A class of recursion operators on a tangent bundle

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Abstract. We generalize the construction of a class of type $(1,1)$ tensor fields R on a tangent bundle which was introduced in a preceding paper. The generalization comes from the fact that, apart from a given Lagrangian, the further data consist of a type $(1,1)$ tensor J along the tangent bundle projection $\tau : TQ \to Q$, rather than a tensor on Q. The main features under investigation are two kinds of recursion properties of R, namely its potential invariance under the flow of the given dynamics and the property of having vanishing Nijenhuis torsion. The theory is applied, in particular, to the case of second-order dynamics coming from a Finsler metric.

1 Introduction

The term *recursion operator* is used in the literature in a number of different contexts, and thus can have quite different meanings. Most often, however, one will use this term when referring to a type $(1,1)$ tensor field, R say, with either (or both) of the following properties: (i) its Nijenhuis torsion $\mathcal{N}_R$ is zero, which is a necessary requirement for example when constructing a Poisson-Nijenhuis structure (see e.g. [12]); (ii) it is invariant under the flow of some given dynamics Γ, i.e. $\mathcal{L}_\Gamma R = 0$, in which case the focus can be on a tensor which maps symmetries of Γ into symmetries. In this paper we shall pay attention to both of these properties.

A major source for many of the ideas to be discussed below is [6], in which the two possible properties of a recursion operator were both found to be relevant in the context of the dynamics of kinetic energy Hamiltonians on the cotangent bundle of a (pseudo-)Riemannian manifold. More precisely, [6] was about (gauged) bi-differential calculi and the natural role they play in the study of bi-Hamiltonian structures. For specific applications, R was taken to be $\widetilde{J}$, the complete lift to the cotangent bundle T^*Q of a type $(1,1)$ tensor field J on Q. The two cases of special interest were the case in which $\widetilde{J}$ is invariant under the flow of a kinetic energy Hamiltonian system and the case in which J is a so-called special conformal Killing tensor or Benenti tensor with respect to a given metric tensor on Q, which plays an important role in the study of Hamiltonian systems of mechanical type

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which are separable in the sense of Hamilton-Jacobi theory. In both cases the Nijenhuis property $\mathcal{N}_R = 0$ came for free.

The matters discussed in [6] are of course limited in that they deal only with systems which are as one might say quadratic in velocities. There are many situations in which one would like to be able to use similar techniques, but which are not subject to that limitation: for example, separable systems in which there are first integrals quartic (say) in velocities; or systems in which the quadratic restriction is replaced by that of being homogeneous of degree two, that is, Finsler structures. The work described in the present paper is part of a programme whose overall objective is adapting [6] to cover more general dynamical systems, which might include such examples. This is far from being trivial, however – the Nijenhuis property $\mathcal{N}_R = 0$ will no longer readily come for free, for example. We shall therefore limit ourselves here, so far as [6] is concerned, to studying tensor fields R which are invariant under the given dynamics, leaving the generalization of the situation which in [6] led to so-called special conformal Killing tensors to a later contribution.

In [17], the first instalment of this programme, we reviewed [6] from a certain kind of tangent bundle perspective, with the purpose of setting the stage for the type of generalization we have in mind. Briefly, if S denotes the canonical almost tangent structure on TQ and J^c the complete lift of J to TQ, it was observed that J^cS provides a kind of alternative almost tangent structure. Thus for any given regular Lagrangian L on TQ it makes sense to consider, in addition to the corresponding symplectic form dd_SL, the 2-form $dd_{J^cS}L$; these two forms give rise in a natural way to a type $(1,1)$ tensor field R on TQ, defined by

$$i_{R(\xi)}dd_SL = i_\xi dd_{J^cS}L, \quad \forall\, \xi \in \mathcal{X}(TQ) \tag{1}$$

(we denote the module of vector fields on a manifold M by $\mathcal{X}(M)$). That this tensor on TQ has an important role to play is suggested by the fact that it is the pullback under the Legendre transform of L of the lift $\widetilde{J}$ of J to T^*Q. The special case in which L is the kinetic energy of a Riemannian metric, or more generally a Lagrangian of mechanical type, then leads to a tangent bundle version of the results in [6], which contains a number of interesting new features.

With a generalization to Finsler spaces, for example, in mind it will obviously not be sufficient simply to replace a Riemannian metric by a Finsler one, say, while keeping J to be a basic tensor field, i.e. a tensor field on Q: it will be necessary to take J to be velocity dependent also, that is, to take it to be a tensor field along the tangent bundle projection. So the main objective of this paper is to generalize the constructions in [17] to the case where J is a tensor field along the tangent bundle projection. When we consider the special case of a Lagrangian coming from a Finsler metric, it will be natural to assume that J is homogeneous of degree zero in the velocities; but we shall not make such an assumption initially. In particular, we shall be concerned with generalizing the definition of R in (1) when J is a tensor field along the tangent bundle projection. For want of a better term we shall call a tensor R defined as in (1) or its generalization an R-tensor.

It is a major component of our approach that we concentrate, certainly so far as intrinsic definitions and coordinate-free calculations are concerned, on the tangent bundle rather than the cotangent bundle. One of the main reasons for this is that the calculus of forms along the tangent bundle projection has been fully developed (and proven to be successful in a number of applications), which is much less the case for a calculus along the cotangent bundle projection. But we have seen in [17] that coordinate calculations tend to be easier on the cotangent bundle side, so we shall try to use the best of both worlds in what follows.

The scheme of the rest of the paper is as follows. As was already mentioned in the final section of [17], the first question to address is whether there are natural generalizations of the complete lift constructions. We therefore begin Section 2 with a discussion of generalizations of the complete lifts J^c and $\widetilde{J}$ to the case in which J is a tensor along the tangent bundle or cotangent bundle projection. The constructions on the tangent bundle side relate to a given dynamics Γ. We show how to generalize the definition (1) of a tensor R associated to J and a given Lagrangian L and investigate its structure and immediate properties. Section 3 is about the conditions for such R to be invariant under the flow of Γ and recalls a number of applications in which such recursion tensors play a distinctive role. The conditions for R to have vanishing Nijenhuis torsion are studied in Section 4. The theory is applied to the particular case of a Finsler Lagrangian in Section 5, and is illustrated on some simple systems in Section 6.

Before closing this introduction, we briefly recall the basics of what one might call SODE-calculus. SODE is an abbreviation for second-order ordinary differential equation. We shall be dealing with systems of second-order ordinary differential equations which can be represented by vector fields Γ on TQ given in terms of base coordinates q^i and corresponding fibre coordinates (velocities) u^i by

$$\Gamma = u^i \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial u^i}$$

for some functions $f^i = f^i(q^j, u^j)$; when we refer below to a dynamical system on a tangent bundle we shall always mean a system of second-order ordinary differential equations of this type, or more often the SODE vector field representing them.

Each SODE defines on TQ a horizontal distribution, or non-linear Ehresmann connection, with connection coefficients $\Gamma^i_j = -\frac{1}{2}\partial f^i/\partial u^j$. We shall denote by $\mathcal{X}(\tau)$ the $C^\infty(TQ)$-module of vector fields along the tangent bundle projection $\tau : TQ \to Q$, that is, sections of the pullback bundle $\tau^*TQ \to TQ$. Each $X \in \mathcal{X}(\tau)$ determines two vector fields on TQ, its horizontal lift X^H where

$$X^H = X^i \left(\frac{\partial}{\partial q^i} - \Gamma^j_i \frac{\partial}{\partial u^j} \right) = X^i H_i,$$

and its vertical lift X^V given by

$$X^V = X^i \frac{\partial}{\partial u^i} = X^i V_i.$$

We can also define horizontal and vertical lifts of a type $(1,1)$ tensor field J along τ by

$$J^H(X^V) = J(X)^V, \qquad J^H(X^H) = J(X)^H, \tag{2}$$

$$J^V(X^V) = 0, \qquad J^V(X^H) = J(X)^V. \tag{3}$$

The curvature $\mathcal{R}$ of the non-linear connection is the vector valued 2-form along τ given by

$$\mathcal{R} = \tfrac{1}{2}\mathcal{R}^i_{jk}\, dq^j \wedge dq^k \otimes \frac{\partial}{\partial q^i}, \qquad \mathcal{R}^i_{jk} := H_k(\Gamma^i_j) - H_j(\Gamma^i_k). \tag{4}$$

Corresponding to the non-linear connection there is a linearized connection, said to be of Berwald type, which can best be interpreted (see e.g. [2]) as a connection on $\tau^*TQ \to TQ$. The main operators associated to this linear connection are a vertical and horizontal covariant derivative, acting on tensor fields along τ, which are determined, for each $X \in \mathcal{X}(\tau)$, by $\mathrm{D}^H_X F = X^H(F)$, $\mathrm{D}^V_X F = X^V(F)$ for their action on functions $F \in C^\infty(TQ)$, by

$$\mathrm{D}^H_X \frac{\partial}{\partial q^i} = X^j \Gamma^k_{ji} \frac{\partial}{\partial q^k}, \qquad \mathrm{D}^V_X \frac{\partial}{\partial q^i} = 0, \qquad \text{where } \Gamma^k_{ji} = \frac{\partial \Gamma^k_j}{\partial u^i}$$

for the action on $\mathcal{X}(\tau)$, and by duality rules for the action on 1-forms along τ. For a full account of the resulting calculus one can consult [14, 15]. For our present needs, however, a number of key relations will generally be sufficient, as was the case for example in the application [16] and in [17]. Most frequently used are bracket relations for vertical and horizontal lifts of vector fields along τ, which read:

$$[X^V, Y^V] = \left(\mathrm{D}^V_X Y - \mathrm{D}^V_Y X\right)^V, \tag{5}$$

$$[X^H, Y^V] = (\mathrm{D}^H_X Y)^V - (\mathrm{D}^V_Y X)^H, \tag{6}$$

$$[X^H, Y^H] = \left(\mathrm{D}^H_X Y - \mathrm{D}^H_Y X\right)^H + \mathcal{R}(X,Y)^V. \tag{7}$$

It will be convenient to set

$$\mathrm{D}^V_X Y - \mathrm{D}^V_Y X = [X,Y]_V, \qquad \mathrm{D}^H_X Y - \mathrm{D}^H_Y X = [X,Y]_H.$$

It is further worthwhile observing that one can introduce a kind of classical tensor calculus notation for the horizontal covariant derivative: taking as example a 2-covariant tensor K along τ, with components K_{ij}, we can put

$$K_{ij|l} := \left(\mathrm{D}^H_{\partial/\partial q^l} K\right)_{ij} = H_l(K_{ij}) - K_{is}\Gamma^s_{lj} - K_{sj}\Gamma^s_{li}.$$

We shall occasionally use such a notation.

