Second-order differential equation fields with symmetry

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Abstract

We examine the reduction of a system of second-order ordinary differential equations which is invariant under the action of a symmetry group. We describe the reduced system, and show how the integral curves of the original system can be reconstructed from the reduced dynamics. We then specialize to invariant Lagrangian systems. We compare and contrast two approaches to reduction in this case. The first leads to the so-called Lagrange-Poincaré equations. The second involves an extension of Routh's reduction procedure to an arbitrary Lagrangian system (that is, one whose Lagrangian is not necessarily the difference of kinetic and potential energies) with a symmetry group which is not necessarily Abelian. Throughout we use a new method of analysis based on adapted frames and associated quasi-velocities.

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1 Introduction

The concept of symmetry plays an important role in a great number of applications in dynamics. Symmetry properties of dynamical systems have been studied intensively in recent years: see for example the survey in the recent monograph [6] by Marsden et al., as well as the more long-established reference [7]. Perhaps the most important aspect of symmetry is its use in reduction. When a dynamical system has a Lie group of symmetries, which is to say that considered as a vector field on some manifold it is invariant under the action of the group on the manifold, then the corresponding equations of motion can be reduced to a new set of equations with fewer unknowns. The working assumption is that the reduced system will be simpler to deal with than the original one.

The bulk of the literature on symmetry in dynamics concentrates on the Hamiltonian description of dynamical systems with symmetry, in which the theory of Poisson manifolds plays the main role. Less well-known is symmetry reduction for Lagrangian systems. It is the latter that is at the core of the present paper.

There are in fact accounts of several different Lagrangian reduction theories to be found in the literature. For example, one distinctive reduction method applies when the configuration space is itself a Lie group; it is called Euler-Poincaré reduction. The particular issue that we will be concerned with, however, is the following. In rough terminology, the invariance of the Lagrangian leads via the Noether theorem to a set of conserved quantities (the components of momentum). There are two alternative broad types of Lagrangian reduction theory, which differ in whether or not the existence of these conserved quantities is explicitly taken into account in the reduction process. The more direct approach, which effectively ignores conservation laws, is called Lagrange-Poincaré reduction (and includes Euler-Poincaré reduction as a special case). Taking account of momentum conservation leads to Routh's procedure. For more details and some comments on the history of these reduction theories, see e.g. [6] and [8].

One main purpose of our paper is to compare and contrast Lagrange-Poincaré reduction and Routh's procedure. Over the last couple of years we have been developing our own techniques for analysing symmetry and reduction of dynamical systems. These techniques, while being well adapted to the discussion of Lagrangian systems, are not restricted to them; in this respect they are different from the techniques usually found in the literature. In fact our techniques are designed to apply to any dynamical system, that is, any vector field, which is invariant under a Lie group.

The basic ideas, which we exploit throughout the paper, are most succinctly explained in the simplest context, that of a first-order dynamical system or plain and unadorned vector field. We discuss this case in the following section. The transition to Lagrangian systems, that is, to dynamical systems of Euler-Lagrange type, is made via the consideration of second-order systems. By a second-order system we mean a second-order differential equation field, that is to say, a vector field on a tangent bundle belonging to that special class whose integral curves satisfy a system of second-order differential equations. We describe the general second-order theory in Section 3. Those particular second-order systems defined by Lagrangians are discussed in Section 4, where the two approaches, Lagrange-Poincaré and Routh, are explained. We end with an example of a second-order system with symmetry, reduced by all three methods.

This paper is in effect a survey and summary of work which has been presented in greater detail in a number of other articles; we draw the reader's attention in particular to [4], [5] and [9].

Throughout the paper, symmetry groups are supposed to act as follows. We have a differentiable manifold M and a connected Lie group G which acts freely and properly to the left on M, so that M is a principal G-bundle. We denote the base by M/G, the projection by $\pi^M : M \to M/G$, and the action by $(x, g) \mapsto \psi^M_g x$. The Lie algebra of G is denoted by $\mathfrak{g}$, and for $\xi \in \mathfrak{g}$, $\tilde{\xi}$ is the fundamental vector field corresponding to ξ (the vector field whose flow is $t \mapsto \psi^M_{\exp(t\xi)}$).

2 The first-order case

Suppose we have a first-order dynamical system, represented by a vector field Z on the manifold M, which is invariant under a symmetry group G acting on M as described above. We can of course express the condition of invariance in terms of the action ψ^M; alternatively, we can note that if Z is invariant then $[\tilde{\xi}, Z] = 0$ for all $\xi \in \mathfrak{g}$, and conversely when this differential condition holds and G is connected (as we assume) then Z is invariant. We will find it convenient always to use the differential version of the condition for invariance.

The invariance of Z implies that it 'passes to the quotient'. That is to say, there is a well-defined vector field on M/G, say $\check{Z}$, which is π^M-related to Z: $\check{Z} = \pi^M_* Z$. We call $\check{Z}$ the reduced dynamical system. Two questions arise:

reduction: how to describe the reduced system $\check{Z}$ explicitly and conveniently;

reconstruction: how, given an integral curve of $\check{Z}$, to obtain an integral curve of Z in a systematic fashion.

2.1 Reduction

In order to formulate a simple description of the reduced dynamics we introduce and work with a local frame $\{E_a, X_i\}$, where the E_a, $a = 1, 2, \ldots, \dim G$, are tangent to the fibres of $\pi^M : M \to M/G$, the X_i, $i = 1, 2, \ldots, \dim(M/G)$ are transverse to the fibres, and all of the members of the basis are G-invariant.

