb}\tau^b_{ca}) = 0$. But the available data on ρ so far only tell us that a sum of four such terms is zero. So again, maybe, by throwing in more conditions, we might be able to ensure that all four parts in the expression (58) vanish separately.

Now consider the hierarchy (44) of further necessary conditions we have obtained. Following the different steps of the calculation that led from (47) to (55), one can show that (48) and (49) for diagonalizable Φ become

$$\sum_{abcd}^{e} \lambda_{(a)}(\lambda_{(b)} - \lambda_{(c)})(\bar{A}_{abb}\tau^b_{dc} + \bar{A}_{acc}\tau^c_{db}) = 0, \tag{60}$$

$$\sum_{abcd}^{e} \lambda^2_{(a)}(\lambda_{(b)} - \lambda_{(c)})(\bar{A}_{abb}\tau^b_{dc} + \bar{A}_{acc}\tau^c_{db}) = 0. \tag{61}$$

Lemma 5.2: If Φ is diagonalizable with distinct eigenvalues, then for ρ to satisfy the hierarchy of conditions (44), it is sufficient that these properties hold for $m = 0,1,2,3$.

Proof: The assumption is that we have for each set of four distinct indices a,b,c,d:

$$\sum_{abcd}^{e} \lambda_{(a)}^{n}(\lambda_{(b)}-\lambda_{(c)})(\bar{A}_{abb}\tau^{b}_{dc}+\bar{A}_{acc}\tau^{c}_{db})=0,$$

and this for $m=0,1,2,3$. Splitting the sum of 12 even permutations into four cyclic sum parts, as was done in (58), we get a homogeneous linear system of four equations for these sums, with the coefficient matrix

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ \lambda_{(a)} & -\lambda_{(b)} & \lambda_{(c)} & -\lambda_{(d)} \\ \lambda^2_{(a)} & -\lambda^2_{(b)} & \lambda^2_{(c)} & -\lambda^2_{(d)} \\ \lambda^3_{(a)} & -\lambda^3_{(b)} & \lambda^3_{(c)} & -\lambda^3_{(d)} \end{pmatrix}.$$

Its determinant, a Vandermonde-type determinant, is equal to the product

$$(\lambda_{(a)}-\lambda_{(b)})(\lambda_{(a)}-\lambda_{(c)})(\lambda_{(a)}-\lambda_{(d)})(\lambda_{(b)}-\lambda_{(c)})(\lambda_{(b)}-\lambda_{(d)})(\lambda_{(c)}-\lambda_{(d)}),$$

and hence is nonzero. It follows that for each set of four distinct indices a,b,c,d, we have

$$\sum_{abc}(\lambda_{(a)}-\lambda_{(b)})(\bar{A}_{daa}\tau^{a}_{cb}+\bar{A}_{dbb}\tau^{b}_{ca})=0. \tag{62}$$

This in turn implies that all further conditions in the hierarchy (44) are automatically satisfied.□

With this result, we are getting as close as we possibly can to concluding that we have sufficiency in this step of the differential ideal process. Indeed, we have now obtained with (62) all the properties that the 2-forms $\Sigma_{a\neq d}\bar{A}_{daa}\phi^{aa}+\alpha_d\phi^{dd}$ must have for belonging to Σ^2, except for those conditions of type (56) for which the cyclic sum over three indices will involve the undetermined function α_d (and this for each fixed d). These missing conditions may cause true obstructions to the existence of a solution for the inverse problem, as for each fixed d, there may, in principle, be three requirements to be satisfied, for only one unknown α_d. But there is no chance of getting more information about such possible obstructions at this level of generality, i.e., without breaking the discussion up into more subcases, because the 2-forms that interest us in an expression like (59) always appear in a wedge product with some ϕ^{dV}, so the functions we called α_d remain completely undetermined.

Let us summarize the situation now. For a 3-form ρ to be in the ideal $\langle\Sigma^2\rangle$, it must satisfy the requirement (40) and the curvature condition (41), but, in fact, also the infinite set of conditions (44) (plus corresponding analogs) that contain the one just mentioned for $m=0$. The special case of diagonalizable Φ has shown that imposing these conditions for $m=0,1,2,3$ is probably the closest we can get to having conditions that are also sufficient. So it is worthwhile exploring what comes out of such conditions when we apply them to exact forms, in terms of possibly new algebraic restrictions on admissible 2-forms.