There is a canonical vector field along τ, the total derivative $\mathbf{T} = u^i \partial/\partial q^i$. Its importance is clear from the fact that $\mathbf{T}^V$ is the Liouville vector field on TQ, so that homogeneity properties in the fibre coordinates will be characterized intrinsically by the $\mathrm{D}^V_{\mathbf{T}}$ operator. Furthermore, $\mathbf{T}^H$ is the horizontal part of the SODE Γ (and will coincide with it in the case of a spray). Thus the following bracket relations, important for calculating Lie derivatives with respect to Γ, are in a way particular cases of the preceding ones:

$$[\Gamma, X^V] = -X^H + (\nabla X)^V, \qquad [\Gamma, X^H] = (\nabla X)^H + \Phi(X)^V. \tag{8}$$

Here Φ, a type $(1,1)$ tensor along τ, is called the *Jacobi endomorphism* and completely determines the curvature (it is equal to $i_{\mathbf{T}}\mathcal{R}$ in the case of a spray), and ∇ is the *dynamical* *covariant derivative*, which on functions acts like Γ and further satisfies $\nabla(\partial/\partial q^i) = \Gamma^j_i \partial/\partial q^j$.

One can also introduce vertical and horizontal exterior derivations on scalar and vector-valued forms. Essentially, they are determined by the following action on functions $F \in C^\infty(TQ)$ and (scalar or vector-valued) 1-forms such as J:

$$d^V F(X) := \mathrm{D}^V_X F, \qquad d^V J(X,Y) := \mathrm{D}^V_X J(Y) - \mathrm{D}^V_Y J(X), \tag{9}$$

with again similar defining relations for d^H.

More results related to the calculus along τ will be recalled when needed.

We also take the opportunity here to recall a few general facts about Lagrangian systems. The Poincaré-Cartan 2-form $\omega_L = dd_S L$ of a Lagrangian L on TQ is entirely determined by a metric tensor field g along τ, where $g = \mathrm{D}^V \mathrm{D}^V L$ is the Hessian of L. Then ω_L is the so-called Kähler lift g^K of g ; ω_L vanishes on two vertical or two horizontal vector fields, while

$$\omega_L(X^V, Y^H) = g(X,Y). \tag{10}$$

For later use, here are the specific properties of g (cf. [15]), known as the Helmholtz conditions, which are (apart from g being symmetric and non-singular) the necessary and sufficient conditions to guarantee that it is indeed the Hessian of a Lagrangian whose Euler-Lagrange equations are equivalent to the given Γ:

$$\nabla g = 0, \quad \mathrm{D}^V_X g(Y,Z) = \mathrm{D}^V_Z g(Y,X), \quad g(\Phi X, Y) = g(X, \Phi Y). \tag{11}$$

In view of the commutator property $[\nabla, \mathrm{D}^V_X] = \mathrm{D}^V_{\nabla X} - \mathrm{D}^H_X$, they further imply that also

$$\mathrm{D}^H_X g(Y,Z) = \mathrm{D}^H_Z g(Y,X). \tag{12}$$

The Poincaré-Cartan 1-form $\theta_L = d_S L$ by the way, being a semi-basic form, can be viewed as a 1-form along τ as well and can then be written as $\theta_L = d^V L$, so that $\theta_L(X^V) = 0$ and $\theta_L(X^H) = \mathrm{D}^V_X L$.

2 R-tensors

In this section we shall propose a generalization of Equation (1), $i_{R(\xi)} dd_S L = i_\xi dd_{J^c S} L$, to the case in which J is a general type $(1,1)$ tensor field along τ; that is to say, we shall define the R-tensor associated with such a tensor field (for a given Lagrangian L). The basic problem is to know what to replace J^c with on the right-hand side, and we discuss this point first. It turns out that it is not necessary to have a Lagrangian for this purpose: a dynamical vector field is enough.

When J is a type $(1,1)$ tensor field along τ and Γ is a given dynamics, there is a natural lift of J to a tensor field $\mathcal{J}_\Gamma J$ on TQ, which was extensively discussed in [14]. One of its properties is that it reduces to a complete lift J^c when J happens to be basic; so this is the natural candidate for attempting to generalize the definition of an R-tensor field.

Before proceeding to the discussion of R-tensors we examine some properties of $\mathcal{J}_\Gamma J$. One may wonder in the first place to what extent $\mathcal{J}_\Gamma J \circ S$ could again provide a kind of alternative almost tangent structure, as $J^c \circ S$ does. Now $\mathcal{J}_\Gamma J$ can be expressed explicitly as follows:

$$\mathcal{J}_\Gamma J = J^H + (\nabla J)^V;$$

since the image of $(\nabla J)^V$ is vertical, only the horizontal part plays a role when composing with S, so this is really a question about $J^H S$. Moreover, it is clear from the defining relations (2), (3), that actually $J^H S = S J^H = J^V$. Obviously $(J^V)^2 = 0$, and the image of J^V coincides with its kernel provided J is non-singular. So $\mathcal{J}_\Gamma J \circ S = J^V$ is indeed an almost tangent structure. In fact since J^V vanishes on vertical vectors, despite appearances its definition doesn't depend on a choice of horizontal distribution (unlike that of J^H).

The canonical almost tangent structure S is integrable, which is to say that its Nijenhuis torsion vanishes. The Nijenhuis torsion of J^V is not always zero, however, as the following result indicates.

Proposition 1. $\mathcal{N}_{J^V} = 0$ *if and only if* $\mathrm{D}^V_{JX} J(Y) - \mathrm{D}^V_{JY} J(X) = 0$.

Proof. It is easy to see that $\mathcal{N}_{J^V}$ gives zero when evaluated on two vertical vector fields or on a horizontal and a vertical one. We further have

$$\begin{aligned}
\mathcal{N}_{J^V}(X^H, Y^H) &= [(JX)^V, (JY)^V] - J^V\Big([(JX)^V, Y^H] + [X^H, (JY)^V]\Big) \\
&= \Big(\mathrm{D}^V_{JX}(JY) - \mathrm{D}^V_{JY}(JX)\Big)^V - J^V\Big((\mathrm{D}^V_{JX} Y)^H - (\mathrm{D}^V_{JY} X)^H\Big) \\
&= (\mathrm{D}^V_{JX} J(Y) - \mathrm{D}^V_{JY} J(X))^V,
\end{aligned}$$

from which the result follows. □

A related question is whether the derivations d_S and d_{J^V} commute, for which the condition is that the Nijenhuis bracket $[J^V, S]$ vanishes.

Proposition 2. $[J^V, S] = 0$ *if and only if* $\mathrm{D}^V_X J(Y) - \mathrm{D}^V_Y J(X) = 0$.

Proof. The proof is a simple computation, completely similar to the one above. □

It is useful at this point to introduce certain tensor fields along the projection τ related to Nijenhuis torsion, which will become important in what follows (and played a relevant role already in the study of decoupling of second-order equations [16]). For a general type $(1,1)$ tensor J along τ we put

$$N^V_J(X,Y) = \mathrm{D}^V_{JX} J(Y) - (J \mathrm{D}^V_X J)(Y), \quad \text{and} \quad \mathcal{N}^V_J(X,Y) = N^V_J(X,Y) - N^V_J(Y,X); \tag{13}$$

we define N^H_J and $\mathcal{N}^H_J$ likewise. It is a simple computation to verify that

$$\mathrm{D}^V_{JX} J(Y) - \mathrm{D}^V_{JY} J(X) = \mathcal{N}^V_J(X,Y) + J(d^V J(X,Y)),$$

so that the following corollary can be drawn from Propositions 1 and 2.

Corollary 1. *The derivations d_S and d_{J^V} constitute a bi-differential calculus if and only if $\mathcal{N}_J^V = 0$ and $d^V J = 0$.*

PROOF. We know that ${d_S}^2 = 0$ and the requirements ${d_{J^V}}^2 = 0$ and $[d_S, d_{J^V}] = 0$ are equivalent to the conditions of the two preceding propositions. The result then readily follows. □

We return to the consideration of R-tensors. We now take Γ to be a (regular) Lagrangian system. We therefore have a symplectic form $\omega_L = dd_S L$ at our disposal, and the suggestion coming from the analysis in [17] is that, using $\omega_1 = dd_{J^V} L$ as a second closed 2-form, the more interesting type $(1,1)$ tensor field R on TQ to look at is defined by

$$i_{R(\xi)} dd_S L = i_\xi dd_{J^V} L, \quad \forall\, \xi \in \mathcal{X}(TQ). \tag{14}$$

This is the definition we shall adopt; however, since for general J we don't have $\mathcal{N}_J^V = 0$, for example, we cannot expect the generalized R-tensor to have the same nice properties as the one for a basic J.

We now set out to characterize R through its action on horizontal and vertical lifts.

We pointed out in the Introduction that the Poincaré-Cartan 1-form $\theta_L = d_S L$ can be written as $\theta_L = d^V L$. Similarly, we have that $d_{J^V} L = J^V(dL) = J^H \theta_L$ is semi-basic, so that the same 1-form, regarded as a form along τ, can equally be written as $J\theta_L$.

Lemma 1. *The closed 2-form $\omega_1 = dd_{J^V} L$ is characterized by $\omega_1(X^V, Y^V) = 0$, and*

$$\begin{aligned} \omega_1(X^V, Y^H) &= \mathrm{D}^V_X (J\theta_L)(Y), \\ \omega_1(X^H, Y^H) &= d^H (J\theta_L)(X, Y). \end{aligned}$$

PROOF. We have

$$\begin{aligned} \omega_1(X^H, Y^H) &= \mathcal{L}_{X^H}\left(\theta_L((JY)^H)\right) - \mathcal{L}_{Y^H}\left(\theta_L((JX)^H)\right) - \theta_L\left(J^H([X^H, Y^H])\right) \\ &= \mathrm{D}^H_X(J\theta_L(Y)) - \mathrm{D}^H_Y(J\theta_L(X)) - \theta_L\left(J(\mathrm{D}^H_X Y - \mathrm{D}^H_Y X)\right) \\ &= \mathrm{D}^H_X(J\theta_L)(Y) - \mathrm{D}^H_Y(J\theta_L)(X), \\ \omega_1(X^V, Y^H) &= \mathcal{L}_{X^V}\left(\theta_L((JY)^H)\right) - \theta_L\left(J^H([X^V, Y^H])\right) \\ &= \mathrm{D}^V_X(J\theta_L(Y)) - \theta_L\left(J(\mathrm{D}^V_X Y)\right) \\ &= \mathrm{D}^V_X(J\theta_L)(Y), \end{aligned}$$

which gives the desired result. □

Using the generalized metric tensor g, we define the transpose of an arbitrary $(1,1)$ tensor K along τ as follows.

Definition 1. *The transpose $\overline{K}$ of K with respect to g is determined by $g(KX, Y) = g(X, \overline{K}Y)$, for all $X, Y \in \mathcal{X}(\tau)$.*

Proposition 3. *For a given type* $(1,1)$ *tensor field* J *along* τ, *let* K *and* U *be defined by*

$$g(KX,Y) = \mathrm{D}^{\mathrm{V}}_Y(J\theta_L)(X), \tag{15}$$
$$g(UX,Y) = d^{\mathrm{H}}(J\theta_L)(X,Y). \tag{16}$$

Then the type $(1,1)$ *tensor field* R *on* TQ *defined by (14) is characterized by*

$$R(X^{\mathrm{V}}) = (\overline{K}X)^{\mathrm{V}}, \tag{17}$$
$$R(X^{\mathrm{H}}) = (KX)^{\mathrm{H}} + (UX)^{\mathrm{V}}. \tag{18}$$

Proof. Observe that $\omega_L(R(X^{\mathrm{V}}), Y^{\mathrm{V}}) = 0$, while in view of the definition of K, U and $\overline{K}$, and the defining relation (14), and using the results of the above lemma, we can write

$$\begin{aligned} \omega_L(R(X^{\mathrm{V}}), Y^{\mathrm{H}}) &= g(\overline{K}X, Y), \\ \omega_L(R(X^{\mathrm{H}}), Y^{\mathrm{V}}) &= -g(KX, Y), \\ \omega_L(R(X^{\mathrm{H}}), Y^{\mathrm{H}}) &= g(UX, Y). \end{aligned}$$

The result now follows from the characterizing properties of ω_L such as (10). □

Note that it follows from the skew-symmetry of the right-hand side in (16) that $\overline{U} = -U$.