We define the E_a as follows. Let $\{e_a\}$ be a basis for $\mathfrak{g}$, $\tilde{e}_a$ the corresponding fundamental vector fields. Then $[\tilde{e}_a, \tilde{e}_b] = -C^c_{ab}\tilde{e}_c$, where the C^c_{ab} are the structure constants of $\mathfrak{g}$ with respect to the given basis. It is clear that in general the $\tilde{e}_a$ are not invariant. We set $E_a = A^b_a\tilde{e}_b$, and enquire under what conditions on the coefficients A^b_a the E_a are invariant. We have

$$[\tilde{e}_a, E_b] = \left(\tilde{e}_a(A^c_b) - C^c_{ad}A^d_b\right)\tilde{e}_c,$$

so the E_a are invariant if and only if

$$\tilde{e}_a(A^c_b) = C^c_{ad}A^d_b.$$

The integrability conditions for these equations are satisfied by virtue of the Jacobi identity. There are therefore local solutions, for which the matrix $A = (A^b_a)$ is non-singular, and for which A is the identity matrix on some specified local section of π^M. Such a local section determines a local trivialization $M \simeq G \times M/G$ of M; identifying the fibres with G, we see that each E_a corresponds to a left-invariant vector field on G. Each $\tilde{e}_a$, on the other hand, corresponds to a right-invariant vector field on G (which explains the sign in the expression for the bracket); $A(g)$ is the matrix of $\mathrm{ad}(g)$ with respect to the basis $\{e_a\}$. In the literature , $\{\tilde{e}_a\}$ is sometimes referred to as the moving frame, $\{E_a\}$ as the body-fixed frame (see for example [1]).

To define the part of the frame transverse to the fibres, we assume that we have at our disposal a principal connection on the principal G-bundle M; we take X_i to be the horizontal lift with respect to the connection of a member of some local basis of vector fields on M/G. In particular, we may (and generally will) take this to be a coordinate basis.

We may now write $Z = Z^a E_a + Z^i X_i$. Since Z, E_a and X_i are all invariant, so also are the coefficients Z^a and Z^i. We may therefore regard the Z^i (in particular) as functions on M/G, and we have

$$\pi^M_* Z = \check{Z} = Z^i \frac{\partial}{\partial x^i},$$

where the x^i are coordinates on M/G. The reduced equations are simply

$$\dot{x}^i = Z^i(x).$$

2.2 Reconstruction

Suppose given an integral curve of $\check{Z}$, say $t \mapsto \check{z}(t)$ (a curve in M/G). We have to find an integral curve of Z, $t \mapsto z(t)$ (a curve in M), over $\check{z}$ (so that $\pi^M \circ z = \check{z}$).

We proceed as follows. Take any lift of $\check{z}$ to M, $t \mapsto \zeta(t)$ (so that $\pi^M \circ \zeta = \check{z}$). Then there is a curve $t \mapsto g(t) \in G$ such that $z(t) = \psi^M_{g(t)}\zeta(t)$. The next questions therefore are how to lift $\check{z}(t)$ to $\zeta(t)$ in a systematic fashion, and having done so, how to find $g(t)$.

Assume as before that we have a principal connection on M, with connection form ω (a $\mathfrak{g}$-valued 1-form on M). Then we can take $\zeta(t)$ to be a horizontal lift of $\check{z}(t)$. We can now derive a differential equation for $g(t)$. First, differentiate the equation for $z(t)$:

$$\dot{z}(t) = \psi^M_{g(t)*}\left(\dot{\zeta}(t) + \widetilde{\vartheta(\dot{g}(t))}|_{\zeta(t)}\right),$$

where ϑ is the Maurer-Cartan form of G (i.e. $g^{-1}\dot{g}$ for a matrix group). We want $z(t)$ to be an integral curve of Z, so

$$\dot{z}(t) = Z_{z(t)} = \psi^M_{g(t)*} Z_{\zeta(t)}$$

by invariance. Thus

$$Z_{\zeta(t)} = \dot{\zeta}(t) + \widetilde{\vartheta(\dot{g}(t))}|_{\zeta(t)}.$$

This formula expresses $Z_{\zeta(t)}$ in terms of its horizontal and vertical components. We pick out the vertical component, or in other words apply ω:

$$\vartheta(\dot{g}(t)) = \omega_{\zeta(t)}(Z).$$

The right-hand side is a curve in $\mathfrak{g}$, so this is an equation in $\mathfrak{g}$, and it has a unique solution for $g(t)$ with $g(0) = \mathrm{id}$. (This is evident for a matrix group, for which the equation is $\dot{g} = g\omega_\zeta(Z)$.)

Then $z(t) = \psi^M_{g(t)}\zeta(t)$ is an integral curve of Z. It is the integral curve through $\zeta(0)$: to find the integral curve over $\check{z}$ through some other point in the fibre over $\check{z}(0)$, say $\psi^M_g\zeta(0)$, we merely have to left translate $z(t)$, that is, take $\psi^M_g z(t)$.

3 The second-order case

A second-order dynamical system determines and is determined by a vector field Γ on a tangent bundle TM, which has the form

$$\Gamma = u^\alpha \frac{\partial}{\partial x^\alpha} + \Gamma^i(x,u)\frac{\partial}{\partial u^\alpha}$$

when expressed in terms of natural coordinates (x^α, u^α); such a vector field is called a second-order differential equation field.

In order to consider symmetries of a second-order differential equation field we must extend the group action from M to TM. Suppose G acts on M as before; then the induced action of G on TM is given by $\psi^{TM}_g(x,u) = (\psi^M_g x, \psi^M_{g*}u)$. (Transformations of TM of this form are sometimes called point transformations.) The fundamental vector fields of the induced action are the complete lifts of the fundamental vector fields of the action on M, which we denote by $\tilde{\xi}^{\mathrm{C}}$. Moreover, TM is a principal G-bundle, and we denote by $\pi^{TM} : TM \to TM/G$ the projection (which is not to be confused with the projection $TM \to M$, which we denote by τ.)

We assume now that the second-order differential equation field Γ is invariant under the induced action of G:

$$[\tilde{\xi}^{\mathrm{C}}, \Gamma] = 0 \quad \text{for all } \xi \in \mathfrak{g}.$$

3.1 Reduction

We will make extensive use of the complete and vertical lifts of vector fields on M to TM: we denote the vertical lift of a vector field X on M by X^{V} (and its complete lift by X^{C} as above). We recall the following formulae for the brackets of such lifts:

$$[X^{\mathrm{C}}, Y^{\mathrm{C}}] = [X,Y]^{\mathrm{C}}, \quad [X^{\mathrm{V}}, Y^{\mathrm{C}}] = [X,Y]^{\mathrm{V}}, \quad [X^{\mathrm{V}}, Y^{\mathrm{V}}] = 0.$$

From these formulae it is clear that the complete and vertical lifts of a G-invariant vector field on M are both invariant under the induced action of G on TM. So if we take an invariant local basis $\{E_a, X_i\}$ on M as before, then $\{E^{\mathrm{C}}_a, X^{\mathrm{C}}_i, E^{\mathrm{V}}_a, X^{\mathrm{V}}_i\}$ is an invariant local basis of vector fields on TM.