Applying (40) to $d\omega$ is an easy computation: as already indicated, it reproduces the requirement (36) we obtained before. The other computations are a lot more involved, so we give a brief indication about the way to proceed. Starting with (44) for $m=0$, the bracket terms of the expansion of $\Sigma^{e}_{X,Y,Z,U}\,d\omega(X^V,R(Y,Z)^V,U^H)$ include, as terms in which X is fixed,

$$-\sum_{Y,Z,U}\{\omega([X^V,R(Z,U)^V],Y^H)+\omega([R(Z,U)^V,Y^H],X^V)+\omega([Y^H,X^V],R(Z,U)^V)\}$$
$$=-\sum_{Y,Z,U}\{\omega(\mathrm{D}^V_X(R(Z,U))^V-(\mathrm{D}^V_{R(Z,U)}X)^V,Y^H)+\omega((\mathrm{D}^V_{R(Z,U)}Y)^H,X^V)-\omega((\mathrm{D}^V_XY)^H,R(Z,U)^V)\},$$

and there are similar terms in which one of the other vector fields is kept fixed each time. If, in the last term on the right, we make use of the property (19), it is easy to see that all terms that arise this way will directly cancel out the terms coming from the expansion of the first term on the right, except those in which the tensor R itself is being derived. The second and third terms on the right will all disappear if the totality of all even sum permutations is taken into account. The only terms that remain then are those involving vertical derivatives of R. But they can be seen to cancel out as well in view of the Bianchi identity,

$$\sum_{X,Y,Z} \mathrm{D}^V R(X,Y,Z) = 0. \tag{63}$$

For the horizontal counterpart (42), something entirely similar happens in view of

$$\sum_{X,Y,Z} \mathrm{D}^H R(X,Y,Z) = 0. \tag{64}$$

We can now more or less see what will happen for the conditions (44) with $m \neq 0$. When, in the terms that have been made explicit above, X is replaced by ΦX, for example, most of the cancellations remain the same, but there will be extra terms in which derivatives of Φ appear; moreover, derivatives of R will now be taken with respect to arguments such as ΦX instead of X and that makes that the Bianchi identity does not directly help. In conclusion, we get the following new requirements:

$$\sum_{X,Y,Z,U}^{e} \omega(\mathrm{D}^V_{\Phi^m X} R(Z,U)^V - (\mathrm{D}^V_{R(Z,U)}\Phi^m)(X)^V, Y^H) = 0, \tag{65}$$

which in the context of the last lemma would be imposed only for $m=1,2,3$, plus analogous conditions with horizontal and vertical lifts or derivatives interchanged.

We have now obtained extra requirements more closely defining the module Σ^2, namely not only the ones just mentioned, but also those that came out of the analysis of the preceding section: (34), (36), and (38), (39). However, continuing further at this level of generality is not profitable.

VI. BEYOND THE DIFFERENTIAL IDEAL: EXAMPLES

In all practical situations, the algebraic conditions discussed so far will establish either that no nondegenerate multiplier exists, or that we have reached the stage of a differential ideal. In the latter case we then construct a Kähler lift g^{K} in the ideal that is closed. In principle, this means addressing the Pfaffian system of Eqs. (7), which is, of course, a particular way of representing a system of partial differential equations for the unknown functions r_k. While it is certainly true that the Cartan-Kähler theory is a powerful vehicle to decide about the integrability of such equations and the generality of their solutions, a drawback of (7) is that it is setup in such a general way as to lose contact with the specific geometrical structure of the inverse problem. For example, the symmetry relating to horizontal and vertical parts suggest splitting (7) into its horizontal and vertical components. But that inevitably must be equivalent to considering the partial differential equations of the inverse problem the way they were encoded geometrically in their most compact form in (2).

In other words, what we are advocating here is that, once we have reached a differential ideal, we go back to the partial differential equations in the original Helmholtz conditions, for example, in the representation

$$D^V_X g(Y,Z) = \mathrm{D}^V_Y g(X,Z), \quad \nabla g = 0, \tag{66}$$

and specifically in that order. Indeed, by splitting the equations for the r_k in that way, we expect that they will become quite tractable: the first set of equations will determine the allowed velocity dependence of the unknown r_k, and $\nabla g=0$ will subsequently further restrict the arbitrary functions that may turn up in solving the first part. It may look rather disappointing that, after all the efforts of the differential ideal process, we now still have to address two of the three Helmholtz conditions (2). But we knew from the beginning that the differential ideal process was not going to solve these equations. The point is that, specifically by the way we have pursued obtaining "efficient" algebraic conditions in that process, the algebraic freedom in the module of admissible 2-forms will likely be restricted in such a way that addressing the equations (66) directly will now become possible.