We will need properties of covariant derivatives of K and U. These will follow directly from their defining relations by making use of the following general commutator relations (see e.g. [15]), which can be seen as defining curvature components of the Berwald-type connection on the pullback bundle $\tau^*TQ \to TQ$ (see e.g. [2]). For arbitrary $X, Y \in \mathcal{X}(\tau)$,

$$[\mathrm{D}^{\mathrm{V}}_X, \mathrm{D}^{\mathrm{V}}_Y] = \mathrm{D}^{\mathrm{V}}_{[X,Y]_V}, \tag{19}$$
$$[\mathrm{D}^{\mathrm{V}}_X, \mathrm{D}^{\mathrm{H}}_Y] = \mathrm{D}^{\mathrm{H}}_{\mathrm{D}^{\mathrm{V}}_X Y} - \mathrm{D}^{\mathrm{V}}_{\mathrm{D}^{\mathrm{H}}_Y X} + \mu_{B(X,Y)}, \tag{20}$$
$$[\mathrm{D}^{\mathrm{H}}_X, \mathrm{D}^{\mathrm{H}}_Y] = \mathrm{D}^{\mathrm{H}}_{[X,Y]_H} + \mathrm{D}^{\mathrm{V}}_{\mathcal{R}(X,Y)} + \mu_{\mathrm{Rie}(X,Y)}. \tag{21}$$

Here B and Rie are type $(1,3)$ tensor fields along τ or, as they appear here, covariant 2-tensors taking values in the module of $(1,1)$-tensors. For a general $(1,1)$ tensor T, μ_T is a derivation of the tensor algebra along τ of degree zero, whose action on functions is zero, while $\mu_T(Z) = TZ$ on vector fields Z and $\mu_T(\alpha) = -T\alpha$ on 1-forms α. To specify now the curvature tensors under consideration, we have for the so-called mixed curvature tensor B that $B(X,Y)Z$ is symmetric in all three arguments and has components $B^i_{jkl} = \Gamma^i_{jkl} = V_k V_l(\Gamma^i_j)$; the tensor Rie on the other hand (which is the Riemann curvature tensor in Riemannian geometry) is defined in general by

$$\mathrm{Rie}(X,Y)Z = -\mathrm{D}^{\mathrm{V}}_Z \mathcal{R}(X,Y). \tag{22}$$

Proposition 4. *We have, for arbitrary* $X, Y, Z \in \mathcal{X}(\tau)$,

$$g(\mathrm{D}^{\mathrm{V}}_Z K(X), Y) - g(\mathrm{D}^{\mathrm{V}}_Y K(X), Z) = 0, \tag{23}$$
$$\mathrm{D}^{\mathrm{V}}_Z g(UX,Y) + g(\mathrm{D}^{\mathrm{V}}_Z U(X), Y) = \mathrm{D}^{\mathrm{H}}_Z g(KY,X) - \mathrm{D}^{\mathrm{H}}_Z g(KX,Y) + g(d^{\mathrm{H}}K(X,Y), Z), \tag{24}$$
$$\sum_{X,Y,Z} \Big(\mathrm{D}^{\mathrm{H}}_X g(UY,Z) + g(\mathrm{D}^{\mathrm{H}}_X U(Y), Z)\Big) = \sum_{X,Y,Z} g(KZ, \mathcal{R}(X,Y)), \tag{25}$$

where $\sum_{X,Y,Z}$ *refers to a cyclic sum over the indicated arguments. Furthermore,*

$$\overline{K} = K \quad \Leftrightarrow \quad d^V(J\theta_L) = 0. \tag{26}$$

Proof. The first property follows immediately from taking a vertical derivative of the defining relation (15) and making use of the 'vertical Helmholtz property' in (11) and the commutator identity (19). For the next two properties, the computations start similarly from the defining relation (16) of U. Taking a D^V derivative, one has to use the commutator relation (20) in the right-hand side: the terms involving the tensor B cancel out in view of its full symmetry and (24) readily follows. For the D^H derivative of (16), the computation is somewhat more involved: one has to apply the commutator (21) a second time after exploiting the skew-symmetry of U, in such a way that a cyclic sum combination appears. On doing so the terms involving Rie cancel out in view of the Bianchi identity $\sum \mathrm{Rie}(X,Y)Z = 0$ and (25) follows. Finally, the characterization of symmetry of K follows directly from the defining relation. □

There are a couple of further consequences which are worth mentioning: one will tell us what the obstruction is for K to be symmetric with respect to $\mathrm{D}^H_X g$; the other shows under what circumstances a property like (23) also holds for the horizontal derivatives of K.

Corollary 2. *For all* $X, Y, Z \in \mathcal{X}(\tau)$, *we have*

$$\begin{aligned} &\mathrm{D}^H_X g(KY,Z) - \mathrm{D}^H_X g(KZ,Y) = \\ &\quad g(\mathrm{D}^V_Z U(X),Y) - g(\mathrm{D}^V_Y U(X),Z) + g(d^H K(X,Z),Y) - g(d^H K(X,Y),Z), \quad (27)\\ &g(\mathrm{D}^H_Z K(X),Y) - g(\mathrm{D}^H_Y K(X),Z) = g(\mathrm{D}^V_Z \nabla K(X),Y) - g(\mathrm{D}^V_Y \nabla K(X),Z). \quad (28)\end{aligned}$$

Proof. If we take (24), interchange Y and Z and subtract, the vertical derivatives of g cancel out in view of their Helmholtz property and (27) follows. On the other hand, acting with ∇ on (24), using $\nabla g = 0$ and the by now familiar commutator of ∇ and D^V easily gives (28). □

In coordinates, using the local basis $\{H_i, V_i\}$ of vector fields and its dual $\{dq^i,\ \eta^i = du^i + \Gamma^i_k dq^k\}$, we have

$$R = K^i_j\, H_i \otimes dq^j + \overline{K}^i_j\, V_i \otimes \eta^j + U^i_j\, V_i \otimes dq^j, \tag{29}$$

where, denoting $V_i(L)$ for shorthand by p_i,

$$K^i_j = g^{ik} V_k(J^l_j p_l), \tag{30}$$

$$U^i_j = g^{ik}[H_j(J^l_k p_l) - H_k(J^l_j p_l)]. \tag{31}$$

It is evident that K and U do not determine J uniquely, or in other words that different Js may give the same R. We shall have occasion to take advantage of this freedom in the choice of J later in the paper.

One could choose to use the momenta p_j as coordinates on TQ rather than the velocities u^i. Then $g^{ik}V_k = \partial/\partial p_i$, and the expression for K becomes

$$K^i_j = \frac{\partial}{\partial p_i}\left(J^l_j p_l\right). \tag{32}$$

It is apparent from this equation that $K^i_j = J^i_j$ when J^i_j is independent of the fibre coordinates.

Symmetry properties with respect to g of course refer to the type $(0,2)$ rather than the type $(1,1)$ representation of the tensor under consideration; that is to say, if we put $K_{ij} = g_{il}K^l_j$, then

$$K_{ij} = \frac{\partial}{\partial u^i}(J^l_j p_l), \tag{33}$$

and the condition (26) for symmetry of K is self-evident. Equally evident then is the property

$$\frac{\partial K_{ij}}{\partial u^l} = \frac{\partial K_{lj}}{\partial u^i}, \tag{34}$$

which is a coordinate form of (23).

The role of $J^l_j p_l$ in the full expression for R has by now become prominent, and this suggests that we should seek to generalize also the notion of complete lift to the cotangent bundle T^*Q of a type $(1,1)$ tensor field J along the cotangent bundle projection $\pi : T^*Q \to Q$. Such a J can act on semi-basic 1-forms on T^*Q, regarded as 1-forms along π, and the canonical 1-form $\theta = p_i dq^i$ is one of those: then $J\theta = J^l_j p_l dq^j$.

Definition 2. *Let J be a type $(1,1)$ tensor field along $\pi : T^*Q \to Q$, then the complete lift $\widetilde{J}$ is the $(1,1)$ tensor on T^*Q defined by*

$$i_{\widetilde{J}(\xi)} d\theta = i_\xi d(J\theta), \quad \forall\, \xi \in \mathcal{X}(T^*Q). \tag{35}$$

Remark: just as with the standard lifting procedures from Q to T^*Q, one can also define the *vertical lift* of a J along π, as being the vector field

$$J^v = J^i_j p_i \frac{\partial}{\partial p_j} \in \mathcal{X}(T^*Q). \tag{36}$$

The right-hand side in the defining relation (35) can then be written also as $i_\xi \mathcal{L}_{J^v} d\theta = i_\xi d i_{J^v} d\theta$ (cf. the definition of complete lift in [4]).

The fact that on TQ the R-tensor of J is a more interesting tensor field to look at than either the generalized complete lift $\mathcal{J}_\Gamma J$ or the horizontal lift J^H is now underscored by the following generalization of Proposition 6 in [17].

Proposition 5. *Let* $\mathrm{Leg} : TQ \to T^*Q$ *denote the Legendre transform defined by the given regular Lagrangian L, then* $\mathrm{Leg}_* R = \widetilde{J}$.

PROOF. As a preliminary remark, starting from a J along τ, we are using the same notation for the corresponding J along π, which is in effect $\mathrm{Leg}_* J$ and is obtained by simply expressing the components $J^i_j(q,u)$ in terms of the cotangent bundle coordinates (q,p). Now, as observed before, the 2-form ω_1 in the right-hand side of (14), if we identify semi-basic forms with forms along τ, can be written with a slight abuse of notation as $d(J\theta_L)$, and it is clear then that its image under Leg_* is just $d(J\theta)$. The statement now immediately follows. □

From the coordinate expression (29) of R and the comment (32) about K, one can in fact immediately surmise that $\widetilde{J}$ must have the form

$$\widetilde{J} = \frac{\partial}{\partial p_i}\left(J^l_j p_l\right)\left(X_i \otimes dq^j + \frac{\partial}{\partial p_j} \otimes \pi_i\right) + \left(X_k(J^l_j p_l) - X_j(J^l_k p_l)\right)\frac{\partial}{\partial p_j} \otimes dq^k, \tag{37}$$

where $X_k = \mathrm{Leg}_* H_k$ and $\mathrm{Leg}_*\eta^j = g^{jk}\pi_k$. For completeness, one can verify that

$$X_k = \frac{\partial}{\partial q^k} - \widetilde{\Gamma}_{lk}\frac{\partial}{\partial p_l}, \quad \text{with} \quad \widetilde{\Gamma}_{lk} = g_{lj}\left(\Gamma^j_k + \frac{\partial^2 H}{\partial p_j \partial q^k}\right),$$

where H of course is the Hamiltonian corresponding to L. Correspondingly, $\pi_k = dp_k + \widetilde{\Gamma}_{kl}dq^l$. It is worthwhile observing that $\widetilde{\Gamma}_{lk} = \widetilde{\Gamma}_{kl}$. In fact, one can easily compute from the definition of the connection coefficients Γ^i_j that a tangent bundle expression for the $\widetilde{\Gamma}_{lk}$ can be written

$$\widetilde{\Gamma}_{lk} = \frac{1}{2}\left(\Gamma(g_{lk}) - \frac{\partial^2 L}{\partial u^k \partial q^l} - \frac{\partial^2 L}{\partial u^l \partial q^k}\right),$$

which is manifestly symmetric.