We now introduce new fibre coordinates with respect to τ, adapted to the invariant basis, which we call quasi-velocities. For any vector field basis $\{Z_\alpha\}$ on M we denote by v^α the components of $u \in T_xM$ with respect to $\{Z_\alpha|_x\}$: so $u = v^\alpha Z_\alpha|_x$. Considered as functions on TM the v^α are fibre coordinates; these are the quasi-velocities corresponding to the basis $\{Z_\alpha\}$. Alternatively, let $\{\theta^\alpha\}$ be the 1-form basis dual to $\{Z_\alpha\}$; each θ^α defines a fibre-linear function on TM, $\hat{\theta}^\alpha$; then $v^\alpha = \hat{\theta}^\alpha$. We denote by (v^a, v^i) the quasi-velocities corresponding to $\{E_a, X_i\}$.

We need expressions for the derivatives of the quasi-velocities with respect to the members of the invariant basis $\{E_a^{\mathrm{C}}, X_i^{\mathrm{C}}, E_a^{\mathrm{V}}, X_i^{\mathrm{V}}\}$. To find them, the following two fomulae are indispensible:

$$Z^{\mathrm{C}}(\hat{\theta}) = \widehat{\mathcal{L}_Z\theta}, \quad Z^{\mathrm{V}}(\hat{\theta}) = \tau^*\theta(Z).$$

For example, we have $\tilde{e}_a^{\mathrm{C}}(v^i) = \widehat{\mathcal{L}_{\tilde{e}_a}\theta^i} = 0$ (since the basis dual to an invariant basis is also invariant).

We also need expressions for the pairwise brackets of $\{E_a, X_i\}$: we have $[E_a, E_b] = C_{ab}^c E_c$, and we set

$$[X_i, X_j] = K_{ij}^a E_a, \qquad [X_i, E_a] = X_i(A_a^b)\tilde{e}_b = \Upsilon_{ia}^b E_b.$$

It is worth noting that since the vector fields X_i and E_a are G-invariant, so are their brackets, and so are the coefficients K_{ij}^a and Υ_{ia}^b.

Let (x^i) be coordinates on M/G, as before. Then we find that

$$\begin{array}{lll}
\tilde{e}_a^{\mathrm{C}}(x^i) = 0, & \tilde{e}_a^{\mathrm{C}}(v^i) = 0, & \tilde{e}_a^{\mathrm{C}}(v^b) = 0, \\
E_a^{\mathrm{C}}(x^i) = 0, & E_a^{\mathrm{C}}(v^i) = 0, & E_a^{\mathrm{C}}(v^b) = \Upsilon_{ia}^b v^i + C_{ac}^b v^c, \\
E_a^{\mathrm{V}}(x^i) = 0, & E_a^{\mathrm{V}}(v^i) = 0, & E_a^{\mathrm{V}}(v^b) = \delta_a^b, \\
X_i^{\mathrm{C}}(x^j) = \delta_i^j, & X_i^{\mathrm{C}}(v^j) = 0, & X_i^{\mathrm{C}}(v^a) = -K_{ij}^a v^j - \Upsilon_{ib}^a v^b, \\
X_i^{\mathrm{V}}(x^j) = 0, & X_i^{\mathrm{V}}(v^j) = \delta_i^j, & X_i^{\mathrm{V}}(v^a) = 0.
\end{array}$$

From the first line, (x^i, v^i, v^a) define coordinates on TM/G. The invariant vector fields of the basis project onto TM/G, and we can read off the coordinate expressions for their projections from the formulae above:

$$\begin{aligned}
\pi_*^{TM} E_a^{\mathrm{C}} &= \left(\Upsilon_{ia}^b v^i + C_{ac}^b v^c\right)\frac{\partial}{\partial v^b}, \quad \pi_*^{TM} E_a^{\mathrm{V}} = \frac{\partial}{\partial v^a}, \\
\pi_*^{TM} X_i^{\mathrm{C}} &= \frac{\partial}{\partial x^i} - \left(K_{ij}^a v^j + \Upsilon_{ib}^a v^b\right)\frac{\partial}{\partial v^b}, \quad \pi_*^{TM} X_i^{\mathrm{V}} = \frac{\partial}{\partial v^i}.
\end{aligned}$$

Since Γ is a second-order differential equation field,

$$\Gamma = v^a E_a^{\mathrm{C}} + v^i X_i^{\mathrm{C}} + \Gamma^a E_a^{\mathrm{V}} + \Gamma^i X_i^{\mathrm{V}}.$$

Each term is invariant, so Γ^a and Γ^i define functions on TM/G. We have

$$\begin{aligned}
\pi_*^{TM}\Gamma = \check{\Gamma} &= v^a(\Upsilon_{ia}^b v^i + C_{ac}^b v^c)\frac{\partial}{\partial v^b} + v^i\frac{\partial}{\partial x^i} \\
&\quad - v^i\left(K_{ij}^a v^j + \Upsilon_{ib}^a v^b\right)\frac{\partial}{\partial v^b} + \Gamma^a\frac{\partial}{\partial v^a} + \Gamma^i\frac{\partial}{\partial v^i} \\
&= v^i\frac{\partial}{\partial x^i} + \Gamma^i\frac{\partial}{\partial v^i} + \Gamma^a\frac{\partial}{\partial v^a}.
\end{aligned}$$

The reduced equations are $\dot{x}^i = v^i$, $\dot{v}^i = \Gamma^i(x^j, v^j, v^b)$, $\dot{v}^a = \Gamma^a(x^j, v^j, v^b)$, or

$$\ddot{x}^i = \Gamma^i(x^j, \dot{x}^j, v^b), \quad \dot{v}^a = \Gamma^a(x^j, \dot{x}^j, v^b);$$

they are of mixed first- and second-order type.