We will finish with two illustrations: one in which the differential ideal process leads to a negative result and one where we reach the final stage and subsequently solve the partial differential equations for the r_k. For these examples, we consider cases where Φ is diagonalizable with distinct eigenvalues and go back to the situation described at the end of Sec. IV. So, if the hope is that we will reach a decisive state at that point, it means that we are in the situation described by Proposition 4.2, so that Σ^1 generates a differential ideal, or that the conditions in that statement lead to the conclusion that no nondegenerate multiplier can exist.

The following is an important preliminary observation. Conditions that involve $\nabla\Phi$, ∇R, or other covariant derivatives of these tensors are likely to produce restrictions that contain the functions $\nabla\lambda_{(a)}$ and even $\nabla\tau^a_{bc}$ (or other derivatives of structure functions) and will prompt for further assumptions about $\nabla\Phi$, for example. The curvature condition (19) is more interesting to look at first, therefore, because it is purely algebraic in the structure functions τ^a_{bc}, as we have seen with (56). It turns out that also (38) is of such a nature in the case of diagonalizable Φ we are considering. Indeed, we have

$$(D^V_{\Phi X_a}\Phi)(X_b) = \lambda_{(a)}{X_a}^V(\lambda_{(b)})X_b + \lambda_{(a)}(\lambda_{(b)} - \lambda_{(c)})\tau^c_{ab}X_c,$$

from which it follows that for a 2-form of type $\omega=\bar{h}_{aa}\phi^{aa}$,

$$\omega((D^V_{\Phi X_a}\Phi(X_b) - D^V_{\Phi X_b}\Phi(X_a))^V, {X_c}^H) = \bar{h}_{bb}\lambda_{(a)}{X_a}^V(\lambda_{(b)})\delta_{bc} - \bar{h}_{aa}\lambda_{(b)}{X_b}^V(\lambda_{(a)})\delta_{ac} + \bar{h}_{cc}(\lambda_{(a)}(\lambda_{(b)} - \lambda_{(c)})\tau^c_{ab} - \lambda_{(b)}(\lambda_{(a)} - \lambda_{(c)})\tau^c_{ba}).$$

But for taking the cyclic sum (38), we need to take a,b,c distinct, so that derivatives of the eigenvalues will disappear. The condition in the end reduces to

$$\sum_{abc}\lambda_{(c)}(\lambda_{(a)} - \lambda_{(b)})(\bar{h}_{aa}\tau^a_{cb} + \bar{h}_{bb}\tau^b_{ca}) = 0, \tag{67}$$

which has a remarkable resemblance to (56). In fact, it is interesting to work out some conclusions from the combination of the conditions (56) and (67). To fix the idea, take $\{a,b,c\}=\{1,2,3\}$ and put, for example, $H_{12}=(\lambda_{(1)}-\lambda_{(2)})G_{12}$, with $G_{12}=(\bar{h}_{11}\tau^1_{32}+\bar{h}_{22}\tau^2_{31})$. Then, it is easy to see that the combination of both conditions is equivalent to requiring that $G_{12}=G_{23}=G_{31}$. So, for example, in dimension 3, we get the following two conditions of curvature type:

$$\bar{h}_{11}\tau^1_{32} + \bar{h}_{22}(\tau^2_{31} - \tau^2_{13}) - \bar{h}_{33}\tau^3_{12} = 0, \tag{68}$$

$$\bar{h}_{11}\tau^1_{23} - \bar{h}_{22}\tau^2_{13} + \bar{h}_{33}(\tau^3_{21} - \tau^3_{12}) = 0, \tag{69}$$

which actually only involve τ^a_{bc} with a,b,c distinct. In dimension 4, there will already be eight of such conditions, coming from the four combinations of three distinct indices: if these are not identically satisfied, chances are small that there will still be a nondegenerate ω.

Our first example is taken from Ref. 7 and is shown there to have no Lagrangian. We wish to confirm that we come to the same conclusion. Consider the four-dimensional system (b constant),

$$\ddot{x} = b\dot{x}\dot{w},$$

$$\ddot{y} = \dot{y}\dot{w},$$

$$\ddot{z} = (1-b)\dot{x}\dot{y} + by\dot{x}\dot{w} - bx\dot{y}\dot{w} + (b+1)\dot{z}\dot{w},$$

$$\ddot{w}=0.$$

With u,v,s,t as the notation for the derivatives, we have

$$4\Phi=\begin{pmatrix} -b^2t^2 & 0 & 0 & b^2ut \\ 0 & -t^2 & 0 & vt \\ a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$a_1=(2b+1)(b+1)tv-(2b+1)byt^2,$$