Before embarking on the two aspects of recursion now, let us state for later use a few more properties of R with respect to the tangent bundle structure on TQ (as encoded by the tensor S).

Proposition 6. $RS = SR \quad \Leftrightarrow \quad K = \overline{K}$.

PROOF. This is a trivial observation from the structure of R. □

Concerning the Nijenhuis bracket of R and S, one can verify easily that $[R,S](X^V, Y^V) = 0$, while

$$\begin{aligned}
[R,S](X^V, Y^H) &= -\left(\mathrm{D}^V_Y \overline{K}(X) + \mathrm{D}^V_X K(Y)\right)^V, \\
[R,S](X^H, Y^H) &= (d^V K(X,Y))^H + ((d^V U - d^H K)(X,Y))^V.
\end{aligned}$$

It looks as though it would be much too strong a condition to expect that $[R,S]$ could be zero, but the following weaker requirement will be useful further on and follows immediately from these relations.

Proposition 7. *$[R,S]$ is a vertical-vector-valued 2-form if and only if $d^V K = 0$.* □

3 Invariant R-tensors

We now turn to the issue of R being a recursion tensor in the sense of being a symmetry generator for Γ.

Theorem 1. $\mathcal{L}_\Gamma R = 0 \;\Leftrightarrow\; K = \overline{K},\;\; U = \nabla K = 0$ *and* $\Phi K = K\Phi$.

Proof. Using the characterization (17,18) of R and the bracket relations (8), it is straightforward to verify that

$$\mathcal{L}_\Gamma R(X^V) = (K-\overline{K})(X)^H + (\nabla\overline{K}+U)(X)^V, \qquad (38)$$

$$\mathcal{L}_\Gamma R(X^H) = (\nabla K - U)(X)^H + (\Phi K - \overline{K}\Phi + \nabla U)(X)^V. \qquad (39)$$

Expressing that the horizontal and vertical parts must vanish separately, the result now immediately follows. □

It is rather remarkable that the only change here with respect to the result for basic J in [17] is that J is replaced by K. Notice also that since U must be zero, invariant R-tensors are of the form $R = K^H$, where K is symmetric, is parallel with respect to the dynamical covariant derivative and commutes with the Jacobi endomorphism Φ.

It is known (see [8]) that an invariant type $(1,1)$ tensor field R on TQ, which is symmetric with respect to ω_L and commutes with S, will give rise to an alternative Lagrangian for Γ, provided that the 2-form $i_R\omega_L$ is closed. We shall see that this theory fits entirely within our present framework. To begin with, we prove an economical version of the way alternative Lagrangians arise in the context of R-tensors.

Proposition 8. *For a given regular Lagrangian L and given type $(1,1)$ tensor J along τ, consider the tensor K defined by (15). Assume K is symmetric, commutes with Φ and satisfies $\nabla K = 0$, and put $g' = K\lrcorner\, g$. Then g' satisfies the Helmholtz conditions (11) and hence, provided that K is non-singular, defines an alternative Lagrangian for Γ.*

Proof. Symmetry of K means the same as saying that g' is symmetric, while the commutativity of K and Φ then implies that also $\Phi\lrcorner\, g'$ is symmetric. $\nabla g' = 0$ trivially follows from $\nabla g = 0$ and $\nabla K = 0$. The vertical Helmholtz property of g, together with (23), finally implies that g' will have the same property. □

It may look a bit odd that there is no mention of U in the above argument. Let us put the point more clearly as follows. Starting from a tensor J along τ, the corresponding R with components K and U is uniquely determined. If K satisfies the above requirements, we have an alternative Lagrangian L', even though the R we started from need not be invariant since U need not be zero. The point is, however, that there is a different tensor then, related to the same K, which is invariant, namely $R' = K^H$. It is the tensor obtained by replacing the ω_1 we first thought of in the definition (14) by $\omega_{L'}$.

It is worth explaining that R' is also an R-tensor in more detail by the following two arguments: (i) with K as the starting point, we discuss what is needed to have that K is derived from a J in such a way that the corresponding U is zero; (ii) we show how an alternative L' gives rise to such a K.

Suppose that the tensor K is symmetric with respect to g and satisfies (23). The latter means (see e.g. (34)) that the covariant form of K comes from some 1-form β, in the sense that $K = \mathrm{D}^V\beta$. The symmetry of K further implies that $\beta = \mathrm{D}^V F$ for some function F, so that K is a Hessian. Having fixed a β, we can clearly find a tensor J, indeed many tensors J, such that $J_i^s p_s = \beta_i$, but the corresponding U does not depend on the freedom in J. The 1-form β itself is determined in the first stage to within an arbitrary 1-form β_0 on the base manifold Q. Assume next that $\nabla K = 0$. Then the property (28) says that $K_{ij|l} = K_{lj|i}$, where $K_{ij} = V_i(\beta_j) = V_j(\beta_i)$, or explicitly

$$H_l(V_j(\beta_i)) - \Gamma^s_{lj}V_s(\beta_i) = H_i(V_j(\beta_l)) - \Gamma^s_{ij}V_s(\beta_l).$$

Interchanging the horizontal and vertical derivatives, it follows that $V_j(H_l(\beta_i) - H_i(\beta_l)) = 0$. Hence $H_l(\beta_i) - H_i(\beta_l)$ are the components of a basic 2-form and thus the freedom of selecting a basic β_0 can be used to cancel them by $d^H\beta_0 = d\beta_0$, which means that the corresponding U then is zero in view of (31). In conclusion, starting from a tensor K, the property (23) ensures that K comes from some J, and if K is symmetric and ∇-parallel, it can always be arranged that the corresponding U is zero. Concerning point (ii) now, if g' is the metric tensor along τ determined by the alternative Lagrangian L' (assumed regular), and we define K by $g' = K \lrcorner\, g$, then K is symmetric and satisfies (23) and $\nabla K = 0$, as a result of the Helmholtz conditions satisfied by both g and g'. Hence, it comes from a J with $U = 0$ and K^H is an R-tensor.

Concerning the other recursion aspect now, the computation of $\mathcal{N}_R$ in all generality, i.e. without linking it to invariance properties of R, is quite tedious and will be addressed in the next section. But for the subclass of horizontal lifts of an arbitrary $(1,1)$ tensor K along τ, which is the situation we encounter here, things are a lot simpler, so we may discuss them already now. Indeed, as was mentioned in [16], we have

$$\mathcal{N}_{K^H}(X^V, Y^V) = \mathcal{N}^V_K(X,Y)^V, \tag{40}$$
$$\mathcal{N}_{K^H}(X^H, Y^V) = N^H_K(X,Y)^V - N^V_K(Y,X)^H, \tag{41}$$
$$\mathcal{N}_{K^H}(X^H, Y^H) = \mathcal{N}^H_K(X,Y)^H + \mathcal{R}_K(X,Y)^V, \tag{42}$$

where the Nijenhuis type tensors along τ were introduced in the previous section and the term related to the curvature $\mathcal{R}$ is defined by

$$\mathcal{R}_K(X,Y) = \mathcal{R}(KX,KY) - K\left(\mathcal{R}(KX,Y) + \mathcal{R}(X,KY)\right) + K^2(\mathcal{R}(X,Y)).$$

So vanishing of $\mathcal{N}_{K^H}$ reduces to three conditions (not five as one might expect), namely

$$N^V_K = 0, \quad N^H_K = 0, \quad \mathcal{R}_K = 0.$$

If K^H is actually the invariant tensor R of Theorem 1, there is a further reduction.

Proposition 9. *Under the conditions of Theorem 1, we have $\mathcal{N}_R = 0$ if and only if $N^V_K = 0$.*

Proof. It was shown in [16] that in all generality, $\nabla N^V_K = N^V_{K,\nabla K} - N^H_K$. We don't need the precise meaning of $N^V_{K,\nabla K}$ right now, because we know that $\nabla K = 0$ in this situation, and it follows that $N^{\scriptscriptstyle V}_{K,\nabla K} = 0$. Thus $N^{\scriptscriptstyle V}_K = 0$ will imply $N^{\scriptscriptstyle H}_K = 0$. Also derived in [16] is an identity which expresses $\mathcal{R}_K$ as a sum of terms, each of which involves either $N^{\scriptscriptstyle V}_K$ or $\Phi K - K\Phi$. Hence, under the present assumptions, $\mathcal{R}_K$ will automatically be zero as well. □

There is an interesting application of such tensors to the characterization of separable Lagrangian equations. Indeed, type $(1,1)$ tensors on TQ which have all the properties encountered in the preceding proposition are getting close to the kind of tensors discussed in [5, 7, 8]. Such tensors must be algebraically diagonalizable and have eigenvalues with even degeneracy (constant degeneracy is understood as being part of the meaning of diagonalizability here). The latter is obvious for our tensors R, since they are of the form $K^{\scriptscriptstyle H}$, so that single eigenvalues of K are double eigenvalues of R. Separability of the given Lagrangian system means that there exists a coordinate transformation on Q such that the system decouples into a number of lower dimensional subsystems in those coordinates. A key role in the discussion of results on separability for second-order differential equations is played by the eigenspaces of the Jacobi endomorphism Φ (see [16]). For the present context, we can state the following result.

Proposition 10. *Suppose that $\mathcal{L}_\Gamma R = 0$ and that K further has the properties $N^{\scriptscriptstyle V}_K = 0$ and $d^{\scriptscriptstyle V}K = 0$. Then, if K is diagonalizable, the given system Γ is separable.*

PROOF. We know that R is invariant, has vanishing Nijenhuis torsion and has doubly degenerate eigenvalues. Moreover, since K is symmetric R commutes with S and since $d^{\scriptscriptstyle V}K = 0$ the Nijenhuis bracket of R and S takes vertical values (see Proposition 7). These are exactly the conditions which are required for the theorem about separability in [7], or better, for the slightly corrected version of this theorem as given in [16]. □

Observe that Corollary 1 implies that under the conditions of this separability statement, the derivations d_S and $d_{K^{\scriptscriptstyle V}}$ on TQ constitute a bi-differential calculus. But we will not pursue this matter further. Instead, let us briefly review the more commonly known application of invariant tensors to the generation of first integrals. In that field also, a bi-differential calculus can play a relevant role, and it is worth trying to understand in detail what the distinctive role in this application is of invariance of R on the one hand and zero torsion on the other.