So far as we are aware, the study of the reduction of general second-order dynamical systems with symmetry by methods similar to ours has been attempted by other authors only for single symmetries (that is, 1-dimensional symmetry groups), in [2]. For a more detailed account of our approach, see [4].

3.2 Reconstruction

In order to carry out reconstruction using the method described in Section 2 we need a principal connection on the bundle $\pi^{TM} : TM \to TM/G$. We have already assumed that we have at our disposal a principal connection on $\pi^M : M \to M/G$. There is in fact a simple method of lifting such a connection to one on π^{TM}. The initial connection is specified by its connection form ω. We show that the pull-back $\tau^*\omega$ of ω to TM by the tangent bundle projectionτ is the connection form of a principal connection on the principal G-bundle π^{TM}. Clearly, $\tau^*\omega$ is a $\mathfrak{g}$-valued 1-form on TM. The action of G on TM is τ-related to the action on M. Likewise, for any $\xi \in \mathfrak{g}$ the fundamental vector field $\tilde{\xi}^{\mathrm{C}}$ corresponding to the action on TM is τ-related to $\tilde{\xi}$, the fundamental vector field corresponding to the action on M. Thus

$$\tau^*\omega(\tilde{\xi}^{\mathrm{C}}) = \omega(\tau_*\tilde{\xi}^{\mathrm{C}}) = \omega(\tilde{\xi}) = \xi,$$

while

$$\psi_g^{TM*}\tau^*\omega = \tau^*\psi_g^{M*}\omega = \mathrm{ad}(g^{-1})\tau^*\omega,$$

as required. The connection defined by $\tau^*\omega$ is called the vertical lift of the original connection, and its connection 1-form is denoted by ω^{V}.

When we use ω^{V} in the reconstruction process, the right-hand side of the reconstruction equation is $\omega^{\mathrm{V}}(\Gamma)$. The special natures of Γ (that it is a second-order differential equation field) and ω^{V} (that it is a vertical lift connection) now come into play. For at any point $u \in TM$, $\omega_u^{\mathrm{V}}(\Gamma) = \omega_{\tau(u)}(\tau_*\Gamma_u) = \omega_{\tau(u)}(u)$; that is to say, $\omega_u^{\mathrm{V}}(\Gamma)$ is just the vertical part of u (considered as an element of $\mathfrak{g}$), and in particular is the same for all G-invariant second-order differential equation fields on TM.

4 Lagrangian systems

We now suppose that we are dealing with a second-order dynamical system Γ defined by a regular Lagrangian L on TM. Thus Γ is the Euler-Lagrange field of L, and satisfies the Euler-Lagrange equations, which in terms of coordinates (x^α, u^α) can be written

$$\Gamma\left(\frac{\partial L}{\partial u^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0.$$

We assume that L is regular, which is to say that its Hessian with respect to the fibre coordinates, the symmetric matrix with entries

$$\frac{\partial^2 L}{\partial u^\alpha \partial u^\beta},$$

is non-singular. Then Γ is uniquely determined by the Euler-Lagrange equations (and the fact that it is a second-order differential equation field). In order to use the methods described in the previous sections we have to express the Euler-Lagrange equations in terms of a vector field basis on M which is not of coordinate type. With respect to the basis $\{Z_\alpha\}$ they take the form

$$\Gamma(Z_\alpha^{\mathrm{V}}(L)) - Z_\alpha^{\mathrm{C}}(L) = 0.$$

Assume that the regular Lagrangian L is G-invariant: $\tilde{\xi}^{\mathrm{C}}(L) = 0$. Then the Euler-Lagrange field Γ is also G-invariant, as one would expect. We wish to carry out a reduction, and to express the reduced equations in terms of an appropriate reduced version of the Lagrangian. As we mentioned in the Introduction, there are in fact two different ways of proceeding.

In the first, which is called Lagrange-Poincaré reduction, we work with the invariant basis $\{E_a, X_i\}$, as before. The Euler-Lagrange equations become

$$\Gamma(X_i^{\mathrm{V}}(L)) - X_i^{\mathrm{C}}(L) = 0, \qquad \Gamma(E_a^{\mathrm{V}}(L)) - E_a^{\mathrm{C}}(L) = 0,$$

and the reduced equations determine a vector field $\check{\Gamma}$ on TM/G.

The second approach could be characterized as making more direct use of the particular properties of the Euler-Lagrange formalism. This time we use a mixed basis $\{\tilde{e}_a, X_i\}$ (mixed in the sense that only part of it is invariant); since $\tilde{e}_a^{\mathrm{C}}(L) = 0$, the Euler-Lagrange equations are

$$\Gamma(X_i^{\mathrm{V}}(L)) - X_i^{\mathrm{C}}(L) = 0, \qquad \Gamma(\tilde{e}_a^{\mathrm{V}}(L)) = 0.$$

Thus the momentum, whose components are $\tilde{e}_a^{\mathrm{V}}(L)$, is conserved. The first step in the reduction process in this case (if 'reduction' is the right word) just consists in restriction to a level set of momentum. The process as a whole is called Routh's procedure; it generalizes the elimination of the momentum conjugate to a cyclic coordinate which was Routh's original version of the procedure [11].