$$a_2=-(b+1)(b+2)ut+b(b+2)xt^2,$$

$$a_3=-(b+1)^2t^2,$$

$$a_4=b(2b+1)yut+(1-b^2)uv-b(b+2)xvt+(b+1)^2st.$$

The eigenvalues and corresponding eigenvectors are

$$-b^2t^2 \quad \text{and} \quad (t,0,(b+1)v-byt,0),$$

$$t^2 \quad \text{and} \quad (0,bt,bxt-(b+1)u,0),$$

$$-(b+1)^2t^2 \quad \text{and} \quad (0,0,1,0),$$

$$0 \quad \text{and} \quad (u,v,s,t).$$

It is easy to compute structure functions, as defined by (51); of the ones with distinct indices, only the following two appear to be nonzero:

$$\tau^3_{21}=(b+1)bt, \quad \tau^3_{12}=-(b+1)t.$$

For having distinct eigenvalues, we have to require that $b\neq 0$ and $b\neq -1$. The eight conditions coming from (56) and (67), applied to the multiplier g_{ab} we are now constructing, then reduce to 2, and they both require that $g_{33}=0$, so that there is indeed no nonsingular multiplier. Of course, it is a fact that, as shown in Ref. 7, the curvature condition by itself already leads to this conclusion, so that there is no real contribution from the extra condition (67).

To get beyond the differential ideal step we choose, as our second example, a three-dimensional system inspired by one of the favorable two-dimensional cases in Ref. 6. Consider the system

$$\ddot{x}=-x,$$

$$\ddot{y}=y^{-1}(1+\dot{y}^2+\dot{z}^2),$$

$$\ddot{z}=0,$$

on an appropriate domain. Denoting the derivatives by u,v,w, Φ this time is given by

$$\Phi = \frac{1}{y^2}\begin{pmatrix} y^2 & 0 & 0 \\ 0 & 2(1+w^2) & -2vw \\ 0 & 0 & 0 \end{pmatrix}.$$

Eigenvalues and corresponding eigenvectors are

$$0 \quad \text{and} \quad (0,vw,1+w^2),$$

$$1 \quad \text{and} \quad (1,0,0),$$

$$2y^{-2}(1+w^2) \quad \text{and} \quad (0,y,0).$$

The eigenvectors X_a are chosen in such a way that $\boldsymbol{\nabla} X_a=0$, which is possible because $\boldsymbol{\nabla}\Phi$ commutes with Φ in this case. The corresponding dual basis of eigenforms is given by

$$\phi^1 = \frac{1}{1+w^2}\theta^3, \quad \phi^2 = \theta^1, \quad \phi^3 = \frac{1}{y}\theta^2 - \frac{vw}{y(1+w^2)}\theta^3.$$

Again, the structure functions τ^a_{bc} are easy to compute and they are zero, except the following:

$$\tau^3_{11} = \frac{v}{y}, \quad \tau^1_{11} = 2w, \quad \tau^3_{31} = w.$$

It is immediately clear that this implies that the conditions (68) and (69) are identically satisfied because all the τ's involved are zero. Going back to the result expressed in Proposition 4.2, we have just dealt with (19) and (38). Let us introduce also horizontal structure functions by $D^H_{X_a}X_b = \nu^c_{ab}X_c$, and observe now that in view of the similarity in structure between (35) and (37), and also between (38) and (39), the explicit form of the conditions (36) and (39), for example for $n=3$, will be the same as (68) and (69), with ν^c_{ab} replacing τ^c_{ab}. But in view of the commutator identity $[\boldsymbol{\nabla},D^V_X]=D^V_{\boldsymbol{\nabla}X}-D^H_X$, and the fact that we chose the X_a to be $\boldsymbol{\nabla}$ invariant, we simply have $\nu^a_{bc}=-\Gamma(\tau^a_{bc})$ here and thus these two other conditions will be identically satisfied as well. It remains to look at (34), but this will hold trivially because $\boldsymbol{\nabla}\Phi$ commutes with Φ. We conclude from Proposition 4.2 that Σ^1 generates a differential ideal.