The equation $\mathcal{L}_\Gamma R = 0$, or essentially $\nabla K = 0$, is a Lax-type equation. It follows that the trace of R (and of all its powers) is a first integral of Γ. In the context of alternative Lagrangians, this geometric set-up explains what is often referred to as the Hojman-Harleston theorem [10]. For a somewhat more general geometric approach to Lax equations, see [1]. The Nijenhuis condition is not required for having first integrals, but it enters the scene when one wishes such integrals to be in involution, i.e. when the issue of complete integrability is at stake. In fact, it was shown in [5], still in the context of alternative Lagrangians but translated to our present set-up, that if $\mathcal{N}_R = 0$ and K has distinct eigenfunctions at each point then these eigenfunctions are in involution. A related issue is the bi-Hamiltonian description, which arises from a Poisson-Nijenhuis structure. There is a somewhat hidden assumption here. Indeed, in order to have a second Poisson structure, originating from the symplectic form ω_L and the tensor R, the so-called Magri-Moroso concomitant must vanish in the first place (see [13, 12]); the Nijenhuis condition then makes the two Poisson structures compatible. Now vanishing of the Magri-Moroso concomitant is equivalent to the 2-form ω_1 on the right-hand side of (14) being closed (see e.g. [6]), and that is automatically satisfied in our present set-up. Another equivalent characterization of this condition was derived in [6] and it implies that, in particular, we will have

$$i_{\mathcal{L}_\Gamma R}\omega_L = -2\, dd_R E_L. \tag{43}$$

This brings us to the subject of bi-differential calculus. Whenever $\mathcal{N}_R = 0$, the derivations d and d_R constitute a bi-differential calculus and this is a useful tool for generating functions (not even first integrals, necessarily) which are in involution, i.e. have vanishing Poisson brackets, with respect to both Poisson structures. The algorithmic process by which such functions are generated (at least locally) requires an initial function f which satisfies $dd_R f = 0$. Obviously, when R is invariant, we have such an initial function since (43) shows that $dd_R E_L = 0$, and the hierarchy of functions in involution will be first integrals.

As we indicated before, we have also other classes of R-tensors in mind for future studies, so it is certainly worthwhile to investigate the vanishing torsion condition in its own right; this will be the subject of the next section.

4 The Nijenhuis torsion of $\widetilde{J}$ and R

We shall approach the computation of the conditions for vanishing Nijenhuis torsion of R in quite a general way.

Let ω be a symplectic 2-form on an even dimensional manifold, and ω_1 any 2-form; define the $(1,1)$-tensor R as before by $i_{R(\xi)}\omega = i_\xi\omega_1$. We shall derive an expression for the Nijenhuis torsion of R in terms of ω and R, under the assumption that ω_1 is closed. The exterior derivative $d\omega_1$ can also be expressed in terms of ω and R; the two expressions have an unexpected affinity. Finally, it will be shown that the condition for the vanishing of the Nijenhuis torsion of R, when ω_1 is closed, can be written $d_R\omega_1 = 0$.

In order to derive the last result we shall need to employ Frölicher-Nijenhuis calculus, and we start by listing some relevant generalities concerning that calculus [9].

It follows from the definition of R that $\omega(R\xi,\eta) = \omega(\xi, R\eta)$, and therefore that $i_R\omega = 2\omega_1$. Observe, however, that this relation cannot be used to define R directly, because one needs to know that R is symmetric with respect to ω before the left-hand side fixes R in view of the non-degeneracy of ω. But it easily further follows now that

$$i_R i_R\omega = 2\, i_{R^2}\omega = 2\, i_R\omega_1.$$

Assume next that $d\omega_1 = 0$ (as well as $d\omega = 0$). Then obviously $di_R\omega = 0$, from which it follows that also $d_R\omega = i_R d\omega - di_R\omega = 0$, and that $d_{R^2}\omega = -di_{R^2}\omega = -di_R\omega_1 = d_R\omega_1$.

In the Frölicher-Nijenhuis classification of i_* and d_* derivations, the commutator of two d_* derivations defines the Nijenhuis bracket of arbitrary vector-valued forms L and M as follows:

$$[d_L, d_M] = d_{[L,M]},$$

and the relation with the Nijenhuis torsion of a type $(1,1)$ tensor field R is that

$$[R,R] = 2\mathcal{N}_R.$$

Finally, the general commutator relation for $[i_L, d_M]$, when applied to the special case that L and M both equal a $(1,1)$-tensor R, yields

$$[i_R, d_R] := i_R d_R - d_R i_R = -i_{[R,R]} + d_{R^2}.$$

It then follows from what precedes that

$$2\, d_R\omega_1 = d_R i_R \omega = 2\, i_{\mathcal{N}_R}\omega - d_{R^2}\omega,$$

or finally

$$2\, i_{\mathcal{N}_R}\omega = 3\, d_R\omega_1. \tag{44}$$

It is clear that $\mathcal{N}_R = 0$ will imply $d_R\omega_1 = 0$, but the fact that these conditions are actually equivalent needs a stronger result, because

$$i_{\mathcal{N}_R}\omega(\xi,\eta,\zeta) = \sum_{\xi,\eta,\zeta} \omega(\mathcal{N}_R(\xi,\eta),\zeta),$$

Thus (44) does not determine $\mathcal{N}_R$, unless we know, what we will show now, that the three terms in the cyclic sum on the right are actually equal.

Proposition 11. *If R is defined by $i_{R(\xi)}\omega = i_\xi\omega_1$, where ω is a symplectic 2-form and ω_1 any 2-form, then*

$$d\omega_1(\xi,\eta,\zeta) = \sum_{\xi,\eta,\zeta} \zeta(\omega(R\xi,\eta)) - \sum_{\xi,\eta,\zeta} \omega(R([\xi,\eta]),\zeta). \tag{45}$$

If in addition $d\omega_1 = 0$ then

$$\omega(\mathcal{N}_R(\xi,\eta),\zeta) = -\sum_{\xi,\eta,\zeta} \zeta(\omega(R\xi,R\eta)) + \sum_{\xi,\eta,\zeta} \omega(R([\xi,\eta]),R\zeta). \tag{46}$$

It follows that when $d\omega_1 = 0$, $\mathcal{N}_R = 0$ if and only if $d_R\omega_1 = 0$.

Proof. The first result follows simply from the identity $d\omega_1(\xi,\eta,\zeta) = \sum \xi(\omega_1(\eta,\zeta)) - \sum \omega_1([\xi,\eta],\zeta)$ and the defining relation for R. To obtain the second result one uses the identity $d\omega(\xi,\eta,\zeta) = \sum \xi(\omega(\eta,\zeta)) - \sum \omega([\xi,\eta],\zeta)$ to express in particular the fact that $d\omega(\xi,R\eta,R\zeta) = 0$. There are two terms on the right-hand side involving derivatives by $R(\cdot)$. Their arguments may be expressed in terms of ω_1, and the closure of ω_1 used to replace each of these terms by five others, none of which involves a derivative by $R(\cdot)$. When the resulting expression is simplified, (46) follows. In particular, (46) implies that the left-hand side $\omega(\mathcal{N}_R(\xi,\eta),\zeta)$ is invariant for cyclic permutations. The final statement now immediately follows from (44). □

The similarity between the expression for $d\omega_1(\xi,\eta,\zeta)$ and the expression for $\omega(\mathcal{N}_R(\xi,\eta),\zeta)$ when $d\omega_1 = 0$ is evident.

We now obtain explicit expressions for the conditions for the vanishing of the Nijenhuis torsions of $\widetilde{J}$ and R, starting with the former.

Now $\widetilde{J}$ is determined by a given J along π and the canonical 1-form θ only, i.e. it does not depend on a given dynamics of Lagrangian or Hamiltonian type. For this reason, there is no advantage to be gained from working in any local frame other than a natural coordinate frame. It is clear from the expression (37), or in fact from a direct interpretation of the definition (35), that in natural bundle coordinates $\widetilde{J}$ will be of the form

$$\widetilde{J} = K^i_j \left(\frac{\partial}{\partial q^i} \otimes dq^j + \frac{\partial}{\partial p_j} \otimes dp_i \right) + M_{kj} \frac{\partial}{\partial p_j} \otimes dq^k, \tag{47}$$

where

$$K^i_j = \frac{\partial}{\partial p_i}(J^s_j p_s), \qquad M_{kj} = \frac{\partial}{\partial q^k}(J^s_j p_s) - \frac{\partial}{\partial q^j}(J^s_k p_s). \tag{48}$$

The following immediate properties of the coefficients of $\widetilde{J}$ will be used below:

$$\frac{\partial K^l_k}{\partial p_j} = \frac{\partial K^j_k}{\partial p_l}, \qquad \frac{\partial M_{jk}}{\partial p_l} = \frac{\partial K^l_k}{\partial q^j} - \frac{\partial K^l_j}{\partial q^k}, \qquad \sum_{i,j,k} \frac{\partial M_{jk}}{\partial q^i} = 0, \tag{49}$$

where $\sum_{i,j,k}$ again refers to a cyclic sum over the indicated indices. In fact, these properties merely express the fact that the 2-form $d(J\theta)$ in the defining relation of $\widetilde{J}$ is closed; that is, they are the coordinate expressions of the first result of the proposition above in this case. They are also directly related to the three properties of Proposition 4 via the Legendre transform.

Theorem 2. *The Nijenhuis tensor of $\widetilde{J}$ vanishes if and only if*

$$A^{ij}_k := K^i_l \frac{\partial K^j_k}{\partial p_l} - K^j_l \frac{\partial K^i_k}{\partial p_l} = 0, \tag{50}$$

$$B^i_{kj} := K^l_k \frac{\partial K^i_j}{\partial q^l} - K^l_j \frac{\partial K^i_k}{\partial q^l} + M_{kl} \frac{\partial K^i_j}{\partial p_l} - M_{jl} \frac{\partial K^i_k}{\partial p_l} + K^i_l \frac{\partial M_{jk}}{\partial p_l} = 0, \tag{51}$$

$$\sum_{i,j,k} C_{ijk} := \sum_{i,j,k} \left(K^l_i \frac{\partial M_{jk}}{\partial q^l} + M_{il} \frac{\partial M_{jk}}{\partial p_l} \right) = 0. \tag{52}$$

Proof. This can be obtained from Proposition 11; alternatively, it can be established by a simple coordinate calculation in which attention must be paid to making appropriate use of the properties (49) for recombining the various coefficients in the right format. One obtains

$$\mathcal{N}_{\widetilde{J}}\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j} \right) = A^{ij}_k \frac{\partial}{\partial p_k},$$

$$\mathcal{N}_{\widetilde{J}}\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial q^j} \right) = A^{ik}_j \frac{\partial}{\partial q^k} + B^i_{kj} \frac{\partial}{\partial p_k},$$

$$\mathcal{N}_{\widetilde{J}}\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j} \right) = B^k_{ij} \frac{\partial}{\partial q^k} + \sum_{i,j,k} C_{ijk} \frac{\partial}{\partial p_k},$$

which implies the stated result. □

The structure of the conditions for the vanishing of $\mathcal{N}_{\widetilde{J}}$ is apparent: they are of the same form as the identities (49) satisfied by the coefficients of $\widetilde{J}$, but with the coordinate vector fields replaced by their images under $\widetilde{J}$. To be precise, they can be cast in the form

$$\begin{aligned}
&\widetilde{J}\left(\frac{\partial}{\partial p_i}\right)(K^j_k) - \widetilde{J}\left(\frac{\partial}{\partial p_j}\right)(K^i_k) = 0,\\
&\widetilde{J}\left(\frac{\partial}{\partial p_i}\right)(M_{jk}) = \widetilde{J}\left(\frac{\partial}{\partial q^j}\right)(K^i_k) - \widetilde{J}\left(\frac{\partial}{\partial q^k}\right)(K^i_j),\\
&\sum_{i,j,k}\widetilde{J}\left(\frac{\partial}{\partial q^i}\right)(M_{jk}) = 0.
\end{aligned}$$

The general observation made after Proposition 11 provides the background explanation for this feature.