4.1 Lagrange-Poincaré reduction

Since it is invariant, L defines a function $\check{L}$ on TM/G. The Euler-Lagrange equations reduce directly to

$$\check{\Gamma}(\check{X}_i^{\mathrm{V}}(\check{L})) - \check{X}_i^{\mathrm{C}}(\check{L}) = 0, \qquad \check{\Gamma}(\check{E}_a^{\mathrm{V}}(\check{L})) - \check{E}_a^{\mathrm{C}}(\check{L}) = 0$$

where $\check{X}_i^{\rm C} = \pi_*^{TM} X_i^{\rm C}$ etc. Using the formulae from the previous section we obtain

$$\check{\Gamma}\left(\frac{\partial \check{L}}{\partial v^i}\right) - \frac{\partial \check{L}}{\partial x^i} = -(K_{ij}^a v^j + \Upsilon_{ib}^a v^b)\frac{\partial \check{L}}{\partial v^a}$$
$$\check{\Gamma}\left(\frac{\partial \check{L}}{\partial v^a}\right) = (\Upsilon_{ia}^b v^i + C_{ac}^b v^c)\frac{\partial \check{L}}{\partial v^b};$$

and as before,

$$\check{\Gamma} = v^i \frac{\partial}{\partial x^i} + \Gamma^i \frac{\partial}{\partial v^i} + \Gamma^a \frac{\partial}{\partial v^a}.$$

These are the Lagrange-Poincaré equations [3], though they are usually written with d/dt in place of $\check{\Gamma}$; see also [9].

4.2 Routh's procedure

We set $p_a = \tilde{e}_a^{\rm C}(L)$. Considered as a vector, (p_a) takes its values in $\mathfrak{g}^*$, the dual of the Lie algebra: it is the (generalized) momentum. Since $\Gamma(p_a) = 0$, the vector field Γ is tangent to the level sets of momentum; we will concentrate on its restriction to one level set, say $N_\mu : p_a = \mu_a$.

We work now with the mixed basis $\{\tilde{e}_a, X_i\}$. The quasi-velocities are $(\tilde{v}^a, v^i)$, where $\tilde{v}^a = A_b^a v^b$; the $\tilde{v}^a$ are not invariant. The pairwise brackets of elements of the basis are

$$[\tilde{e}_a, X_i] = 0, \qquad [X_i, X_j] = R_{ij}^a \tilde{e}_a, \quad R_{ij}^a = A_a^b K_{ij}^b.$$

(The expression for $[X_i, X_j]$ identifies the R_{ij}^a as the components of curvature of the connection on π^M, regarded as a $\mathfrak{g}$-valued 2-form on M/G.)

The derivatives of the quasi-velocities are

$$\begin{array}{ll} X_i^{\rm C}(v^j) = 0, & X_i^{\rm C}(\tilde{v}^a) = -R_{ij}^a v^j, \\ X_i^{\rm V}(v^j) = \delta_i^j, & X_i^{\rm V}(\tilde{v}^a) = 0, \\ \tilde{e}_a^{\rm C}(v^i) = 0, & \tilde{e}_a^{\rm C}(\tilde{v}^b) = C_{ac}^b \tilde{v}^c, \\ \tilde{e}_a^{\rm V}(v^i) = 0, & \tilde{e}_a^{\rm V}(\tilde{v}^b) = \delta_a^b. \end{array}$$

Set $g_{ab} = \tilde{e}_a^{\rm V}(p_b) = \tilde{e}_a^{\rm V}(\tilde{e}_b^{\rm V}(L))$. Since vertical lifts commute, $g_{ba} = g_{ab}$. We assume that the symmetric matrix (g_{ab}) is everywhere non-singular. Then $\tilde{e}_a^{\rm V}$ is transverse to N_μ, and in principle we can solve the equations $p_a = \mu_a$ for $\tilde{v}^a$. Thus restricting to a level set of momentum is a form of reduction, in the sense that by doing so we reduce the number of variables, and presumably thereby the difficulty of the problem. It is however a somewhat different form of reduction from those discussed so far: reduction by restriction rather than projection.

The g_{ab} are in fact components of the Hessian of L. The Hessian of L can be defined in a coordinate-independent way as the symmetric covariant 2-tensor g along τ given by

$$g(u, v) = u^{\rm V}(v^{\rm V}(L)),$$

for any vectors u, v at the same point of M. We have $g_{ab} = g(\tilde{e}_a, \tilde{e}_b)$.

We may use g_{ab} to define vector fields tangent to N_μ. Denote by (g^{ab}) the matrix inverse to (g_{ab}). For any vector field Y on TM, the vector field $Y - g^{ab}Y(p_a)\tilde{e}_b^{\mathrm{V}}$ annihilates p_a and is therefore tangent to each level set of momentum. In particular, set

$$\begin{aligned} \bar{X}_i^{\mathrm{C}} &= X_i^{\mathrm{C}} - g^{ab}X_i^{\mathrm{C}}(p_b)\tilde{e}_a^{\mathrm{V}} = X_i^{\mathrm{C}} - P_i^a\tilde{e}_a^{\mathrm{V}} \\ \bar{X}_i^{\mathrm{V}} &= X_i^{\mathrm{V}} - g^{ab}X_i^{\mathrm{V}}(p_b)\tilde{e}_a^{\mathrm{V}} = X_i^{\mathrm{V}} - Q_i^a\tilde{e}_a^{\mathrm{V}}. \end{aligned}$$

Then $\bar{X}_i^{\mathrm{C}}$, $\bar{X}_i^{\mathrm{V}}$ are tangent to N_μ. We have

$$\bar{X}_i^{\mathrm{C}}(\tilde{v}^a) = -R_{ij}^a v^j - P_i^a, \quad \bar{X}_i^{\mathrm{V}}(\tilde{v}^a) = -Q_i^a.$$

We now derive some Euler-Lagrange-like equations which determine the restriction of Γ to the level set of momentum N_μ. These equations involve, not the Lagrangian itself, but a modification of it called the Routhian, which is given by $\mathcal{R} = L - p_a\tilde{v}^a$. Now

$$\begin{aligned} X_i^{\mathrm{C}}(L) &= \bar{X}_i^{\mathrm{C}}(L) + P_i^a p_a = \bar{X}_i^{\mathrm{C}}(L) - p_a(\bar{X}_i^{\mathrm{C}}(\tilde{v}^a) + R_{ij}^a v^j) \\ &= \bar{X}_i^{\mathrm{C}}(\mathcal{R}) - p_a R_{ij}^a v^j; \\ X_i^{\mathrm{V}}(L) &= \bar{X}_i^{\mathrm{V}}(L) + Q_i^a p_a = \bar{X}_i^{\mathrm{V}}(L) - p_a\bar{X}_i^{\mathrm{V}}(\tilde{v}^a) \\ &= \bar{X}_i^{\mathrm{V}}(\mathcal{R}). \end{aligned}$$

But $\Gamma(X_i^{\mathrm{V}}(L)) - X_i^{\mathrm{C}}(L) = 0$, and Γ is tangent to N_μ. So if $\mathcal{R}^\mu$ is the restriction of the Routhian to N_μ we have

$$\Gamma(\bar{X}_i^{\mathrm{V}}(\mathcal{R}^\mu)) - \bar{X}_i^{\mathrm{C}}(\mathcal{R}^\mu) = -\mu_a R_{ij}^a v^j.$$

These are the required equations; we call them the generalized Routh equations.