The admissible g's are of the form $g=\Sigma_k r_k \phi^k \otimes \phi^k$ and following the scheme explained at the beginning of this section, we now start looking at the equations to be satisfied by the r_k. For the vertical closure conditions in (66), it is convenient to re-express g in the standard basis of θ^i, from which we learn that

$$g_{11} = r_2, \quad g_{22} = \frac{r_3}{y^2}, \quad g_{33} = \frac{1}{(1+w^2)^2}\left(r_1 + r_3\frac{v^2w^2}{y^2}\right),$$

$$g_{12} = g_{13} = 0, \quad g_{23} = -r_3\frac{vw}{y^2(1+w^2)}.$$

The vertical closure conditions then are

$$\frac{\partial g_{11}}{\partial v} = \frac{\partial g_{11}}{\partial w} = 0, \quad \frac{\partial g_{22}}{\partial u} = \frac{\partial g_{33}}{\partial u} = 0,$$

$$\frac{\partial g_{22}}{\partial w} = \frac{\partial g_{23}}{\partial v}, \quad \frac{\partial g_{33}}{\partial v} = \frac{\partial g_{23}}{\partial w},$$

or translated into equations for the r_k,

$$\frac{\partial r_2}{\partial v} = \frac{\partial r_2}{\partial w} = 0, \quad \frac{\partial r_3}{\partial u} = \frac{\partial r_1}{\partial u} = 0,$$

$$(1+w^2)\frac{\partial r_3}{\partial w} + vw\frac{\partial r_3}{\partial v} = - r_3 w,$$

and

$$y^2\frac{\partial r_1}{\partial v} = - v^2w^2\frac{\partial r_3}{\partial v} - vw(1+w^2)\frac{\partial r_3}{\partial w} - r_3 v(1+w^2).$$

The solution of these equations is quite straightforward. We have that r_2 can depend on u only and, of course, also arbitrarily on x,y,z,t for the moment. r_3, on the other hand, cannot depend on u and using the method of characteristics on the other equation that involves r_3 only gives

$$r_3 = \frac{1}{v}\chi(\xi), \quad \text{with } \xi = \frac{v}{\sqrt{1+w^2}},$$

where χ is an arbitrary function of the indicated argument, which again can further depend on x,y,z,t. The last equation and the fact that r_1 cannot depend on u either, then produces

$$r_1 = - \frac{\sqrt{1+w^2}}{y^2}\psi(\xi) + \zeta(w),$$

where $\psi' = \chi$ and, like ψ, the arbitrary ζ can further depend on x,y,z,t.

It remains to impose that ∇g must be zero. Since $\nabla \phi^a = 0$ by construction, this simply means that the r_k must be first integrals. The conclusion for r_2 is immediately that it cannot depend on y and z and simply must be a first integral of the equation $\ddot{x} = -x$. Equating $\Gamma(r_3) = 0$, we see that χ can no longer depend on x and further must satisfy

$$\left(\xi + \frac{1}{\xi}\right)\frac{\partial \chi}{\partial \xi} + y\frac{\partial \chi}{\partial y} + \frac{y}{v}\left(w\frac{\partial \chi}{\partial z} + \frac{\partial \chi}{\partial t}\right) - \left(1 + \frac{1}{\xi^2}\right)\chi = 0.$$

Now every function that depends on v and w through the variable ξ only, gives zero when acted upon by the vector field $(1+w^2)\partial/\partial w + vw\,\partial/\partial v$. Applying this operator to the left-hand side of the above equation, it follows that we must have

$$\frac{\partial \chi}{\partial z} - w\frac{\partial \chi}{\partial t} = 0,$$

and repeating the same process subsequently implies that χ cannot depend on z and t at all. The reduced equation for χ then can easily be solved by the method of characteristics again and yields

$$\chi(\xi,y) = \xi\chi_0\left(\frac{\sqrt{1+\xi^2}}{y}\right),$$

where χ_0 is an arbitrary function of the indicated single argument. With this further restriction, one can verify that the first term in the expression for r_1 is a first integral, and if the same must hold for ζ, this function cannot depend on x and y and simply has to be a first integral of the equation $\ddot{z} = 0$. In summary, the general solution for the r_k is given by

$$r_1 = - \frac{\sqrt{1+w^2}}{y^2}\psi(\xi,y) + \zeta(w,z,t),$$

$$r_2 = \sigma(u,x,t),$$

$$r_3 = \frac{1}{v}\chi(\xi,y) = \frac{\xi}{v}\chi_0\left(\frac{\sqrt{1+\xi^2}}{y}\right), \quad \xi = \frac{v}{\sqrt{1+w^2}},$$

where χ_0 is further arbitrary, $\partial\psi/\partial\xi=\chi$, ζ is a first integral of the equation $\ddot{z}=0$, and σ a first integral of the equation $\ddot{x}=-x$.

In a forthcoming paper, we plan to apply these techniques more systematically to the identification of a number of classes of three dimensional (and possibly higher dimensional) systems for which a multiplier exists. The last example here, for example, belongs to a class that is characterized by the fact that two of the eigenform codistributions of a diagonalizable Φ are integrable, and the third one is not, and this is one of the cases we shall be able to treat in all generality, even in any dimension.

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