We now come back to the situation on the tangent bundle, where we have the tools to approach the question in an intrinsic way. So, take ω now to be the closed 2-form $\omega_L = g^{K}$ on TQ and $\omega_1 = d(J\theta_L)$. In principle one should evaluate $\mathcal{N}_R$ on all combinations of horizontal and vertical lifts and identify each time the horizontal and vertical component of the result; but the cyclic sum invariance of $\omega(\mathcal{N}_R(\xi,\eta),\zeta)$ means that, for example, $\omega(\mathcal{N}_R(X^{H},Y^{V}),Z^{V})$ will follow from $\omega(\mathcal{N}_R(Y^{V},Z^{V}),X^{H})$; furthermore, it is easy to see from the expression in Proposition 11 that $\omega(\mathcal{N}_R(X^{V},Y^{V}),Z^{V}) = 0$. Thus in the end only three components need to be computed, which is in agreement with the coordinate results in Theorem 2.

Theorem 3. *Let R be defined by $i_{R(\xi)}\omega_L = i_\xi d(J\theta_L)$ and thus be characterized as in Proposition 3. Then, the necessary and sufficient conditions for $\mathcal{N}_R$ to vanish are:*

$$\mathrm{D}^{V}_{KX}K(Y) - K(\mathrm{D}^{V}_{X}K(Y)) = 0, \quad \textit{or equivalently} \quad \mathcal{N}^{V}_{K} = 0, \tag{53}$$

$$\mathcal{N}^{H}_{K}(X,Y) + \mathrm{D}^{V}_{UX}K(Y) - \mathrm{D}^{V}_{UY}K(X) = 0, \tag{54}$$

$$\begin{aligned}
&\sum_{X,Y,Z}\Big(g(d^{H}K(UY,Z),X) + g(d^{H}K(Y,UZ),X) + g(d^{H}K(Y,Z),UX)\Big)\\
&\quad - \sum_{X,Y,Z}\Big(g(\mathrm{D}^{V}_{Y}U(UZ),X) - g(\mathrm{D}^{V}_{Z}U(UY),X) + g(d^{H}U(Y,Z),KX)\Big)\\
&\qquad\qquad = \sum_{X,Y,Z} g(\mathcal{R}(Y,Z),K^2X).
\end{aligned} \tag{55}$$

Proof. In agreement with what was said above, we need to compute only, for example,

$$\omega_L(\mathcal{N}_R(X^{V},Y^{V}),Z^{H}), \quad \omega_L(\mathcal{N}_R(X^{H},Y^{V}),Z^{H}), \quad \omega_L(\mathcal{N}_R(X^{H},Y^{H}),Z^{H}).$$

Considering the relation (46) with $\xi = X^{V}, \eta = Y^{V}, \zeta = Z^{H}$, making use of the defining relations (17,18) of R and (10), the first sum on the right readily reduces to $-\mathrm{D}^{V}_{X}(g(Y,K^2Z)) + \mathrm{D}^{V}_{Y}(g(X,K^2Z))$. In evaluating such expressions, there is no need to take account of terms which involve derivatives of vector field arguments: we know that these will always cancel out in the end since we are computing a tensorial quantity. In fact, the terms of the second cyclic sum in (46) will exactly take care of these cancellations in this case. Terms involving derivatives of g cancel out in view of one of the Helmholtz properties (11), there remains:

$$g(X, \mathrm{D}^{V}_{Y} K^2(Z)) - g(Y, \mathrm{D}^{V}_{X} K^2(Z)) = 0.$$

One can easily eliminate g from this expression by making appropriate use of (23) after expanding the derivatives of K^2; what follows is the first of the conditions (53). It does not look very obvious that this is actually equivalent to

$$\mathcal{N}^{V}_{\overline{K}}(X,Y) := \mathrm{D}^{V}_{\overline{K}X}\overline{K}(Y) - (\overline{K}\mathrm{D}^{V}_{X}\overline{K})(Y) - \mathrm{D}^{V}_{\overline{K}Y}\overline{K}(X) + (\overline{K}\mathrm{D}^{V}_{Y}\overline{K})(X) = 0.$$

To see that, one has to lower an index by g again, use (23) to arrive at an expression like $g(Z, \mathrm{D}^{V}_{\overline{K}X} K(Y)) - g(X, \mathrm{D}^{V}_{\overline{K}Z} K(Y))$, then take the derivatives outside g to enable switching from K to $\overline{K}$, and continue making use of the vertical Helmholtz condition and property (23) until all terms are expressed in terms of $\overline{K}$. We leave the details to the reader. It is important, however, to be aware of this rather surprising equivalence in (53), because $\mathcal{N}^{V}_{\overline{K}} = 0$ is the condition one would arrive at if the line of computation which led to (40) would be generalized.

The computation of $\omega_L(\mathcal{N}_R(X^H, Y^V), Z^H)$ runs in a very similar way. Again, the second cyclic sum in the right-hand side of (46) takes care of the necessary cancellations to arrive at a tensorial expression. Elimination of derivatives of g requires making use of the horizontal Helmholtz condition (12) this time and of the property (24). The condition (54) then quite easily follows.

Consider finally $\omega_L(\mathcal{N}_R(X^H, Y^H), Z^H)$. The first cyclic sum in (46) becomes

$$\sum_{X,Y,Z} \mathrm{D}^{H}_{X}(g(UZ, KY) - g(UY, KZ)).$$

The terms involving derivatives of g can be written in the form

$$\sum_{X,Y,Z} \Big(\mathrm{D}^{H}_{Z} g(UY, KX) - \mathrm{D}^{H}_{X} g(UY, KZ)\Big) = \sum_{X,Y,Z} \Big(\mathrm{D}^{H}_{UY} g(Z, KX) - \mathrm{D}^{H}_{UY} g(X, KZ)\Big),$$

in view of the Helmholtz property, after which they can be replaced by algebraic terms through (24) (or better its consequence (27)). It is then easy to see that, together with the remaining terms of the first cyclic sum, they make up the first two lines in the expression for (55). The right-hand side in this expression directly comes from what remains to be considered in the second cyclic sum of (46). □

It is quite easy to see that (53) and (54) have the following meaning in terms of components with respect to the basis $\{H_i, V_i\}$:

$$\overline{K}^s_t V_s(K^j_k) - K^j_s V_t(K^s_k) = 0, \tag{56}$$
$$K^l_k H_l(K^i_j) - K^l_j H_l(K^i_k) + K^i_l \Big(H_j(K^l_k) - H_k(K^l_j)\Big) + U^l_k V_l(K^i_j) - U^l_j V_l(K^i_k) = 0, \tag{57}$$

and that these correspond to the first two cotangent bundle expressions we obtained in Theorem 2. A corresponding version of the third condition, derived directly from

Equation (55) with the aid of (27), can be written

$$\sum_{i,j,k}\Big(H_k(U^l_i K_{lj} - U^l_j K_{li}) - \mathcal{R}^l_{ij}K_{lm}K^m_k\Big) = 0. \tag{58}$$

Finally we remark that one can manipulate (55) further to eliminate g from it as well (i.e. to raise an index, so to speak). One will need the property (25) in this process; but this is a quite tedious exercise and results in an expression which is not very transparent.

5 Application: the Finsler case

We are now in a position to generalize interesting results of [6] and [17] from the pseudo-Riemannian to the Finsler case. So, without changing notations, it will from now on be understood that the tangent bundle TQ has its zero section removed. For our present purposes there is no need to enter into much detail of Finsler geometry; it will be sufficient that we assume that the given non-degenerate Lagrangian is homogeneous of degree two in the fibre coordinates. Since this implies that the Lagrangian is equal to its corresponding energy function (and therefore is a first integral), we shall call it E. The corresponding generalized metric $g = \mathrm{D}^V\mathrm{D}^V E$ is homogeneous of degree zero and the second order vector field Γ is a spray. In such a context, the natural thing to do is to assume then that J, the type $(1,1)$ tensor field along τ we start from, also is homogeneous of degree zero. Indeed, we then immediately recover the Riemannian situation when 'homogeneous of degree zero' is specialized to 'independent of the velocities'.

As said in the introduction, the operator which characterizes homogeneity of tensor fields along τ is $\mathrm{D}^V_{\mathbf{T}}$. For a good overview and later use, let us list a number of interesting relations and properties which (not always exclusively) apply in the Finsler case.

Lemma 2. *When the Lagrangian E is homogeneous of degree 2, we have (X and Y being arbitrary vector fields along τ)*

$$\nabla\mathbf{T} = 0, \quad \mathrm{D}^V_X\mathbf{T} = X, \quad \mathrm{D}^H_X\mathbf{T} = 0, \tag{59}$$

$$\nabla g = 0, \quad \mathrm{D}^V_{\mathbf{T}} g = 0, \quad \mathrm{D}^V_X g(\mathbf{T}, Y) = 0, \quad \mathrm{D}^H_X g(\mathbf{T}, Y) = 0, \tag{60}$$

$$\Gamma(E) = d^H E = 0, \quad \theta_E = \mathbf{T}\lrcorner\, g, \quad \nabla\theta_E = 0, \quad \mathrm{D}^V_X\theta_E = X\lrcorner\, g, \quad \mathrm{D}^H_X\theta_E = 0. \tag{61}$$

PROOF. Concerning equations (59), $\nabla\mathbf{T} = 0$ is the homogeneity property which indicates that we have a spray. The second equality in (59) is always true and the third then follows from the commutator $[\nabla, \mathrm{D}^V_X] = \mathrm{D}^V_{\nabla X} - \mathrm{D}^H_X$. For (60), $\nabla g = 0$ is one of the general Helmholtz properties (11), the second expresses that g is homogeneous of degree zero, and the other two then are a direct consequence of (11,12). Finally, $\Gamma(E) = \nabla E = 0$, since E is a first integral; $\mathrm{D}^H_X E = 0$ or equivalently $d^H E = 0$ then follows from the same commutator relation; $g(X, \mathbf{T}) = \mathrm{D}^V_X\mathrm{D}^V_{\mathbf{T}}E - \mathrm{D}^V_{\mathrm{D}^V_X\mathbf{T}}E = 2\mathrm{D}^V_X E - \mathrm{D}^V_X E = \theta_E(X)$, i.e. $\theta_E = \mathbf{T}\lrcorner\, g$, from which the remaining three equations immediately follow by taking the appropriate derivative ($\mathrm{D}^V_X\theta_E = X\lrcorner\, g$ in fact always holds). □

Note in passing that $\nabla \mathbf{T} = 0$ implies that $\nabla \equiv \mathrm{D}^{H}_{\mathbf{T}}$ and that $\Phi = i_{\mathbf{T}}\mathcal{R}$.