The generalized Routh equations may appear to be straightforwardly second-order differential equations, unlike the other reduced equations for second-order differential equation fields, which are mixed first- and second-order equations. This appearance is deceptive. In the first place, the generalized Routh equations (when expressed explicitly as differential equations) are equations on N_μ, not TM/G as is the case for the other reduced equations. Now N_μ can be locally identified with $M \times_{M/G} T(M/G)$. For local coordinates on N_μ we may take (x^i, θ^a, v^i), where (θ^a) are fibre coordinates on M, so that (x^i, θ^a) are coordinates on M and (x^i, v^i) coordinates on $T(M/G)$. The quasi-coordinates (v^i, v^a) on TM are linear combinations of $\dot{x}^i$ and $\dot{\theta}^a$, and in fact $v^i = \dot{x}^i$. So we can express v^a in terms of $\dot{x}^i$ and $\dot{\theta}^a$. On TM the resulting expression is an identity; but on restricting to N_μ, the equations $p_a = \mu_a$, when expressed in this way in terms of $\dot{x}^i$ and $\dot{\theta}^a$, become additional implicit first-order differential equations, which we may regard as equations for the $\dot{\theta}^a$ (since the equations $v^i = \dot{x}^i$ are already subsumed in the representation of the generalized Routh equations as second-order equations).

The level set N_μ is not in general G-invariant: $\tilde{\xi}^{\mathrm{C}}$ is not in general tangent to N_μ. In fact

$$\tilde{\xi}^{\mathrm{C}}(p_a) = \xi^b\tilde{e}_b^{\mathrm{C}}(p_a) = \xi^b[\tilde{e}_b^{\mathrm{C}}, \tilde{e}_a^{\mathrm{V}}](L) = \xi^b C_{ab}^c p_c.$$

Thus $\tilde{\xi}^C$ will be tangent to N_μ if and only if $\xi^b C^c_{ab}\mu_c = 0$. The set of $\xi \in \mathfrak{g}$ which satisfy this condition forms a subalgebra $\mathfrak{g}_\mu$ of $\mathfrak{g}$. It is in fact the algebra of G_μ, the isotropy group of $\mu \in \mathfrak{g}^*$ under the coadjoint action of G in $\mathfrak{g}^*$. Now $\Gamma|_{N_\mu}$, $\mathcal{R}^\mu$, and the generalized Routh equations, are all invariant by G_μ. We can therefore carry out a further reduction, by G_μ, in the manner described earlier, to obtain a reduced system on N_μ/G_μ. The resulting reduced equations have been called the Lagrange-Routh equations [8]. We do not give the derivation here, but refer the reader to [5], as well as [8], for the details. In fact [8] contains an extensive discussion of the background to Routh's procedure and its modern generalization. The methods used in this paper are quite different from ours, however, and it deals only with so-called simple mechanical systems. For a more detailed account of all aspects of our approach see [5].

4.3 Reconstruction

The same method of reconstruction as was described for second-order differential equations in the previous section, namely using the vertical lift connection, can be used for Lagrange-Poincaré reduction. For Routh's procedure it is necessary to carry out reconstruction only for the final stage of reduction by G_μ: an integral curve of the restriction of Γ to N_μ is, after all, an integral curve of Γ. It is not so obvious how to adapt the vertical lift connection to this situation, though it can be done. We will now describe an alternative way of constructing a connection, which is based more closely on the fact that we are dealing with a Lagrangian system, and applies more-or-less directly to both reconstruction problems.

We consider first the case of a simple mechanical system, which is one in which the Lagrangian takes the simple form $L = T - V$ where T is a kinetic energy function derived from a Riemannian metric g, and V a function on M defining the potential energy. The symmetry group G consists of all isometries of the metric g leaving V invariant. Then the distribution on M consisting of all vectors orthogonal (with respect to g) to the fibres of $\pi^M : M \to M/G$ is G-invariant, and defines a principal connection (of which it is the horizontal distribution). This is the so-called mechanical connection on π^M. It can be lifted to a principal connection on $\pi^{TM} : TM \to TM/G$, as before. For the vertical lift of the mechanical connection, $v \in T_uTM$ is horizontal just when $g_{\tau(u)}(\tau_* v, \tilde{\xi}) = 0$ for all $\xi \in \mathfrak{g}$. This connection can be used for reconstruction in the Lagrange-Poincaré case. For Routh's procedure we define the required connection by saying that $v \in T_uN_\mu$ is horizontal just when $g_{\tau(u)}(\tau_* v, \tilde{\xi}) = 0$ for all $\xi \in \mathfrak{g}_\mu$.

In general, we can use the Hessian of L in place of the Riemannian metric. This doesn't give a connection on M, but does give connections on $TM \to TM/G$ and $N_\mu \to N_\mu/G_\mu$. Indeed, since we have (wittingly) used the same symbol, g, for both the metric in the case of a simple mechanical system and the Hessian in general, the definitions are almost identical: the only difference is that in general g is not projectable. For the Lagrange-Poincaré case, we say that $v \in T_uTM$ is horizontal just when $g_u(\tau_* v, \tilde{\xi}) = 0$ for all $\xi \in \mathfrak{g}$. For the Routhian case, we say that $v \in T_uN_\mu$ is horizontal just when $g_u(\tau_* v, \tilde{\xi}) = 0$

for all $\xi \in \mathfrak{g}_\mu$. Both of these specifications define principal connections, which we call collectively the generalized mechanical connection. A fuller account of this construction can be found in [9].

5 An example: Wong's equations

In this final section we determine the reduced equations for an interesting second-order differential equation field, namely the geodesic field for a Riemannian manifold on which a group G acts freely and properly to the left as isometries. We make the further stipulation that the vertical part of the metric (that is, its restriction to the fibres of $\pi^M : M \to M/G$) comes from a bi-invariant metric on G. The reduced equations in such a case are known as Wong's equations [3, 10]. We will derive the reduced equations by each of the three methods discussed above.

We will denote the metric by g. The fact that the symmetry group acts as isometries means that the fundamental vector fields $\tilde{\xi}$ are Killing fields: $\mathcal{L}_{\tilde{\xi}} g = 0$. It follows that the components of g with respect to the members of an invariant basis $\{E_a, X_i\}$ are themselves invariant. We have a small notational problem to deal with here: we will need to distinguish between the components of g with respect to the fundamental vector fields $\tilde{e}_a$ and those with respect to the E_a. We will set $g(\tilde{e}_a, \tilde{e}_b) = g_{ab}$, as before. For $g(E_a, E_b)$ we will write h_{ab}. We set $g(X_i, X_j) = g_{ij}$. We will use the mechanical connection, which means that $g(\tilde{e}_a, X_i) = 0$. Since both h_{ab} and g_{ij} are G-invariant functions, they pass to the quotient; in particular, the g_{ij} are the components with respect to the coordinate fields of a metric on M/G, the reduced metric.

The further assumption about the vertical part of the metric has the following implications. It means in the first place that $\mathcal{L}_{E_c} g(E_a, E_b) = 0$ (as well as $\mathcal{L}_{\tilde{e}_c} g(E_a, E_b) = 0$), and secondly that the h_{ab} must be independent of the coordinates x^i on M/G, which is to say that they must be constants. From the first condition, taking into account the bracket relations $[E_a, E_b] = C^c_{ab} E_c$, we easily find that the h_{ab} must satisfy $h_{ad} C^d_{bc} + h_{bd} C^d_{ac} = 0$. It is implicit in our choice of an invariant basis that we are working in a local trivialization of $M \to M/G$. Then $\tilde{e}_a$, E_a and A^b_a are all objects defined on the G factor, and so are independent of the x^i. We may write

$$X_i = \frac{\partial}{\partial x^i} - \gamma^a_i E_a$$

for some coefficients γ^a_i which are clearly G-invariant; moreover

$$[X_i, E_a] = \gamma^c_i C^b_{ac} E_b = \Upsilon^b_{ia} E_b.$$

Thus $\Upsilon^b_{ia} = \gamma^c_i C^b_{ac}$, and therefore $h_{ac} \Upsilon^c_{ib} + h_{bc} \Upsilon^c_{ib} = 0$.

The second-order differential equation field Γ of interest is the geodesic field of the Riemannian metric g. To find the reduced equations by the direct method we have to express Γ in terms of the invariant basis, and for this purpose we need the connection coefficients of the Levi-Civita connection in terms of this basis. Using the data above in the standard Koszul formulae for the Levi-Civita connection coefficients of g with respect to the basis $\{E_a, X_i\}$ we find that the only non-zero coefficients are Γ^i_{jk}, which are just the connection coefficients of the Levi-Civita connection of the reduced metric g_{ij}, and

$$\Gamma^a_{bc} = \tfrac{1}{2}C^a_{bc}, \quad \Gamma^a_{jb} = \Upsilon^a_{jb}, \quad \Gamma^a_{jk} = \tfrac{1}{2}K^a_{jk}, \quad \Gamma^i_{jb} = -\tfrac{1}{2}g^{ik}h_{bc}K^c_{jk} = \Gamma^i_{bj}.$$

It follows that

$$\begin{aligned}\Gamma &= v^i X_i^{\mathrm{C}} + v^a E_a^{\mathrm{C}} \\ &\quad - \left(\Gamma^i_{jk}v^j v^k + (\Gamma^i_{jb} + \Gamma^i_{bj})v^j v^b + \Gamma^i_{bc}v^b v^c\right) X_i^{\mathrm{V}} \\ &\quad - \left(\Gamma^a_{jk}v^j v^k + (\Gamma^a_{jb} + \Gamma^a_{bj})v^j v^b + \Gamma^a_{bc}v^b v^c\right) E_a^{\mathrm{V}} \\ &= v^i X_i^{\mathrm{C}} + v^a E_a^{\mathrm{C}} - \left(\Gamma^i_{jk}v^j v^k - g^{ik}h_{bc}K^c_{jk}v^j v^b\right) X_i^{\mathrm{V}} - \Upsilon^a_{jb}v^j v^b E_a^{\mathrm{V}}.\end{aligned}$$

The reduced vector field on TM/G is therefore

$$\check{\Gamma} = v^i \frac{\partial}{\partial x^i} - \left(\Gamma^i_{jk}v^j v^k - g^{ik}h_{bc}K^c_{jk}v^j v^b\right) \frac{\partial}{\partial v^i} - \Upsilon^a_{jb}v^j v^b \frac{\partial}{\partial v^a}$$

and the reduced equations are

$$\begin{aligned}\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k &= g^{ik}h_{bc}K^c_{jk}\dot{x}^j v^b \\ \dot{v}^a + \Upsilon^a_{jb}\dot{x}^j v^b &= 0.\end{aligned}$$

These are Wong's equations. (The form of the second of these equations suggests that the Υ^a_{jb} should be regarded as connection coefficients. It is indeed the case that they are: the connection in question is that induced by ω on the adjoint bundle, that is, the vector bundle associated with the principal G-bundle π^M by the adjoint action of G on $\mathfrak{g}$.)