The next thing to analyse is the effect of assuming that the J we start from is homogeneous of degree zero, i.e. $\mathrm{D}^{V}_{\mathbf{T}} J = 0$.

Proposition 12. *If g and J are homogeneous of degree 0, then K is homogeneous of degree 0 and U is homogeneous of degree 1. Moreover, we have $K\theta_E = J\theta_E$ and the defining relation of U simplifies to*

$$g(UX, Y) = g(\mathbf{T}, d^{H}J(X, Y)) = g(\mathbf{T}, d^{H}K(X, Y)). \tag{62}$$

PROOF. From the defining relation $g(KX, Y) = \mathrm{D}^{V}_{Y}(J\theta_E)(X)$ and the fact that $\mathrm{D}^{V}_{\mathbf{T}} g = \mathrm{D}^{V}_{\mathbf{T}} J = 0$, it follows that

$$\begin{aligned} g(\mathrm{D}^{V}_{\mathbf{T}} K(X), Y) &= \mathrm{D}^{V}_{\mathbf{T}} \mathrm{D}^{V}_{Y}(J\theta_E)(X) - \mathrm{D}^{V}_{\mathrm{D}^{V}_{\mathbf{T}} Y}(J\theta_E)(X) \\ &= \mathrm{D}^{V}_{Y} \mathrm{D}^{V}_{\mathbf{T}}(J\theta_E)(X) - \mathrm{D}^{V}_{\mathrm{D}^{V}_{Y} \mathbf{T}}(J\theta_E)(X) = 0, \end{aligned}$$

since $J\theta_E$ is homogeneous of degree 1 and $\mathrm{D}^{V}_{Y}\mathbf{T} = Y$. That $\mathrm{D}^{V}_{\mathbf{T}} U = U$ follows in the same way from taking the $\mathrm{D}^{V}_{\mathbf{T}}$ derivative of (16) and using the appropriate commutation property for $\mathrm{D}^{V}_{\mathbf{T}}$ and D^{H}_{X}. But in fact, it is also obvious from the coordinate expression (31) or directly from the intrinsic definition, if we observe first that horizontal derivatives preserve the order of homogeneity (and vertical ones of course reduce the order by one). Taking $Y = \mathbf{T}$ in the defining relation of K, we immediately have that $g(KX, \mathbf{T}) = g(JX, \mathbf{T})$ or $K\theta_E = J\theta_E$. Finally, the simplification in the defining relation for U immediately follows from the fact that $\mathrm{D}^{H}_{X}\theta_E = 0$, so that $d^{H}(J\theta_E)(X, Y) = \theta_E(d^{H}J(X, Y))$. □

Commutation relations such as (20, 21) are interesting tools in obtaining further properties of interest. Consider for example the important property $\mathrm{D}^{H}_{X}\theta_E = 0$ of a Finsler system as starting point. By taking a further vertical and horizontal covariant derivative, it then easily follows from (20, 21) and $\mathrm{D}^{V}_{X}\theta_E = X \lrcorner\, g$ that

$$\begin{aligned} g(\mathbf{T}, B(X, Y)Z) &= \mathrm{D}^{H}_{X} g(Y, Z), & (63) \\ g(\mathbf{T}, \mathrm{Rie}(X, Y)Z) &= g(\mathcal{R}(X, Y), Z). & (64) \end{aligned}$$

We now come back to the two aspects of recursion under study and investigate what the homogeneity properties of the Finsler case can do to simplify the conditions for vanishing $\mathcal{L}_\Gamma R$ or $\mathcal{N}_R$.

Theorem 4. *Assume that g and J are homogeneous of degree 0. Then, if K is symmetric and $\nabla K = 0$, we have automatically that $U = 0$ and $\Phi K = K\Phi$. In other words, the necessary and sufficient conditions for having $\mathcal{L}_\Gamma R = 0$ (see Theorem 1) reduce to $K = \overline{K}$ and $\nabla K = 0$.*

PROOF. We know from Proposition 4 and the homogeneity that $K = \overline{K}$ implies $d^{V}(J\theta_E) = d^{V}(K\theta_E) = 0$. Since $[\nabla, d^{V}] = -d^{H}$, it then follows from the assumption $\nabla K = 0$ and the property $\nabla\theta_E = 0$ that also $d^{H}(K\theta_E) = d^{H}(J\theta_E) = 0$, whence $U = 0$.

Showing that Φ will commute with K can be done by a kind of integrability analysis, similar to the procedure which was followed for the Riemannian case in Appendix A of [17]. A much simpler proof, however, goes as follows. The property (33), which roughly expresses that K comes from a J, plus (34), ensure for a symmetric K that K_{ij} is a Hessian of some function, and we can actually determine such a function in the Finsler case. Indeed, from the symmetry of K and the homogeneity, we have that

$$\frac{\partial(J^l_j p_l u^j)}{\partial u^k} = J^l_k p_l + u^j \frac{\partial J^l_j p_l}{\partial u^k} = J^l_k p_l + u^j \frac{\partial J^l_k p_l}{\partial u^j} = 2J^l_k p_l,$$

so that K_{ij} is the Hessian of the function

$$k := \tfrac{1}{2} J^l_j p_l u^j = \tfrac{1}{2} K^l_j p_l u^j \quad \text{or in intrinsic terms} \quad k = \tfrac{1}{2}(K\theta_E)(\mathbf{T}). \tag{65}$$

It follows from $\nabla\mathbf{T} = 0$, $\nabla\theta_E = 0$ and $\nabla K = 0$ that k is a first integral. Moreover, the above computation expresses that $d^{\scriptscriptstyle V} k = K\theta_E$ and thus

$$0 = \nabla d^{\scriptscriptstyle V} k = d^{\scriptscriptstyle V} \nabla k - d^{\scriptscriptstyle H} k = -d^{\scriptscriptstyle H} k.$$

But in the case of a spray, as was already shown by Klein [11], $d^{\scriptscriptstyle H} k = 0$ is a necessary and sufficient for k to be a Lagrangian for the system. Hence its Hessian K will commute with Φ. □

The conditions for vanishing Nijenhuis torsion also simplify in the Finsler case.

Theorem 5. *If g and J are homogeneous of degree 0, we have $\mathcal{N}_R = 0$ if and only if the coefficients A^{ij}_k and B^i_{kj} (see (50) and (51)) vanish, or equivalently (53) and (54) hold true.*

Proof. We go back to the equivalent calculation of $\mathcal{N}_{\widetilde{J}}$ on T^*Q, knowing that by homogeneity: $J^l_j p_l = K^l_j p_l$ and $p_i \partial K^i_j / \partial p_k = 0$. Multiplying condition (51) by p_i, we thus get:

$$K^l_k \frac{\partial(K^i_j p_i)}{\partial q^l} - K^l_j \frac{\partial(K^i_k p_i)}{\partial q^l} + K^i_l p_i \frac{\partial M_{jk}}{\partial p_l} = 0.$$

Taking a further derivative with respect to q^m, it follows that

$$\begin{aligned} &\frac{\partial K^l_k}{\partial q^m}\frac{\partial(K^i_j p_i)}{\partial q^l} - \frac{\partial K^l_j}{\partial q^m}\frac{\partial(K^i_k p_i)}{\partial q^l} + \frac{\partial(K^i_l p_i)}{\partial q^m}\frac{\partial M_{jk}}{\partial p_l} \\ &K^l_k \frac{\partial^2(K^i_j p_i)}{\partial q^l \partial q^m} - K^l_j \frac{\partial^2(K^i_k p_i)}{\partial q^l \partial q^m} + K^i_l p_i \frac{\partial^2 M_{jk}}{\partial p_l \partial q^m} = 0. \end{aligned}$$

Taking now a cyclic sum over j, k, m, the last term disappears in view of (49), while the first and second line then can be recast exactly in the form of, respectively, the second and first term in (52). Hence, the third condition in Theorem 2, in the Finsler case, is automatically satisfied in view of the second. □

6 Illustrative examples and conclusions

We have introduced a class of type $(1,1)$ tensor fields R on a tangent bundle TQ which are constructed out of a given Lagrangian system and a $(1,1)$ tensor J along the projection $\tau : TQ \to Q$. One of the interesting points is that such R-tensors arise from the pullback under the Legendre transform of the complete lift $\widetilde{J}$ of a tensor along the cotangent bundle projection $\pi : T^*Q \to Q$. Our main achievement is that we have unraveled in a precise way the different requirements which have to be met for R to be invariant under the given dynamics, or to have vanishing Nijenhuis torsion, or to have both properties. By way of direct application, we have seen how such conditions reduce or simplify in the particular case of Lagrangian equations, coming from the energy function of a Finsler metric. This is a generalization of the more common kinetic energy Lagrangians associated to a Riemannian or pseudo-Riemannian metric. But we would like to emphasize here that our present general results are also relevant for the Riemannian situation. Indeed, it is quite common to look in the Riemannian case only at recursion tensors which are natural lifts of tensors on the base manifold, and the point is that this is often too restrictive: i.e. even in that situation, there can be features which require the introduction of tensors whose components depend non-linearly on the fibre coordinates of TQ or T^*Q.

In order to illustrate the practical applicability of the various conditions we identified, we choose to show how one can make constructive use of them in constructing recursion-type tensors related to some simple dynamics. Naturally, the simple classical system par excellence for testing new developments is the harmonic oscillator. So consider first the Lagrangian

$$L = \tfrac{1}{2}(u_1^2 + u_2^2) - \tfrac{1}{2}(q_1^2 + q_2^2).$$

The metric is the Euclidian one and $\Phi = -1$, so that any choice for K will commute with it. Most of the relevant conditions we have met are conditions on K rather than on J, but it is the property (34) which will ensure that K comes from some J. We wish to construct some invariant R-tensors here which will give rise to alternative Lagrangians.

Let us first make K symmetric by choosing simply $K_{12} = 0$. Then (34) further requires that K_{11} is independent of u_2 and K_{22} independent of u_1, and imposing $\nabla K = 0$ requires that they must be first integrals. We can take, for example

$$K_{11} = u_1^2 + q_1^2, \qquad K_{22} = u_2^2 + q_2^2.$$

According to Proposition 8, K^H will be an invariant tensor and will give rise to an alternative Lagrangian, which is easily found to be

$$L' = \tfrac{1}{12}(u_1^4 + u_2^4) + \tfrac{1}{2}(q_1^2u_1^2 + q_2^2u_2^2) - \tfrac{1}{4}(q_1^4 + q_2^4).$$

This is perhaps nothing very surprising, but observe that even for such a quite trivial example, we need a theory in which the tensor J as well as K are tensor fields along τ. A tensor J which gives rise to the above K in the sense of (33) is given by, for example, $J^i_i = (q_i^2 + \frac{1}{3}u_i^2)$ ($J^i_j = 0$ for $i \neq j$), and the corresponding U as defined by (16) is easily seen to be zero. Moreover, $N^V_K = 0$, so that K^H has vanishing Nijenhuis torsion as well.