The geodesic equations may also be derived from the Lagrangian

$$L = \tfrac{1}{2}g_{\alpha\beta}u^\alpha u^\beta = \tfrac{1}{2}g_{ij}v^i v^j + \tfrac{1}{2}h_{ab}v^a v^b.$$

It is of course G-invariant. We may therefore apply Lagrange-Poincaré reduction, which gives the reduced equations

$$\begin{aligned}\frac{d}{dt}(g_{ij}v^j) - \tfrac{1}{2}\frac{\partial g_{jk}}{\partial x^i}v^j v^k &= -(K^a_{ij}v^j + \Upsilon^a_{ib}v^b)h_{ac}v^c \\ \frac{d}{dt}(h_{ab}v^b) &= (\Upsilon^b_{ia}v^i + C^b_{ac}v^c)h_{bd}v^d.\end{aligned}$$

Now $\Upsilon^a_{ib}h_{ac}$ is skew-symmetric in b and c, and $C^b_{ac}h_{bd}$ is skew-symmetric in c and d, so the final terms in each equation vanish identically, and we may write the equations in the form

$$\begin{aligned}g_{ij}\left(\ddot{x}^j + \Gamma^j_{kl}\dot{x}^k\dot{x}^l\right) &= -h_{bc}K^c_{ij}\dot{x}^j v^b \\ h_{ab}\left(\dot{v}^b + \Upsilon^b_{ic}\dot{x}^i v^c\right) &= 0,\end{aligned}$$

using the skew-symmetry of $\Upsilon^c_{ib}h_{ac}$ again in the second equation. These equations are equivalent to the ones obtained by the direct method (K^c_{ij} is of course skew-symmetric in its lower indices).

In order to use Routh's procedure we must rewrite the Lagrangian in terms of the quasi-velocities associated with the mixed basis: it is given by

$$L = \tfrac{1}{2}g_{ij}v^iv^j + \tfrac{1}{2}g_{ab}\tilde{v}^a\tilde{v}^b.$$

The momentum is given by $p_a = g_{ab}\tilde{v}^b$, and the Routhian by

$$\mathcal{R} = L - p_a\tilde{v}^a = \tfrac{1}{2}g_{ij}v^iv^j - \tfrac{1}{2}g^{ab}p_ap_b.$$

The next problem is to calculate $\bar{X}_i^{\mathrm{V}}(\mathcal{R})$ and $\bar{X}_i^{\mathrm{C}}(\mathcal{R})$. In fact, it is easy to see that $\bar{X}_i^{\mathrm{V}}(\mathcal{R}) = g_{ij}v^j$. The calculation of $\bar{X}_i^{\mathrm{C}}(\mathcal{R})$ reduces to the calculation of $X_i(g_{ij})$ and $X_i(g^{ab})$. The first is straightforward. For the second, we note that $g_{ab} = \bar{A}^c_a\bar{A}^d_bh_{cd}$, where $(\bar{A}^b_a)$ is the matrix inverse to (A^b_a); since the right-hand side is independent of the x^i, so is g_{ab}, and so equally is g^{ab}. It follows that

$$\bar{X}_i^{\mathrm{C}}(\mathcal{R}) = \tfrac{1}{2}\frac{\partial g_{jk}}{\partial x^i}v^jv^k - \tfrac{1}{2}\gamma^c_iE_c(g^{ab})p_ap_b.$$

Now $E_c(g^{ab}) = -A^d_c(g^{ae}C^b_{de} + g^{be}C^a_{de})$, from Killing's equations. Using the relation between g_{ab} and h_{ab}, and the fact that ad is a Lie algebra homomorphism, we find that

$$E_c(g^{ab}) = -A^a_dA^b_e(h^{df}C^e_{cf} + h^{ef}C^d_{cf}).$$

The expression in the brackets vanishes, as follows easily from the properties of h_{ab}. Thus the generalized Routh equation is

$$\frac{d}{dt}(g_{ij}v^j) - \tfrac{1}{2}\frac{\partial g_{jk}}{\partial x^i}v^jv^k = g_{ij}\left(\dot{v}^j + \Gamma^j_{kl}v^kv^l\right) = -\mu_aR^a_{ij}v^j.$$

But $R^a_{ij} = A^a_bK^b_{ij}$, and $\mu_a = g_{ab}\tilde{v}^b = \bar{A}^c_ah_{bc}v^b$, so $\mu_aR^a_{ij} = h_{bc}K^c_{ij}v^b$. The generalized Routh equation is therefore equivalent to

$$g_{ij}\left(\ddot{x}^j + \Gamma^j_{kl}\dot{x}^k\dot{x}^l\right) = -h_{bc}K^c_{ij}\dot{x}^jv^b$$

again. On the other hand, the constancy of μ_a gives

$$h_{bc}\frac{d}{dt}(\bar{A}^c_av^b) = 0.$$

If we are to understand this equation in the present context, we evidently need to calculate $\dot{A}^b_a$. Now

$$\dot{A}^b_a = v^iX_i(A^b_a) + \tilde{v}^c\tilde{e}_c(A^b_a) = v^i\Upsilon^c_{ia}A^b_c + \tilde{v}^cC^b_{cd}A^d_a.$$

It follows that

$$\begin{aligned} h_{bc}\frac{d}{dt}(\bar{A}^c_a) &= -h_{bc}\bar{A}^d_a\bar{A}^c_e\dot{A}^e_d = -h_{bc}\bar{A}^d_a\bar{A}^c_e(v^i\Upsilon^f_{id}A^e_f + \tilde{v}^fC^e_{fg}A^g_d) \\ &= -h_{bc}\bar{A}^d_a(v^i\Upsilon^c_{id} + v^eC^c_{ed}), \end{aligned}$$

where in the last step we have again used the fact that ad is a Lie algebra homomorphism. Now from the skew-symmetry properties of h_{ab} we obtain

$$h_{bc}\frac{d}{dt}(\bar{A}^c_a) = h_{cd}\bar{A}^d_a(v^i\Upsilon^c_{ib} + v^e C^c_{eb}),$$

and therefore

$$h_{bc}\frac{d}{dt}(\bar{A}^c_a v^b) = h_{cd}\bar{A}^d_a(\dot{v}^c + \Upsilon^c_{ib}v^i v^b).$$

The first-order part of Wong's equations is thus equivalent to the constancy of momentum.

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