Another symmetric K, which has all the properties of the preceding one, is given by

$$K_{11} = K_{22} = u_1u_2 + q_1q_2, \qquad K_{12} = K_{21} = \tfrac{1}{2}(u_1^2 + u_2^2 + q_1^2 + q_2^2).$$

So again, $R = K^H$ satisfies $\mathcal{L}_\Gamma R = 0$ and $\mathcal{N}_R = 0$, and the corresponding Lagrangian is found to be

$$L' = \tfrac{1}{2}u_1u_2\left(\tfrac{1}{3}(u_1^2 + u_2^2) + q_1^2 + q_2^2\right) + \tfrac{1}{2}q_1q_2(u_1^2 + u_2^2 - q_1^2 - q_2^2).$$

For a different example, we start from the Lagrangian $L = \frac{1}{2}(q_1^2u_1^2 + u_2^2)$, which means that

$$\Gamma = u_1\frac{\partial}{\partial q_1} + u_2\frac{\partial}{\partial q_2} - \frac{u_1^2}{q_1}\frac{\partial}{\partial u_1}.$$

The only non-zero connection coefficient is $\Gamma_1^1 = u_1/q_1$ (and $\Phi = 0$ so that no restrictions can come from the commutation requirement in some of the propositions).

Suppose that this time our priority is to construct a tensor R with vanishing torsion. Then, it may be advantageous to work with the conditions of Theorem 2 on the cotangent bundle (which can be regarded also as conditions on TQ, but expressed in the variables (q,p)), but we will further assume from the outset that K is symmetric. Recall the rather remarkable fact that for symmetric K, $\mathcal{N}_K^V = 0$ (which is the same as $A_k^{ij} = 0$ in the variables (q,p) and involves 2 requirements in dimension 2) is actually equivalent to the, in principle, stronger condition $N_K^V = 0$ (which consists of 6 requirements in dimension 2). From the symmetry of K, it follows that we must have $K_1^2 = q_1^2\,K_2^1$. Then (32), which expresses that K comes from some J, implies the existence of some function F such that

$$J_1^s p_s = \frac{\partial F}{\partial p_1} \quad \text{and} \quad q_1^2\,J_2^s p_s = \frac{\partial F}{\partial p_2},$$

and it follows that we will have

$$K_1^1 = \frac{\partial^2 F}{\partial p_1^2}, \quad K_2^1 = q_1^{-2}\frac{\partial^2 F}{\partial p_1 \partial p_2}, \quad K_1^2 = \frac{\partial^2 F}{\partial p_1 \partial p_2}, \quad K_2^2 = q_1^{-2}\frac{\partial^2 F}{\partial p_2^2}.$$

Using this information it is easy to see that the two independent conditions $A_1^{12} = A_2^{12} = 0$ express that the ratio $(K_1^1 - K_2^2)/K_1^2$ must be independent of the p_i, provided K_1^2 is not zero. So there are two cases to be considered. The case $K_1^2 = 0$ is not very interesting; if we look for an illustration in the Finsler class, for example, the homogeneity requirement leads to the conclusion that F must actually be quadratic in the p_i, say $F = \frac{1}{2}(h_1p_1^2 + q_1^2h_2p_2^2)$ with arbitrary $h_i(q)$. Suppose we then further impose $\nabla K = 0$ again. Then it readily follows that the h_i must be constants and the final conclusion is that $L' = \frac{1}{2}(aq_1^2u_1^2 + bu_2^2)$ is a two-parameter family of Lagrangians for the given system. If we take $K_1^2 \neq 0$ now, a subcase is clearly given by $K_2^2 = K_1^1$. This gives a wave-type equation for F, with general solution $F = F_1(P_1, q) + F_2(P_2, q)$, where $P_1 = p_1 - q_1p_2$, $P_2 = p_1 + q_1p_2$. Again, the homogeneity requirement for a Finsler situation makes it Riemannian and if we then proceed in the same way by imposing $\nabla K = 0$, we end up with the discovery of a somewhat less trivial two-parameter family of alternative Lagrangians, namely

$$L' = \tfrac{1}{2}\left(a(q_1u_1 - u_2)^2 + b(q_1u_1 + u_2)^2\right).$$

In both cases treated so far, the two other conditions of Theorem 2 are satisfied as well, because the requirement $\nabla K = 0$ has brought us back to the situation covered by Proposition 9.

We will proceed in the same way now for the general case, which will bring us to a rather exotic recursion tensor and a corresponding non-trivial Lagrangian. So, in general, from the condition $\mathcal{N}_K^V = 0$ we can put

$$K_2^1 = q_1^{-2} K_1^2, \qquad K_2^2 = K_1^1 - f(q) K_1^2,$$

where the last relation is actually a second-order partial differential equation for F. Imposing $\nabla K = 0$ it immediately follows that $f(q)$ must be q_1^{-1}, that K_1^1 must be a first integral, F_1 say, and that $K_1^2 = q_1 F_2$, where F_2 also is an as yet undetermined first integral. In an attempt to circumvent the difficult issue of solving the equation for F, observe that $K_1^2 = q_1 F_2$ implies that $\partial F/\partial p_2 = q_1 \int F_2 dp_1$, wherein we omit additive functions depending on only one of the p_i because these will lead to terms in the solution which were identified in the first case. If we use this in the expression for K_2^2 in terms of F, introduce the auxiliary function

$$\xi = \int \frac{\partial F_2}{\partial p_2} dp_1,$$

and now re-express that K_2^2 must be a first integral, it follows that ξ must solve the linear first-order equation

$$q_1 p_1 \frac{\partial \xi}{\partial q_1} + q_1^3 p_2 \frac{\partial \xi}{\partial q_2} + p_1^2 \frac{\partial \xi}{\partial p_1} = p_1 \xi.$$

Using the method of characteristics, the general solution of this equation is found to be

$$\xi = p_1 \eta(x_1, x_2, x_3), \quad \text{with} \quad x_1 = p_1/q_1,\ x_2 = p_2,\ x_3 = q_2 - \tfrac{1}{2}(p_2/p_1) q_1^3,$$

where η is an arbitrary function of the indicated arguments and these x_i all are first integrals. It follows that $K_2^2 = q_1^{-1}\xi = x_1 \eta$. Since F_2 must itself be a first integral (and is not allowed to depend on time) it must actually be a function of the x_i as well, and the definition of ξ implies that

$$\frac{\partial F_2}{\partial p_2} = \frac{\partial \xi}{\partial p_1} = \eta + x_1 \eta_{x_1} + \tfrac{1}{2}(x_2/x_1)\eta_{x_3} q_1^2.$$

Acting with Γ on both sides, and intertwining Γ with $\partial/\partial p_2$ in the left-hand side, it follows that $(F_2)_{x_3} = -x_2 \eta_{x_3}$, and thus $F_2 = -x_2 \eta + \zeta(x_1, x_2)$ for some arbitrary ζ. Returning with this information to the preceding equation, we get the restriction

$$\zeta_{x_2} = 2\eta + x_1 \eta_{x_1} + x_2 \eta_{x_2}.$$

Taking a derivative with respect to x_3, we get a first-order partial differential equation for η_{x_3} which is easy to solve; after integration with respect to x_3 one learns that η must be of the form

$$\eta = x_2^{-2} \phi(x_2 x_1^{-1}, x_3),$$

for some as yet arbitrary ϕ. In fact there is an extra freedom for adding a function of x_1 and x_2, but that can be absorbed into ζ. Moreover, the preceding equation now implies that ζ cannot depend on x_2 and so we omit it (as an additive function of only one of the p_i). We have now come to a stage where we know that $F_2 = -x_2\eta$ and

$$K^2_2 = x_1\eta, \quad K^2_1 = q_1 F_2, \quad K^1_2 = q_1^{-1} F_2, \quad K^1_1 = (x_1 - x_2)\eta,$$

with η as described above. To find further specifications about η we re-impose now that K must satisfy

$$\frac{\partial K^1_2}{\partial p_2} = \frac{\partial K^2_2}{\partial p_1}, \qquad \frac{\partial K^1_1}{\partial p_2} = \frac{\partial K^2_1}{\partial p_1}.$$

The first condition appears to be satisfied automatically, but the second gives an equation for ϕ, with coefficients which can be expressed in terms of x_1 and x_2, except for a factor q_1^2 in the coefficient of ϕ_{x_3}. It then follows, for example from acting with Γ on the equation, that ϕ cannot depend on x_3, in other words must be a function of $x := x_2/x_1$ only, and the condition reduces to

$$(x - x^2 - x^3)\phi' = (2 - x)\phi.$$

The solution of this equation is $\phi(x) = x^2(x^2 + x - 1)^{-1}$. We thus have found the following type $(1,1)$ tensor

$$K^1_1 = (x_1 - x_2)y^{-1}, \quad K^2_2 = x_1 y^{-1}, \quad K^1_2 = -q_1^{-1} x_2 y^{-1}, \quad K^2_1 = -q_1 x_2 y^{-1},$$

where we have put $y = x_2^2 + x_1x_2 - x_1^2$ for shorthand; rather surprisingly, K is homogeneous of degree -1 in the p_i. This K by construction satisfies all requirements for having that $R = K^H$ is Γ-invariant and has vanishing Nijenhuis torsion again. It is the Hessian of a Lagrangian which will be homogeneous of degree 1 and non-degenerate, but we don't have an explicit expression for this Lagrangian. Observe finally that one can easily check that also $d^V K = 0$. This means that we are actually in the situation of Proposition 10, so that the system is separable. This is not so surprising, of course, since the given system is given as decoupled equations. But in fact, the conclusion we reach here is not so trivial: it means that the given system will also separate in entirely different coordinates, namely coordinates in which K diagonalizes and which are guaranteed to exist by the theory in [16]. But we will not pursue this issue further.

To conclude now: there are a number of interesting applications in which type $(1,1)$ tensor fields can play a distinctive role. In the present paper, we have focused on the question of invariance of such tensors under a given Lagrangian flow, for its obvious applications to recursion procedures for symmetries, or the generation of first integrals, and even for less obvious applications such as the question of decoupling of second-order equations, as briefly documented in Section 3. Of course, whenever type $(1,1)$ tensors are part of a theory, one is bound to study the effect of vanishing Nijenhuis torsion. Not unexpectedly, as we have seen in Section 4, this is a rather more complicated issue than in the case of J living on Q, but still there are interesting simplifications occurring in the number of conditions. This is even more so in the Finslerian case, which we have explored as a particular case of the general theory, but at the same time for a direct generalization of the results we discussed for (pseudo-)Riemannian spaces in [17].

We plan to study another subclass of such R-tensors in a forthcoming contribution, with the purpose of generalizing, again from basic tensor fields to tensor fields along the projection, the constructions which led to a gauged bi-differential calculus in [6] and were related, for example, to the study of projective equivalence in [3]. The results we obtained here about Nijenhuis torsion will of course be directly applicable also to this entirely different problem.

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